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ON THE L^2 INEQUALITIES INVOLVING TRIGONOMETRIC POLYNOMIALS AND THEIR DERIVATIVES

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ABSTRACT. In this note we study the upper bound of the integral

$$\int_0^\pi (t^{(k)}(x))^2 w(x)\,dx$$

where $t(x)$ is a trigonometric polynomial with real coefficients such that $\|t\|_\infty \le 1$ and $w(x)$ is a nonnegative function defined on $[0, \pi]$. When $w(x) = \sin^j x$, where j is a positive integer, we obtain the exact upper bound for the above integral.

1. INTRODUCTION

The purpose of this note is to investigate the following quantity:

$$\sup_{\|t\|_\infty \le 1} \int_0^\pi (t_n^{(k)}(x))^2 w(x)\,dx \tag{1}$$

where $t_n(x)$ is a trigonometric polynomial of degree n with real coefficients such that

$$\|t_n\|_\infty = \max_{0\le x\le\pi} |t_n(x)| \le 1,$$

and $w(x)$ is a nonnegative function defined on $[0, \pi]$.

When $w(x) = \sin x$ and $k = 1$, Varma ([6] and [7]) obtained the exact bound for (1). In the case $w(x) = \sin^2 x$ and $k = 1$, the sharp bound was found by Shen [3]. As Shen pointed out his method fails for $w(x) = \sin^3 x$. However, Varma, Mills, and Smith [8] managed to get the exact bounds for $w(x) = \sin^3 x$, $k = 1$, and $w(x) = \sin x$, $k = 2$ and 3, when $t_n(x)$ is a real even trigonometric polynomial of degree n. For the applications of these inequalities, please see [6], [7], and [8]. In this note we apply a new technique to make uniform the proofs and generalize the results in [3], [6], and [8] to the weight functions $w(x) = \sin^j x$, where j is a positive integer, and the higher derivative cases.

For convenience, we introduce some necessary notations first. Let $\mathbf{T}_n$ be the collection of all real trigonometric polynomials of degree $\le n$ bounded by 1 on

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the interval $[0, \pi]$. Denote

$$\mathbf{T}_n^e := \{t_n(x) : t_n(x) \text{ is an even function in } \mathbf{T}_n\},$$
$$\mathbf{T}_n^0 := \{t_n(x) : t_n(x) \text{ is an odd function in } \mathbf{T}_n\},$$

and

$$\mathbf{N} := \{\text{all positive integers}\}.$$

Among other results, in this note we establish the following

Theorem. *Let $t_n(x) \in \mathbf{T}_n$ and $k \in \mathbf{N}$, $k \geq 2$. Then for $n \geq 4$,*

$$\int_0^\pi (t_n^{(k)}(x))^2 \sin^4 x \, dx \leq \frac{\pi}{16} n^{2k} \left(1 - \frac{16}{n^2}\right) \tag{2}$$

with equality if and only if $t_n(x) = \cos n(x - x_0)$.

This is Theorem 4 in Section 3. The analogue of the above theorem for $w(x) = \sin^3 x$ is proved in Section 3 too. In fact, the technique used in the proof of the above theorem also works in the cases $w(x) = \sin^j x$, $j \geq 5$. We discuss this in Section 4. In Section 2, we investigate the cases $w(x) = \sin x$ and $w(x) = \sin^2 x$. The results in [3], [6] and [8] are generalized to the higher derivative case. These inequalities are used later in Section 3.

2. The cases $w(x) = \sin x$, $\sin^2 x$

In this section we consider the cases $w(x) = \sin x$, $\sin^2 x$. The exact bounds of (1) are obtained in Theorem 1 and 2, which are the extensions of the results in [6] and [3]. In Section 3, we apply Theorem 2 to prove the theorem stated in the introduction.

In the proofs of our theorems, we need the following lemma. A similar lemma was stated in [3], and the same process was used in [7] and [8]. For the sake of completeness, we provide a proof here.

