

ON CLOSED MINIMAL SUBMANIFOLDS IN PINCHED RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper, we first prove a generalized Simons integral inequality for closed minimal submanifolds in a Riemannian manifold. Second, we prove a pinching theorem for closed minimal submanifolds in a complete simply connected pinched Riemannian manifold, which generalizes the results obtained by S. S. Chern, M. do Carmo, and S. Kobayashi and A. M. Li and J. M. Li respectively. Finally, we obtain a distribution theorem for the square norm of the second fundamental form of M under the assumption that M is a minimal submanifold with parallel second fundamental form in a Riemannian manifold.

1. INTRODUCTION

Let M^n be an n -dimensional oriented closed minimal submanifold in an $(n+p)$ -dimensional manifold N^{n+p} . We denote the square norm of the second fundamental form of M by S . In the case that the ambient manifold N is the Euclidean sphere $S^{n+p}(1)$, it is well known [2] that if $S \leq n/(2 - 1/p)$ on M , then either M is the unit sphere $S^n(1)$, one of the Clifford minimal hypersurfaces in $S^{n+1}(1)$, or the Veronese surface in $S^4(1)$. Further discussions in this regard have been carried out by many other authors ([3, 5, 8, 9, 12], etc.). Recently, A. M. Li and J. M. Li [6] have improved the pinching constant above to $\frac{2}{3}n$ for the case $p \geq 3$. But all these results were obtained under the assumption that the ambient manifolds possess very nice symmetry.

The aim of the present paper is to establish a generalized Simons integral inequality for minimal submanifolds in a Riemannian manifold, and prove a pinching theorem for minimal submanifolds in a complete simply connected pinched Riemannian manifold, which does not possess symmetry in general. The proof uses some equations and inequalities naturally associated to the second fundamental form of M , the curvature tensor of N , and their covariant derivatives. Since we do not assume that N^{n+p} is a sphere, the maximum principle and the estimate for ΔS in [2, 6] cannot be applied here, and the trick of constructing a differentiable 1-form and using integral estimates seems essential. Finally, a distribution theorem for S is obtained under the assumption

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that M is a minimal submanifold with parallel second fundamental form in a Riemannian manifold.

2. PRELIMINARIES

Let M^n be an n -dimensional Riemannian manifold immersed in an $(n+p)$ -dimensional Riemannian manifold N^{n+p} . We shall make use of the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n + p, \quad 1 \leq i, j, k, \dots \leq n, \\ n + 1 \leq \alpha, \beta, \gamma, \dots \leq n + p.$$

Choose a local field of orthonormal frames $\{e_A\}$ in N such that, restricted to M , the e_i 's are tangent to M . Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the field of dual frames and the connection 1-forms of N respectively. Restricting these forms to M , we have

$$(2.1) \quad \omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha,$$

$$(2.2) \quad h = \sum_{\alpha, i, j} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \quad \xi = \frac{1}{n} \sum_{\alpha, i} h_{ii}^\alpha e_\alpha,$$

$$(2.3) \quad R_{ijkl} = K_{ijkl} + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha),$$

$$(2.4) \quad R_{\alpha\beta kl} = K_{\alpha\beta kl} + \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta),$$

where $h, \xi, R_{\alpha\beta kl}, R_{ijkl}$, and K_{ABCD} are the second fundamental form, the mean curvature vector, the normal curvature tensor, the curvature tensor of M , and the curvature tensor of N respectively. We define

$$S = \|h\|^2, \quad H = \|\xi\|, \quad H_\alpha = (h_{ij}^\alpha)_{n \times n}.$$

M is called minimal if H vanishes identically. Therefore, if M is minimal, its scalar curvature is given by

$$R = \sum_{i, j} K_{ijij} - S.$$

Now we define the covariant derivatives of h_{ij}^α , denoted by h_{ijk}^α and h_{ijkl}^α respectively, as

$$\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum_s h_{sj}^\alpha \omega_{is} + \sum_s h_{is}^\alpha \omega_{js} + \sum_\beta h_{ij}^\beta \omega_{\alpha\beta}, \\ \sum_l h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha + \sum_s h_{sjk}^\alpha \omega_{is} + \sum_s h_{isk}^\alpha \omega_{js} + \sum_s h_{ijs}^\alpha \omega_{ks} + \sum_\beta h_{ijk}^\beta \omega_{\alpha\beta}.$$

