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ON CLOSED MINIMAL SUBMANIFOLDS IN PINCHED RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper, we first prove a generalized Simons integral inequality for closed minimal submanifolds in a Riemannian manifold. Second, we prove a pinching theorem for closed minimal submanifolds in a complete simply connected pinched Riemannian manifold, which generalizes the results obtained by S. S. Chern, M. do Carmo, and S. Kobayashi and A. M. Li and J. M. Li respectively. Finally, we obtain a distribution theorem for the square norm of the second fundamental form of M under the assumption that M is a minimal submanifold with parallel second fundamental form in a Riemannian manifold.

1. INTRODUCTION

Let M^n be an n-dimensional oriented closed minimal submanifold in an $(n+p)$-dimensional manifold N^{n+p}. We denote the square norm of the second fundamental form of M by S. In the case that the ambient manifold N is the Euclidean sphere $S^{n+p}(1)$, it is well known [2] that if $S \leq n/(2-1/p)$ on M, then either M is the unit sphere $S^n(1)$, one of the Clifford minimal hypersurfaces in $S^{n+1}(1)$, or the Veronese surface in $S^4(1)$. Further discussions in this regard have been carried out by many other authors ([3, 5, 8, 9, 12], etc.). Recently, A. M. Li and J. M. Li [6] have improved the pinching constant above to $\frac{2}{3}n$ for the case $p \geq 3$. But all these results were obtained under the assumption that the ambient manifolds possess very nice symmetry.

The aim of the present paper is to establish a generalized Simons integral inequality for minimal submanifolds in a Riemannian manifold, and prove a pinching theorem for minimal submanifolds in a complete simply connected pinched Riemannian manifold, which does not possess symmetry in general. The proof uses some equations and inequalities naturally associated to the second fundamental form of M, the curvature tensor of N, and their covariant derivatives. Since we do not assume that N^{n+p} is a sphere, the maximum principle and the estimate for ΔS in [2, 6] cannot be applied here, and the trick of constructing a differentiable 1-form and using integral estimates seems essential. Finally, a distribution theorem for S is obtained under the assumption

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that M is a minimal submanifold with parallel second fundamental form in a Riemannian manifold.

2. Preliminaries

Let M^n be an n-dimensional Riemannian manifold immersed in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. We shall make use of the following convention on the range of indices:

$$1 \le A, B, C, \ldots \le n+p, \qquad 1 \le i, j, k, \ldots \le n,$$
$$n+1 \le \alpha, \beta, \gamma, \ldots \le n+p.$$

Choose a local field of orthonormal frames $\{e_A\}$ in N such that, restricted to M, the e_i's are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the field of dual frames and the connection 1-forms of N respectively. Restricting these forms to M, we have

$$\omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j, \qquad h^\alpha_{ij} = h^\alpha_{ji}, \tag{2.1}$$

$$h = \sum_{\alpha, i, j} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha, \qquad \xi = \frac{1}{n}\sum_{\alpha, i} h^\alpha_{ii} e_\alpha, \tag{2.2}$$

$$R_{ijkl} = K_{ijkl} + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}), \tag{2.3}$$

$$R_{\alpha\beta kl} = K_{\alpha\beta kl} + \sum_i (h^\alpha_{ik}h^\beta_{il} - h^\alpha_{il}h^\beta_{ik}), \tag{2.4}$$

where h, ξ, $R_{\alpha\beta kl}$, R_{ijkl}, and K_{ABCD} are the second fundamental form, the mean curvature vector, the normal curvature tensor, the curvature tensor of M, and the curvature tensor of N respectively. We define

$$S = \|h\|^2, \qquad H = \|\xi\|, \qquad H_\alpha = (h^\alpha_{ij})_{n\times n}.$$

M is called minimal if H vanishes identically. Therefore, if M is minimal, its scalar curvature is given by

$$R = \sum_{i,j} K_{ijij} - S.$$

Now we define the covariant derivatives of h^α_{ij}, denoted by h^α_{ijk} and h^α_{ijkl} respectively, as

