TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 4, April 1995

THE EXPOSED POINTS OF THE SET OF INVARIANT MEANS

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ABSTRACT. Let G be a σ-compact infinite locally compact group, and let LIM be the set of left invariant means on $L^\infty(G)$. We prove in this paper that if G is amenable as a discrete group, then LIM has no exposed points. We also give another proof of the Granirer theorem that the set $LIM(X, G)$ of G-invariant means on $L^\infty(X, \beta, p)$ has no exposed points, where G is an amenable countable group acting ergodically as measure-preserving transformations on a nonatomic probability space (X, β, p).

1. Introduction and Notations

Let G be a locally compact group with a fixed left Haar measure λ. If G is compact, we assume $\lambda(G) = 1$. Let $L^p(G)$ be the associated real Lebesgue spaces $(1 \le p \le \infty)$. For $f \in L^\infty(G)$ and $x \in G$, the left translation of f by x is defined by ${}_xf(y) = f(xy)$, $y \in G$. A mean on $L^\infty(G)$ is a positive functional on $L^\infty(G)$ with $m(1) = 1$. A left invariant mean is a mean with $m({}_xf) = m(f)$ for any $x \in G$ and $f \in L^\infty(G)$. The set of left invariant mean on $L^\infty(G)$ is denoted by LIM.

If $LIM \neq \phi$, we say that G is amenable. Let G_d be the same algebraic group as G with a discrete topological structure. Then G is amenable if G_d is amenable. Properties of amenable groups and left invariant means can be found in Greenleaf [9], Paterson [10] and Pier [11].

When G is amenable, LIM, as a w^*-compact convex subset of $L^\infty(G)^*$, is the w^*-closed convex hull of all its extreme points. It is natural to ask how many exposed points LIM has. Granirer [4] studied intensively the existence of exposed points of LIM for a countable amenable semigroup (also see Chou [1]). In particular, he proved by using very general theorems that LIM has exposed points if and only if G has finite left ideals for a countable amenable semigroup G [4, Corollary 4.1]. Yang [15] proved that if G is a infinite amenable discrete group, then LIM has no exposed points.

In this paper, we prove that LIM has no exposed points for any σ-compact locally compact group which is amenable as a discrete group. The idea of the proof is to "split" a nonnegative function in $L^\infty(G)$ by a category argument,

Received by the editors January 5, 1994; originally communicated to the *Proceedings of the AMS* by J. Marshall Ash.

1991 *Mathematics Subject Classification*. Primary 43A07.

Key words and phrases. Locally compact groups, amenable groups, invariant means, the exposed points.

This research is supported by an NSERC grant.

the technique used by Rosenblatt [12]. We also adapt this technique to prove the Granirer theorem of [5] and [6] in a different way that the set $LIM(X, G)$ of G-invariant means on $L^\infty(X, \beta, p)$ has no exposed points, where G is an amenable countable group acting ergodically as measure-preserving transformations on a nonatomic probability space (X, β, p). He derives it using very general theorems. See Chou [2] and Rosenblatt [13] for details of the study of the set $LIM(X, G)$.

The author would like to thank Professor E.E. Granirer for pointing out that Theorem 2 in this paper is a special case of his general theorems in [5] and [6] and for many valuable conversations.

2. Exposed points of LIM

In this section we will be concerned with LIM for a locally compact group and will prove our first main result. We need the following, probably known, proposition for which we were unable to find a reference.

Proposition 1. *Let G be a nondiscrete locally compact group, and let K be a compact subset of G. If $f \in L^\infty(G)$ and $\lambda\{t \in G : f(t) \neq 0\}$ is finite, then the function defined by*

$$F(x_1, x_2, \dots, x_n) = \lambda\left\{t \in G : \frac{1}{n}\sum_{i=1}^{n} {}_{x_i}f(t) > a\right\}$$

is lower semicontinuous on K^n, where a is a constant.