Then we have

$$(2.5) \quad h_{ijk}^\alpha - h_{ikj}^\alpha = K_{\alpha ikj},$$

and the Ricci formula

$$(2.6) \quad h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_s h_{sj}^\alpha R_{sikl} + \sum_s h_{is}^\alpha R_{sjkl} + \sum_\beta h_{ij}^\beta R_{\beta\alpha kl}.$$

Considering K_{aijk} as a section of $T^\perp(M) \otimes T^*(M) \otimes T^*(M) \otimes T^*(M)$, we also define its covariant derivative K_{aijkl} as

$$\sum_l K_{aijkl} \omega_l = dK_{aijk} + \sum_s K_{asjk} \omega_{is} + \sum_s K_{aisk} \omega_{js} + \sum_s K_{aijs} \omega_{ks} + \sum_\beta K_{\beta ijk} \omega_{\alpha\beta}.$$

M is called a submanifold with parallel second fundamental form if $h_{ijk}^\alpha \equiv 0$ for all i, j, k, α . The Laplacian Δh_{ij}^α of the second fundamental form h is defined by $\Delta h_{ij}^\alpha = \sum_k h_{ijkk}^\alpha$. In the next section, we sometimes also use $\nabla_k h_{ij}^\alpha$ to denote h_{ijk}^α , etc.

For a matrix $A = (a_{ij})_{n \times n}$ we denote by $N(A)$ the square norm of A , i.e., $N(A) = \text{tr}(A^t A) = \sum_{i,j} a_{ij}^2$. Then $N(A) = N(TA^t T)$, for each orthogonal $(n \times n)$ -matrix T .

Proposition 1 (see [2, 6]). *Let $A_{n+1}, A_{n+2}, \dots, A_{n+p}$ be symmetric $(n \times n)$ -matrices. Denote $S_{\alpha\beta} = \text{tr}(A_\alpha^t A_\beta)$, $S_\alpha = S_{\alpha\alpha} = N(A_\alpha)$, $S = \sum_\alpha S_\alpha$. Then*

$$(2.7) \quad \sum_{\alpha, \beta} N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha, \beta} S_{\alpha\beta}^2 \leq \left(1 + \frac{1}{2} \text{sgn}(p-1)\right) S^2,$$

where $\text{sgn}(\cdot)$ is the standard sign function, and the equality holds if and only if at most two matrices A_α and A_β are not zero and these two matrices can be transformed simultaneously by an orthogonal matrix into scalar multiples of \tilde{A}_α and \tilde{A}_β respectively, where

$$\tilde{A}_\alpha = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array} \right), \quad \tilde{A}_\beta = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right).$$

Proposition 2 (see [4]). *Let N be an $(n+p)$ -dimensional Riemannian manifold. If $a \leq K_N \leq b$ at a point $x \in N$, then, at this point,*

- (i) $|K_{ACBC}| \leq \frac{1}{2}(b-a)$, for $A \neq B$.
- (ii) $|K_{ABCD}| \leq \frac{2}{3}(b-a)$, for A, B, C, D distinct with each other.

3. INEQUALITIES AND PINCHING THEOREMS

From now on, we assume that M^n is a minimal submanifold in N^{n+p} . By (2.5), (2.6), and the minimality of M , we have

$$(3.1) \quad \begin{aligned} \Delta h_{ij}^\alpha &= - \sum_k (K_{\alpha k i k j} + K_{\alpha i j k k}) + \sum_{k,m} h_{mi}^\alpha R_{m k j k} \\ &+ \sum_{k,m} h_{km}^\alpha R_{m i j k} + \sum_{k,\beta} h_{ki}^\beta R_{\alpha \beta k j}. \end{aligned}$$

Substituting (2.3) and (2.4) into the above, (3.1) becomes

$$\begin{aligned} \Delta h_{ij}^\alpha &= - \sum_k (K_{\alpha k i k j} + K_{\alpha i j k k}) + \sum_{m,k} (h_{mi}^\alpha K_{m k j k} + h_{mk}^\alpha K_{m i j k}) + \sum_{k,\beta} h_{ki}^\beta K_{\alpha \beta k j} \\ &+ \sum_{m,k,\beta} (h_{mi}^\alpha h_{mj}^\beta h_{kk}^\beta + 2h_{km}^\alpha h_{ki}^\beta h_{mj}^\beta - h_{km}^\alpha h_{km}^\beta h_{ij}^\beta - h_{mi}^\alpha h_{mk}^\beta h_{kj}^\beta - h_{mj}^\alpha h_{ki}^\beta h_{mk}^\beta). \end{aligned}$$