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_s h^\alpha_{sj}\omega_{is} + \sum_s h^\alpha_{is}\omega_{js} + \sum_\beta h^\beta_{ij}\omega_{\alpha\beta},$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_s h^\alpha_{sjk}\omega_{is} + \sum_s h^\alpha_{isk}\omega_{js} + \sum_s h^\alpha_{ijs}\omega_{ks} + \sum_\beta h^\beta_{ijk}\omega_{\alpha\beta}.$$

Then we have

$$h^\alpha_{ijk} - h^\alpha_{ikj} = K_{\alpha ikj}, \tag{2.5}$$

and the Ricci formula

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_s h^\alpha_{sj}R_{sikl} + \sum_s h^\alpha_{is}R_{sjkl} + \sum_\beta h^\beta_{ij}R_{\beta\alpha kl}. \tag{2.6}$$

Considering $K_{\alpha ijk}$ as a section of $T^{\perp}(M)\otimes T^*(M)\otimes T^*(M)\otimes T^*(M)$, we also define its covariant derivative $K_{\alpha ijkl}$ as

$$\sum_l K_{\alpha ijkl}\omega_l = dK_{\alpha ijk} + \sum_s K_{\alpha sjk}\omega_{is} + \sum_s K_{\alpha isk}\omega_{js} + \sum_s K_{\alpha ijs}\omega_{ks} + \sum_\beta K_{\beta ijk}\omega_{\alpha\beta}.$$

M is called a submanifold with parallel second fundamental form if $h^\alpha_{ijk}\equiv 0$ for all i, j, k, α. The Laplacian Δh^α_{ij} of the second fundamental form h is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. In the next section, we sometimes also use $\nabla_k h^\alpha_{ij}$ to denote h^α_{ijk}, etc.

For a matrix $A=(a_{ij})_{n\times n}$ we denote by $N(A)$ the square norm of A, i.e., $N(A) = \operatorname{tr}(A^tA) = \sum_{i,j} a^2_{ij}$. Then $N(A) = N(TA^tT)$, for each orthogonal $(n\times n)$-matrix T.

Proposition 1 (see [2, 6]). *Let $A_{n+1}, A_{n+2}, \dots, A_{n+p}$ be symmetric $(n\times n)$-matrices. Denote $S_{\alpha\beta} = \operatorname{tr}(A^t_\alpha A_\beta)$, $S_\alpha = S_{\alpha\alpha} = N(A_\alpha)$, $S=\sum_\alpha S_\alpha$. Then*

$$\sum_{\alpha,\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha,\beta} S^2_{\alpha\beta} \le \left(1+\frac{1}{2}\operatorname{sgn}(p-1)\right)S^2, \tag{2.7}$$

where $\operatorname{sgn}(\cdot)$ is the standard sign function, and the equality holds if and only if at most two matrices A_α and A_β are not zero and these two matrices can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\widetilde{A}_\alpha$ and $\widetilde{A}_\beta$ respectively, where

$$\widetilde{A}_\alpha = \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right), \qquad \widetilde{A}_\beta = \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right).$$

Proposition 2 (see [4]). *Let N be an $(n+p)$-dimensional Riemannian manifold. If $a\le K_N\le b$ at a point $x\in N$, then, at this point,*

(i) $|K_{ACBC}|\le \frac{1}{2}(b-a)$, *for $A\ne B$.*
(ii) $|K_{ABCD}|\le \frac{2}{3}(b-a)$, *for A, B, C, D distinct with each other.*