Proof. First let us prove that $\int_G |{}_xf - f|\,dt \to 0$ as $x \to e$. If $f = 1_E$, then $\int_G |{}_xf - f|\,dt = \lambda(x^{-1}E\Delta E) \to 0$ as $x \to e$ since the map $x \to \lambda(x^{-1}E \cap E)$ is continuous from K to R. For any f with $\lambda\{t \in G : f(t) \neq 0\}$ finite and an $\epsilon > 0$, choose a simple function $\varphi = \sum_{p=1}^{q} a_p 1_{E_p}$ such that $\|f - \varphi\|_1 < \epsilon$. There exists an open neighborhood U of e such that $\sum_{p=1}^{q} |a_p| \int_G |{}_x1_{E_p} - 1_{E_p}|\,dt < \epsilon$ for any $x \in U$. Hence, for every $x \in U$,

$$\int_G |{}_xf - f|\,dt \leq \int_G |{}_xf - {}_x\varphi|\,dt + \int_G |{}_x\varphi - \varphi|\,dt + \int_G |\varphi - f|\,dt \leq 3\epsilon.$$

Let $u^\alpha = (u_1^\alpha, u_2^\alpha, \dots, u_n^\alpha)$ be a net and $u = (u_1, u_2, \dots, u_n) \in K^n$ with $u^\alpha \to u$ in K^n. If there is an $\epsilon_0 > 0$ such that $F(u^\alpha) < F(u) - \epsilon_0$, then we can find a $\delta > 0$ such that $F(u^\alpha) < \lambda\{t \in G : \frac{1}{n}\sum_{i=1}^{n} {}_{u_i}f(t) > a + \delta\} - \epsilon_0$ for every α. Thus

$$\int_G \left|\frac{1}{n}\sum_{i=1}^{n} {}_{u_i^\alpha}f - \frac{1}{n}\sum_{i=1}^{n} {}_{u_i}f\right| dt \leq \frac{1}{n}\sum_{i=1}^{n} \int_G |{}_{u_i^\alpha}f - {}_{u_i}f|\,dt \to 0$$

when $u^\alpha \to u$ in K^n. On the other hand,

$$\int_G \left|\frac{1}{n}\sum_{i=1}^{n} {}_{u_i^\alpha}f - \frac{1}{n}\sum_{i=1}^{n} {}_{u_i}f\right| dt \geq \int_{B_\alpha} \left|\frac{1}{n}\sum_{i=1}^{n} {}_{u_i^\alpha}f - \frac{1}{n}\sum_{i=1}^{n} {}_{u_i}f\right| dt \geq \delta\lambda(B_\alpha) \geq \delta\epsilon_0,$$

where

$$B_\alpha = \left\{t \in G : \frac{1}{n}\sum_{i=1}^{n} {}_{u_i}f(t) > a + \delta\right\} \sim \left\{t \in G : \frac{1}{n}\sum_{i=1}^{n} {}_{u_i^\alpha}f(t) > a\right\}$$

with $\lambda(B_\alpha) \geq \lambda\{t \in G : \frac{1}{n}\sum_{i=1}^{n} {}_{u_i}f(t) > a+\delta\} - F(u^\alpha) > \epsilon_0$. This is a contradiction. Therefore the function F from K^n to R is lower semicontinuous. □

To prove our result, we will need the following two lemmas.

Lemma 2. *Let G be a locally compact group and let $f \in L^\infty(G)$ be a function with $0 \leq f \leq 1$ and $\lambda\{x \in G : f(x) \neq 0\} < \infty$. If f_k is a sequence of functions in $L^\infty(G)$ with $0 \leq f_k \leq f$ $(k = 0, 1, 2, \ldots)$, $f_k \to f_0$ in $\|\cdot\|_1$-norm, then $\inf_{(x_1, x_2, \ldots, x_n) \in K^n} F_{f_k}(x_1, x_2, \ldots, x_n) = 0$ for any $k \geq 1$ implies $\inf_{(x_1, x_2, \ldots, x_n) \in K^n} F_{f_0}(x_1, x_2, \ldots, x_n) = 0$, where K is a compact subset of G and $F_h(x_1, x_2, \ldots, x_n) = \lambda\{t \in G : \frac{1}{n}\sum_{i=1}^{n} {}_{x_i}h(t) > a\}$ for any $h \in L^\infty(G)$.*

