THE EXPOSED POINTS OF THE SET OF INVARIANT MEANS

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Abstract. Let $G$ be a $\sigma$-compact infinite locally compact group, and let $LIM$ be the set of left invariant means on $L^\infty(G)$. We prove in this paper that if $G$ is amenable as a discrete group, then $LIM$ has no exposed points. We also give another proof of the Granirer theorem that the set $LIM(X, G)$ of $G$-invariant means on $L^\infty(X, \beta, p)$ has no exposed points, where $G$ is an amenable countable group acting ergodically as measure-preserving transformations on a nonatomic probability space $(X, \beta, p)$.

1. Introduction and Notations

Let $G$ be a locally compact group with a fixed left Haar measure $\lambda$. If $G$ is compact, we assume $\lambda(G) = 1$. Let $L^p(G)$ be the associated real Lebesgue spaces ($1 \leq p \leq \infty$). For $f \in L^\infty(G)$ and $x \in G$, the left translation of $f$ by $x$ is defined by $xf(y) = f(xy)$, $y \in G$. A mean on $L^\infty(G)$ is a positive functional on $L^\infty(G)$ with $m(1) = 1$. A left invariant mean is a mean with $m(xf) = m(f)$ for any $x \in G$ and $f \in L^\infty(G)$. The set of left invariant mean on $L^\infty(G)$ is denoted by $LIM$.

If $LIM \neq \phi$, we say that $G$ is amenable. Let $G_d$ be the same algebraic group as $G$ with a discrete topological structure. Then $G$ is amenable if $G_d$ is amenable. Properties of amenable groups and left invariant means can be found in Greenleaf [9], Paterson [10] and Pier [11].

When $G$ is amenable, $LIM$, as a $w^*$-compact convex subset of $L^\infty(G)^*$, is the $w^*$-closed convex hull of all its extreme points. It is natural to ask how many exposed points $LIM$ has. Granirer [4] studied intensively the existence of exposed points of $LIM$ for a countable amenable semigroup (also see Chou [1]). In particular, he proved by using very general theorems that $LIM$ has exposed points if and only if $G$ has finite left ideals for a countable amenable semigroup $G$ [4, Corollary 4.1]. Yang [15] proved that if $G$ is a finite amenable discrete group, then $LIM$ has no exposed points.

In this paper, we prove that $LIM$ has no exposed points for any $\sigma$-compact locally compact group which is amenable as a discrete group. The idea of the proof is to "split" a nonnegative function in $L^\infty(G)$ by a category argument,
the technique used by Rosenblatt [12]. We also adapt this technique to prove the Granirer theorem of [5] and [6] in a different way that the set \( \text{LIM}(X, G) \) of \( G \)-invariant means on \( L^\infty(X, \beta, p) \) has no exposed points, where \( G \) is an amenable countable group acting ergodically as measure-preserving transformations on a nonatomic probability space \( (X, \beta, p) \). He derives it using very general theorems. See Chou [2] and Rosenblatt [13] for details of the study of the set \( \text{LIM}(X, G) \).

The author would like to thank Professor E.E. Granirer for pointing out that Theorem 2 in this paper is a special case of his general theorems in [5] and [6] and for many valuable conversations.

2. Exposed points of \( \text{LIM} \)

In this section we will be concerned with \( \text{LIM} \) for a locally compact group and will prove our first main result. We need the following, probably known, proposition for which we were unable to find a reference.

**Proposition 1.** Let \( G \) be a nondiscrete locally compact group, and let \( K \) be a compact subset of \( G \). If \( f \in L^\infty(G) \) and \( \lambda\{t \in G : f(t) \neq 0\} \) is finite, then the function defined by

\[
F(x_1, x_2, \ldots, x_n) = \lambda\left\{ t \in G : \frac{1}{n} \sum_{i=1}^{n} x_i f(t) > a \right\}
\]

is lower semicontinuous on \( K^n \), where \( a \) is a constant.

