TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 5, May 1995

EXAMPLES OF $B(D, \lambda)$-REFINABLE AND WEAK $\overline{\theta}$-REFINABLE SPACES

STEPHEN H. FAST AND J. C. SMITH

ABSTRACT. In 1980, J. C. Smith asked for examples which would demonstrate the relationships between the properties $B(D, \lambda)$-refinability, $B(D, \omega_0)$-refinability, and weak $\overline{\theta}$-refinability. This paper gives such examples in the class of T_4 spaces.

1. INTRODUCTION AND DEFINITIONS

In [5], J. C. Smith asks for an example showing the relationship between $B(D, \lambda)$-refinability and $B(D, \omega_0)$-refinability where λ represents a countable ordinal. Smith also conjectured that weak $\overline{\theta}$-refinability is strictly weaker than $B(D, \omega_0)$-refinability. This paper gives examples demonstrating the relationship between these properties.

In §2 we construct a T_4 space $K^*_{\omega_1}$ such that for each $\alpha < \omega_1$, $K^*_{\omega_1}$ has a closed subspace that is $B(D, \alpha)$-refinable but not $B(D, \beta)$-refinable for any $\beta < \alpha$. We then show that $K^*_{\omega_0+1}$ is weak $\overline{\theta}$-refinable, implying that weak $\overline{\theta}$-refinability is strictly weaker than $B(D, \omega_0)$-refinability.

Definition 1. A space is said to be *mesocompact* if every open cover of the space has an open refinement $\mathscr{V}$ such that any compact subset of the space meets only finitely many members of $\mathscr{V}$.

Definition 2. A space is said to be *metacompact* if every open cover of the space has a point-finite open refinement.

Definition 3. A space X is said to be $B(D, \lambda)$ *-refinable* provided that for every open cover $\mathscr{U} = \{V_\delta | \delta \in \Lambda\}$ of X there exists a family $\{\mathscr{B}_\alpha : \alpha \in \lambda\}$ of partial refinements of $\mathscr{U}$ such that the following conditions hold:

(1) $\bigcup\{\mathscr{B}_\alpha : \alpha \in \lambda\}$ is a refinement of $\mathscr{U}$;
(2) $\bigcup_{\gamma<\alpha}(\bigcup \mathscr{B}_\gamma)$ is closed in X for every $\alpha \in \lambda$;
(3) $\mathscr{B}_\alpha$ is a relatively discrete closed collection in $X - \bigcup(\bigcup_{i<\alpha} \mathscr{B}_i)$.

In this case $\mathscr{B} = \bigcup\{\mathscr{B}_\alpha | \alpha \in \lambda\}$ is said to be a $B(D, \lambda)$ *refinement of* $\mathscr{U}$. Note that if we define for each $\alpha \in \lambda$ the family $\mathscr{H}_\alpha = \{H_\delta | \delta \in \Lambda\}$ where

Received by the editors August 10, 1990 and, in revised form, February 11, 1994; originally communicated to the *Proceedings of the AMS* by D. K. Burke.

1991 *Mathematics Subject Classification.* Primary 54D18, 54D20, 54G20; Secondary 54D15.

Key words and phrases. $B(D, \lambda)$-refinable, weak $\overline{\theta}$-refinable, metacompact, mesocompact, θ-refinable, compact, Bing space, collectionwise T_2, normal, standard subbasic funnel.

0002-9947/95 \$1.00 + \$.25 per page

$H_\delta = \bigcup\{B \in \mathscr{B}_\alpha | B \subseteq U_\delta$ and $B \not\subset U_\gamma$ for $\gamma < \delta\}$, then $\mathscr{H} = \bigcup\{\mathscr{H}_\alpha | \alpha \in \lambda\}$ is also a $B(D, \lambda)$ refinement of $\mathscr{U}$. We say that $\mathscr{H}$ is the *amalgamation* of $\mathscr{B}$. Thus we may henceforth assume that all $B(D, \lambda)$ refinements are amalgamated. Note that each $\mathscr{H}_\alpha$ is a one-to-one partial refinement of $\mathscr{U}$.

