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HYPERSURFACES IN SPACE FORMS SATISFYING THE CONDITION $\Delta x = Ax + B$

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ABSTRACT. In this work we study and classify pseudo-Riemannian hypersurfaces in pseudo-Riemannian space forms which satisfy the condition $\Delta x = Ax + B$, where A is an endomorphism, B is a constant vector, and x stands for the isometric immersion. We prove that the family of such hypersurfaces consists of open pieces of minimal hypersurfaces, totally umbilical hypersurfaces, products of two nonflat totally umbilical submanifolds, and a special class of quadratic hypersurfaces.

0. INTRODUCTION

Let x be an isometric immersion of a hypersurface M_s^n in $\mathbb{R}_t^{n+1}$ and assume there exist an endomorphism A of $\mathbb{R}_t^{n+1}$ and a constant vector B in $\mathbb{R}_t^{n+1}$ such that $\Delta x = Ax + B$. We ask the following question: "*What is the geometric meaning involved in that algebraic condition*?" This question was first studied in the Euclidean case by Chen and Petrovic [4], Dillen, Pas, and Verstraelen [5], and Hasanis and Vlachos [7], who obtained some interesting classification theorems. Recently, Park [10], following closely the ideas in [1] and [2], has considered that condition with $B = 0$ for hypersurfaces in Euclidean spherical and hyperbolic spaces.

To study that question in its full generality, it seemed natural to us to begin with Lorentzian surfaces [1]. Later, in [2], in order to generalize the above papers we gave a classification theorem for pseudo-Euclidean hypersurfaces. Actually, we proved that the only hypersurfaces satisfying the matricial condition on the Laplacian are open pieces of minimal hypersurfaces, totally umbilical hypersurfaces and pseudo-Riemannian products of a totally umbilical and a totally geodesic submanifold.

This paper arises as a natural continuation of [1] and [2], taking now a nonflat pseudo-Riemannian space form as the ambient space. Here, we analyze the isometric immersions x of a hypersurface M_s^n of $\overline{M}_\nu^{n+1}$ satisfying $\Delta x = Ax + B$, where $\overline{M}_\nu^{n+1}$ is the pseudo-Euclidean sphere $\mathbb{S}_\nu^{n+1} \subset \mathbb{R}_\nu^{n+2}$ or the pseudo-Euclidean hyperbolic space $\mathbb{H}_\nu^{n+1} \subset \mathbb{R}_{\nu+1}^{n+2}$.

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In this new situation, the codimension of the manifold M_s^n in the pseudo-Euclidean space where it is lying is two, so that one hopes to find a richer family of examples satisfying the asked condition. On the other hand, although the proofs given in [2] do not work here, we follow the techniques developed there.

Before referring to the main result, we wish to point out that a lot of hypersurfaces having nondiagonalizable shape operator are given. This property makes substantially different this case and that treated in [2].

The main result of this paper states that the only hypersurfaces M_s^n of $\overline{M}_\nu^{n+1}$ satisfying the matricial condition on the Laplacian are open pieces of minimal hypersurfaces, totally umbilical hypersurfaces, pseudo-Riemannian products of two nonflat totally umbilical submanifolds and quadratic hypersurfaces defined by $\{x \in \mathbb{R}_t^{n+2} : \langle x, x\rangle = \pm 1, \langle Lx, x\rangle = c\}$, where L is a selfadjoint endomorphism of $\mathbb{R}_t^{n+2}$ with minimal polynomial μ_L of degree two, and c is a real constant such that $\mu_L(kc) \neq 0$.