Proof. Let $\inf_{(x_1, x_2, \ldots, x_n) \in K^n} F_{f_0}(x_1, x_2, \ldots, x_n) = \epsilon_0 > 0$. Then for any $x = (x_1, x_2, \ldots, x_n) \in K^n$, there is an $i_x \in N$ such that $\lambda\{t \in G : \frac{1}{n}\sum_{i=1}^{n} {}_{x_i}f_0(t) > a + \frac{1}{i_x}\} > \frac{\epsilon_0}{2}$. Since the map $(y_1, y_2, \ldots, y_n) \to \lambda\{t \in G : \frac{1}{n}\sum_{i=1}^{n} {}_{y_i}f_0(t) > a + \frac{1}{i_x}\}$ from K^n to R is lower semicontinuous by Proposition 1, there exists an open neighborhood U_x of x in K^n such that $\lambda\{t \in G : \frac{1}{n}\sum_{i=1}^{n} {}_{y_i}f_0(t) > a + \frac{1}{i_x}\} > \frac{\epsilon_0}{2}$ for any $y = (y_1, y_2, \ldots, y_n) \in U_x$. Let $U_{x^{(1)}}, U_{x^{(2)}}, \ldots, U_{x^{(p)}}$ be a cover of K^n. Then for any $y \in K^n$, $\lambda\{t \in G : \frac{1}{n}\sum_{i=1}^{n} {}_{y_i}f_0(t) > a + \frac{1}{i_0}\} > \frac{\epsilon_0}{2}$, where $i_0 = \max\{i_{x^{(1)}}, i_{x^{(2)}}, \ldots, i_{x^{(p)}}\}$.

By the hypothesis and the fact that K^n is compact, for each $k \in N$, we can choose an $x^k = (x_1^k, x_2^k, \ldots, x_n^k) \in K^n$ such that $\lambda\{t \in G : \frac{1}{n}\sum_{i=1}^{n} {}_{x_i^k}f_k(t) > a\} = 0$. Then

$$\int_G \left| \frac{1}{n}\sum_{i=1}^{n} {}_{x_i^k}f_0(t) - \frac{1}{n}\sum_{i=1}^{n} {}_{x_i^k}f_k(t) \right| dt \leq \|f_0 - f_k\|_1.$$

On the other hand,

$$\begin{aligned}
&\int_G \left| \frac{1}{n}\sum_{i=1}^{n} {}_{x_i^k}f_0(t) - \frac{1}{n}\sum_{i=1}^{n} {}_{x_i^k}f_k(t) \right| dt \\
&= \int_{\{t\in G \,:\, \frac{1}{n}\sum_{i=1}^{n} {}_{x_i^k}f_k(t) \leq a\}} \left| \frac{1}{n}\sum_{i=1}^{n} {}_{x_i^k}f_0(t) - \frac{1}{n}\sum_{i=1}^{n} {}_{x_i^k}f_k(t) \right| dt \\
&\geq \frac{1}{i_0}\lambda\left\{ t \in G : \frac{1}{n}\sum_{i=1}^{n} {}_{x_i^k}f_0(t) > a + \frac{1}{i_0} \right\} \\
&\geq \frac{1}{i_0}\frac{\epsilon_0}{2}.
\end{aligned}$$

This contradicts to that $\|f_k - f_0\|_1 \to 0$. □

The following lemma is a consequence of Lemma 6A and Lemma 6C of Talagrand [14].

Lemma 3. *Let G be a σ-compact nondiscrete locally compact group. If G is amenable as a discrete group and $f \in L^\infty(G)$, then for any $\epsilon > 0$ there is an open subset Ω of G and an $m_0 \in LIM$ such that $\lambda(\Omega) < \epsilon$, $m_0(1_\Omega) = 1$, and $m_0(f) = \mathrm{Sup}\{m(f) : m \in LIM\}$.*

Proof. It follows from step 1 of the proof of Theorem 6D in [14] that for any positive integer n, there exists an open set Ω_n and an $m_n \in LIM$ such that

$\lambda(\Omega_n) < \frac{\epsilon}{2^n}$, $m_n(1_{\Omega_n}) = 1$, and $m_n(f) \geq \operatorname{Sup}\{m(f) : m \in LIM\} - \frac{1}{n}$ (in fact, the condition of that ν is a topologically left invariant is not used and $\lambda(\Omega)$ can be made as small as we want in the proof of step 1).