**Proof.** First let us prove that \( \int_G |x f - f| \, dt \to 0 \) as \( x \to e \). If \( f = 1_E \), then \( \int_G |x f - f| \, dt = \lambda(x^{-1} E \Delta E) \to 0 \) as \( x \to e \) since the map \( x \to \lambda(x^{-1} E \cap E) \) is continuous from \( K \) to \( R \). For any \( f \) with \( \lambda\{t \in G : f(t) \neq 0\} \) finite and an \( \epsilon > 0 \), choose a simple function \( \varphi = \sum_{p=1}^{q} a_p 1_{E_p} \) such that \( \|f - \varphi\|_1 < \epsilon \). There exists an open neighborhood \( U \) of \( e \) such that \( \sum_{p=1}^{q} |a_p| \int_G 1_{E_p} - 1_{E_p} \, dt < \epsilon \) for any \( x \in U \). Hence, for every \( x \in U \),

\[
\int_G |x f - f| \, dt \leq \int_G |x f - x \varphi| \, dt + \int_G |x \varphi - \varphi| \, dt + \int_G |\varphi - f| \, dt \leq 3\epsilon.
\]

Let \( u^\alpha = (u_1^\alpha, u_2^\alpha, \ldots, u_n^\alpha) \) be a net and \( u = (u_1, u_2, \ldots, u_n) \in K^n \) with \( u^\alpha \to u \) in \( K^n \). If there is an \( \epsilon_0 > 0 \) such that \( F(u^\alpha) < F(u) - \epsilon_0 \), then we can find a \( \delta > 0 \) such that \( F(u^\alpha) < \lambda\{t \in G : \frac{1}{n} \sum_{i=1}^{n} u_i f(t) > a + \delta\} - \epsilon_0 \) for every \( \alpha \). Thus

\[
\int_G \left| \frac{1}{n} \sum_{i=1}^{n} u_i^\alpha f - \frac{1}{n} \sum_{i=1}^{n} u_i f \right| \, dt \leq \frac{1}{n} \sum_{i=1}^{n} \int_G |u_i^\alpha f - u_i f| \, dt \to 0
\]

when \( u^\alpha \to u \) in \( K^n \). On the other hand,

\[
\int_G \left| \frac{1}{n} \sum_{i=1}^{n} u_i^\alpha f - \frac{1}{n} \sum_{i=1}^{n} u_i f \right| \, dt \geq \int_{B_\alpha} \left| \frac{1}{n} \sum_{i=1}^{n} u_i^\alpha f - \frac{1}{n} \sum_{i=1}^{n} u_i f \right| \, dt \geq \delta \lambda(B_\alpha) \geq \delta \epsilon_0,
\]

where

\[
B_\alpha = \left\{ t \in G : \frac{1}{n} \sum_{i=1}^{n} u_i f(t) > a + \delta \right\} \sim \left\{ t \in G : \frac{1}{n} \sum_{i=1}^{n} u_i^\alpha f(t) > a \right\}
\]
with \( \lambda(B_\alpha) \geq \lambda\{t \in G : \frac{1}{n} \sum_{i=1}^{n} w_i f(t) > a + \delta \} - F(u^\alpha) > \epsilon_0 \). This is a contradiction. Therefore the function \( F \) from \( K^n \) to \( R \) is lower semicontinuous. \( \square \)

To prove our result, we will need the following two lemmas.

**Lemma 2.** Let \( G \) be a locally compact group and let \( f \in L^\infty(G) \) be a function with \( 0 \leq f \leq 1 \) and \( \lambda\{x \in G : f(x) \neq 0\} < \infty \). If \( f_k \) is a sequence of functions in \( L^\infty(G) \) with \( 0 \leq f_k \leq f \) (\( k = 0, 1, 2, \ldots \)), \( f_k \to f_0 \) in \( \| \cdot \|_1 \)-norm, then \( \inf_{(x_1, x_2, \ldots, x_n) \in K^n} F_k(x_1, x_2, \ldots, x_n) = 0 \) for any \( k \geq 1 \) implies \( \inf_{(x_1, x_2, \ldots, x_n) \in K^n} F_0(x_1, x_2, \ldots, x_n) = 0 \), where \( K \) is a compact subset of \( G \) and \( F_h(x_1, x_2, \ldots, x_n) = \lambda\{t \in G : \frac{1}{n} \sum_{i=1}^{n} x_i h(t) > a\} \) for any \( h \in L^\infty(G) \).

