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GROUPOIDS ASSOCIATED WITH ENDOMORPHISMS

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ABSTRACT. To a compact Hausdorff space which covers itself, we associate an r-discrete locally compact Hausdorff groupoid. Its $\mathbf{C}^*$-algebra carries an action of the circle allowing it to be regarded as a crossed product by an endomorphism and as a generalization of the Cuntz algebra O_p. We consider examples related to coverings of the circle and of a Heisenberg 3-manifold.

Given a compact Hausdorff space X and a continuous surjective map $\sigma: X \to X$, we want to associate to the pair (X, σ) a $\mathbf{C}^*$-algebra $\mathbf{C}^*(X, \sigma)$ which, in the case σ is a homeomorphism, coincides with the well-known crossed product $C(X) \times_\sigma \mathbf{Z}$.

In this paper we consider the case of a covering $\sigma: X \to X$ and $\mathbf{C}^*(X, \sigma)$ will be defined via a groupoid that generalizes the construction of the Cuntz groupoid O_p considered by Renault in [Re]. In that case X is a unilateral sequence space and σ is the unilateral shift. We start with some definitions and notation that make sense in greater generality.

Definition. Given a set X and a map $\sigma: X \to X$, let

$$\Gamma = \{(x, n, y) \in X \times \mathbf{Z} \times X \mid \exists k, l \geq 0, n = l - k, \sigma^k x = \sigma^l y\}$$

with the set of composable pairs

$$\Gamma^{(2)} = \{((x, n, y), (w, m, z)) \in \Gamma \times \Gamma \mid w = y\}.$$

The multiplication and inversion are given by

$$(x, n, y)(y, m, z) = (x, m+n, z) \quad \text{and} \quad (x, n, y)^{-1} = (y, -n, x).$$

The range and source maps are

$$r(x, n, y) = (x, 0, x), \quad s(x, n, y) = (y, 0, y).$$

With these operations Γ becomes a groupoid with the unit space identified with X. The equivalence relation associated to Γ will be denoted by R, where

$$R = \{(x, y) \in X \times X \mid \exists k, l \geq 0, \sigma^k x = \sigma^l y\},$$

and the isotropy group bundle is

$$I = \{(x, n, x) \in X \times \mathbf{Z} \times X \mid \exists k, l \geq 0, n = l - k, \sigma^k x = \sigma^l x\}.$$

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For $k \geq 0$ we consider

$$R_k := \{(x, y) \in X \times X \mid \sigma^k x = \sigma^k y\}$$

and

$$R_\infty := \bigcup_{k \geq 0} R_k .$$

Note that $R_\infty \subset R$.

Theorem 1. *Suppose $\sigma: X \to X$ is a covering. Then Γ, defined above, carries a topology making it an r-discrete locally compact Hausdorff groupoid with the Haar system given by the counting measures.*

Proof. Consider on R_k the induced topology from $X \times X$. Then R_k is a principal r-discrete groupoid since σ^k is a local homeomorphism (see [Ku1]). Consider on $R_\infty = \bigcup_{k \geq 0} R_k$ the inductive limit topology. Since the R_k are compact as topological spaces, R_∞ is a hyperfinite relation in the terminology of Kumjian (see [Ku2]). Hence R_∞ is a principal r-discrete groupoid and X, identified with the diagonal in $X \times X$, is open in R_∞. Let

$$\widetilde{X} = \left\{ (x_i) \in \prod_{i \geq 0} X_i \mid X_i = X,\ \sigma(x_{i+1}) = x_i\ \forall i \geq 0 \right\}$$

be the projective limit space and let $\pi_0: \widetilde{X} \to X$ the projection onto the first component. With the product topology, $\widetilde{X}$ becomes a compact Hausdorff space, and π_0 a continuous open map. The map σ induces a homeomorphism $\tilde{\sigma}: \widetilde{X} \to \widetilde{X}$,