1. Preliminaries

Let $\mathbb{R}_t^{n+2}$ be the $(n+2)$-dimensional pseudo-Euclidean space whose metric tensor is given by

$$ds^2 = -\sum_{i=1}^{t} dx^i \otimes dx^i + \sum_{j=t+1}^{n+2} dx^j \otimes dx^j,$$

where $(x_1, \ldots, x_{n+2})$ is the standard coordinate system. For each $k \neq 0$, let $\overline{M}_\nu^{n+1}(k)$ be the complete and simply connected space with constant sectional curvature $\operatorname{sign}(k)/k^2$. A model for $\overline{M}_\nu^{n+1}(k)$ is the pseudo-Euclidean sphere $\mathbb{S}_\nu^{n+1}(k)$ if $k > 0$ and the pseudo-Euclidean hyperbolic space $\mathbb{H}_\nu^{n+1}(k)$ if $k < 0$, where $\mathbb{S}_\nu^{n+1}(k) = \{x \in \mathbb{R}_\nu^{n+2} : \langle x, x\rangle = k^2\}$ and $\mathbb{H}_\nu^{n+1}(k) = \{x \in \mathbb{R}_{\nu+1}^{n+2} : \langle x, x\rangle = -k^2\}$, $\langle\ ,\ \rangle$ standing for the indefinite inner product in the pseudo-Euclidean space. Throughout this paper we will assume, without loss of generality, that $k^2 = 1$.

Let M_s^n be a pseudo-Riemannian hypersurface in $\overline{M}_\nu^{n+1}$ and let ∇ ($\overline{\nabla}$ and $\tilde{\nabla}$) denote the Levi-Civita connection on M_s^n ($\overline{M}_\nu^{n+1}$ and $\mathbb{R}_t^{n+2}$, respectively). We will also denote by N the unit normal vector field to M_s^n in $\overline{M}_\nu^{n+1}$. Let H' and H be the mean curvature vector fields of M_s^n in $\overline{M}_\nu^{n+1}$ and $\mathbb{R}_t^{n+2}$, respectively. Thus we may write $H' = \alpha N$, α being the mean curvature of M_s^n in $\overline{M}_\nu^{n+1}$, and

$$H = H' - kx = \alpha N - kx. \tag{1.1}$$

Let $x : M_s^n \to \overline{M}_\nu^{n+1}$ be an isometric immersion satisfying the condition

$$\Delta x = Ax + B, \tag{1.2}$$

where A is an endomorphism of $\mathbb{R}_t^{n+2}$ and B a constant vector in $\mathbb{R}_t^{n+2}$. Taking covariant derivative in (1.2) and using the Laplace-Beltrami equation $\Delta x = -nH$ and the Weingarten formula we get $AX = nS_H X - nD_X H$, for any vector field X tangent to M_s^n, where D denotes the normal connection on M_s^n and S_ξ the Weingarten endomorphism associated to a normal vector field

ξ. Then by (1.1) we have $D_X H = X(\alpha)N$ and $S_H X = \alpha SX + kX$, where, for short, we have written S for the Weingarten endomorphism S_N. From now on, we will call S the shape operator of M_s^n. Now from the above formulae we deduce that

$$AX = n(\alpha SX + kX) - nX(\alpha)N. \tag{1.3}$$

From (1.2), taking into account the Laplace-Beltrami equation and (1.1), we obtain the following equation:

$$Ax = -n\alpha N + nkx - B. \tag{1.4}$$

By applying the Laplacian on both sides of (1.2) and using again that $\Delta x = -nH$, we find $AH = \Delta H$; that along with (1.1) leads to $\alpha AN = \Delta H + kAx$. Now, bringing here (1.4) and the formula for ΔH obtained in [3, Lemma 3]

$$\Delta H = 2S(\nabla\alpha) + n\varepsilon\alpha\nabla\alpha + (\Delta\alpha + \varepsilon\alpha|S|^2 + nk\alpha)N - nk(k + \varepsilon\alpha^2)x,$$

where $\nabla\alpha$ stands for the gradient of α, $\varepsilon = \langle N, N\rangle$ and $|S|^2 = \operatorname{trace}(S^2)$, we get the following equation

$$\alpha AN = 2S(\nabla\alpha) + n\varepsilon\alpha\nabla\alpha + (\Delta\alpha + \varepsilon\alpha|S|^2)N - nk\varepsilon\alpha^2 x - kB. \tag{1.5}$$

2. Some examples

In this paper we wish to classify the hypersurfaces M_s^n in $\overline{M}_\nu^{n+1}$ whose isometric immersion satisfies the condition (1.2). In order to get such a classification we need some examples.