**Proof.** Let \( \inf_{(x_1, x_2, \ldots, x_n) \in K^n} F_0(x_1, x_2, \ldots, x_n) = \epsilon_0 > 0 \). Then for any \( x = (x_1, x_2, \ldots, x_n) \in K^n \), there is an \( i_x \in N \) such that \( \lambda\{t \in G : \frac{1}{n} \sum_{i=1}^{n} x_i f_0(t) > a + \frac{1}{i_x} \} > \frac{\epsilon_0}{2} \). Since the map \( (y_1, y_2, \ldots, y_n) \mapsto \lambda\{t \in G : \frac{1}{n} \sum_{i=1}^{n} y_i f_0(t) > a + \frac{1}{i_0} \} \) from \( K^n \) to \( R \) is lower semicontinuous by Proposition 1, there exists an open neighborhood \( U_x \) of \( x \) in \( K^n \) such that \( \lambda\{t \in G : \frac{1}{n} \sum_{i=1}^{n} y_i f_0(t) > a + \frac{1}{i_0} \} > \frac{\epsilon_0}{2} \) for any \( y = (y_1, y_2, \ldots, y_n) \in U_x \). Let \( U_{x(1)}, U_{x(2)}, \ldots, U_{x(n)} \) be a cover of \( K^n \). Then for any \( y \in K^n \), \( \lambda\{t \in G : \frac{1}{n} \sum_{i=1}^{n} y_i f_0(t) > a + \frac{1}{i_0} \} > \frac{\epsilon_0}{2} \), where \( i_0 = \max\{i_{x(1)}, i_{x(2)}, \ldots, i_{x(n)}\} \).

By the hypothesis and the fact that \( K^n \) is compact, for each \( k \in N \), we can choose an \( x^k = (x^k_1, x^k_2, \ldots, x^k_n) \in K^n \) such that \( \lambda\{t \in G : \frac{1}{n} \sum_{i=1}^{n} x^k_i f(t) > a\} = 0 \). Then

\[
\int_G \left| \frac{1}{n} \sum_{i=1}^{n} x^k_i f_0(t) - \frac{1}{n} \sum_{i=1}^{n} x^k_i f_k(t) \right| dt \leq \|f_0 - f_k\|_1.
\]

On the other hand,

\[
\int_G \left| \frac{1}{n} \sum_{i=1}^{n} x^k_i f_0(t) - \frac{1}{n} \sum_{i=1}^{n} x^k_i f_k(t) \right| dt
\]

\[
= \int_{t \in G : \frac{1}{n} \sum_{i=1}^{n} x^k_i f_k(t) \leq a} \left| \frac{1}{n} \sum_{i=1}^{n} x^k_i f_0(t) - \frac{1}{n} \sum_{i=1}^{n} x^k_i f_k(t) \right| dt
\]

\[
\geq \frac{1}{i_0} \lambda\left\{ t \in G : \frac{1}{n} \sum_{i=1}^{n} x^k_i f_0(t) > a + \frac{1}{i_0} \right\}
\]

\[
\geq \frac{1}{i_0} \frac{\epsilon_0}{2}.
\]

This contradicts to that \( \|f_k - f_0\|_1 \to 0 \). \( \square \)

The following lemma is a consequence of Lemma 6A and Lemma 6C of Talagrand [14].

**Lemma 3.** Let \( G \) be a \( \sigma \)-compact nondiscrete locally compact group. If \( G \) is amenable as a discrete group and \( f \in L^\infty(G) \), then for any \( \epsilon > 0 \) there is an open subset \( \Omega \) of \( G \) and an \( m_0 \in LIM \) such that \( \lambda(\Omega) < \epsilon \), \( m_0(1_\Omega) = 1 \), and \( m_0(f) = \text{Sup}\{m(f) : m \in LIM\} \).

**Proof.** It follows from step 1 of the proof of Theorem 6D in [14] that for any positive integer \( n \), there exists an open set \( \Omega_n \) and an \( m_n \in LIM \) such that
Let $m_0$ be a $w^*$-limit point of $\{m_n\}$ and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. Then $\lambda(\Omega) < \epsilon$, $m_0 \in \text{LIM}$ with $m_0(1_\Omega) = 1$, and $m_0(f) = \text{Sup}\{m(f) : m \in \text{LIM}\}$. □

Now we are ready to prove our first main result concerning the exposed points of \text{LIM} for a locally compact group.

**Theorem 1.** Let $G$ be a $\sigma$-compact infinite locally compact group. If $G$ is amenable as a discrete group, then \text{LIM} has no exposed points.