$$\tilde{\sigma}(x_0, x_1, x_2, \ldots) = (\sigma x_0, x_0, x_1, \ldots),$$

with inverse

$$\tilde{\sigma}^{-1}(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots),$$

such that $\pi_0 \circ \tilde{\sigma} = \sigma \circ \pi_0$ (see [Br]). Let $\widetilde{R}_\infty = (\pi_0 \times \pi_0)^{-1}(R_\infty)$ equipped with the preimage of the inductive limit topology. Then

$$\widetilde{R}_\infty = \{((x_i), (y_i)) \in \widetilde{X} \times \widetilde{X} \mid \exists k \geq 0,\ \sigma^k x_0 = \sigma^k y_0\}$$

is a principal r-discrete groupoid with unit space identified with $\widetilde{X}$. The map $\tilde{\sigma}$ induces an automorphism of $\widetilde{R}_\infty$ by the formula

$$\tilde{\sigma}((x_i), (y_i)) = (\tilde{\sigma}(x_i), \tilde{\sigma}(y_i)),$$

and one can consider the semidirect product $\widetilde{\Gamma} = \widetilde{R}_\infty \times_{\tilde{\sigma}} \mathbf{Z}$ (see [Re]). Two elements of $\widetilde{\Gamma}$, written as $((x_i), n, (y_i))$ and $((w_i), m, (v_i))$ with $((x_i), (y_i))$, $((w_i), (v_i)) \in \widetilde{R}_\infty$, are composable iff $(w_i) = \tilde{\sigma}^{-n}(y_i)$, $(v_i) = \tilde{\sigma}^{-n}(z_i)$ and then the product is given by the formula

$$((x_i), n, (y_i)) \cdot (\tilde{\sigma}^{-n}(y_i), m, \tilde{\sigma}^{-n}(z_i)) = ((x_i), n+m, (z_i)).$$

The inverse of $((x_i), n, (y_i))$ is $(\tilde{\sigma}^{-n}(y_i), -n, \tilde{\sigma}^{-n}(x_i))$. With the induced topology from $\widetilde{R}_\infty \times \mathbf{Z}$, $\widetilde{\Gamma}$ becomes an r-discrete locally compact groupoid. Note that $\Gamma = \pi(\widetilde{\Gamma})$, where

$$\pi: \widetilde{X} \times \mathbf{Z} \times \widetilde{X} \to X \times \mathbf{Z} \times X$$

is given by the formula

$$\pi((x_i), y, (y_i)) = (x_0, n, \pi_0(\tilde{\sigma}^{-n}(y_i))),$$

and that π is a groupoid homomorphism. With the quotient topology, Γ becomes an r-discrete locally compact groupoid. It turns out that a basis of open sets for the topology of Γ consists of the subsets of the form $U \times \{n\} \times V$, where $n \in \mathbf{Z}$, U, V are open in X, $\sigma^n|_V : V \to \tau(U)$ (respectively $\sigma^{-n}|_U : U \to \rho(V)$) is a homeomorphism for $n \geq 0$ (respectively $n \leq 0$), and τ, ρ are suitable deck transformations for suitable powers of σ. The equivalence relation R with the trace of this topology was considered also by Vershik and Arzumanian (see [ArVe]). Note that, by construction, Γ has a (left) Haar system given by the counting measures. □

Definition. If X is compact and $\sigma: X \to X$ is a covering map, we define $\mathbf{C}^*(X, \sigma)$ to be $\mathbf{C}^*(\Gamma)$.