Lemma 1. *Let $t_n(x) \in \mathbf{T}_n$, $k \in \mathbf{N}$, and $w(x) \in C^2[0, \pi]$. Then*

$$\begin{aligned}
&\int_0^\pi (t_n^{(k)}(x))^2 \left(2w(x) + \frac{1}{2n^2} w''(x)\right) dx \\
&\quad = A + B + \frac{1}{2} \int_0^\pi [n^2 (t_n^{(k-1)}(x))^2 + (t_n^{(k)}(x))^2](w(x) + w''(x)/n^2) \, dx \\
&\qquad + \frac{1}{2n^2} \int_0^\pi [n^2 (t_n^{(k)}(x))^2 + (t_n^{(k+1)}(x))^2] w(x) \, dx \\
&\qquad - \frac{1}{2n^2} \int_0^\pi [n^2 t_n^{(k-1)}(x) + t_n^{(k+1)}(x)]^2 w(x) \, dx,
\end{aligned} \tag{3}$$

where $A = t_n^{(k-1)}(x) t_n^{(k)}(x) w(x)|_0^\pi$, and $B = -\frac{1}{2}(t_n^{(k-1)}(x))^2 w'(x)|_0^\pi$.

Proof. Since

$$\begin{aligned}
&\int_0^\pi (t_n^{(k)}(x))^2 w(x) dx \\
&\quad = t_n^{(k-1)}(x) t_n^{(k)}(x) w(x)|_0^\pi \\
&\qquad - \int_0^\pi t_n^{(k-1)}(x) t_n^{(k+1)}(x) w(x) dx - \int_0^\pi t_n^{(k-1)}(x) t_n^{(k)}(x) w'(x) \, dx,
\end{aligned}$$

and

$$\begin{aligned}&\int_0^\pi t_n^{(k-1)}(x)t_n^{(k)}(x)w'(x)\,dx\\&\qquad=(t_n^{(k-1)}(x))^2w'(x)|_0^\pi-\int_0^\pi(t_n^{(k-1)}(x))^2w''(x)\,dx\\&\qquad\quad-\int_0^\pi t_n^{(k-1)}(x)t_n^{(k)}(x)w'(x)\,dx,\end{aligned}$$

we have

$$\begin{aligned}&\int_0^\pi(t_n^{(k)}(x))^2w(x)\,dx\\&\quad=\frac{1}{2}\int_0^\pi(t_n^{(k-1)}(x))^2w''(x)\,dx\\&\qquad+A+B-\int_0^\pi t_n^{(k-1)}(x)t_n^{(k+1)}(x)w(x)\,dx\\&\quad=\frac{1}{2n^2}\int_0^\pi[n^2(t_n^{(k-1)}(x))^2+(t_n^{(k)}(x))^2]w''(x)\,dx\\&\qquad+A+B-\frac{1}{2n^2}\int_0^\pi(t_n^{(k)}(x))^2w''(x)\,dx\\&\qquad-\frac{1}{2n^2}\int_0^\pi[n^2t_n^{(k-1)}(x)+t_n^{(k+1)}(x)]^2w(x)\,dx\\&\qquad+\frac{n^2}{2}\int_0^\pi(t_n^{(k-1)}(x))^2w(x)\,dx+\frac{1}{2n^2}\int_0^\pi(t_n^{(k+1)}(x))^2w(x)\,dx.\end{aligned}$$

Now, we rewrite

$$\begin{aligned}&\frac{n^2}{2}\int_0^\pi(t_n^{(k-1)}(x))^2w(x)\,dx+\frac{1}{2n^2}\int_0^\pi(t_n^{(k+1)}(x))^2w(x)\,dx\\&\quad=\frac{1}{2}\int_0^\pi[n^2(t_n^{(k-1)}(x))^2+(t_n^{(k)}(x))^2]w(x)\,dx-\int_0^\pi(t_n^{(k)}(x))^2w(x)\,dx\\&\qquad+\frac{1}{2n^2}\int_0^\pi[n^2(t_n^{(k)}(x))^2+(t_n^{(k+1)}(x))^2]w(x)\,dx,\end{aligned}$$

from which we have (3). □

Theorem 1. *Let* $t_n(x)\in\mathbf{T}_n$ *and* $k\in\mathbf{N}$. *Then*

$$\int_0^\pi(t_n^{(k)}(x))^2\sin x\,dx\le n^{2k}\left(1+\frac{1}{4n^2-1}\right)\tag{4}$$

with equality if and only if $t_n^{(k)}(x)=\pm n^k\sin nx$.