3. Inequalities and pinching theorems

From now on, we assume that M^n is a minimal submanifold in N^{n+p}. By (2.5), (2.6), and the minimality of M, we have

$$\begin{aligned}\Delta h^\alpha_{ij} = &-\sum_k (K_{\alpha kikj} + K_{\alpha ijkk}) + \sum_{k,m} h^\alpha_{mi}R_{mkjk}\\ &+\sum_{k,m} h^\alpha_{km}R_{mijk} + \sum_{k,\beta} h^\beta_{ki}R_{\alpha\beta kj}.\end{aligned} \tag{3.1}$$

Substituting (2.3) and (2.4) into the above, (3.1) becomes

$$\begin{aligned}\Delta h^\alpha_{ij} = &-\sum_k (K_{\alpha kikj} + K_{\alpha ijkk}) + \sum_{m,k} (h^\alpha_{mi}K_{mkjk} + h^\alpha_{mk}K_{mijk}) + \sum_{k,\beta} h^\beta_{ki}K_{\alpha\beta kj}\\ &+\sum_{m,k,\beta}(h^\alpha_{mi}h^\beta_{mj}h^\beta_{kk} + 2h^\alpha_{km}h^\beta_{ki}h^\beta_{mj} - h^\alpha_{km}h^\beta_{km}h^\beta_{ij} - h^\alpha_{mi}h^\beta_{mk}h^\beta_{kj} - h^\alpha_{mj}h^\beta_{ki}h^\beta_{mk}).\end{aligned}$$

Therefore,

$$
\begin{aligned}
\frac{1}{2}\Delta S &= \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 + \sum_{i,j,\alpha} h^\alpha_{ij}\Delta h^\alpha_{ij} \\
&= \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 - \sum_{i,j,k,\alpha} (h^\alpha_{ij}K_{\alpha kikj} + h^\alpha_{ij}K_{\alpha ijkk}) \\
&\quad + \sum_{i,j,k,m,\alpha} (h^\alpha_{mj}h^\alpha_{ij}K_{mkik} + h^\alpha_{mk}h^\alpha_{ij}K_{mijk}) \\
&\quad + \sum_{i,j,k,\alpha,\beta} h^\alpha_{ij}h^\beta_{ki}K_{\alpha\beta kj} - \sum_{i,j,k,l,\alpha,\beta} h^\alpha_{ij}h^\alpha_{kl}h^\beta_{ij}h^\beta_{kl} \\
&\quad - \sum_{i,j,k,l,\alpha,\beta} (h^\alpha_{ik}h^\beta_{jk} - h^\alpha_{jk}h^\beta_{ik})(h^\alpha_{il}h^\beta_{jl} - h^\alpha_{jl}h^\beta_{il}).
\end{aligned}
\tag{3.2}
$$

Put

$$S_{\alpha\beta} = \sum_{i,j} h^\alpha_{ij}h^\beta_{ij}.$$

Then the $(p\times p)$-matrix $(S_{\alpha\beta})$ is symmetric and can be assumed to be diagonal for a suitable choice of $\{e_\alpha\}$, i.e.,

$$S_{\alpha\beta} = S_\alpha\delta_{\alpha\beta} \quad \text{for all } \alpha,\ \beta.$$

By the definition, $S = \sum_\alpha S_\alpha$. From (3.2) we have

Lemma 1. *Denote*

$$
\begin{aligned}
A &= -\sum_{\alpha,\beta} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_\alpha S_\alpha^2, \\
B &= \sum_{i,j,k,m,\alpha} (h^\alpha_{mj}h^\alpha_{ij}K_{mkik} + h^\alpha_{mk}h^\alpha_{ij}K_{mijk}) + \sum_{i,j,k,\alpha,\beta} h^\alpha_{ij}h^\beta_{ki}K_{\alpha\beta kj}, \\
C &= \sum_{i,j,k,\alpha} (h^\alpha_{ijk})^2 - \sum_{i,j,k,\alpha} (h^\alpha_{ij}K_{\alpha kikj} + h^\alpha_{ij}K_{\alpha ijkk}).
\end{aligned}
$$

Then

$$\tfrac{1}{2}\Delta S = A + B + C. \tag{3.3}$$

Let $a(x)$ and $b(x)$ denote the infimum and the supremum of the sectional curvature of N at a point x respectively. Now we derive a lower bound for B in terms of a, b, and S.