2.1. Minimal hypersurfaces M_s^n in $\overline{M}_\nu^{n+1}$ obviously satisfy (1.2). Indeed, by using (1.1) we have $H = -kx$ and $\Delta x = nkx$. So we can take $A = nkI_{n+2}$ and $B = 0$.

2.2. Let M_s^n be a totally umbilical hypersurface in $\overline{M}_\nu^{n+1}$. Taking into account the classification theorem for such hypersurfaces (see, for example, [9, Theorem 1.4]) we get, according to whether $\langle H, H\rangle$ is positive, negative, or zero, M_s^n is an open piece of a pseudo-Euclidean sphere $\mathbb{S}_s^n(r)$, a pseudo-Euclidean hyperbolic space $\mathbb{H}_s^n(r)$ or $\mathbb{R}_s^n$. In the last case, the immersion $f : \mathbb{R}_s^n \to \mathbb{R}_{s+1}^{n+2}$ is given by $f(u) = (q(u), u_1, \dots, u_n, q(u))$, where $q(u) = a\langle u, u\rangle + \langle b, u\rangle + c$, $a \neq 0$. The pseudo-Euclidean spheres and pseudo-Euclidean hyperbolic spaces both satisfy the condition (1.2). Indeed, by considering φ as the standard immersion of $\mathbb{S}_s^n(r)$ or $\mathbb{H}_s^n(r)$ in a hyperplane $\mathbb{R}_{s'}^{n+1}$ of $\mathbb{R}_t^{n+2}$, we know from [2] that $\Delta\varphi = L\varphi$, L being an endomorphism of $\mathbb{R}_{s'}^{n+1}$. The $(n+1)\times(n+1)$ matrix L and the immersion φ become an $(n+2)\times(n+2)$ matrix A (filling with zeros) and an immersion x in $\mathbb{R}_t^{n+2}$, respectively, in a natural way and so we get (1.2) with $B = 0$. Therefore the most interesting case is that with $\langle H, H\rangle = 0$. Now we can choose a point p in $\mathbb{R}_{s+1}^{n+2}$ such that $\langle f-p, f-p\rangle = \pm 1$ and then $x = f-p$ is an immersion from $\mathbb{R}_s^n$ in $\overline{M}_\nu^{n+1}$ with $\Delta x = -2n(a, 0, \dots, 0, a)$. Thus this hypersurface satisfies (1.2) with $A = 0$ and $B = (-2na, 0, \dots, 0, -2na)$. Furthermore, from the equation $\Delta x = -n\alpha N + nkx$, we easily obtain that its constant mean curvature α is given by $\alpha^2 = 1$.

2.3. Let $x : M_s^m \to \mathbb{R}_t^{m+1}$ and $y : M_{s'}'^{m'} \to \mathbb{R}_{t'}^{m'+1}$ be two isometric immersions satisfying the condition (1.2) and let $z = x \times y$ be the natural isometric immersion from the pseudo-Riemannian product $M_s^m \times M_{s'}'^{m'}$ in $\mathbb{R}_{t+t'}^{m+m'+2}$. If $\Delta x = Ax + B$ and $\Delta' y = A'y + B'$, then we can consider $\widetilde{A} = \operatorname{diag}[A, A']$ and $\widetilde{B} = (B, B')$. Thus it is easy to show that $\widetilde{\Delta} z = \widetilde{A} z + \widetilde{B}$. Then from [2, §2], we can construct the following examples of hypersurfaces in $\overline{M}_\nu^{n+1}$ satisfying the condition (1.2):

(a) $\mathbb{S}_u^p(r) \times \mathbb{S}_{s-u}^{n-p}(\sqrt{1-r^2}) \subset \mathbb{S}_s^{n+1}$, with $0 < r < 1$ and $r \neq \sqrt{p/n}$, whose constant mean curvature is given by $\alpha^2 = (nr^2 - p)^2/(n^2 r^2 (1-r^2))$;

(b) $\mathbb{H}_u^p(-r) \times \mathbb{H}_{s-u}^{n-p}(-\sqrt{1-r^2}) \subset \mathbb{H}_{s+1}^{n+1}$, with $0 < r < 1$ and $r \neq \sqrt{p/n}$, $\alpha^2 = (nr^2 - p)^2/(n^2 r^2 (1-r^2))$;