Observe that, in the case σ is 1-1, Γ is the transformation group groupoid $X \times \mathbf{Z}$ and $\mathbf{C}^*(\Gamma) \simeq C(X) \times_\sigma \mathbf{Z}$. But note that when σ is not 1-1 (the case in which we are interested), $\mathbf{C}^*(\Gamma)$ is more closely allied to a crossed product of $\mathbf{Z}_+$ on the noncommutative algebra $\mathbf{C}^*(R_\infty)$. To see this and in order to study the structure of $\mathbf{C}^*(X, \sigma)$, let $c: \Gamma \to \mathbf{Z}$,

$$c(x, n, y) = -n.$$

Then c is a 1-cocycle, which induces an S^1-action on $\mathbf{C}^*(\Gamma)$ via the formula

$$(\lambda f)(x, n, y) = \lambda^{-n} f(x, n, y), \qquad f \in C_c(\Gamma), \lambda \in S^1,$$

where $C_c(\Gamma)$ are the compactly supported functions on Γ. The fixed point algebra is $\mathbf{C}^*(R_\infty)$, which appears as an inductive limit of $\mathbf{C}^*(R_k)$, $k \geq 0$. Kumjian observed in [Ku1] that $\mathbf{C}^*(R_k) \simeq C(X) \times G_k$, where G_k is the group of deck transformations of the covering $\sigma^k: X \to X$.

Definition. The *orbit* of $x \in X$ is defined to be

$$O(x) = \bigcup_{k \geq 0} \sigma^{-k} \sigma^k x,$$

where

$$\sigma^{-k} y = \{z \in X \mid \sigma^k z = y\}.$$

We say that σ *minimal* if each orbit is dense and that σ is *essentially free* if

$$\{x \in X \mid \forall k, l \geq 0, \sigma^k x = \sigma^l x \Rightarrow k = l\}$$

is dense in X.

Proposition. *If σ is minimal, then $\mathbf{C}^*(R_\infty)$ is simple. Moreover, if σ is essentially free, then $\mathbf{C}^*(\Gamma)$ is also simple.*

Proof. Since σ is minimal and

$$O(x) = \{y \in X \mid (x, y) \in R_\infty\} \subset \{y \in X \mid (x, y) \in R\},$$

it follows that there are no nontrivial open invariant subsets. If σ is also essentially free, then the groupoid Γ is essentially principal in the sense of Renault (see Definition II.4.3 of [Re]). Now apply Proposition II.4.6 of [Re], where the ideals of an essentially principal groupoid are characterized. □

The covering map σ induces a *-endomorphism of $\mathbf{C}^*(R_\infty)$, denoted also by σ, via the formula

$$(\sigma f)(x, y) = \frac{1}{p} f(\sigma x, \sigma y), \qquad f \in C_c(R_\infty),$$

where p is the index of the covering.

Indeed the fact that this gives an endomorphism follows from these calculations:

$$(\sigma(fg))(x, y) = \frac{1}{p}(fg)(\sigma x, \sigma y) = \frac{1}{p} \sum_{z \sim \sigma x} f(\sigma x, z) g(z, \sigma y)$$

and

$$\begin{aligned}((\sigma f)(\sigma g))(x, y) &= \sum_{x \sim w} (\sigma f)(x, w)(\sigma g)(w, y) \\ &= \frac{1}{p^2} \sum_{x \sim w} f(\sigma x, \sigma w) g(\sigma w, \sigma y) = \frac{1}{p} \sum_{\sigma x \sim z} f(\sigma x, z) g(z, \sigma y),\end{aligned}$$

where $z = \sigma w$ and $x \sim y$ means $(x, y) \in R_\infty$.

One can check easily that the induced endomorphism is never unital (unless σ is a homeomorphism). However, more important for our purposes, σ is induced by a nonunitary isometry and this allows us to represent $\mathbf{C}^*(X, \sigma)$ as the crossed product of $\mathbf{C}^*(R_\infty)$ by this endomorphism in the sense of Paschke (see [Pa1, Pa2]). In his notation, $B = \mathbf{C}^*(\Gamma)$, $A = \mathbf{C}^*(R_\infty)$ and the spectral subspace E_1 is generated by functions supported on $\{(x, n, y) \in \Gamma \mid n = -1\}$. Let

$$v(x, n, y) = \begin{cases} 1/\sqrt{p} & \text{if } n = -1 \text{ and } y = \sigma x, \\ 0 & \text{otherwise.} \end{cases}$$