Proof. Since $w(x)=\sin x$ and $t_n(x)\in\mathbf{T}_n$, and by the Bernstein inequality (see [2]), we have

$$A=0\quad\text{and}\quad B\le n^{2(k-1)}.$$

(Notice that $B\le n^{2(k-1)}$ with equality if and only if $t_n^{(k-1)}(x)=\pm n^{k-1}\cos nx$.)

$$2w(x)+\frac{1}{2n^2}w''(x)=2\left(1-\frac{1}{4n^2}\right)\sin x,$$

and

$$w(x) + \frac{w''(x)}{n^2} = \left(1 - \frac{1}{n^2}\right) \sin x.$$

Combining the well-known inequality of G. Szegő [4, p. 64] and the Bernstein inequality, we have

$$n^2 (t_n^{(j)}(x))^2 + (t_n^{(j+1)}(x))^2 \leq n^{2(j+1)}, \qquad j = 0, 1, \dots, \tag{5}$$

with equality if and only if $t_n(x) = \pm \cos n(x - x_0)$. Then from (3) we obtain that

$$\begin{aligned} 2\left(1 - \frac{1}{4n^2}\right) & \int_0^\pi (t_n^{(k)}(x))^2 \sin x \, dx \\ & \leq n^{2(k-1)} + \frac{n^{2k}}{2}\left(1 - \frac{1}{n^2}\right) \int_0^\pi \sin x \, dx + \frac{n^{2(k+1)}}{2n^2} \int_0^\pi \sin x \, dx \\ & = 2n^{2k}, \end{aligned}$$

which implies (4). □

If we restrict $t_n(x)$ to be an even or odd trigonometric polynomial, then we can obtain a smaller bound. More precisely, we have

Theorem 1A. *Let* $t_n(x) \in \mathbf{T}_n^e$ *and let* k *be an even integer. Then*

$$\int_0^\pi (t_n^{(k)}(x))^2 \sin x \, dx \leq n^{2k} \left(1 - \frac{1}{4n^2 - 1}\right) \tag{6}$$

with equality if and only if $t_n(x) = \pm \cos nx$.

Proof. Notice that $t_n^{(k-1)}(x)$ is an odd trigonometric polynomial in this case; thus $B = 0$. From (3), we have

$$\begin{aligned} 2\left(1 - \frac{1}{4n^2}\right) & \int_0^\pi (t_n^{(k)}(x))^2 \sin x \, dx \\ & \leq \frac{n^{2k}}{2}\left(1 - \frac{1}{n^2}\right) \int_0^\pi \sin x \, dx + \frac{n^{2(k+1)}}{2n^2} \int_0^\pi \sin x \, dx \\ & = n^{2k}\left(2 - \frac{1}{n^2}\right), \end{aligned}$$

from which we complete the proof of the theorem. □

Remark 1. The referee kindly pointed out the following proof of the above theorem. Set $s_n(x) = t_n^{(k-2)}(x)/n^{k-2}$; then $s_n(x) \in \mathbf{T}_n^e$. Now applying Varma, Mills and Smith [8] (Lemma 2.1) to $s_n(x)$ we have (6). The same procedure works for Theorem 1B (applying the result in [6] (Lemma 1)) and Theorem 2 (using the result in [3]) as the referee observed.

Similarly, one can establish

Theorem 1B. *Let* $t_n(x) \in \mathbf{T}_n^0$ *and let* k *be an odd integer. Then*

$$\int_0^\pi (t_n^{(k)}(x))^2 \sin x \, dx \leq n^{2k} \left(1 - \frac{1}{4n^2 - 1}\right) \tag{7}$$

with equality if and only if $t_n(x) = \pm \sin nx$.

When $w(x) = \sin^2 x$, the argument in [3] can be utilized to prove the following theorem. We shall apply another technique, which is easier and more direct in the author's opinion. The same idea is used in Section 3.

Theorem 2. *Let $t_n(x) \in \mathbf{T}_n$ and $k \in \mathbf{N}$. Then for $n \geq 2$,*

$$\int_0^\pi (t_n^{(k)}(x))^2 \sin^2 x\,dx \leq \frac{\pi}{4} n^{2k} \tag{8}$$

with equality if and only if $t_n(x) = \cos n(x - x_0)$.