(c) $\mathbb{S}_u^p(r) \times \mathbb{H}_{s-u}^{n-p}(-\sqrt{-1+r^2}) \subset \mathbb{S}_{s+1}^{n+1}$, with $r > 1$,

$$\alpha^2 = (nr^2 - p)^2/(n^2 r^2 (r^2 - 1));$$

(d) $\mathbb{S}_u^p(r) \times \mathbb{H}_{s-u}^{n-p}(-\sqrt{1+r^2}) \subset \mathbb{H}_s^{n+1}$, with $r > 0$,

$$\alpha^2 = (nr^2 + p)^2/(n^2 r^2 (1 + r^2));$$

where $1 \leq p \leq n-1$ and $0 \leq u \leq s$. We will refer to these examples as the *pseudo-Riemannian nonminimal standard products.*

2.4. The hypersurfaces in examples 2.2 and 2.3 have diagonalizable shape operator. However, it seems natural thinking of hypersurfaces with nondiagonalizable shape operator satisfying (1.2) into indefinite ambient spaces. Let L be a selfadjoint endomorphism of $\mathbb{R}_t^{n+2}$; that is, $\langle Lx, y\rangle = \langle x, Ly\rangle$ for all $x, y \in \mathbb{R}_t^{n+2}$. Let $f : \overline{M}_\nu^{n+1} \to \mathbb{R}$ be the quadratic function defined by $f(x) = \langle Lx, x\rangle$ and assume that the minimal polynomial of L is given by $\mu_L(t) = t^2 + at + b$, $a, b \in \mathbb{R}$. Then by computing the gradients, at each point $x \in \overline{M}_\nu^{n+1}$, we have $\tilde{\nabla} f(x) = 2Lx$ and $\overline{\nabla} f(x) = 2Lx - 2kf(x)x$. If $\widetilde{\Delta}$ and $\overline{\Delta}$ denote the Laplacian operators on $\mathbb{R}_t^{n+2}$ and $\overline{M}_\nu^{n+1}$, respectively, a straightforward computation yields $\widetilde{\Delta} f(x) = -2\operatorname{trace}(L)$ and $\overline{\Delta} f(x) = -2\operatorname{trace}(L) - 2k(n+1)f(x)$.

Consider the level set $M = f^{-1}(c)$ for a real constant c. Then at a point x in M we have $\langle \overline{\nabla} f(x), \overline{\nabla} f(x)\rangle = 4\langle L^2 x, x\rangle - 4kf(x)^2 = -4k\mu_L(kc)$, and so f is an isoparametric function (see [6]). Thus the level hypersurfaces $\{f^{-1}(c)\}_{c \in I}$, where $I \subset \{c \in \mathbb{R} : \mu_L(kc) \neq 0\}$ is connected, form an isoparametric family in the classical sense. The shape operator of M_s^n is given by

$$SX = -\frac{1}{|\overline{\nabla} f|}\overline{\nabla}_X(\overline{\nabla} f) = \frac{1}{|\mu_L(kc)|^{1/2}}(kcX - LX)$$

and a messy computation gives

$$\operatorname{tr}(S) = \frac{nkc - \operatorname{tr}(L) - a}{|\mu_L(kc)|^{1/2}}.$$

Then the mean curvature α is given by

$$\alpha = \frac{\varepsilon}{n}\operatorname{tr}(S) = \delta \frac{a + \operatorname{tr}(L) - nkc}{nk|\mu_L(kc)|^{1/2}},$$

where δ stands for the sign of $\mu_L(kc)$. Therefore we get

$$H' = \frac{a + \operatorname{tr} L - knc}{kn\mu_L(kc)}(Lx - kcx),$$

from which we deduce, by using $\Delta x = -n(H' - kx)$, that $\Delta x = Ax$, where A is given by

$$A = \frac{knc - a - \operatorname{tr} L}{k\mu_L(kc)} L + \frac{c \operatorname{tr} L + (n+1)ac + knb}{\mu_L(kc)} I_{n+2}.$$