Therefore,

$$\begin{aligned}
 \frac{1}{2}\Delta S &= \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha \\
 &= \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - \sum_{i,j,k,\alpha} (h_{ij}^\alpha K_{\alpha k i k j} + h_{ij}^\alpha K_{\alpha i j k k}) \\
 (3.2) \quad &+ \sum_{i,j,k,m,\alpha} (h_{mj}^\alpha h_{ij}^\alpha K_{m k i k} + h_{mk}^\alpha h_{ij}^\alpha K_{m i j k}) \\
 &+ \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta K_{\alpha \beta k j} - \sum_{i,j,k,l,\alpha,\beta} h_{ij}^\alpha h_{kl}^\beta h_{ij}^\beta h_{kl}^\beta \\
 &- \sum_{i,j,k,l,\alpha,\beta} (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta)(h_{il}^\alpha h_{jl}^\beta - h_{jl}^\alpha h_{il}^\beta).
 \end{aligned}$$

Put

$$S_{\alpha\beta} = \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta.$$

Then the $(p \times p)$ -matrix $(S_{\alpha\beta})$ is symmetric and can be assumed to be diagonal for a suitable choice of $\{e_\alpha\}$, i.e.,

$$S_{\alpha\beta} = S_\alpha \delta_{\alpha\beta} \quad \text{for all } \alpha, \beta.$$

By the definition, $S = \sum_\alpha S_\alpha$. From (3.2) we have

Lemma 1. Denote

$$\begin{aligned}
 A &= - \sum_{\alpha,\beta} N(H_\alpha H_\beta - H_\beta H_\alpha) - \sum_\alpha S_\alpha^2, \\
 B &= \sum_{i,j,k,m,\alpha} (h_{mj}^\alpha h_{ij}^\alpha K_{m k i k} + h_{mk}^\alpha h_{ij}^\alpha K_{m i j k}) + \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta K_{\alpha \beta k j}, \\
 C &= \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - \sum_{i,j,k,\alpha} (h_{ij}^\alpha K_{\alpha k i k j} + h_{ij}^\alpha K_{\alpha i j k k}).
 \end{aligned}$$

Then

$$(3.3) \quad \frac{1}{2}\Delta S = A + B + C.$$

Let $a(x)$ and $b(x)$ denote the infimum and the supremum of the sectional curvature of N at a point x respectively. Now we derive a lower bound for B in terms of a , b , and S .

Lemma 2. $B \geq nbS - [n + \frac{2}{3}(p-1)(n-1)^{1/2}](b-a)S$.

Proof. Fix a vector e_α . Let $\{e_i\}$ be a frame diagonalizing the matrix (h_{ij}^α) such that

$$h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}, \quad 1 \leq i, j \leq n.$$

Then

$$\begin{aligned}
 (3.4) \quad &\sum_{i,j,k,m} h_{mj}^\alpha h_{ij}^\alpha K_{m k i k} + \sum_{i,j,k,m} h_{mk}^\alpha h_{ij}^\alpha K_{m i j k} + \sum_{i,j,k,\beta} h_{ij}^\alpha h_{ki}^\beta K_{\alpha \beta k j} \\
 &= \sum_{i,k} (\lambda_i^\alpha)^2 K_{i k i k} + \sum_{i,k} \lambda_k^\alpha \lambda_i^\alpha K_{k i i k} + \sum_{i,k,\beta} h_{ki}^\beta \lambda_i^\alpha K_{\alpha \beta k i}.
 \end{aligned}$$

By Proposition 2, we have

$$|K_{\alpha\beta ki}| \leq \frac{2}{3}(b - a) \quad \text{for } \alpha \neq \beta, i \neq k.$$