Lemma 2. $B \ge nbS - [n + \frac{2}{3}(p-1)(n-1)^{1/2}](b-a)S$.

Proof. Fix a vector e_α. Let $\{e_i\}$ be a frame diagonalizing the matrix (h^α_{ij}) such that

$$h^\alpha_{ij} = \lambda^\alpha_i\delta_{ij}, \qquad 1 \le i,\ j \le n.$$

Then

$$
\begin{aligned}
&\sum_{i,j,k,m} h^\alpha_{mj}h^\alpha_{ij}K_{mkik} + \sum_{i,j,k,m} h^\alpha_{mk}h^\alpha_{ij}K_{mijk} + \sum_{i,j,k,\beta} h^\alpha_{ij}h^\beta_{ki}K_{\alpha\beta kj} \\
&\quad = \sum_{i,k} (\lambda^\alpha_i)^2 K_{ikik} + \sum_{i,k} \lambda^\alpha_k\lambda^\alpha_i K_{kiik} + \sum_{i,k,\beta} h^\beta_{ki}\lambda^\alpha_i K_{\alpha\beta ki}.
\end{aligned}
\tag{3.4}
$$

By Proposition 2, we have

$$|K_{\alpha\beta ki}| \leq \tfrac{2}{3}(b-a) \quad \text{for } \alpha \neq \beta,\ i \neq k.$$

Hence, for fixed α, one sees

$$\begin{aligned}\sum_{i,k,\beta} h_{ki}^{\alpha}\lambda_i^{\alpha}K_{\alpha\beta ki} &\geq -\sum_{i\neq k,\beta\neq\alpha} \tfrac{2}{3}(b-a)|h_{ki}^{\beta}\lambda_i^{\alpha}| \\ &\geq -\sum_{i\neq k,\beta\neq\alpha} \tfrac{1}{3}(b-a)[(n-1)^{1/2}(h_{ki}^{\beta})^2+(n-1)^{-1/2}(\lambda_i^{\alpha})^2] \\ &\geq -\tfrac{1}{3}(n-1)^{1/2}(b-a)\sum_{\beta\neq\alpha}\operatorname{tr}H_{\beta}^2 \\ &\quad -\tfrac{1}{3}(n-1)^{1/2}(p-1)(b-a)\operatorname{tr}H_{\alpha}^2.\end{aligned} \tag{3.5}$$

On the other hand, we have

$$\begin{aligned}&\sum_{i,k}(\lambda_i^{\alpha})^2K_{ikik}+\sum_{i,k}\lambda_k^{\alpha}\lambda_i^{\alpha}K_{kiik} \\ &= \frac{1}{2}\sum_{i,k}(\lambda_i-\lambda_k)^2K_{ikik} \geq \frac{1}{2}a\sum_{i,k}(\lambda_i-\lambda_k)^2 = na\operatorname{tr}H_{\alpha}^2.\end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we obtain

$$\begin{aligned}B &\geq \sum_{\alpha}[na\operatorname{tr}H_{\alpha}^2-\tfrac{1}{3}(n-1)^{1/2}(b-a)\sum_{\beta\neq\alpha}\operatorname{tr}H_{\beta}^2 \\ &\qquad -\tfrac{1}{3}(n-1)^{1/2}(p-1)(b-a)\operatorname{tr}H_{\alpha}^2] \\ &= nbS-[n+\tfrac{2}{3}(p-1)(n-1)^{1/2}](b-a)S.\end{aligned} \tag{3.7}$$

This proves Lemma 2.

We shall next estimate the integral of C.

Lemma 3. $\int_M C \geq -\frac{1}{72}pn(n-1)(26n-25)\int_M(b-a)^2$.