3. First characterization results

The aim of this section is to show that a hypersurface M_s^n of $\overline{M}_\nu^{n+1}$ satisfying the condition (1.2) has to be of constant mean curvature. To do that, let $\mathscr{W}$ be the open set of regular points of α^2, which we may assume a nonempty set. From (1.3) we have $\langle AX, x\rangle = 0$, for any vector field X tangent to M_s^n. Taking covariant derivative there we get

$$\langle A\sigma(X, Y), x\rangle = -\langle AX, Y\rangle, \tag{3.1}$$

for all tangent vectors X and Y, where σ represents the second fundamental form of M_s^n in $\mathbb{R}_t^{n+2}$, which is given by

$$\sigma(X, Y) = \varepsilon\langle SX, Y\rangle N - k\langle X, Y\rangle x. \tag{3.2}$$

Now equation (3.1), jointly with (1.3) and (3.2), leads to

$$\varepsilon\langle SX, Y\rangle\langle AN, x\rangle - k\langle X, Y\rangle\langle Ax, x\rangle = -n\alpha\langle SX, Y\rangle - nk\langle X, Y\rangle.$$

Bringing here the formulae for Ax and AN given in (1.4) and (1.5), respectively, a straightforward computation yields

$$\langle SX - \varepsilon\alpha X, Y\rangle\langle B, x\rangle = 0, \tag{3.3}$$

at the points of $\mathscr{W}$. This equation is the key to the following result.

Lemma 3.1. *Let $x : M_s^n \to \overline{M}_\nu^{n+1}$ be a hypersurface such that $\Delta x = Ax + B$. If M_s^n has nonconstant mean curvature, then $B = 0$.*

Proof. Let us consider the set $\mathscr{U} = \{p \in \mathscr{W} : \langle B, x\rangle(p) \neq 0\}$ and assume it is a nonempty set. Then at the points of $\mathscr{U}$, from (1.3) and (3.3), we have

$$AX = n(\varepsilon\alpha^2 + k)X - nX(\alpha)N. \tag{3.4}$$

Since $n \geq 2$, we can always find a vector field X such that $X(\alpha) = \langle X, \nabla\alpha\rangle = 0$. This shows, by using (3.4), that $n(\varepsilon\alpha^2 + k)$ is an eigenvalue of A and therefore locally constant on $\mathscr{U}$, which is a contradiction. Hence $\mathscr{U} = \varnothing$ and $\langle B, x\rangle = 0$ on $\mathscr{W}$. Taking covariant derivative here we deduce that B has no tangent component and therefore we get $B = \varepsilon\langle B, N\rangle N$ and $\langle B, N\rangle = 0$, because $\mathscr{W}$ is not empty.

Next we are going to make some computations before stating the main result of this section. From equation (1.3) it is easy to see that

$$\langle AX, Y\rangle = \langle X, AY\rangle, \tag{3.5}$$

for all tangent vector fields X and Y. Taking covariant derivative here and using the Gauss formula jointly with (3.5), we find

$$\begin{aligned}\langle A\sigma(X, Z)Y\rangle - \langle A\sigma(Y, Z), X\rangle \\ = \langle\sigma(X, Z), AY\rangle - \langle\sigma(Y, Z), AX\rangle .\end{aligned} \tag{3.6}$$

By (3.2) and (1.3), the equations (3.6) becomes

$$\begin{aligned}\varepsilon\langle SX, Z\rangle\langle AN, Y\rangle - k\langle X, Z\rangle\langle Ax, Y\rangle \\ - \varepsilon\langle SY, Z\rangle\langle AN, X\rangle + k\langle Y, Z\rangle\langle Ax, X\rangle \\ = -nY(\alpha)\langle SX, Z\rangle + nX(\alpha)\langle SY, Z\rangle .\end{aligned} \tag{3.7}$$

Finally, by Lemma 3.1, (1.4) and (1.5), from (3.7) we obtain

$$TX(\alpha)SY = TY(\alpha)SX, \tag{3.8}$$

where T means the selfadjoint operator defined by $TX = n\alpha X + \varepsilon SX$. This equation becomes the crucial point to show the next result.