**Proof.** When $G$ is discrete, it is proved by Yang [15] that \text{LIM} has no exposed points. Assume that $G$ is nondiscrete. Since $G$ is $\sigma$-compact, there is a sequence of subsets $\{K_n : n \in \mathbb{N}\}$ such that $G = \bigcup_{n=1}^{\infty} K_n$, where $K_n$ is compact and $K_n \subseteq K_{n+1}$ ($n = 1, 2, \ldots$). Assume that $m_0 \in \text{LIM}$ is an exposed point of \text{LIM}. Then there is an $f_0 \in L^1(G)$ such that

\[ (*) \quad m_0(f_0) > m(f_0) \quad \text{for any } m \in \text{LIM} \text{ and } m \neq m_0. \]

We are going to show that we can choose $f_0$ as above such that $0 \leq f_0 \leq 1$ and $f_0 \in L^1(G)$. Let $f_1 = \frac{f_0 + \|f_0\|_\infty}{\|f_0 + \|f_0\|_\infty\|_\infty}$ . Then $f_1$ also satisfies (*) since $f_1 \geq 0$ and $m(1) = 1$ for all $m \in \text{LIM}$. Thus, $m_0(f_1) > 0$ by the fact that $\text{LIM} \neq \{m_0\}$ (see [7]). By Lemma 3, there exists an open subset $\Omega$ of $G$ and an $m_1 \in \text{LIM}$ such that $\lambda(\Omega) < 1$, $m_1(1_\Omega) = 1$, and $m_1(f_1) = \text{Sup}\{m(f_1) : m \in \text{LIM}\}$. Hence $m_1(f_1) = m_0(f_1)$ and $m_1 = m_0$ by (*). Let $g = f_11_\Omega$. Then $g$ satisfies (*). In fact, for any $m \in \text{LIM} \sim \{m_0\}$, $m(g) = m(f_11_\Omega) \leq m(f_1) < m_0(f_1) = m_0(g)$ since $m_0(f_11_{G^\sim}) = m_1(f_11_{G^\sim}) = 0$. Note that $g \geq 0$ and $g \in L^\infty(G) \cap L^1(G)$. Let $X = \{f \in L^\infty(G) : 0 \leq f \leq g\}$ and $a = m_0(g)$. Then $(X, ||\cdot||_1)$ is a complete metric space and $a > 0$.

Let $n \in \mathbb{N}$ and $n > 0$ be fixed. For any $p, q \in \mathbb{N}$, put

$$X_{p, q} = \left\{ f \in X : \exists x_1, x_2, \ldots, x_p \in K_q \right\}.$$

with

$$\lambda \left\{ t \in G : \frac{1}{p} \sum_{i=1}^{p} x_i f(t) > a - \frac{1}{n} \right\} = 0.$$

At first, each $X_{p, q}$ is closed. In fact, let $f_k \in X_{p, q}$ and $f_k \to f$ in $(X, ||\cdot||_1)$. By Lemma 2,

$$\inf_{(x_1, x_2, \ldots, x_p) \in K^p_q} \lambda \left\{ t \in G : \frac{1}{p} \sum_{i=1}^{p} x_i f(t) > a - \frac{1}{n} \right\} = 0.$$

By Lemma 1, the map $(x_1, x_2, \ldots, x_p) \rightarrow \lambda \left\{ t \in G : \frac{1}{p} \sum_{i=1}^{p} x_i f(t) > a - \frac{1}{n} \right\}$ from $K^p_q$ to $R$ is lower semicontinuous. Since $K^p_q$ is compact, there exists $(x_1, x_2, \ldots, x_p) \in K^p_q$ such that $\lambda \left\{ t \in G : \frac{1}{p} \sum_{i=1}^{p} x_i f(t) > a - \frac{1}{n} \right\} = 0$. Therefore, $f \in X_{p, q}$.