Definition 4. Let $\mathscr{H} = \{H_\alpha : \alpha \in A\}$ be a collection of subsets of a space X. Then for $x \in X$ we define $\mathrm{ord}(x, \mathscr{H}) = |\alpha \in A : x \in H_\alpha|$.

Definition 5. A space X is said to be *θ-refinable* provided that for every open cover $\mathscr{U}$ of X there exists a sequence $\{\mathscr{S}_n : n \in \omega_0\}$ of open refinements of $\mathscr{U}$ such that for each $x \in X$ there exists some $n_x \in \omega_0$ such that $0 < \mathrm{ord}(x, \mathscr{S}_{n_x}) < \infty$.

Definition 6. A space X is said to be *weak $\overline{\theta}$-refinable* provided that for every open cover $\mathscr{U}$ of X there exists a sequence $\{\mathscr{S}_n : n \in \omega_0\}$ of open partial refinements of $\mathscr{U}$ such that the following conditions hold:

(1) $\bigcup\{\mathscr{S}_n : n \in \omega_0\}$ is an open refinement of $\mathscr{U}$;

(2) for each $x \in X$ there exists some $n_x \in \omega_0$ such that $0 < \mathrm{ord}(x, \mathscr{S}_{n_x}) < \infty$;

(3) for each $x \in X$ there exists some $k_x \in \omega_0$ such that $x \notin \bigcup\{\bigcup \mathscr{S}_n : n \geq k_x\}$.

Note that mesocompact $\to$ metacompact $\to$ θ-refinable $\to$ $B(D, \omega_0)$-refinable $\to$ weak $\overline{\theta}$-refinable.

It is known that the first three implications are not reversible [4]. This paper shows that the last implication is not reversible in T_4 spaces.

2. Construction of the space $K^*_{\omega_1}$

In [1], R. H. Bing gave an example of a normal topological space that is not collectionwise T_2. We call such an example a *Bing space* and give the construction of such a space below.

Let $Q_1 = P(\omega_1) =$ the set of all subsets of ω_1. Let $\widetilde{G}_1 = \{f \in \prod_{q \in Q_1}\{0, 1\} | f \in \Pi^{-1}_{\{\alpha\}}(1)$ for some positive, finite number of $\alpha \in \omega_1\}$, where $\{0, 1\}$ is the two-point discrete space. Note that $f \in \Pi^{-1}_{\{\alpha\}}(1)$ if and only if $f(\{\alpha\}) = 1$. For $\alpha \in \omega_1$, define f_α by

$$f_\alpha(q) = \begin{Bmatrix} 1 \text{ if } \alpha \in q \\ 0 \text{ if } \alpha \notin q \end{Bmatrix}.$$

Define $K_1 = \{f_\alpha | \alpha \in \omega_1\}$ so that $K_1 \subseteq \widetilde{G}_1$. Let $G_1 = K_1 \cup \{f \in \widetilde{G}_1 - K_1 | f(q) = 1$ for only finitely many $q \in Q_1\}$. Topologize G_1 by adding to the induced Tychonoff product topology all singleton sets $\{g\}$ where $g \in G_1 - K_1$. Let $K^*_2 = G_1$ with the topology described above. For each $\alpha < \omega_1$, let $U^2_\alpha = \pi^{-1}_{\{\alpha\}}(1)$. We call such a set U^2_α a standard subbasic open funnel. Since $\mathscr{U}^2 = \{U^2_\alpha | \alpha < \omega_1\}$ covers K^*_2, we call $\mathscr{U}^2$ the standard open cover of K^*_2. Let $Q_2 = P(G_1 - K_1)$. Using Q_2, construct K_2 and the Bing space G_2 as in the construction of the Bing space G_1 using Q_1. Let $K^*_3 = G_1 \cup G_2$, identifying $G_1 - K_1$ with K_2 by the bijection $\varphi : (G_1 - K_1) \to K_2$ defined by the following:

$\varphi(f) = g_f \in K_2$, where

$$g_f(q) = \begin{Bmatrix} 1 \text{ if } f \in q \\ 0 \text{ if } f \notin q \end{Bmatrix} \text{ for all } f \in G_1 - K_1 .$$