Proof. Note that

$$\begin{aligned}&-\sum_{i,j,k,\alpha}(h_{ik}^{\alpha}K_{\alpha jijk}+h_{ij}^{\alpha}K_{\alpha ijkk}) \\ &= -\sum_{i,j,k,\alpha}\nabla_k(h_{ik}^{\alpha}K_{\alpha jij}+h_{ij}^{\alpha}K_{\alpha ijk})+\sum_{i,j,k,\alpha}(h_{ikk}^{\alpha}K_{\alpha jij}+h_{ijk}^{\alpha}K_{\alpha ijk}).\end{aligned}$$

We define a differentiable 1-form as

$$\omega = \sum_{i,j,k,\alpha}(h_{ik}^{\alpha}K_{\alpha jij}+h_{ij}^{\alpha}K_{\alpha ijk})\omega_k. \tag{3.8}$$

It follows that

$$\operatorname{div}\omega = \sum_{i,j,k,\alpha}\nabla_k(h_{ik}^{\alpha}K_{\alpha jij}+h_{ij}^{\alpha}K_{\alpha ijk}).$$

Thus

$$C = \sum_{i,j,k,\alpha}(h_{ijk}^{\alpha})^2+\sum_{i,j,k,\alpha}(h_{ikk}^{\alpha}K_{\alpha jij}+h_{ijk}^{\alpha}K_{\alpha ijk})-\operatorname{div}\omega.$$

Since M is minimal, we have

$$\sum_i h^\alpha_{iij} = 0 \quad \text{for all } j\,,\ \alpha\,. \tag{3.9}$$

From (2.5), (3.9), and Proposition 2, we have

$$\begin{aligned}\sum_{i,j,k,\alpha} h^\alpha_{ikk}K_{\alpha jij} &= \sum_{i,j,k,\alpha}(h^\alpha_{kki} - K_{\alpha kik})K_{\alpha jij} = -\sum_{i,\alpha}\left(\sum_j K_{\alpha jij}\right)^2 \\ &\geq -\tfrac{1}{4}pn(n-1)^2(b-a)^2\,.\end{aligned} \tag{3.10}$$

On the other hand, by Proposition 2, we have

$$\begin{aligned}&\sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 + \sum_{i,j,k,\alpha} h^\alpha_{ijk}K_{\alpha ijk} \\ &\quad\geq -\frac{1}{4}\sum_{i,j,k,\alpha}(K_{\alpha ijk})^2 \\ &\quad\geq -\frac{1}{4}\sum_\alpha \sum_{i,j,k \text{ distinct}}(K_{\alpha ijk})^2 - \frac{1}{2}\sum_\alpha\sum_{i\neq j}(K_{\alpha iji})^2 \\ &\quad\geq -\frac{1}{9}pn(n-1)(n-2)(b-a)^2 - \frac{1}{8}pn(n-1)(b-a)^2\,.\end{aligned} \tag{3.11}$$

So

$$C \geq -\frac{1}{72}pn(n-1)(26n-25)(b-a)^2 - \operatorname{div}\omega\,, \tag{3.12}$$

and by using Green's divergence theorem, we get

$$\int_M C \geq -\frac{1}{72}pn(n-1)(26n-25)\int_M (b-a)^2\,. \tag{3.13}$$

Lemma 3 follows.