Then $v \in E_1$,

$$(v^* v)(x, n, y) = \begin{cases} 1 & \text{if } x = y \text{ and } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus v is an isometry in B and

$$(vv^*)(x, n, y) = \begin{cases} 1/p & \text{if } \sigma x = \sigma y \text{ and } n = 0, \\ 0 & \text{otherwise} \end{cases}$$

is a projection in A, different from 1. Moreover, $\sigma f = v f v^*$. These calculations lead to

Theorem 2. *Suppose $\sigma : X \to X$ is minimal. Then we have an exact sequence*

$$\begin{aligned}\cdots \to K_0(\mathbf{C}^*(R_\infty)) &\xrightarrow{\sigma_* - \mathrm{id}} K_0(\mathbf{C}^*(R_\infty)) \xrightarrow{i_*} K_0(\mathbf{C}^*(X, \sigma)) \to \\ &\to K_1(\mathbf{C}^*(R_\infty)) \xrightarrow{\sigma_* - \mathrm{id}} K_1(\mathbf{C}^*(R_\infty)) \xrightarrow{i_*} K_1(\mathbf{C}^*(X, \sigma)) \to \cdots\end{aligned}$$

where $i : \mathbf{C}^*(R_\infty) \hookrightarrow \mathbf{C}^*(X, \sigma)$ is the inclusion map and id *is the identity on K-theory.*

Proof. Using the fact that $\sigma f = v f v^*$, it follows that $\sigma(\mathbf{C}^*(R_\infty))$ is a corner in $\mathbf{C}^*(R_\infty)$. Since $\mathbf{C}^*(R_\infty)$ is simple, it is a full corner and we may apply Theorem 4.1 of [Pa2]. Notice that

$$\sigma : \mathbf{C}^*(R_\infty) \to \sigma(\mathbf{C}^*(R_\infty))$$

is an isomorphism, therefore σ_* is an isomorphism at the level of K- theory. □

We turn now to some examples that illustrate the applicability and the limitations of our analysis.

Example 1. Let $X = \prod_{i \geq 0} X_i$, where $X_i = \{1, 2, \ldots, p\}$ for each $i \geq 0$, with product topology, and let $\sigma: X \to X$ be the unilateral shift,

$$\sigma(x_0, x_1, \ldots) = (x_1, x_2, \ldots).$$

Then $\sigma: X \to X$ is a p-fold covering. In this case

$$\mathbf{C}^*(R_k) \simeq M_{p^k}(C(X)),$$

R_∞ is the Glimm equivalence relation,

$$\mathbf{C}^*(R_\infty) \simeq UHF(p^\infty),$$

and $\mathbf{C}^*(X, \sigma)$ is the Cuntz algebra O_p. Note that σ is essentially free since the points in X with trivial isotropy are the nonperiodic sequences, which are dense in X. The fact that $UHF(p^\infty)$ and O_p are simple is well-known.

Example 2. Let $A = (A(i, j))$ be a $p \times p$ matrix of 0's and 1's, let X be as in example 1 and let

$$X_A = \{(x_i) \in X \mid A(x_i, x_{i+1}) = 1\}.$$

Define $\sigma_A: X_A \to X_A$ to be the restriction of σ to X_A. For simplicity, we require that

$$\sum_{i=1}^{p} A(i, j) = q \quad \forall j$$

for some $q \geq 2$, so that σ_A is a q-fold covering. (Without this condition, we would consider different q's for different pieces of X_A). Then

$$\mathbf{C}^*(R_k) \simeq M_{q^k}(C(X_A)),$$

$\mathbf{C}^*(R_\infty)$ is an AF-algebra, and $\mathbf{C}^*(X_A, \sigma_A)$ is the Cuntz-Krieger algebra O_A. If the matrix A is aperiodic, then σ_A is minimal and essentially free, so that $\mathbf{C}^*(R_\infty)$ and O_A are simple (again a well-known result).