Topologize K_3^* as follows. If A is an open subset of K_2^*, then $A \cup \{A_f | f \in K_2 \cap A\}$, where each A_f is a set containing f that is open in G_2, is a basic open set in K_3^*. In addition, any open set $B \subseteq G_2$ is open in K_3^*. For each $\alpha < \omega_1$, let $U_\alpha^3 = U_\alpha^2 \cup \{f | f \in \pi_{\{k\}}^{-1}(1)$ for some $k \in K_2 \cup U_\alpha^2\}$. Note that $U^3 = \{U_\alpha^3 | \alpha < \omega_1\}$ is the standard open cover of K_3^*. We now construct by transfinite recursion the space $K_{\omega_1}^*$.

Let $\gamma < \omega_1$.

Case (1). If $\gamma = \beta + 1$, where β is a successor ordinal, construct K_γ^* from K_β^* as K_3^* was constructed from K_2^* above. For each $\alpha < \omega_1$, let $U_\alpha^\gamma = U_\alpha^\beta \cup \{f | f \in \pi_{\{k\}}^{-1}(1)$ for some $k \in K_\beta \cap U_\alpha^\beta\}$. Note that $\mathscr{U}^\gamma = \{U_\alpha^\gamma | \alpha < \omega_1\}$ is the standard open cover of K_γ^*.

Case (2). If γ is a limit ordinal, let $K_\gamma^* = \bigcup\{K_\delta^* | \delta < \gamma\}$ with the natural identification of levels and topology.

Case (3). If γ is the successor of a limit ordinal, say $\gamma = \beta + 1$, let $\psi : \omega \to \beta$ be an increasing cofinal map where each $\psi(n)$ is a successor ordinal. Let $K_\gamma = \{\varphi : \omega \to K_\beta^* | \varphi(n) \in K_{\psi(n)}$ for every $n \in \omega$ and a tail of the image of φ is contained in some member of the standard open cover $\mathscr{U}^\beta$ of $K_\beta^*\}$. Define $K_\gamma^* = K_\beta^* \cup K_\gamma$ topologized as follows: If set B is open in K_β^*, then $B \cup \{\varphi \in K_\gamma |$ a tail of the image of φ is in $B\}$ is an open set in K_γ^*. In addition, all singleton sets $\{\varphi\}$ for $\varphi \in K_\gamma$ are open.

Define the space $K_{\omega_1}^* = \bigcup\{K_\gamma^* | \gamma < \omega_1\}$ with the natural identification of the levels and topology. Note that $\mathscr{U} = \{U_\delta | \delta < \omega_1\}$ is an open cover of $K_{\omega_1}^*$, which we call the standard open cover, that each K_γ^* is a closed subspace of $K_{\omega_1}^*$, that each K_γ is a relatively closed, discrete subspace of $K_{\omega_1}^* - \bigcup\{K_\beta^* | \beta < \gamma\}$, and that $K_{\omega_1}^*$ is T_4. To see that $K_{\omega_1}^*$ is normal, suppose that A and B are disjoint nonempty subsets of $K_{\omega_1}^*$. Then $A \cap (K_\gamma \cup K_{\gamma+1})$ and $B \cap (K_\gamma \cap K_{\gamma+1})$ are disjoint nonempty subsets of $K_\gamma \cup K_{\gamma+1}$, which is a normal Bing space, and can be separated by disjoint open sets, say A^* and B^*. Since $A^* \cap K_{\gamma+1}$ and $B^* \cap K_{\gamma+1}$ are disjoint closed subsets of $K_{\gamma+1} \cup K_{\gamma+2}$, a straightforward induction can be used to construct disjoint open subsets of $K_{\omega_1}^*$ separating A and B.

We now state our main result.

Theorem 1. *For every countable ordinal α, there exists a T_4, $B(D, \alpha)$-refinable space which is not $B(D, \beta)$-refinable for any $\beta < \alpha$.*

Proof. Since each K_α is a relatively closed discrete subspace of $K_{\omega_1}^* - (\bigcup_{\gamma<\alpha} K_\gamma | \gamma$ is a successor ordinal), it is clear that K_α^* is $B(D, \alpha)$-refinable.