Hence, for fixed α , one sees

$$\begin{aligned} \sum_{i, k, \beta} h_{ki}^\alpha \lambda_i^\alpha K_{\alpha\beta ki} &\geq - \sum_{i \neq k, \beta \neq \alpha} \frac{2}{3}(b - a) |h_{ki}^\beta \lambda_i^\alpha| \\ (3.5) \quad &\geq - \sum_{i \neq k, \beta \neq \alpha} \frac{1}{3}(b - a) [(n - 1)^{1/2} (h_{ki}^\beta)^2 + (n - 1)^{-1/2} (\lambda_i^\alpha)^2] \\ &\geq -\frac{1}{3}(n - 1)^{1/2}(b - a) \sum_{\beta \neq \alpha} \text{tr } H_\beta^2 \\ &\quad - \frac{1}{3}(n - 1)^{1/2}(p - 1)(b - a) \text{tr } H_\alpha^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{i, k} (\lambda_i^\alpha)^2 K_{ikik} + \sum_{i, k} \lambda_k^\alpha \lambda_i^\alpha K_{kii k} \\ (3.6) \quad &= \frac{1}{2} \sum_{i, k} (\lambda_i - \lambda_k)^2 K_{ikik} \geq \frac{1}{2} a \sum_{i, k} (\lambda_i - \lambda_k)^2 = na \text{tr } H_\alpha^2. \end{aligned}$$

Substituting (3.5) and (3.6) into (3.4), we obtain

$$\begin{aligned} B &\geq \sum_{\alpha} [na \text{tr } H_\alpha^2 - \frac{1}{3}(n - 1)^{1/2}(b - a) \sum_{\beta \neq \alpha} \text{tr } H_\beta^2 \\ (3.7) \quad &\quad - \frac{1}{3}(n - 1)^{1/2}(p - 1)(b - a) \text{tr } H_\alpha^2] \\ &= nbS - [n + \frac{2}{3}(p - 1)(n - 1)^{1/2}](b - a)S. \end{aligned}$$

This proves Lemma 2.

We shall next estimate the integral of C .

Lemma 3. $\int_M C \geq -\frac{1}{72}pn(n - 1)(26n - 25) \int_M (b - a)^2.$

Proof. Note that

$$\begin{aligned} - \sum_{i, j, k, \alpha} (h_{ik}^\alpha K_{\alpha j i k} + h_{ij}^\alpha K_{\alpha i j k k}) \\ &= - \sum_{i, j, k, \alpha} \nabla_k (h_{ik}^\alpha K_{\alpha j i j} + h_{ij}^\alpha K_{\alpha i j k}) + \sum_{i, j, k, \alpha} (h_{ikk}^\alpha K_{\alpha j i j} + h_{ijk}^\alpha K_{\alpha i j k}). \end{aligned}$$

We define a differentiable 1-form as

$$(3.8) \quad \omega = \sum_{i, j, k, \alpha} (h_{ik}^\alpha K_{\alpha j i j} + h_{ij}^\alpha K_{\alpha i j k}) \omega_k.$$

It follows that

$$\text{div } \omega = \sum_{i, j, k, \alpha} \nabla_k (h_{ik}^\alpha K_{\alpha j i j} + h_{ij}^\alpha K_{\alpha i j k}).$$

Thus

$$C = \sum_{i, j, k, \alpha} (h_{ijk}^\alpha)^2 + \sum_{i, j, k, \alpha} (h_{ikk}^\alpha K_{\alpha j i j} + h_{ijk}^\alpha K_{\alpha i j k}) - \text{div } \omega.$$

Since M is minimal, we have

$$(3.9) \quad \sum_i h_{ij}^\alpha = 0 \quad \text{for all } j, \alpha.$$

From (2.5), (3.9), and Proposition 2, we have

$$(3.10) \quad \sum_{i,j,k,\alpha} h_{ikk}^\alpha K_{\alpha j i j} = \sum_{i,j,k,\alpha} (h_{k k i}^\alpha - K_{\alpha k i k}) K_{\alpha j i j} = - \sum_{i,\alpha} \left(\sum_j K_{\alpha j i j} \right)^2 \geq -\frac{1}{4} p n (n-1)^2 (b-a)^2.$$

On the other hand, by Proposition 2, we have

$$(3.11) \quad \begin{aligned} & \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,k,\alpha} h_{ijk}^\alpha K_{\alpha i j k} \\ & \geq -\frac{1}{4} \sum_{i,j,k,\alpha} (K_{\alpha i j k})^2 \\ & \geq -\frac{1}{4} \sum_{\alpha} \sum_{i,j,k \text{ distinct}} (K_{\alpha i j k})^2 - \frac{1}{2} \sum_{\alpha} \sum_{i \neq j} (K_{\alpha i j i})^2 \\ & \geq -\frac{1}{9} p n (n-1)(n-2)(b-a)^2 - \frac{1}{8} p n (n-1)(b-a)^2. \end{aligned}$$

So

$$(3.12) \quad C \geq -\frac{1}{72} p n (n-1)(26n-25)(b-a)^2 - \text{div } \omega,$$

and by using Green's divergence theorem, we get

$$(3.13) \quad \int_M C \geq -\frac{1}{72} p n (n-1)(26n-25) \int_M (b-a)^2.$$

Lemma 3 follows.