Proposition 3.2. *Let $x : M_s^n \to \overline{M}_\nu^{n+1}$ be an isometric immersion such that $\Delta x = Ax + B$. Then M_s^n has constant mean curvature.*

Proof. From Lemma 3.1 we can assume $B = 0$ and then equation (3.8) holds on $\mathscr{W}$. First, suppose that $T(\nabla\alpha) \neq 0$ at the points of $\mathscr{W}$. Then there is a vector field X tangent to M_s^n such that $TX(\alpha) \neq 0$, so that by using (3.8) we find that $\operatorname{rank} S = 1$ at the points of $\mathscr{W}$. Therefore, we can choose a local orthonormal frame $\{E_1, \dots, E_n\}$ such that $SE_1 = n\varepsilon\alpha E_1$, $SE_i = 0$, $i = 2, \dots, n$, and $\varepsilon_i = \langle E_i, E_i\rangle$. Also from (3.8) we have that $E_i(\alpha) = 0$, $i = 2, \dots, n$, and using again (1.3), (1.4), and (1.5), we get

$$\begin{aligned}AE_1 &= n(k + \varepsilon n\alpha^2)E_1 - nE_1(\alpha)N, \\ AE_i &= nkE_i, \qquad i = 2, \dots, n, \\ AN &= 3n\varepsilon\varepsilon_1 E_1(\alpha)E_1 + \left\{\frac{\Delta\alpha}{\alpha} + \varepsilon n^2\alpha^2\right\}N - nk\varepsilon\alpha x, \\ Ax &= -n\alpha N + nkx .\end{aligned}$$

Therefore, $\operatorname{span}\{E_1, N, x\}$ is an invariant subspace under A and the characteristic polynomial $p_A(t)$ of A is given by $p_A(t) = (t - nk)^{n-1}p_{A^*}(t)$, where A^* stands for $A|_{\operatorname{span}\{E_1, N, x\}}$. Then $p_{A^*}(t)$ is constant and we can find three real constants λ_1, λ_2, and λ_3 (which are nothing but the invariants associated to A^*) such that

$$\begin{aligned}\lambda_1 &= \frac{\Delta\alpha}{\alpha} + 2n\{k + \varepsilon n\alpha^2\}, \\ \lambda_2 &= n(2k + \varepsilon n\alpha^2)\left(\frac{\Delta\alpha}{\alpha} + \varepsilon n^2\alpha^2\right) + 3n^2\varepsilon\varepsilon_1 E_1(\alpha)^2 \\ &\quad + n^2k(k + \varepsilon n\alpha^2) - kn^2\varepsilon\alpha^2, \\ \lambda_3 &= n^2k(k + \varepsilon n\alpha^2)\left(\frac{\Delta\alpha}{\alpha} + \varepsilon n^2\alpha^2\right) \\ &\quad - n^3k\varepsilon\alpha^2(k + \varepsilon n\alpha^2) + 3n^3\varepsilon\varepsilon_1 kE_1(\alpha)^2 .\end{aligned}$$

Then we obtain

$$nk\lambda_2 = \lambda_3 + n^3(k + \varepsilon n\alpha^2) + n^2\left(\frac{\Delta\alpha}{\alpha} + \varepsilon n^2\alpha^2\right) + n^4k\alpha^4$$

and

$$\frac{\Delta\alpha}{\alpha} = \lambda_1 - 2n(k + \varepsilon n\alpha^2).$$

The last two equations allow us to write $n^4\alpha^4 = kp_{A^*}(kn)$ and so α is locally constant on $\mathscr{W}$, which is a contradiction.