To prove that K_α^* is not $B(D, \beta)$-refinable for any $\beta < \alpha$, our inductive proof proceeds in the following way. Let $\{\mathscr{B}_\mu | \mu < \alpha\}$ be a collection of partial refinements of $\mathscr{U}$ such that the following conditions hold:

(1) $\bigcup_{\gamma<\mu}(\bigcup \mathscr{B}_\gamma)$ is closed in $K_{\omega_1}^*$ for every $\mu < \alpha$;

(2) $\mathscr{B}_\mu$ is a relatively discrete, closed collection in $K^*_{\omega_1} - (\bigcup_{\gamma<\mu}(\bigcup \mathscr{B}_\gamma))$ for each $\mu < \alpha$.

First, for every limit ordinal $\alpha < \omega_1$ and $x \in (U_\delta \cap K_{\alpha+1}) - \bigcup_{\mu<\alpha}(\bigcup \mathscr{B}_\mu)$, let $S_{(x,\delta)}$ be an open funnel about x such that

$$S_{(x,\delta)} \subseteq U_\delta - \left(\left(\bigcup_{\mu<\alpha} K_\mu\right) \cup \left(\bigcup_{\mu<\alpha}\left(\bigcup \mathscr{B}_\mu\right)\right)\right).$$

For every $\alpha < \omega_1$, we will show that there exists a $\delta_\alpha < \omega_1$ such that for every $\delta \geq \delta_\alpha$ there exists a $x(\alpha, \delta) \in K_{\alpha+1} \cap U_\delta$ and an open set $V(\alpha, \delta)$ about $x(\alpha, \delta)$ such that the following conditions hold:

(1) $\gamma < \alpha \Rightarrow \delta_\gamma < \delta_\alpha$;

(2) $\gamma < \alpha$ and $\delta \geq \delta_\alpha \Rightarrow V(\alpha, \delta) \subseteq V(\gamma, \delta)$;

(3) $\delta \geq \delta_\alpha \Rightarrow x(\alpha, \delta) \notin (\bigcup_{\sigma<\delta} U_\sigma) \cup (\bigcup_{\mu<\alpha}(\bigcup \mathscr{B}_\mu))$;

(4) $\delta \geq \delta_\alpha \Rightarrow V(\alpha, \delta) \cap (\bigcup_{\mu<\alpha}(\bigcup \mathscr{B}_\mu)) = \varnothing$;

(5) For every limit ordinal $\tau > \alpha$, $\delta \geq \delta_\alpha$ and $x \in K_{\tau+1} \cap V(\alpha, \delta) \Rightarrow S_{(x,\delta)} \subseteq V(\alpha, \delta)$.

For $\alpha = 0$, let $\delta_\alpha = 0$ and $x(\alpha, \delta) = f_\delta$ for every $\delta < \omega_1$. Let $V(\alpha, \delta) = U_\delta$ for each $\delta < \omega_1$. Then conditions (1)–(5) are satisfied for $\alpha = 0$.

Suppose that $\alpha > 0$ and that conditions (1)–(5) above hold for every $\beta < \alpha$. We show that these conditions hold for α.

I. *Case* 1. α is a limit ordinal.

Since $\alpha < \omega_1$, choose $\psi < \omega_1$ such that $\psi > \delta_\beta$ for every $\beta < \alpha$. Hence for every $\delta > \psi$, there exists for each $\beta < \alpha$, a $x(\beta, \delta) \in K_{\beta+1} \cap U_\delta$ and an open set $V(\beta, \delta)$ about $x(\beta, \delta)$ such that the inductive conditions (1)–(5) hold. Let $\delta_\alpha = \psi$.

For each $\delta > \psi$, let $x(\alpha, \delta) = \varphi_\delta \in K_{\alpha+1}$, where $\varphi_\delta(\beta) = x(\beta, \delta)$ for every β in the image of ψ_α (see above the construction Case 3 of $K_{\gamma+1}$, where γ is a limit ordinal). By inductive conditions (2) and (3), each $x(\alpha, \delta)$ is well defined and $x(\alpha, \delta) \notin (\bigcup_{\sigma<\delta} U_\sigma) \cup (\bigcup_{\mu<\alpha}(\bigcup \mathscr{B}_\mu))$. By the choice of ψ it follows that $\gamma < \alpha \Rightarrow \delta_\gamma < \psi$. Since if $\delta \geq \psi$, we have $x(\alpha, \delta) \notin \bigcup_{\mu<\alpha}(\bigcup \mathscr{B}_\mu)$, by the definition of the sets $V(\gamma, \delta)$, we can choose an open set $V(\alpha, \delta)$ about $x(\alpha, \delta)$ such that (2), (4), and (5) hold.