Now we define

$$\begin{aligned} D(n, p) &= n + \frac{2}{3}(p-1)(n-1)^{1/2}, \\ E(n, p) &= \frac{1}{72} p n (n-1)(26n-25). \end{aligned}$$

Theorem 1 (Generalized Simons inequality). *Let M^n be an n -dimensional oriented closed minimal submanifold in an $(n+p)$ -dimensional Riemannian manifold N^{n+p} . Denote the infimum and the supremum of the sectional curvature of N at a point x by $a(x)$ and $b(x)$ respectively. Then*

$$\int_M [nbS - (1 + \frac{1}{2} \text{sgn}(p-1))S^2 - D(n, p)(b-a)S - E(n, p)(b-a)^2] \leq 0.$$

Proof. Combining Proposition 1, Lemma 1 and 2, we obtain

$$(3.14) \quad \frac{1}{2} \Delta S \geq nbS - (1 + \frac{1}{2} \text{sgn}(p-1))S^2 - [n + \frac{2}{3}(p-1)(n-1)^{1/2}](b-a)S + C.$$

Integrating both sides of (3.14) and applying Lemma 3, we have

$$(3.15) \quad \int_M [nbS - (1 + \frac{1}{2} \text{sgn}(p-1))S^2 - D(n, p)(b-a)S - E(n, p)(b-a)^2] \leq 0.$$

This completes the proof of Theorem 1.

Denote

$$\alpha(n, p) = \frac{1}{12}[pn(n-1)(52n-50)]^{1/2},$$

$$\beta(n, p) = n + \frac{2}{3}(p-1)(n-1)^{1/2} + \frac{1}{12}[pn(n-1)(52n-50)]^{1/2}.$$

We are now in a position to prove

Theorem 2. *There is a number $\delta(n, p)$ with $0 < \delta(n, p) < 1$ such that if there exists an oriented closed minimal submanifold M^n in a complete simply connected manifold N^{n+p} with $\delta(n, p) \leq K_N \leq 1$ and*

$$\alpha(n, p)(1 - c) \leq S \leq n - \frac{1}{3}n \operatorname{sgn}(p - 1) - \beta(n, p)(1 - c),$$

where c is the infimum of the sectional curvature of N , then either M is the unit sphere $S^n(1)$, one of the Clifford minimal hypersurfaces $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$, $k = 1, 2, \dots, n - 1$, in $S^{n+1}(1)$, or the Veronese surface in $S^4(1)$. Moreover, $N = S^{n+p}(1)$.

Proof. Since

$$c \leq a(x) \leq b(x) \leq 1,$$

(3.15) gives

$$(3.16) \quad \int_M [nS - (1 + \frac{1}{2} \operatorname{sgn}(p - 1))S^2 - D(n, p)(1 - c)S - E(n, p)(1 - c)^2] \leq 0.$$

Take

$$\delta(n, p) = 1 - n(3 - \operatorname{sgn}(p - 1))(3D(n, p) + 6E^{1/2}(n, p))^{-1}.$$

Then

$$\alpha(n, p)(1 - c) \leq n - \frac{1}{3}n \operatorname{sgn}(p - 1) - \beta(n, p)(1 - c).$$

From the assumption

$$(3.17) \quad \alpha(n, p)(1 - c) \leq S \leq n - \frac{1}{3}n \operatorname{sgn}(p - 1) - \beta(n, p)(1 - c),$$

we see that

$$(3.18) \quad nS - (1 + \frac{1}{2} \operatorname{sgn}(p - 1))S^2 - D(n, p)(1 - c)S - E(n, p)(1 - c)^2 \geq 0.$$

Therefore, all inequalities in (3.10), (3.11), (3.15), and (3.18) are actually equalities. This implies $1 - c = b - a = 0$ and N is a complete simply connected Riemannian manifold with constant curvature 1. Hence $N = S^{n+p}(1)$. This together with (3.16) and (3.18) gives

$$S = 0 \quad \text{or} \quad S = n - \frac{1}{3}n \operatorname{sgn}(p - 1).$$

Furthermore, the previous inequalities become equalities, and it is not hard to see from Proposition 1 that either M is the unit sphere $S^n(1)$, one of the Clifford hypersurfaces $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n})$, $k = 1, 2, \dots, n - 1$, or the Veronese surface. This proves Theorem 2.