Let m_0 be a w^*-limit point of $\{m_n\}$ and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. Then $\lambda(\Omega) < \epsilon$, $m_0 \in LIM$ with $m_0(1_\Omega) = 1$, and $m_0(f) = \operatorname{Sup}\{m(f) : m \in LIM\}$. □

Now we are ready to prove our first main result concerning the exposed points of LIM for a locally compact group.

Theorem 1. *Let G be a σ-compact infinite locally compact group. If G is amenable as a discrete group, then LIM has no exposed points.*

Proof. When G is discrete, it is proved by Yang [15] that LIM has no exposed points. Assume that G is nondiscrete. Since G is σ-compact, there is a sequence of subsets $\{K_n : n \in N\}$ such that $G = \bigcup_{n=1}^{\infty} K_n$, where K_n is compact and $K_n \subseteq K_{n+1}$ $(n = 1, 2, \ldots)$. Assume that $m_0 \in LIM$ is an exposed point of LIM. Then there is an $f_0 \in L^\infty(G)$ such that

$$(*) \qquad m_0(f_0) > m(f_0) \quad \text{for any } m \in LIM \text{ and } m \neq m_0.$$

We are going to show that we can choose f_0 as above such that $0 \leq f_0 \leq 1$ and $f_0 \in L^1(G)$. Let $f_1 = \frac{f_0 + \|f_0\|_\infty}{\|f_0 + \|f_0\|_\infty\|_\infty}$. Then f_1 also satisfies $(*)$ since $f_1 \geq 0$ and $m(1) = 1$ for all $m \in LIM$. Thus, $m_0(f_1) > 0$ by the fact that $LIM \neq \{m_0\}$ (see [7]). By Lemma 3, there exists an open subset Ω of G and an $m_1 \in LIM$ such that $\lambda(\Omega) < 1$, $m_1(1_\Omega) = 1$, and $m_1(f_1) = \operatorname{Sup}\{m(f_1) : m \in LIM\}$. Hence $m_1(f_1) = m_0(f_1)$ and $m_1 = m_0$ by $(*)$. Let $g = f_1 1_\Omega$. Then g satisfies $(*)$. In fact, for any $m \in LIM \sim \{m_0\}$, $m(g) = m(f_1 1_\Omega) \leq m(f_1) < m_0(f_1) = m_0(g)$ since $m_0(f_1 1_{G\sim\Omega}) = m_1(f_1 1_{G\sim\Omega}) = 0$. Note that $g \geq 0$ and $g \in L^\infty(G) \bigcap \mathrm{L}^1(G)$. Let $X = \{f \in L^\infty(G) : 0 \leq f \leq g\}$ and $a = m_0(g)$. Then $(X, \|\cdot\|_1)$ is a complete metric space and $a > 0$.

Let $n \in N$ and $n > 0$ be fixed. For any $p, q \in N$, put

$$X_{p,q} = \left\{ f \in X : \exists x_1, x_2, \ldots, x_p \in K_q \right.$$
$$\left. \text{with } \lambda\left\{ t \in G : \frac{1}{p} \sum_{i=1}^{p} {}_{x_i}f(t) > a - \frac{1}{n} \right\} = 0 \right\}.$$

At first, each $X_{p,q}$ is closed. In fact, let $f_k \in X_{p,q}$ and $f_k \to f$ in $(X, \|\cdot\|_1)$. By Lemma 2,

$$\inf_{(x_1, x_2, \ldots, x_p) \in K_q^p} \lambda\left\{ t \in G : \frac{1}{p} \sum_{i=1}^{p} {}_{x_i}f(t) > a - \frac{1}{n} \right\} = 0.$$

By Lemma 1, the map $(x_1, x_2, \ldots, x_p) \to \lambda\{t \in G : \frac{1}{p} \sum_{i=1}^{p} {}_{x_i}f(t) > a - \frac{1}{n}\}$ from K_q^p to R is lower semicontinuous. Since K_q^p is compact, there exists $(x_1, x_2, \ldots, x_p) \in K_q^p$ such that $\lambda\{t \in G : \frac{1}{p} \sum_{i=1}^{p} {}_{x_i}f(t) > a - \frac{1}{n}\} = 0$. Therefore $f \in X_{p,q}$.