Also, $X_{p, q}$ is nowhere dense. In fact, for any $f \in X$ and any $\epsilon > 0$, by Lemma 3 there is an open subset $\Omega_1$ of $G$ and an $m_1 \in \text{LIM}$ such that $\lambda(\Omega_1) < \epsilon$, $m_1(1_{\Omega_1}) = 1$, and $m_1(g) = \text{Sup}\{m(g) : m \in \text{LIM}\}$. Since
g satisfies (*), \( m_1 = m_0 \). Let \( f^* = g 1_{\Omega_1} + f 1_{G \setminus \Omega_1} \). Then \( f^* \in X \) and 
\[ ||f^* - f||_1 = ||g 1_{\Omega_1} - f 1_{\Omega_1}|| < 2\varepsilon. \]
Since \( m_0(f^*) = m_0(g 1_{\Omega_1}) = m_0(g) = a > a - \frac{1}{n} \), \( \lambda(t \in G : \frac{1}{p} \sum_{i=1}^{p} x_i f^*(t) > a - \frac{1}{n}) \neq 0 \) for any \((x_1, x_2, \ldots, x_p) \in K_q^p\).
Hence \( f^* \notin X_{p,q} \).

For any \( p, q \in N \), let \( X_{p,q}^c = \{ f \in X : g - f \notin X_{p,q} \} \). Then \( X_{p,q}^c \) and 
\( X_{p,q} \) are isometric in \((X, ||\cdot||_1)\). So \( X_{p,q}^c \) is also nowhere dense in \((X, ||\cdot||_1)\). Hence there exists an \( f \in X \sim \bigcup_{p,q} \left( X_{p,q} \cup X_{p,q}^c \right) \) by the completeness of \( X \).

For any \( x_1, x_2, \ldots, x_p \in G \), there is \( q \in N \) such that \( x_1, x_2, \ldots, x_p \in K_q \). Thus, \( \lambda(t \in G : \frac{1}{p} \sum_{i=1}^{p} x_i (f(t) > a - \frac{1}{n}) \neq 0 \) since \( f \notin X_{p,q} \). There exists \( m_n \in \text{LIM} \) such that \( m_n(f) > a - \frac{1}{n} \) by Proposition 3 of [7]. Similarly, since for any \( x_1, x_2, \ldots, x_p \in G \), \( \lambda(t \in G : \frac{1}{p} \sum_{i=1}^{p} x_i (g-f)(t) > a - \frac{1}{n}) \neq 0 \), there exists \( M_n \in \text{LIM} \) such that \( M_n(g-f) > a - \frac{1}{n} \). Let \( m \) and \( M \) be \( \omega^* \) limit points of \( m_n \) and \( M_n \), respectively. Then \( m, M \in \text{LIM} \) and \( m(f) \geq a \) and 
\( M(g-f) \geq a \). Since \( 0 \leq f \leq g \) and \( 0 \leq g-f \leq g \), \( m(g) \geq a \) and \( M(g) \geq a \). Hence \( m = M = m_0 \) by (*) and since \( a = m_0(g) = \text{Sup}\{m(g) : m \in \text{LIM}\} \). Therefore \( M(g-f) = 0 \). This contradicts \( a > 0 \). \( \square \)

3. Exposed points of \( \text{LIM}(X,G) \)

In this section we are going to prove an analogue of Theorem 1 for groups acting ergodically as measure-preserving transformations on a nonatomic probability space \((X, \beta, \mu)\).

Let \((X, \beta, \mu)\) be a nonatomic probability space, \( G \) a group, and \((s,x) \rightarrow sx \) a measure-preserving ergodic action of \( G \) on \((X, \beta, \mu)\). Then \( G \) also acts on \( L^\infty(X, \beta, \mu) : (sf)(x) = f(sx), f \in L^\infty(X, \beta, \mu), s \in G, \) and \( x \in X \). A positive linear functional of norm 1 on \( L^\infty(X, \beta, \mu) \) is said to be \( G \)-invariant mean if \( m(sf) = m(f) \) for \( s \in G \) and \( f \in L^\infty(X, \beta, \mu) \). The set of \( G \)-invariant means is denoted by \( \text{LIM}(X,G) \).

It is natural to ask how big the set \( \text{LIM}(X,G) \) is. When \( G \) is a countable amenable semigroup, del Junco and Rosenblatt [3] proved \( \text{LIM}(X,S) \) contains more than one element. Chou [2] showed that the cardinality of \( \text{LIM}(X,G) \) is at least \( 2^c \) for any countable amenable group, where \( c \) is the cardinality of the continuum. Our Theorem 2 shows that \( \text{LIM}(X,G) \) does not have exposed points in the case that \( G \) is an amenable countable group acting ergodically as measure-preserving transformations on a nonatomic probability space. This theorem was proved by Granirer in Theorem 3 in [5] and Theorem 2.6 in [6] without the assumptions of the ergodical acting and the measure-preserving transformations. Here we will give a different and direct proof.