Example 3. Let $X = \mathbf{T}$, $\sigma: \mathbf{T} \to \mathbf{T}$, $\sigma(x) = x^p$, $p \geq 2$. In this case $\mathbf{C}^*(R_k) \simeq M_{p^k}(C(\mathbf{T}))$ (see [Ku1]), and $\mathbf{C}^*(R_\infty)$ is the Bunce-Deddens algebra of type p^∞. It is known that

$$K_0(\mathbf{C}^*(R_\infty)) \simeq \mathbf{Z}[1/p], \quad K_1(\mathbf{C}^*(R_\infty)) \simeq \mathbf{Z}$$

(see [B1]) and it is easy to see that σ is minimal. Thus one may apply the exact sequence in Theorem 2 to compute the K-theory of $\mathbf{C}^*(\mathbf{T}, \sigma)$. In order to determine σ_*, observe that $\sigma(\mathbf{C}^*(R_k)) \subset \mathbf{C}^*(R_{k+1})$ and it will be enough to know the maps at the level of K-theory induced by

$$\mathbf{C}^*(R_0) = C(\mathbf{T}) \xrightarrow{\sigma} M_p(C(\mathbf{T})) = \mathbf{C}^*(R_1).$$

Borrowing notation from [Ku1], $l^2(\sigma)$ is a $C(\mathbf{T})$-Hilbert module via the inner product

$$(f|h)(x) = \sum_{\sigma y = x} \overline{f}(y) h(y), \qquad f, h \in C(\mathbf{T}),$$

with basis $g_j(x) = x^j/\sqrt{p}$, $j = 0, 1, \dots, p-1$. Furthermore, $\mathbf{C}^*(R_1) \simeq \mathscr{K}(l^2(\sigma))$. Since $f \in C(R_1)$ may be viewed as an element of $\mathscr{K}(l^2(\sigma))$ via the formula

$$(fg)(x) = \sum_y f(x, y)g(y),\ g \in l^2(\sigma),$$

one can identify f with the matrix (f_{lj}), where $f_{lj} = (g_l | f g_j)$. So, letting $x = e(t) = \exp(2\pi i t)$, we see that

$$f_{lj}(e(t)) = \frac{1}{p} \sum_{k,m=0}^{p-1} e((t(j-l) + k(j-l) + mj)/p) f(e((t+k)/p), \\ e((t+m+k)/p)).$$

Following these identifications, the projection $\sigma(1) = vv^*$ corresponds to $e_{11} \otimes 1$ in $M_p(C(\mathbf{T}))$, so that

$$\mathbf{Z} = K_0(C(\mathbf{T})) \xrightarrow{\sigma_*} K_0(M_p(C(\mathbf{T}))) = \mathbf{Z}$$

is the identity map. If we denote by u the unitary

$$u(x, n, y) = \begin{cases} x & \text{if } y = x \text{ and } n = 0, \\ 0 & \text{otherwise}, \end{cases}$$

which generates $C(\mathbf{T})$, then vuv^* corresponds in $M_p(C(\mathbf{T}))$ to $e_{11} \otimes u$ so that

$$\mathbf{Z} = K_1(C(\mathbf{T})) \xrightarrow{\sigma_*} K_1(M_p(C(\mathbf{T}))) = \mathbf{Z}$$

is again the identity map. Therefore $\sigma_* \colon K_0(\mathbf{C}^*(R_\infty)) \to K_0(\mathbf{C}^*(R_\infty))$ is the multiplication by $1/p$, and $\sigma_* \colon K_1(\mathbf{C}^*(R_\infty)) \to K_1(\mathbf{C}^*(R_\infty))$ is the identity map. We get the exact sequences

$$0 \to \mathbf{Z}_{p-1} \to K_0(\mathbf{C}^*(\mathbf{T}, \sigma)) \to \mathbf{Z} \to 0$$

and

$$0 \to \mathbf{Z} \to K_1(\mathbf{C}^*(\mathbf{T}, \sigma)) \to 0,$$

from which it follows that

$$K_0(\mathbf{C}^*(\mathbf{T}, \sigma)) \simeq \mathbf{Z} \oplus \mathbf{Z}_{p-1} \quad \text{and} \quad K_1(\mathbf{C}^*(\mathbf{T}, \sigma)) \simeq \mathbf{Z}.$$