Also, $X_{p,q}$ is nowhere dense. In fact, for any $f \in X$ and any $\epsilon > 0$, by Lemma 3 there is an open subset Ω_1 of G and an $m_1 \in LIM$ such that $\lambda(\Omega_1) < \epsilon$, $m_1(1_{\Omega_1}) = 1$, and $m_1(g) = \operatorname{Sup}\{m(g) : m \in LIM\}$. Since

g satisfies $(*)$, $m_1 = m_0$. Let $f^* = g1_{\Omega_1} + f1_{G\sim\Omega_1}$. Then $f^* \in X$ and $\|f^* - f\|_1 = \|g1_{\Omega_1} - f1_{\Omega_1}\|_1 < 2\epsilon$. Since $m_0(f^*) = m_0(g1_{\Omega_1}) = m_0(g) = a > a - \frac{1}{n}$, $\lambda\{t \in G : \frac{1}{p}\sum_{i=1}^{p} {}_{x_i}f^*(t) > a - \frac{1}{n}\} \neq 0$ for any $(x_1, x_2, \ldots, x_p) \in K_q^p$. Hence $f^* \notin X_{p,q}$.

For any $p, q \in N$, let $X_{p,q}^c = \{f \in X : g - f \in X_{p,q}\}$. Then $X_{p,q}$ and $X_{p,q}^c$ are isometric in $(X, \|\cdot\|_1)$. So $X_{p,q}^c$ is also nowhere dense in $(X, \|\cdot\|_1)$. Hence there exists an $f \in X \sim \bigcup_{p,q}(X_{p,q} \cup X_{p,q}^c)$ by the completeness of X.

For any $x_1, x_2, \ldots, x_p \in G$, there is $q \in N$ such that $x_1, x_2, \ldots, x_p \in K_q$. Thus, $\lambda\{t \in G : \frac{1}{p}\sum_{i=1}^{p} {}_{x_i}f(t) > a - \frac{1}{n}\} \neq 0$ since $f \notin X_{p,q}$. There exists $m_n \in LIM$ such that $m_n(f) > a - \frac{1}{n}$ by Proposition 3 of [7]. Similarly, since for any $x_1, x_2, \ldots, x_p \in G$, $\lambda\{t \in G : \frac{1}{p}\sum_{i=1}^{p} {}_{x_i}(g - f)(t) > a - \frac{1}{n}\} \neq 0$, there exists $M_n \in LIM$ such that $M_n(g - f) > a - \frac{1}{n}$. Let m and M be w^* limit points of m_n and M_n, respectively. Then $m, M \in LIM$ and $m(f) \geq a$ and $M(g-f) \geq a$. Since $0 \leq f \leq g$ and $0 \leq g-f \leq g$, $m(g) \geq a$ and $M(g) \geq a$. Hence $m = M = m_0$ by $(*)$ and since $a = m_0(g) = \operatorname{Sup}\{m(g) : m \in LIM\}$. Therefore $M(g - f) = 0$. This contradicts $a > 0$. □

3. Exposed points of $LIM(X, G)$

In this section we are going to prove an analogue of Theorem 1 for groups acting ergodically as measure-preserving transformations on a nonatomic probability space (X, β, p).

Let (X, β, p) be a nonatomic probability space, G a group, and $(s, x) \to sx$ a measure-preserving ergodic action of G on (X, β, p). Then G also acts on $L^\infty(X, \beta, p)$: $(sf)(x) = f(sx)$, $f \in L^\infty(X, \beta, p)$, $s \in G$, and $x \in X$. A positive linear functional of norm 1 on $L^\infty(X, \beta, p)$ is said to be G-invariant mean if $m(sf) = m(f)$ for $s \in G$ and $f \in L^\infty(X, \beta, p)$. The set of G-invariant means is denoted by $LIM(X, G)$.

It is natural to ask how big the set $LIM(X, G)$ is. When G is a countable amenable semigroup, del Junco and Rosenblatt [3] proved $LIM(X, S)$ contains more than one element. Chou [2] showed that the cardinality of $LIM(X, G)$ is at least 2^c for any countable amenable group, where c is the cardinality of the continuum. Our Theorem 2 shows that $LIM(X, G)$ does not have exposed points in the case that G is an amenable countable group acting ergodically as measure-preserving transformations on a nonatomic probability space. This theorem was proved by Granirer in Theorem 3 in [5] and Theorem 2.6 in [6] without the assumptions of the ergodical acting and the measure-preserving transformations. Here we will give a different and direct proof.