**Lemma 4.** Let \( G \) be a group acting ergodically as measure-preserving transformations on a nonatomic probability space \((X, \beta, \mu)\). If \( m \in \text{LIM}(X,G) \) and \( f \in L^\infty(X) \) with \( 0 \leq f \leq 1 \), then for any \( x_1, x_2, \ldots, x_n \in G, \epsilon > 0, \) and \( \delta > 0 \) there exists a subset \( V \) of \( X \) such that \( \mu(V) < \epsilon \) and 
\[ \mu \left\{ t \in X : \frac{1}{n} \sum_{i=1}^{n} x_i (f 1_{V})(t) > m(f) - \delta \right\} \neq 0. \]

**Proof.** Let \( a = m(f) \). Since \( m(\frac{1}{n} \sum_{i=1}^{n} x_i f) = a \), \( p \{ t \in X : \frac{1}{n} \sum_{i=1}^{n} x_i f(t) > a - \delta \} > 0 \). Hence there is a subset \( J \subseteq \{1, 2, \ldots, n\} \) and \( a_i \) for each \( i \in J \).
such that $\frac{1}{n} \sum_{i \in J} a_i > a - \delta$ and $p(\bigcap_{i \in J} \{ t \in X : x_i f(t) > a_i \}) > 0$. Let $E_{a_i} = \{ t \in X : f(t) > a_i \}$. Then $\{ t \in X : x_i f(t) > a_i \} = \{ t \in X : x_i t \in E_{a_i} \}$, which is denoted by $x_i^{-1} E_{a_i}$. Hence $p(\bigcap_{i \in J} x_i^{-1} E_{a_i}) > 0$. Since $X$ is nonatomic, there exists $A \subseteq \bigcap_{i \in J} x_i^{-1} E_{a_i}$ such that $0 < p(A) < \frac{1}{n} \epsilon$. Let $V = \bigcup_{i \in J} x_i A$. Then $0 < p(V) < \epsilon$. If $t \in A$, then $x_i t \in V \cap E_{a_i}$ for each $i \in J$. Hence $x_i(fV)(t) = fV(x_i t) f(x_i t) > a_i$, i.e. $A \subseteq \bigcap_{i \in J} \{ t \in X : x_i (fV)(t) > a_i \}$ and $0 < p(A)$.

The following lemma is due to Granirer. See [7, Proposition 3] and [8, Proposition 5] for its proof.

**Lemma 5.** Let $G$ be a group acting ergodically as measure-preserving transformations on a nonatomic probability space $(X, \beta, \mu)$. If $m \in LIM(X, G)$ and $f \in L^\infty(X)$, then

$$\sup \{ m(f) : m \in LIM(X, G) \} = \inf \text{ess sup}_t \left[ \frac{1}{n} \sum_{i=1}^{n} x_i f(t) \right].$$

**Theorem 2** (Granirer). If $G$ is a amenable countable group acting ergodically as measure-preserving transformations on a nonatomic probability space $(X, \beta, \mu)$, then the set $LIM(X, G)$ of $G$-invariant means on $L^\infty(X)$ has no exposed points.

**Proof.** Let $G = \bigcup_{n=1}^{\infty} K_n$, where each $K_n$ is a finite subset of $G$ and $K_n \subseteq K_{n+1}$ $(n = 1, 2, \ldots)$. Let $m_0 \in LIM(X, G)$ be an exposed point of $LIM(X, G)$. Then there is an $f_0 \in L^\infty(X)$ such that

$$(*)_n \quad m_0(f_0) > m(f_0) \quad \text{for any } m \in LIM(X, G) \text{ and } m \neq m_0.$$