Finally, assume now there is a point p in $\mathscr{W}$ such that $T(\nabla\alpha)(p) = 0$. Note that from (1.3) and (1.5) we have in $\mathscr{W}$, $\forall i = 1, \ldots, n$,

$$\tag{3.9} \begin{aligned} \langle AE_i, N\rangle &= -n\varepsilon E_i(\alpha), \\ \langle E_i, AN\rangle &= \frac{2\varepsilon}{\alpha}\langle T(\nabla\alpha), E_i\rangle - n\varepsilon E_i(\alpha). \end{aligned}$$

It follows that, at p,

$$\tag{3.10} \langle AE_i, N\rangle = \langle E_i, AN\rangle.$$

From (1.3), (1.4), (1.5), (3.1), and (3.10) we deduce that A is a selfadjoint endomorphism of $\mathbb{R}_t^{n+2}$, and thus equation (3.10) remains valid at every point in $\mathscr{W}$. In turn, from (3.10) $T(\nabla\alpha) = 0$ on $\mathscr{W}$ and so $\nabla\alpha$ is an eigenvector of S with associated eigenvalue $-n\varepsilon\alpha$. If $\langle\nabla\alpha, \nabla\alpha\rangle = \nabla\alpha(\alpha) = 0$, from (1.3) we could write $A(\nabla\alpha) = n(k - n\varepsilon\alpha^2)\nabla\alpha$; then $n(k - n\varepsilon\alpha^2)$ should be an eigenvalue of A and α must be locally constant on $\mathscr{W}$, which cannot hold. Therefore, we can choose a local orthonormal frame $\{E_1, \ldots, E_n\}$ with E_1 parallel to $\nabla\alpha$ such that

$$\begin{aligned} AE_1 &= n(k - n\varepsilon\alpha^2)E_1 - nE_1(\alpha)N, \\ AE_i &= n(\alpha SE_i + kE_i), \qquad i = 2, \ldots, n, \\ AN &= -n\varepsilon\varepsilon_1 E_1(\alpha)E_1 + \left\{\frac{\Delta\alpha}{\alpha} + \varepsilon|S|^2\right\}N - nk\varepsilon\alpha x, \\ Ax &= -n\alpha N + nkx. \end{aligned}$$

Since $SE_i \in \operatorname{span}\{E_2, \ldots, E_n\}$, $i = 2, \ldots, n$, we may write S^* for the endomorphism S restricted to $\operatorname{span}\{E_2, \ldots, E_n\}$. Working on the characteristic polynomials, from the above equations we can deduce that

$$p_A(t) = (n\alpha)^{n-1}q(t)p_{S^*}\left(\frac{t - nk}{n\alpha}\right),$$

where $q(t)$ is a polynomial of degree three. Let $\{r_1, \ldots, r_n\}$ be the possibly complex roots of $p_S(t)$, with $r_1 = -n\varepsilon\alpha$ and $\{r_2, \ldots, r_n\}$ the roots of $p_{S^*}(t)$. Then the functions $nk + n\alpha r_j$, $j = 2, \ldots, n$, are roots of $p_A(t)$ and therefore they are locally constant on $\mathscr{W}$. Thus from the formula $\sum_{j=2}^n(nk + n\alpha r_j) = n(n-1)k + n\alpha\sum_{j=2}^n r_j = n(n-1)k + n\alpha(\operatorname{tr}S - r_1) = n(n-1)k + 2\varepsilon n^2\alpha$, we obtain that α is locally constant on $\mathscr{W}$, which is a contradiction.

Summarizing, we have that $\mathscr{W}$ has to be empty; i.e., M_s^n has constant mean curvature.

4. Main results

We have just proved that the hypersurfaces M_s^n of $\overline{M}_\nu^{n+1}$ satisfying the condition (1.2) have constant mean curvature. In this section, we wish to give a classification theorem of such a class of hypersurfaces. To do that, we recall the following definition. A hypersurface M_s^n is said to be *isoparametric* if the characteristic polynomial $p_S(t)$ of its shape operator S is the same at all points of M_s^n. When S is diagonalizable (for example, in the definite case) that means that the principal curvatures of M_s^n, as well as its multiplicities, are constant. Our first main result reads as follows.