The isotropy groups

$$I_x = \{n \in \mathbf{Z} | x^{p^k} = x^{p^l},\ l - k = n\}$$

are trivial for a dense set of x in $\mathbf{T}$, therefore σ is essentially free and $\mathbf{C}^*(\mathbf{T}, \sigma)$ is a simple $\mathbf{C}^*$-algebra. Since the corresponding isometry v is nonunitary, it follows that $\mathbf{C}^*(\mathbf{T}, \sigma)$ is infinite.

One can consider coverings of $\mathbf{T}^q$ for $q \geq 2$ to get other simple infinite $\mathbf{C}^*$-algebras.

Example 4. Consider the real Heisenberg group of dimension 3, N. We write the elements of N as triples $(x, y, z) \in \mathbf{R}^3$ with multiplication

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + xy').$$

Consider also the discrete Heisenberg group

$$L = \{(x, y, z) \in N \mid x, y, z \in \mathbf{Z}\},$$

and let $X = N/L$. This is a compact 3-manifold (a circle bundle over $\mathbf{T}^2$). The elements of X will be written as $[x, y, z]$, $x, y, z \in [0, 1)$. Let

$$s: N \to N, \quad s(x, y, z) = (px, qy, pqz),$$

where p, $q \geq 2$ are integers. Since $s(L) \subset L$, we get a map

$$\sigma: X \to X, \quad \sigma[x, y, z] = [px, qy, pqz],$$

which is a p^2q^2-fold covering (see [Sh]). Invoking the methods of Kumjian [Ku1] again, we observe that a basis for the $C(X)$-Hilbert module $l^2(\sigma)$ is given by

$$g_{l,m,n}[x, y, z] = \frac{1}{pq}e(lx + my + nz), \qquad \begin{aligned} l &= 0, \dots, p-1, \\ m &= 0, \dots, q-1, \\ n &= 0, \dots, pq-1, \end{aligned}$$

where $e(a) = \exp(2\pi i a)$. This allows us to identify $\mathbf{C}^*(R_1)$ with $M_{p^2q^2}(C(X))$. Therefore $\mathbf{C}^*(R_\infty)$ will be an inductive limit of matrix algebras over $C(X)$.

Observe that σ is minimal and essentially free, so that $\mathbf{C}^*(R_\infty)$ and $\mathbf{C}^*(X, \sigma)$ are simple. It is known that the computation of $K_*(C(X))$ may be reduced to the computation of $K_*(\mathbf{C}^*(L))$, which is done by Anderson and Paschke (see [AnPa]). The computation of the K-theory of $\mathbf{C}^*(X, \sigma)$ using Theorem 2 can proceed, at least in principle. However at the moment, the details appear quite complicated.

Remark. If $\sigma: X \to X$ is a branched covering like $z \to z^p$ on the closed unit disc $\mathbf{D}^2$, one can define a locally compact topology on the groupoid Γ using the procedure described above. However, it will no longer be r-discrete. The reason is that the family of measures

$$\lambda^{(0,0)}(f) = 2f(0, 0), \quad \lambda^{(x,x)}(f) = f(x, x) + f(x, -x), \qquad x \neq 0,$$

is a Haar system for

$$R_1 = \{(x, y) \in \mathbf{D}^2 \times \mathbf{D}^2 \mid x^2 = y^2\}$$

which does not consist of counting measures (see [Re]).

Notice that $\mathbf{D}^2 \setminus \{0\}$ is an open invariant subset of the unit space $\mathbf{D}^2$. Groupoids associated with branched coverings will be discussed elsewhere.

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