Before we consider the case where α is not a limit ordinal, we need the lemma below.

Lemma 1. *Let $\mathscr{U}$ be the standard subbasic open cover of $K^*_{\omega_1}$. Fix $\beta < \omega_1$ where β is a successor ordinal. Suppose that X is a closed subset of $K^*_{\omega_1}$ and $\mathscr{B} = \{B_{f_\alpha} | f_\alpha \in K_1\}$ is a relatively discrete closed collection in $K^*_{\omega_1} \backslash X$ which partially refines $\mathscr{U}$ such that the following conditions hold*:

(A) *There exists an $\alpha_1 \in \omega_1$ such that for every $\gamma > \alpha_1$ there exists some*

(1) $g(\beta, \gamma) \in U_\gamma \cap K_\beta \backslash (X \cup (\bigcup\{U_\tau | \tau < \gamma\}))$

and

(2) *for each $g(\beta, \gamma)$ in* (1) *we can choose a funnel $V(\beta, \gamma) \subseteq U_\gamma$ about $g(\beta, \gamma)$ such that $V(\beta, \gamma) \cap X = \varnothing$ and $V(\beta, \gamma)$ hits at most one member of $\mathscr{B}$; i.e. $V(\beta, \gamma) \cap B \neq \varnothing$ iff $g(\beta, \gamma) \in B$ for each $B \in \mathscr{B}$.* (*)

(B) *Then there exists an $\alpha_2 \in \omega_1$ such that for every $\rho > \alpha_2$ we have*

$$\left[V(\beta, \rho) \backslash \left(\left(\bigcup \mathscr{B}\right) \cup \left(\bigcup\{U_\tau | \tau < \rho\}\right)\right)\right] \cap K_{\beta+1} \neq \varnothing.$$

Remark. Note that in (A), if (1) holds, then (2) follows.

Proof. Assume (A) and suppose (B) is false; that is, no such α_2 exists. Choose $\gamma_0 > \alpha_1$ and a funnel $V(\beta, \gamma_0)$ such that

$$\left[V(\beta, \gamma_0)\setminus\left(\left(\bigcup\mathscr{B}\right) \cup (\{U_{f_\tau}|\tau < \gamma_0\})\right)\right] \cap K_{\beta+1} = \varnothing.$$

By our supposition we can choose $\gamma_1 > \gamma_0$ such that $\gamma_1 > \tau$ if $V(\beta, \gamma_0) \cap B_{f_\tau} \neq \varnothing$ and

$$\left[V(\beta, \gamma_1)\setminus\left(\bigcup\mathscr{B}\right) \cup \left(\bigcup\{U_{f_\tau}|\tau < \gamma_1\}\right)\right] \cap K_{\beta+1} = \varnothing.$$

Assume that for $\rho < \Gamma$ γ_ρ has been chosen such that the following conditions hold:

(i) $\gamma_\rho > \gamma_\delta$ if $\delta < \rho$;
(ii) $\gamma_\rho > \tau$ if $V(\beta, \gamma_\delta) \cap B_{f_\tau} \neq \varnothing$ for any $\delta < \rho$;
(iii) $[V(\beta, \gamma_\rho)\setminus((\bigcup\mathscr{B}) \cup (\bigcup\{U_{f_\tau}|\tau < \gamma_\rho\}))] \cap K_{\beta+1} = \varnothing$.

By our supposition there exists

$$\gamma_\Gamma > \sup(\{\gamma_\rho|\rho < \Gamma\} \cup \{\tau \in \omega_1|B_{f_\tau} \cap V(\beta, \gamma_\rho) = \varnothing \text{ for some } \rho < \Gamma\})$$

such that (i), (ii), and (iii) hold. Thus we can continue the induction on ω_1.