Proof. Let $w(x) = \sin^2 x$ in Lemma 1. Notice that $A = B = 0$ in this case, and

$$w''(x) = (\sin^2 x)'' = 2 - 4\sin^2 x.$$

From (3) we have

$$\begin{aligned}
&2\left(1 - \frac{1}{n^2}\right) \int_0^\pi (t_n^{(k)}(x))^2 \sin^2 x\,dx \\
&= -\frac{1}{n^2} \int_0^\pi (t_n^{(k)}(x))^2\,dx \\
&\quad + \frac{1}{2} \int_0^\pi [n^2 (t_n^{(k-1)}(x))^2 + (t_n^{(k)}(x))^2] \left(\sin^2 x + \frac{2 - 4\sin^2 x}{n^2}\right) dx \\
&\quad + \frac{1}{2n^2} \int_0^\pi [n^2 (t_n^{(k)}(x))^2 + (t_n^{(k+1)}(x))^2] \sin^2 x\,dx \\
&\quad - \frac{1}{2n^2} \int_0^\pi [n^2 t_n^{(k-1)}(x) + t_n^{(k+1)}(x)]^2 \sin^2 x\,dx \\
&\leq -\frac{1}{n^2} \int_0^\pi (t_n^{(k)}(x))^2 dx + \frac{n^{2k}}{2} \left(1 - \frac{4}{n^2}\right) \int_0^\pi \sin^2 x\,dx \\
&\quad + \frac{1}{n^2} \int_0^\pi [n^2 (t_n^{(k-1)}(x))^2 + (t_n^{(k)}(x))^2]\,dx + \frac{n^{2k}}{2} \int_0^\pi \sin^2 x\,dx \\
&= \frac{\pi}{4}\left(1 - \frac{4}{n^2}\right) n^{2k} + \int_0^\pi (t_n^{(k-1)}(x))^2\,dx + \frac{\pi}{4} n^{2k} \\
&\leq \frac{\pi}{4}\left(2 - \frac{4}{n^2}\right) n^{2k} + \frac{\pi}{2} n^{2(k-1)}.
\end{aligned}$$

We used the inequality in [5, Theorem 1] in the last step. This completes the proof of Theorem 2. □

Remark 2. The inequality which we used in the proof was also proved by Kristiansen [1, Corollary 2]. However, part of Corollary 1 in [1] is not valid as $f(x) = 1 - x^2$ is a counterexample in the L^2 norm case.

Remark 3. We can apply the technique in the proof of Theorem 1 to establish the following: If $t_n(x) \in \mathbf{T}_n^e \cup \mathbf{T}_n^0$ and $k \in \mathbf{N}$, then

$$\int_0^\pi (t_n^{(k)}(x))^2 dx \leq \frac{\pi}{2} n^{2k}$$

with equality if and only if $t_n(x) = \pm\cos nx$ or $\pm\sin nx$. This inequality is much weaker than the result in [1] and [5].

3. The cases $w(x) = \sin^3 x$, $\sin^4 x$

The main object in this section is to prove the theorem stated in Section 1. We first establish the inequality when $w(x) = \sin^3 x$.

Theorem 3A. *Let* $t_n(x) \in \mathbf{T}_n^e$, $k \geq 2$, *and let* k *be an odd integer. Then for* $n \geq 3$,

$$\int_0^\pi (t_n^{(k)}(x))^2 \sin^3 x\, dx \leq \frac{2}{3} n^{2k} \left(1 - \frac{9}{(4n^2-1)(4n^2-9)}\right) \tag{9}$$

with equality if and only if $t_n(x) = \pm \cos nx$.

Remark 4. As the referee pointed out, we can use the same procedure mentioned after the proof of Theorem 1A to prove the above theorem (applying the result of [8] (Lemma 2.2)). We can also prove it using Theorem 1A and (3) as we demonstrate in the proof of Theorem 3B.

From Theorem 1B and Lemma 1, we can prove

Theorem 3B. *Let* $t_n(x) \in \mathbf{T}_n^0$, $k \geq 2$, *and let* k *be an even integer. Then for* $n \geq 3$,

$$\int_0^\pi (t_n^{(k)}(x))^2 \sin^3 x\, dx \leq \frac{2}{3} n^{2k} \left(1 - \frac{9}{(4n^2-1)(4n^2-9)}\right) \tag{10}$$

with equality if and only if $t_n(x) = \pm \sin nx$.