Now we define

$$D(n\,,p) = n + \frac{2}{3}(p-1)(n-1)^{1/2}\,,$$
$$E(n\,,p) = \frac{1}{72}pn(n-1)(26n-25)\,.$$

Theorem 1 (Generalized Simons inequality). *Let M^n be an n-dimensional oriented closed minimal submanifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. Denote the infimum and the supremum of the sectional curvature of N at a point x by $a(x)$ and $b(x)$ respectively. Then*

$$\int_M [nbS - (1 + \tfrac{1}{2}\operatorname{sgn}(p-1))S^2 - D(n\,,p)(b-a)S - E(n\,,p)(b-a)^2] \leq 0\,.$$

Proof. Combining Proposition 1, Lemma 1 and 2, we obtain

$$\tfrac{1}{2}\Delta S \geq nbS - (1+\tfrac{1}{2}\operatorname{sgn}(p-1))S^2 - [n + \tfrac{2}{3}(p-1)(n-1)^{1/2}](b-a)S + C\,. \tag{3.14}$$

Integrating both sides of (3.14) and applying Lemma 3, we have

$$\int_M [nbS - (1+\tfrac{1}{2}\operatorname{sgn}(p-1))S^2 - D(n\,,p)(b-a)S - E(n\,,p)(b-a)^2] \leq 0\,. \tag{3.15}$$

This completes the proof of Theorem 1.

Denote

$$\alpha(n, p) = \tfrac{1}{12}[pn(n-1)(52n-50)]^{1/2},$$
$$\beta(n, p) = n + \tfrac{2}{3}(p-1)(n-1)^{1/2} + \tfrac{1}{12}[pn(n-1)(52n-50)]^{1/2}.$$

We are now in a position to prove

Theorem 2. *There is a number $\delta(n,p)$ with $0 < \delta(n,p) < 1$ such that if there exists an oriented closed minimal submanifold M^n in a complete simply connected manifold N^{n+p} with $\delta(n,p) \le K_N \le 1$ and*

$$\alpha(n, p)(1-c) \le S \le n - \tfrac{1}{3} n \operatorname{sgn}(p-1) - \beta(n, p)(1-c),$$

where c is the infimum of the sectional curvature of N, then either M is the unit sphere $S^n(1)$, one of the Clifford minimal hypersurfaces $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$, $k = 1, 2, \dots, n-1$, in $S^{n+1}(1)$, or the Veronese surface in $S^4(1)$. Moreover, $N = S^{n+p}(1)$.

Proof. Since

$$c \le a(x) \le b(x) \le 1,$$

(3.15) gives

$$\int_M [nS - (1 + \tfrac{1}{2}\operatorname{sgn}(p-1))S^2 - D(n,p)(1-c)S - E(n,p)(1-c)^2] \le 0. \tag{3.16}$$

Take

$$\delta(n, p) = 1 - n(3 - \operatorname{sgn}(p-1))(3D(n,p) + 6E^{1/2}(n,p))^{-1}.$$

Then

$$\alpha(n, p)(1-c) \le n - \tfrac{1}{3} n \operatorname{sgn}(p-1) - \beta(n, p)(1-c).$$

From the assumption

$$\alpha(n, p)(1-c) \le S \le n - \tfrac{1}{3} n \operatorname{sgn}(p-1) - \beta(n, p)(1-c), \tag{3.17}$$

we see that

$$nS - (1 + \tfrac{1}{2}\operatorname{sgn}(p-1))S^2 - D(n,p)(1-c)S - E(n,p)(1-c)^2 \ge 0. \tag{3.18}$$

Therefore, all inequalities in (3.10), (3.11), (3.15), and (3.18) are actually equalities. This implies $1 - c = b - a = 0$ and N is a complete simply connected Riemannian manifold with constant curvature 1. Hence $N = S^{n+p}(1)$. This together with (3.16) and (3.18) gives

$$S = 0 \quad \text{or} \quad S = n - \tfrac{1}{3} n \operatorname{sgn}(p-1).$$

Furthermore, the previous inequalities become equalities, and it is not hard to see from Proposition 1 that either M is the unit sphere $S^n(1)$, one of the Clifford hypersurfaces $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$, $k = 1, 2, \dots, n-1$, or the Veronese surface. This proves Theorem 2.

Remark 1. Theorem 2 can be considered as a generalization of the main theorems of [2, 6] as well as a pinching theorem for ambient manifolds.