Since the singletons $\{g(\beta, \gamma_\delta)|\delta \in \omega_1\}$ cannot be separated by pairwise disjoint open sets in $G_\beta = K_\beta \cup K_{\beta+1}$, and since $V(\beta, \gamma_\delta) \cap G_\beta$ is open in G_β for every $\delta \in \omega_1$, there exists $\delta_1, \delta_2 \in \omega_1$ $(\delta_1 < \delta_2)$, such that $V(\beta, \gamma_{\delta_1}) \cap V(\beta, \gamma_{\delta_2}) \cap K_{\beta+1} \neq \varnothing$. Now $G_\beta \cap V(\beta, \gamma_{\delta_1}) \cap V(\beta, \gamma_{\delta_2}) = \bigcap_{j=1}^{k} \Pi_{q_k}^{-1}(t_k)$ for some $q_1, q_2, \dots, q_k \in Q_\beta$, and each t_j has the value 1 or 0. Since $g(\beta, \gamma_{\delta_1}) \in V(\beta, \gamma_{\delta_1})$ and $g(\beta, \gamma_{\delta_2}) \in V(\beta, \gamma_{\delta_2})$, by (A) if any $q_j = \{h\}$, where $h \in \bigcup\{U_{f_\tau}|\tau < \gamma_{\delta_1}\} \cap K_\beta$, it follows that $t_j = 0$. Hence $[V(\beta, \gamma_{\delta_1}) \cap V(\beta, \gamma_{\delta_2})\setminus(\bigcup\{U_{f_\tau}|\tau < \gamma_{\delta_1}\})] \cap K_{\beta+1} \neq \varnothing$. Choose $x \in [V(\beta, \gamma_{\delta_1}) \cap V(\beta, \gamma_{\delta_2})\setminus (\bigcup\{U_{f_\tau}|\tau < \gamma_{\delta_1}\})] \cap K_{\beta+1}$. By (iii) above, we must have that $x \in \bigcup\mathscr{B}$. Choose $B_x \in \mathscr{B}$ with $x \in B_x$. Then

$$(**) \qquad x \in V(\beta, \gamma_{\delta_1}) \cap V(\beta, \gamma_{\delta_2}) \cap B_x.$$

By (A-2) above, $g(\beta, \gamma_{\delta_1}) \in B_x$. Thus it follows from (ii) and the assumption (A) that $g(\beta, \gamma_{\delta_2}) \notin B_x$. However, by (A-2) we have that $V(\beta, \gamma_{\delta_2}) \cap B_x = \varnothing$, contradicting $(**)$. Therefore, the lemma is proved.

We now consider the case where α is a successor ordinal.

II. *Case* 2. $\alpha = \beta + 1$.

Since β satisfies the inductive hypothesis (1)–(5), condition (A) of Lemma 1 is satisfied. To see this, let $X = \bigcup_{\mu<\alpha}(\bigcup \beta_\mu)$ and let $\mathscr{B} = \mathscr{B}_\alpha$. Then the α_1 of Lemma 1, condition (A) is fulfilled by δ_β, since for each $\delta \geq \delta_\beta$ the element $x(\beta, \delta) \in (U_{f_\delta} \cap K_\alpha)\setminus((\bigcup_{\sigma<\delta} U_\sigma) \cup (\bigcup_{\mu<\alpha}(\bigcup \beta_\mu)))$ by inductive hypothesis (3).

Hence by Lemma 1, there exists $\delta_\alpha < \omega_1$ satisfying condition (3). Clearly, we can choose $\delta_\alpha > \delta_\beta$, so (1) is satisfied. Since $\mathscr{B}_\alpha$ is a relatively closed, discrete collection in $K_{\omega_1}^* - (\bigcup_{\mu<\alpha}(\bigcup \beta_\mu))$, we can choose an open set $V(\alpha, \delta)$ about each $x(\alpha, \delta)$ such that conditions (2), (4), and (5) are satisfied. The proof of Theorem 1 is now complete.