Remark 1. Theorem 2 can be considered as a generalization of the main theorems of [2, 6] as well as a pinching theorem for ambient manifolds.

Theorem 3. Let M^n be an oriented closed minimal submanifold with parallel second fundamental form in a Riemannian manifold N^{n+p} . Then

(i) $S \leq pnd + F(n, p)(d - c)$, where $F(n, p) = \frac{2}{3}p(p - 1)(n - 1)^{1/2}$ and d is the supremum of the sectional curvature of N ,

(ii) if $\delta'(n, p) < K_N \leq 1$, here

$$\delta'(n, p) = 1 - n(3 - \operatorname{sgn}(p - 1))[3n + 2(p - 1)(n - 1)^{1/2}]^{-1},$$

then either M is totally geodesic or $n - \frac{1}{3}n \operatorname{sgn}(p - 1) - D(n, p)(1 - c) \leq S \leq pn + F(n, p)(1 - c)$.

Proof. From the proof of Lemma 3 we have

$$C = - \sum_{i, j, k, \alpha} \nabla_k (h_{ik}^\alpha K_{\alpha j i j} + h_{ij}^\alpha K_{\alpha i j k}).$$

It is easy to see from (2.5) that $K_{\alpha i j k} = 0$, for all i, j, k, α . So

$$(3.19) \quad C = 0.$$

Since $\frac{1}{2}\Delta S = \sum (h_{ijk}^\alpha)^2 + \sum h_{ij}^\alpha \Delta h_{ij}^\alpha = 0$, S is a constant. This together with (3.3) and (3.9) implies

$$(3.20) \quad A + B = 0.$$

Obviously,

$$(3.21) \quad \sum_{\alpha, \beta} N(H_\alpha H_\beta - H_\beta H_\alpha) + \sum_{\alpha} S_\alpha^2 \geq S^2/p.$$

For fixed α , similar to the estimate of lower bound for B , we have

$$\begin{aligned} \text{LHS of (3.4)} &= \sum_{i, k} (\lambda_i^\alpha)^2 K_{ikik} + \sum_{i, k} \lambda_k^\alpha \lambda_i^\alpha K_{kii k} + \sum_{i, k, \beta} h_{ki}^\beta \lambda_i^\alpha K_{\alpha \beta ki} \\ &\leq nd \operatorname{tr} H_\alpha^2 + \frac{1}{3}(p - 1)(n - 1)^{1/2}(d - c) \operatorname{tr} H_\alpha^2 \\ &\quad + \frac{1}{3}(n - 1)^{1/2}(d - c) \sum_{\beta \neq \alpha} \operatorname{tr} H_\beta^2. \end{aligned}$$

This gives

$$(3.22) \quad B \leq ndS + \frac{2}{3}(p - 1)(n - 1)^{1/2}(d - c)S.$$

It follows from (3.20), (3.21), and (3.22) that

$$ndS + \frac{2}{3}(p - 1)(n - 1)^{1/2}(d - c)S \geq S^2/p.$$

This yields

$$(3.23) \quad S \leq pnd + \frac{2}{3}p(p - 1)(n - 1)^{1/2}(d - c).$$

If $\delta'(n, p) < K_N \leq 1$, it is not hard to see from the definition of $\delta'(n, p)$ that

$$(3.24) \quad n - \frac{1}{3}n \operatorname{sgn}(p - 1) - D(n, p)(1 - c) > 0.$$

By (3.20), Proposition 1, and Lemma 2, we get

$$(3.25) \quad nS - (1 + \frac{1}{3} \operatorname{sgn}(p - 1))S^2 - D(n, p)(1 - c)S \leq 0,$$

which together with (3.23) implies that either $S = 0$ or $n - \frac{1}{3}n \operatorname{sgn}(p - 1) - D(n, p)(1 - c) \leq S \leq pn + F(n, p)(1 - c)$. This completes the proof of Theorem 3.

Remark 2. When $p = 1$, the constant $\delta'(n, p)$ equals zero, which is independent of dimension. In this case, we have that either $S = 0$ or $nc \leq S \leq n$.

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