Theorem 4.1. *Let $x : M_s^n \to \overline{M}_\nu^{n+1}$ be an isometric immersion satisfying $\Delta x = Ax + B$. Then M_s^n is a minimal or an isoparametric hypersurface.*

Proof. Let M_s^n be a hypersurface of $\overline{M}_\nu^{n+1}$ satisfying (1.2). By Proposition 3.2 we can assume that the mean curvature α of M_s^n in $\overline{M}_\nu^{n+1}$ is a nonzero constant and so equation (3.3) works here. If M_s^n is not totally umbilical in $\overline{M}_\nu^{n+1}$, then we have $\langle B, x\rangle = 0$ and, as in Lemma 3.1, $B = 0$. Now, from (1.3), (1.4), and (1.5) we get

$$\begin{aligned} AX &= n(\alpha SX + kX), \\ AN &= \varepsilon|S|^2 N - nk\varepsilon\alpha x, \\ Ax &= -n\alpha N + nkx. \end{aligned} \tag{4.1}$$

Working again on the characteristic polynomials $p_A(t)$ and $p_S(t)$, as in §3, we deduce that $p_S(t)$ is constant on M_s^n and the proof finishes.

Next with the aim of getting a complete classification of those hypersurfaces M_s^n of $\overline{M}_\nu^{n+1}$ satisfying (1.2), some easy computations are needed. From Theorem 4.1, M_s^n is an isoparametric hypersurface provided that the (constant) mean curvature α is not zero, and thus $|S|^2$ is also constant. Taking the covariant derivative in the expression of AN in (4.1) we have $\tilde\nabla_X(AN) = -\varepsilon|S|^2 SX - nk\varepsilon\alpha X$ and $\tilde\nabla_X(AN) = A(\tilde\nabla_X N) = -n(\alpha S^2X + kSX)$, from which we obtain

$$S^2 + \frac{nk - \varepsilon|S|^2}{n\alpha} S - k\varepsilon I = 0, \tag{4.2}$$

where I stands for the identity operator on the tangent bundle of M_s^n. We have just seen that if M_s^n is not totally umbilical, then $B = 0$ and thus equations (4.1) and (4.2) allow us to write

$$A^2 - (\varepsilon|S|^2 + kn)A + n\varepsilon k(|S|^2 - n\alpha^2)I_{n+2} = 0, \tag{4.3}$$

and furthermore, from (4.1), A is a selfadjoint endomorphism of $\mathbb{R}_t^{n+2}$. If S is diagonalizable, from (4.2), M_s^n has exactly two constant principal curvatures. By using now similar arguments as in Theorem 2.5 of [11] and Lemma 2 of [8], we deduce that M_s^n is an open piece of a pseudo-Riemannian product of two nonflat totally umbilical submanifolds. If S is not diagonalizable, from (4.3), the minimal polynomial $\mu_A(t)$ of A is given by $\mu_A(t) = t^2 + at + b$, with $a = -(\varepsilon|S|^2 + kn)$ and $b = n\varepsilon k(|S|^2 - n\alpha^2)$. Since $\langle Ax, x\rangle = n$ is constant on M_s^n and $\mu_A(kn) = -n^2\alpha^2\varepsilon k \neq 0$, then M_s^n is an open piece of a quadratic hypersurface as in example 2.4. Summing up, we have proved the following theorem.

Theorem 4.2. *Let $x : M_s^n \to \overline{M}_\nu^{n+1}$ be an isometric immersion. Then $\Delta x = Ax + B$ if and only if M_s^n is an open piece of one of the following hypersurfaces in $\overline{M}_\nu^{n+1}$;*

(1) *a minimal hypersurface,*
(2) *a totally umbilical hypersurface,*
(3) *a pseudo-Riemannian nonminimal standard product,*
(4) *a quadratic hypersurface as in example* 2.4, *with nondiagonalizable shape operator* $(a^2 - 4b \leq 0)$.

As a consequence, we obtain the classification theorem for hypersurfaces in $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$, which generalizes Theorem 1.3 in [10].

Corollary 4.3. *Let* $x : M^n \to \overline{M}^{n+1}$ *be a nonminimal hypersurface. Then* $\Delta x = Ax + B$ *if and only if* M^n *is an open piece of one of the following hypersurfaces:*
(1) *a totally umbilical hypersurface,*
(2) *a product* $\overline{M}^p(r_1) \times \mathbb{S}^{n-p}(r_2)$.

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