Theorem 2. *G_n is metacompact for each $n \in N$.*

Proof. Let $\mathscr{U}$ be any open cover of G_n. For each $f \in K_n$, choose a member U_f of $\mathscr{U}$ that contains f. For each $f \in K_n$, let V_f represent the standard funnel about f. Then $\{U_f \cap V_f | f \in K_1\} \cup \{\{g\} | g \in G_n \backslash K_n\}$ is a point-finite refinement of $\mathscr{U}$.

Theorem 3. *$K^*_{\omega_0}$ is metacompact.*

Proof. Let $\mathscr{V} = \{V_\alpha | \alpha \in A\}$ be an open cover of $K^*_{\omega_0}$. Then, since G_1 is metacompact, $\mathscr{V}$ has a 1-1, open in $K^*_{\omega_0}$, partial refinement $\mathscr{T} = \{T_\alpha | \alpha \in A\}$ which is point-finite on G_1 and covers G_1. For each $x \in G_n \cap T_\alpha$, let $C(x, n, \alpha)$ be the intersection of the standard funnel about x with T_α. Define the open set S^1_α inductively as follows:

(1) $S^1_\alpha \cap G_1 = T_\alpha \cap G_1$;
(2) for $n > 1$, $S^1_\alpha \cap G_n = (S^1_\alpha \cap K_n) \cup \{y \in G_n | y \in C(x, n, \alpha)$ for some $x \in S^1_\alpha \cap K_n\}$.

Note that $\mathscr{S}^1 = \{S^1_\alpha | \alpha \in A\}$ is an open in $K^*_{\omega_0}$, point-finite, partial refinement of $\mathscr{V}$ which covers G_1. To see this, suppose $x \in K_3$. Then $x \in \Pi^{-1}_q(1)$ for only finitely many q in $P(K_2)$. Hence x is a member of only finitely many standard funnels about elements in K_2. Since $K_2 \subseteq G_1$, $\mathscr{S}^1$ is point-finite on K_2. Thus, by (2) it must be the case that $\mathscr{S}^1$ is point-finite on K_3. Continuing in this way it follows that $\mathscr{S}^1$ is point-finite on K_n for every n.

Next, since $K_1 \cup K_2 \cup \cdots \cup K_{n-1}$ is a closed subset of $K^*_{\omega_0}$, and since G_n is metacompact for each n, we can construct a 1-1 point-finite open partial refinement $\mathscr{S}^n$ of $\mathscr{V}$ that covers K_n and misses $K_1 \cup K_2 \cup \cdots \cup K_{n-1}$. It then follows that $\mathscr{S} = \bigcup\{\mathscr{S}^n : n \in N\}$ is a point-finite open refinement of $\mathscr{V}$. Hence, $K^*_{\omega_0}$ is metacompact.

To show that $K^*_{\omega_0}$ is mesocompact, it suffices to show that every compact subset of $K^*_{\omega_0}$ is finite. In [2], J. R. Boone shows that every compact subset of G_1 is finite. We now extend this result to obtain the following lemma.

Lemma 2. *For each $n \in N$, if C is a compact subset of $K^*_{\omega_0}$, then $C \cap (\bigcup\{G_i | i \leq n\})$ is finite.*

Proof. The proof is by induction on N. For $n = 1$ observe that $G_1 \cap C$ is closed in $K^*_{\omega_0}$ and therefore compact. Suppose that we can choose distinct elements $f_1, f_2, f_3, \ldots, f_n, \ldots$ in $C \cap G_1$. Since K_1 is discrete, we may assume that each f_i belongs to $K_2 \cap C$. Since $|\{q \in Q_1 | f_i(q) \neq 0$ for some $i \in N\}| \leq \aleph_0$, for each $f \in K_1$ we can choose a basic open funnel V_f about f that misses $\{f_1, f_2, f_3, \ldots\}$, since $|\{q \in Q_1 | f(q) \neq 0\}| > \aleph_0$. Thus $\{V_f | f \in K_1 \cap C\} \cup \bigcup\{\{f\} : f \in C \cap K_2\}$ is an open cover (open in G_1) of $C \cap G_1$ with no finite subcover, contradicting the compactness of $C \cap G_1$. Hence $C \cap G_1$ must be finite.