Lemma 4. *Let G be a group acting ergodically as measure-preserving transformations on a nonatomic probability space (X, β, p). If $m \in LIM(X, G)$ and $f \in L^\infty(X)$ with $0 \leq f \leq 1$, then for any $x_1, x_2, \ldots, x_n \in G$, $\epsilon > 0$, and $\delta > 0$ there exists a subset V of X such that $p(V) < \epsilon$ and*

$$p\left\{t \in X : \frac{1}{n}\sum_{i=1}^{n} {}_{x_i}(f1_V)(t) > m(f) - \delta\right\} \neq 0.$$

Proof. Let $a = m(f)$. Since $m(\frac{1}{n}\sum_{i=1}^{n} {}_{x_i}f) = a$, $p\{t \in X : \frac{1}{n}\sum_{i=1}^{n} {}_{x_i}f(t) > a - \delta\} > 0$. Hence there is a subset $J \subseteq \{1, 2, \ldots, n\}$ and a_i for each $i \in J$

such that $\frac{1}{n}\sum_{i\in J} a_i > a-\delta$ and $p(\bigcap_{i\in J}\{t\in X : {}_{x_i}f(t) > a_i\}) > 0$. Let $E_{a_i} = \{t\in X : f(t) > a_i\}$. Then $\{t\in X : {}_{x_i}f(t) > a_i\} = \{t\in X : x_it\in E_{a_i}\}$, which is denoted by $x_i^{-1}E_{a_i}$. Hence $p(\bigcap_{i\in J} x_i^{-1}E_{a_i}) > 0$. Since X is nonatomic, there exists $A\subseteq \bigcap_{i\in J} x_i^{-1}E_{a_i}$ such that $0 < p(A) < \frac{1}{n}\epsilon$. Let $V = \bigcup_{i\in J} x_iA$. Then $0 < p(V) < \epsilon$. If $t\in A$, then $x_it\in V\cap E_{a_i}$ for each $i\in J$. Hence ${}_{x_i}(f1_V)(t) = 1_V(x_it)f(x_it) > a_i$, i.e. $A\subseteq \bigcap_{i\in J}\{t\in X : {}_{x_i}(f1_V)(t) > a_i\}$ and

$$\begin{aligned} 0 &< p(A) \\ &\le p\left\{t\in X : \frac{1}{n}\sum_{i\in J} {}_{x_i}(f1_V)(t) > a-\delta\right\} \\ &\le p\left\{t\in X : \frac{1}{n}\sum_{i=1}^{n} {}_{x_i}(f1_V)(t) > a-\delta\right\}. \quad \square \end{aligned}$$

The following lemma is due to Granirer. See [7, Proposition 3] and [8, Proposition 5] for its proof.

Lemma 5. *Let G be a group acting ergodically as measure-preserving transformations on a nonatomic probability space (X, β, p). If $m\in LIM(X, G)$ and $f\in L^\infty(X)$, then*

$$\sup\{m(f) : m\in LIM(X, G)\} = \inf_{x_1, x_2, \dots, x_n\in G} \operatorname*{ess\,sup}_t \left[\frac{1}{n}\sum_{i=1}^{n} {}_{x_i}f(t)\right].$$

Theorem 2 (Granirer). *If G is a amenable countable group acting ergodically as measure-preserving transformations on a nonatomic probability space (X, β, p), then the set $LIM(X, G)$ of G-invariant means on $L^\infty(X, \beta, p)$ has no exposed points.*

Proof. Let $G = \bigcup_{n=1}^\infty K_n$, where each K_n is a finite subset of G and $K_n\subseteq K_{n+1}$ $(n = 1, 2, \dots)$. Let $m_0\in LIM(X, G)$ be an exposed point of $LIM(X, G)$. Then there is an $f_0\in L^\infty(X)$ such that

(*) $\qquad m_0(f_0) > m(f_0)$ for any $m\in LIM(X, G)$ and $m\neq m_0$.