Proof. Since there is no preliminary result available, we apply Theorem 1B and (3) to prove this theorem directly. Let $w(x) = \sin^3 x$; then $A = B = 0$. And notice that

$$(\sin^3 x)'' = 6 \sin x - 9 \sin^3 x,$$
$$2w(x) + \frac{1}{2n^2} w''(x) = \left(2 - \frac{9}{2n^2}\right) \sin^3 x + \frac{3}{n^2} \sin x,$$

and

$$w(x) + \frac{w''(x)}{n^2} = \left(1 - \frac{9}{n^2}\right) \sin^3 x + \frac{6}{n^2} \sin x.$$

Therefore, from (3) we have

$$\begin{aligned}
&\int_0^\pi (t_n^{(k)}(x))^2 \left[\left(2 - \frac{9}{2n^2}\right) \sin^3 x + \frac{3}{n^2} \sin x\right] dx \\
&\quad = \frac{1}{2} \int_0^\pi [n^2 (t_n^{(k-1)}(x))^2 + (t_n^{(k)}(x))^2] \left[\left(1 - \frac{9}{n^2}\right) \sin^3 x + \frac{6}{n^2} \sin x\right] dx \\
&\quad + \frac{1}{2n^2} \int_0^\pi [n^2 (t_n^{(k)}(x))^2 + (t_n^{(k+1)}(x))^2] \sin^3 x\, dx \\
&\quad - \frac{1}{2n^2} \int_0^\pi [n^2 t_n^{(k-1)}(x) + t_n^{(k+1)}(x)]^2 \sin^3 x\, dx,
\end{aligned}$$

which implies

$$\begin{aligned}
&\left(2 - \frac{9}{2n^2}\right) \int_0^\pi (t_n^{(k)}(x))^2 \sin^3 x\, dx \\
&\quad \leq \frac{n^{2k}}{2} \left(1 - \frac{9}{n^2}\right) \int_0^\pi \sin^3 x\, dx + 3 \int_0^\pi (t_n^{(k-1)}(x))^2 \sin x\, dx \\
&\quad + \frac{n^{2k}}{2} \int_0^\pi \sin^3 x\, dx \\
&\quad \leq \frac{2}{3} n^{2k} \left(2 - \frac{9}{2n^2} - \frac{9}{2n^2(4n^2-1)}\right).
\end{aligned}$$

In the last inequality we applied Theorem 1B. This gives us the desired result. □

Instead of using inequalities in Theorems 1A and 1B, if we use the inequality in Theorem 1, then we obtain

Theorem 3. *Let* $t_n(x) \in \mathbf{T}_n$ *and* $k \geq 2$, $k \in \mathbf{N}$. *Then for* $n \geq 3$,

$$\int_0^\pi (t_n^{(k)}(x))^2 \sin^3 x\, dx \leq \frac{2}{3} n^{2k} \left(1 + \frac{9}{(4n^2-1)(4n^2-9)}\right) \tag{11}$$

with equality if and only if $t_n^{(k-1)}(x) = \pm n^{k-1} \sin nx$.

When $w(x) = \sin^4 x$, the same argument works. In this case, notice that

$$(\sin^4 x)'' = 12 \sin^2 x - 16 \sin^4 x,$$

$$2w(x) + \frac{1}{2n^2} w''(x) = \left(2 - \frac{8}{n^2}\right) \sin^4 x + \frac{6}{n^2} \sin^2 x,$$

and

$$w(x) + \frac{w''(x)}{n^2} = \left(1 - \frac{16}{n^2}\right) \sin^4 x + \frac{12}{n^2} \sin^2 x.$$

Then using Lemma 1 and Theorem 2, we obtain

Theorem 4. *Let* $t_n(x) \in \mathbf{T}_n$ *and* $k \in \mathbf{N}$, $k \geq 2$. *Then for* $n \geq 4$,

$$\int_0^\pi (t_n^{(k)}(x))^2 \sin^4 x\, dx \leq \frac{3\pi}{16} n^{2k} \tag{12}$$

with equality if and only if $t_n(x) = \cos n(x - x_0)$.