Assume that for all $k < n$, $C \cap [\bigcup\{G_i : i \leq k\}]$ is finite.

By inductive hypothesis, $C \cap (\bigcup\{G_i | i \leq n-1\})$ is compact and therefore finite. Suppose we can choose distinct elements $f_1, f_2, f_3, \ldots$ in $C \cap K_{n+1}$. Since $|\{q \in Q_n | f_i(q) \neq 0$ for some $i \in N\}| \leq \aleph_0$, for each $f \in C \cap (\bigcup\{G_i | i \leq n-1\})$ we can choose a funnel V_f about f which misses $\{f_1, f_2, f_3, \ldots\} \subseteq C$. Thus $\{V_f | f \in C \cap (\bigcup\{G_i | i \leq n-1\})\} \cup \bigcup\{\{f\} : f \in C \cap K_{n+1}\}$ is an open cover of $C \cap (\bigcup\{G_i | i \leq n\})$ with no finite subcover, in contradiction to the compactness of $C \cap (\bigcup\{G_i | i \leq n\})$, so the lemma follows.

Lemma 3. *If C is a compact subset of $K^*_{\omega_0}$, then C is finite.*

Proof. Suppose C is an infinite compact subset of $K^*_{\omega_0}$. Then by Lemma 2 there must exist a sequence $f : N \to C$ such that $f(n) \notin \bigcup\{K_j | f(i) \in K_j$ for some $i < n\}$. Since $\{f(n) | n \in N\}$ is a closed subset of C, it is compact. It is easy to show that this sequence is also discrete and thus cannot be compact. Hence the lemma is proved.

From Theorem 3 and Lemma 3 above we now have the following.

Theorem 4. *$K^*_{\omega_0}$ is mesocompact.*

Theorem 5. (i) *$K^*_{\omega_0}$ is $B(D, \omega_0)$-refinable but not $B(D, n)$-refinable for any n.*
(ii) *$K^*_{\omega_0+1}$ is $B(D, \omega_0 + 1)$-refinable but not $B(D, \omega_0)$-refinable.*
(iii) *$K^*_{\omega_0+1}$ is weak $\overline{\theta}$-refinable and T_4.*

Proof. The proof of (i) and (ii) follow directly from Theorem 1. Let $\mathscr{U}$ be an open cover of $K^*_{\omega_0+1}$. Since $K^*_{\omega_0}$ is metacompact, $\mathscr{U}$ has a partial refinement $\mathscr{S}_1$, which is point-finite on $K^*_{\omega_0}$. Next, let $\mathscr{S}_2$ be the collection of singleton subsets of $K^*_{\omega_0+1} \backslash K^*_{\omega_0}$. Then $\mathscr{S}_1 \cup \mathscr{S}_2$ is a weak $\overline{\theta}$-refinement of $\mathscr{U}$.

Question. If we modify the construction of each G_n so that

$$G_n = \left\{ f \in \prod_{q \in Q_n} \{0, 1\} | f \in \Pi^{-1}_{\{\alpha\}}(1) \text{ for some } \alpha \in \omega_1 \right\},$$

would the space $K^*_{\omega_0+1}$ remain weak $\overline{\theta}$-refinable? The authors conjecture that it would not. If not, then $K^*_{\omega_0+1}$ is a T_4, $B(D, \omega_0 + 1)$-refinable space that is not weak $\overline{\theta}$-refinable.

Ackowledgment

The authors would like to thank the referees for several helpful comments concerning the paper.

References

1. R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. **3** (1951), 175–186.
2. J. R. Boone, *Some characterizations of paracompactness in k-spaces*, Fund. Math. **72** (1971), 145–153.
3. E. Michael, *Point-finite and locally finite coverings*, Canad. J. Math. **7** (1955), 275–279.
4. R. H. Price, *The property $B(P, \alpha)$-refinability and its relationship to generalized paracompact topological spaces*, Ph.D. dissertation, Virginia Polytechnic Inst. and State Univ., 1987.
5. J. C. Smith, *Irreducible spaces and property b_1*, Topology Proc. **5** (1980), 187–200.

Department of Mathematics, Bluefield College, Bluefield, Virginia 24605-1799

Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061-0123