Let $g = \frac{f_0+\|f_0\|_\infty}{\|f_0+\|f_0\|_\infty\|_\infty}$. Then g also satisfies $(*)$ since $m(1) = 1$ for any $m\in LIM(X, G)$. Note that $g\in L^\infty(X)\cap L^1(X)$ and $g\ge 0$. Thus $m(g)\ge 0$ for any $m\in LIM(X, G)$. By $(*)$ and the fact that $M(X, G)$ contains more than one element (see del Junco and Rosenblatt [3]), $m_0(g) > 0$. Let $a = m_0(g)$ and $Y = \{f\in L^\infty(X) : 0\le f\le g\}$. Then $(Y, \|\cdot\|_1)$ is a complete metric space and $a > 0$. For any $n\in N$ and $\delta > 0$, set

$$X_n = \left\{f\in Y : \exists x_1, x_2, \dots, x_k\in K_n \text{ with } p\left\{t\in X : \frac{1}{k}\sum_{i=1}^{k} {}_{x_i}f(t) > a-\delta\right\} = 0\right\}.$$

At first, each Y_n is closed. In fact, let $f_k\in Y_n$ and $f_k\to f$ in $(Y, \|\cdot\|_1)$. We can assume that $f_k\to f$ a.e. [p]. So for any $x_1, x_2, \dots, x_r\in K_n$

$$\frac{1}{r}\sum_{i=1}^{r} {}_{x_i}f_k(t) \to \frac{1}{r}\sum_{i=1}^{r} {}_{x_i}f(t) \quad \text{a.e. [p] as } k\to\infty.$$

Also, for each k, there are $x_1, x_2, \dots, x_r\in K_n$ such that

$$p\left\{t\in X : \frac{1}{r}\sum_{i=1}^{r} {}_{x_i}f_k(t) > a-\delta\right\} = 0.$$

Since K_n is finite, there are $x_1, x_2, \ldots, x_r \in K_n$ such that

$$p\left\{t \in X : \frac{1}{r}\sum_{i=1}^{r} {}_{x_i}f(t) > a - \delta\right\} = 0.$$

Thus, $f \in Y_n$. Therefore Y_n is closed.

Also, for any $f \in Y$ and any $\epsilon > 0$, for any $x_1, x_2, \ldots, x_r \in K_n$, by Lemma 4, there is a subset V of X such that $p(V) < \epsilon$ and

$$p\left\{t \in X : \frac{1}{r}\sum_{i=1}^{r} {}_{x_i}(g1_V)(t) > a - \delta\right\} > 0.$$

Let $f^* = g1_V + f1_{X\sim V}$. Then $f^* \in Y$ and $\|f^* - f\|_1 = \|g1_V - f1_V\|_1 < 2\epsilon$. Since $f1_{X\sim V} \geq 0$,

$$0 < p\left\{t \in X : \frac{1}{r}\sum_{i=1}^{r} {}_{x_i}(g1_V)(t) > a - \delta\right\} \leq p\left\{t \in X : \frac{1}{r}\sum_{i=1}^{r} {}_{x_i}f^*(t) > a - \delta\right\}.$$

Hence $f^* \notin Y_n$ and Y_n is nowhere dense.

For any $n \in N$, let $Y_n^c = \{f \in Y : g - f \in Y_n\}$. Then Y_n and Y_n^c are isometric in $(Y, \|\cdot\|_1)$. So Y_n^c is also nowhere dense in $(Y, \|\cdot\|_1)$. Hence there exists an $f \in Y \sim \bigcup_n (Y_n \cup Y_n^c)$ by the completeness of Y.

For any $x_1, x_2, \ldots, x_n \in G$, since $p\{t \in X : \frac{1}{n}\sum_{i=1}^{n} {}_{x_i}f(t) > a - \delta\} > 0$, by Lemma 5 there exist $m_\delta \in LIM(X, G)$, such that $m_\delta(f) > a - \delta$. Let m be the w^* limit point of $\{m_\delta\}$. Then $m \in LIM(X, G)$ and $m(f) \geq a$. Similarly, since for any $x_1, x_2, \ldots, x_n \in G$, $p\{t \in X : \frac{1}{n}\sum_{i=1}^{n} {}_{x_i}(g - f)(t) > a - \delta\} > 0$, there exists $M \in LIM(X, G)$ such that $M(g - f) \geq a$. Since $0 \leq f \leq g$, $m(g) \geq a$ and $M(g) \geq a$. By $(*)$, $m = M = m_0$. So $M(g - f) = 0$. This contradicts $a > 0$. □

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