Theorem 3. *Let M^n be an oriented closed minimal submanifold with parallel second fundamental form in a Riemannian manifold N^{n+p}. Then*

(i) $S \le pnd + F(n, p)(d - c)$, *where* $F(n, p) = \frac{2}{3}p(p-1)(n-1)^{1/2}$ *and* d *is the supremum of the sectional curvature of* N,

(ii) *if* $\delta'(n, p) < K_N \le 1$, *here*

$$\delta'(n, p) = 1 - n(3 - \operatorname{sgn}(p-1))[3n + 2(p-1)(n-1)^{1/2}]^{-1},$$

then either M *is totally geodesic or* $n - \frac{1}{3}n \operatorname{sgn}(p-1) - D(n, p)(1-c) \le S \le pn + F(n, p)(1-c)$.

Proof. From the proof of Lemma 3 we have

$$C = -\sum_{i,j,k,\alpha} \nabla_k (h^\alpha_{ik} K_{\alpha jij} + h^\alpha_{ij} K_{\alpha ijk}).$$

It is easy to see from (2.5) that $K_{\alpha ijk} = 0$, for all i, j, k, α. So

$$C = 0. \tag{3.19}$$

Since $\frac{1}{2}\Delta S = \sum (h^\alpha_{ijk})^2 + \sum h^\alpha_{ij}\Delta h^\alpha_{ij} = 0$, S is a constant. This together with (3.3) and (3.9) implies

$$A + B = 0. \tag{3.20}$$

Obviously,

$$\sum_{\alpha,\beta} N(H_\alpha H_\beta - H_\beta H_\alpha) + \sum_\alpha S_\alpha^2 \ge S^2/p. \tag{3.21}$$

For fixed α, similar to the estimate of lower bound for B, we have

$$\begin{aligned} LHS \text{ of } (3.4) &= \sum_{i,k} (\lambda^\alpha_i)^2 K_{ikik} + \sum_{i,k} \lambda^\alpha_k \lambda^\alpha_i K_{kiik} + \sum_{i,k,\beta} h^\beta_{ki} \lambda^\alpha_i K_{\alpha\beta ki} \\ &\le nd \operatorname{tr} H_\alpha^2 + \tfrac{1}{3}(p-1)(n-1)^{1/2}(d-c)\operatorname{tr} H_\alpha^2 \\ &\quad + \tfrac{1}{3}(n-1)^{1/2}(d-c)\sum_{\beta\ne\alpha} \operatorname{tr} H_\beta^2. \end{aligned}$$

This gives

$$B \le n\,dS + \tfrac{2}{3}(p-1)(n-1)^{1/2}(d-c)S. \tag{3.22}$$

It follows from (3.20), (3.21), and (3.22) that

$$n\,dS + \tfrac{2}{3}(p-1)(n-1)^{1/2}(d-c)S \ge S^2/p.$$

This yields

$$S \le pnd + \tfrac{2}{3}p(p-1)(n-1)^{1/2}(d-c). \tag{3.23}$$

If $\delta'(n, p) < K_N \le 1$, it is not hard to see from the definition of $\delta'(n, p)$ that

$$n - \tfrac{1}{3}n \operatorname{sgn}(p-1) - D(n, p)(1-c) > 0. \tag{3.24}$$

By (3.20), Proposition 1, and Lemma 2, we get

$$nS - (1 + \tfrac{1}{S}\operatorname{sgn}(p-1))S^2 - D(n, p)(1-c)S \le 0, \tag{3.25}$$

which together with (3.23) implies that either $S = 0$ or $n - \frac{1}{3} n \operatorname{sgn}(p-1) - D(n, p)(1-c) \leq S \leq pn + F(n, p)(1-c)$. This completes the proof of Theorem 3.

Remark 2. When $p = 1$, the constant $\delta'(n, p)$ equals zero, which is independent of dimension. In this case, we have that either $S = 0$ or $nc \leq S \leq n$.

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