4. The cases $w(x) = \sin^j x$, $j \geq 5$

In the cases $w(x) = \sin^j x$, $j \geq 5$, we can still apply the argument in the proof of Theorem 2. In general, when $w(x) = \sin^j x$, we have

$$(\sin^j x)'' = j(j-1) \sin^{j-2} x - j^2 \sin^j x,$$

$$2w(x) + \frac{1}{2n^2} w''(x) = \left(2 - \frac{j^2}{2n^2}\right) \sin^j x + \frac{j(j-1)}{2n^2} \sin^{j-2} x,$$

and

$$w(x) + \frac{1}{n^2} w''(x) = \left(1 - \frac{j^2}{n^2}\right) \sin^j x + \frac{j(j-1)}{n^2} \sin^{j-2} x.$$

From Lemma 1 and mathematical induction, we can prove the corresponding inequalities. But these inequalities become more and more complicated. We do not formulate them in this note. However, we can state them in the following fashion, which can be proved by induction.

Theorem 5. *Let* $t_n(x) \in \mathbf{T}_n$, *let* j *be an odd integer, and* $k \geq \frac{j+1}{2}$, $k \in \mathbf{N}$. *Then for* $n \geq j$,

(13)

$$\int_0^\pi (t_n^{(k)}(x))^2 \sin^j x\, dx \leq n^{2k} \int_0^\pi \sin^2 \left(nx + \left(1 - (-1)^{(j-1)/2}\right) \frac{\pi}{4}\right) \sin^j x\, dx$$

with equality if and only if $t_n^{(k-(j-1)/2)}(x) = \pm n^{k-(j-1)/2} \sin nx$.

Proof. We use mathematical induction on j. When $j = 1$, the theorem is valid, which is Theorem 1. Suppose the theorem is true for $j-2$. Let $w(x) = \sin^j x$, and note that $A = B = 0$. From (3) we have

$$\int_0^\pi (t_n^{(k)}(x))^2 \left[\left(2 - \frac{j^2}{2n^2}\right) \sin^j x + \frac{j(j-1)}{2n^2} \sin^{j-2} x\right] dx$$
$$\leq \frac{1}{2}\int_0^\pi [n^2(t_n^{(k-1)}(x))^2 + (t_n^{(k)}(x))^2]\left[\left(1 - \frac{j^2}{n^2}\right) \sin^j x + \frac{j(j-1)}{n^2} \sin^{j-2} x\right] dx$$
$$+ \frac{1}{2n^2}\int_0^\pi [n^2(t_n^{(k)}(x))^2 + (t_n^{(k+1)}(x))^2] \sin^j x \, dx,$$

which implies

$$\left(2 - \frac{j^2}{2n^2}\right)\int_0^\pi (t_n^{(k)}(x))^2 \sin^j x \, dx$$
$$\leq \frac{n^{2k}}{2}\left(1 - \frac{j^2}{n^2}\right)\int_0^\pi \sin^j x \, dx + \frac{n^{2k}}{2}\int_0^\pi \sin^j x \, dx$$
$$+ \frac{j(j-1)}{2}\int_0^\pi (t_n^{(k-1)}(x))^2 \sin^{j-2} x \, dx$$
$$\leq \frac{n^{2k}}{2}\left(1 - \frac{j^2}{n^2}\right)\int_0^\pi \sin^j x \, dx + \frac{n^{2k}}{2}\int_0^\pi \sin^j x \, dx$$
$$+ \frac{j(j-1)}{2} n^{2(k-1)} \int_0^\pi \left[\sin\left(nx + (1 - (-1)^{(j-3)/2})\frac{\pi}{4}\right)\right]^2 \sin^{j-2} x \, dx.$$

The last inequality becomes equality if and only if

$$t_n^{(k-1-(j-3)/2)}(x) = \pm n^{k-1-(j-3)/2} \sin nx,$$

that is

$$t_n^{(k-(j-1)/2)}(x) = \pm n^{k-(j-1)/2} \sin nx.$$

But in this case, the last second inequality becomes equality too. This completes the proof. □

Theorem 6. *Let* $t_n(x) \in \mathbf{T}_n$, *let* j *be an even integer, and* $k \geq \frac{j}{2}$, $k \in \mathbf{N}$. *Then for* $n \geq j$,

$$\int_0^\pi (t_n^{(k)}(x))^2 \sin^j x \, dx \leq \int_0^\pi (\cos^{(k)} nx)^2 \sin^j x \, dx \tag{14}$$

with equality if and only if $t_n(x) = \cos n(x - x_0)$.

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