Let $g = \frac{f_0 + \| f_0 \|_{L^\infty}}{\| f_0 + \| f_0 \|_{L^\infty}}$. Then $g$ also satisfies $(*)_n$ since $m(1) = 1$ for any $m \in LIM(X, G)$. Note that $g \in L^\infty(X) \cap L^1(X)$ and $g \geq 0$. Thus $m(g) \geq 0$ for any $m \in LIM(X, G)$. By $(*)_n$ and the fact that $M(X, G)$ contains more than one element (see del Junco and Rosenblatt [3]), $m_0(g) > 0$. Let $a = m_0(g)$ and $Y = \{ f \in L^\infty(X) : 0 \leq f \leq g \}$. Then $(Y, \| \cdot \|_1)$ is a complete metric space and $a > 0$. For any $n \in N$ and $\delta > 0$, set

$$X_n = \left\{ f \in Y : \exists x_1, x_2, \ldots, x_k \in K_n \text{ with } p \left\{ t \in X : \frac{1}{k} \sum_{i=1}^{k} x_i f(t) > a - \delta \right\} = 0 \right\}.$$ 

At first, each $Y_n$ is closed. In fact, let $f_k \in Y_n$ and $f_k \to f$ in $(Y, \| \cdot \|_1)$. We can assume that $f_k \to f$ a.e. $[\mu]$. So for any $x_1, x_2, \ldots, x_r \in K_n$

$$\frac{1}{r} \sum_{i=1}^{r} x_i f_k(t) - \frac{1}{r} \sum_{i=1}^{r} x_i f(t) \quad \text{a.e. } [\mu] \text{ as } k \to \infty.$$

Also, for each $k$, there are $x_1, x_2, \ldots, x_r \in K_n$ such that

$$p \left\{ t \in X : \frac{1}{r} \sum_{i=1}^{r} x_i f_k(t) > a - \delta \right\} = 0.$$
Since $K_n$ is finite, there are $x_1, x_2, \ldots, x_r \in K_n$ such that

$$p \left\{ t \in X : \frac{1}{r} \sum_{i=1}^{r} x_i f(t) > a - \delta \right\} = 0.$$  

Thus, $f \in Y_n$. Therefore $Y_n$ is closed.

Also, for any $f \in Y$ and any $\epsilon > 0$, for any $x_1, x_2, \ldots, x_r \in K_n$, by Lemma 4, there is a subset $V$ of $X$ such that $p(V) < \epsilon$ and

$$p \left\{ t \in X : \frac{1}{r} \sum_{i=1}^{r} x_i (g_1 V)(t) > a - \delta \right\} > 0.$$  

Let $f^* = g_1 V + f^1 X_{X \rightarrow V}$. Then $f^* \in Y$ and $\|f^* - f\|_1 = \|g_1 V - f^1 V\|_1 < 2\epsilon$.

Since $f^1 X_{X \rightarrow V} \geq 0$,

$$0 < p \left\{ t \in X : \frac{1}{r} \sum_{i=1}^{r} x_i (g_1 V)(t) > a - \delta \right\} \leq p \left\{ t \in X : \frac{1}{r} \sum_{i=1}^{r} x_i f^*(t) > a - \delta \right\}.$$  

Hence $f^* \notin Y_n$ and $Y_n$ is nowhere dense.

For any $n \in \mathbb{N}$, let $Y_n^c = \{ f \in Y : g - f \notin Y_n \}$. Then $Y_n$ and $Y_n^c$ are isometric in $(Y, \| \cdot \|_1)$. So $Y_n^c$ is also nowhere dense in $(Y, \| \cdot \|_1)$. Hence there exists an $f \in Y \sim \bigcup_n (Y_n \cup Y_n^c)$ by the completeness of $Y$.

For any $x_1, x_2, \ldots, x_n \in G$, since $p\{t \in X : \frac{1}{n} \sum_{i=1}^{n} x_i f(t) > a - \delta\} > 0$, by Lemma 5 there exist $m_\delta \in \text{LIM}(X, G)$, such that $m_\delta(f) > a - \delta$. Let $m$ be the $w^*$ limit point of $\{m_\delta\}$. Then $m \in \text{LIM}(X, G)$ and $m(f) \geq a$. Similarly, since for any $x_1, x_2, \ldots, x_n \in G$, $p\{t \in X : \frac{1}{n} \sum_{i=1}^{n} x_i (g - f)(t) > a - \delta\} > 0$, there exists $M \in \text{LIM}(X, G)$ such that $M(g - f) \geq a$. Since $0 \leq f \leq g$, $m(g) \geq a$ and $M(g) \geq a$. By (*), $m = M = m_0$. So $M(g - f) = 0$. This contradicts $a > 0$. \qed

**References**


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