EXAMPLES OF $B(D, \lambda)$-REFINABLE AND WEAK $\theta$-REFINABLE SPACES

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Abstract. In 1980, J. C. Smith asked for examples which would demonstrate the relationships between the properties $B(D, \lambda)$-refinability, $B(D, \omega_0)$-refinability, and weak $\theta$-refinability. This paper gives such examples in the class of $T_4$ spaces.

1. INTRODUCTION AND DEFINITIONS

In [5], J. C. Smith asks for an example showing the relationship between $B(D, \lambda)$-refinability and $B(D, \omega_0)$-refinability where $\lambda$ represents a countable ordinal. Smith also conjectured that weak $\theta$-refinability is strictly weaker than $B(D, \omega_0)$-refinability. This paper gives examples demonstrating the relationship between these properties.

In §2 we construct a $T_4$ space $K^*_\omega_1$ such that for each $\alpha < \omega_1$, $K^*_\omega_1$ has a closed subspace that is $B(D, \alpha)$-refinable but not $B(D, \beta)$-refinable for any $\beta < \alpha$. We then show that $K^*_\omega_0 + 1$ is weak $\theta$-refinable, implying that weak $\theta$-refinability is strictly weaker than $B(D, \omega_0)$-refinability.

Definition 1. A space is said to be mesocompact if every open cover of the space has an open refinement $\mathcal{V}$ such that any compact subset of the space meets only finitely many members of $\mathcal{V}$.

Definition 2. A space is said to be metacompact if every open cover of the space has a point-finite open refinement.

Definition 3. A space $X$ is said to be $B(D, \lambda)$-refinable provided that for every open cover $\mathcal{U} = \{U_\delta : \delta \in \Lambda\}$ of $X$ there exists a family $\{\mathcal{B}_\alpha : \alpha \in \lambda\}$ of partial refinements of $\mathcal{U}$ such that the following conditions hold:

1. $\bigcup\{\mathcal{B}_\alpha : \alpha \in \lambda\}$ is a refinement of $\mathcal{U}$;
2. $\bigcup_{\gamma < \alpha} (\bigcup \mathcal{B}_\gamma)$ is closed in $X$ for every $\alpha \in \lambda$;
3. $\mathcal{B}_\alpha$ is a relatively discrete closed collection in $X - \bigcup_{i < \alpha} \mathcal{B}_i$.

In this case $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha | \alpha \in \lambda\}$ is said to be a $B(D, \lambda)$ refinement of $\mathcal{U}$. Note that if we define for each $\alpha \in \lambda$ the family $\mathcal{H}_\alpha = \{H_\delta : \delta \in \Lambda\}$ where

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$H_\delta = \bigcup \{ B \in B_\alpha | B \subseteq U_\delta \text{ and } B \not\subset U_\gamma \text{ for } \gamma < \delta \}, \text{ then } \mathcal{H} = \bigcup \{ \mathcal{H}_\alpha | \alpha \in \lambda \} \text{ is also a } B(D, \lambda) \text{ refinement of } \mathcal{U}. \text{ We say that } \mathcal{H} \text{ is the amalgamation of } B. \text{ Thus we may henceforth assume that all } B(D, \lambda) \text{ refinements are amalgamated. Note that each } \mathcal{H}_\alpha \text{ is a one-to-one partial refinement of } \mathcal{U}.

**Definition 4.** Let $\mathcal{H} = \{ H_\alpha : \alpha \in A \}$ be a collection of subsets of a space $X$. Then for $x \in X$ we define \( \text{ord}(x, \mathcal{H}) = |\alpha \in A : x \in H_\alpha| \).

**Definition 5.** A space $X$ is said to be $\theta$-refinable provided that for every open cover $\mathcal{U}$ of $X$ there exists a sequence $\{ \mathcal{S}_n : n \in \omega_0 \}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$ there exists some $n_x \in \omega_0$ such that $0 < \text{ord}(x, \mathcal{S}_{n_x}) < \infty$.

**Definition 6.** A space $X$ is said to be weak $\overline{\theta}$-refinable provided that for every open cover $\mathcal{U}$ of $X$ there exists a sequence $\{ \mathcal{S}_n : n \in \omega_0 \}$ of open partial refinements of $\mathcal{U}$ such that the following conditions hold:

1. $\bigcup \{ \mathcal{S}_n : n \in \omega_0 \}$ is an open refinement of $\mathcal{U}$;
2. for each $x \in X$ there exists some $n_x \in \omega_0$ such that $0 < \text{ord}(x, \mathcal{S}_{n_x}) < \infty$;
3. for each $x \in X$ there exists some $k_x \in \omega_0$ such that $x \notin \bigcup \{ \{ \mathcal{S}_n : n \geq k_x \} \}$.

Note that mesocompact $\rightarrow$ metacompact $\rightarrow$ $\theta$-refinable $\rightarrow B(D, \omega_0)$-refinable $\rightarrow$ weak $\overline{\theta}$-refinable.

It is known that the first three implications are not reversible [4]. This paper shows that the last implication is not reversible in $T_4$ spaces.

**2. Construction of the space $K^*_\omega$**

In [1], R. H. Bing gave an example of a normal topological space that is not collectionwise $T_2$. We call such an example a Bing space and give the construction of such a space below.

Let $Q_1 = P(\omega_1) = \text{ the set of all subsets of } \omega_1$. Let $\tilde{G}_1 = \{ f \in \prod_{\alpha \in \omega_1} \{ 0, 1 \} | f \in \Pi_{\{ \alpha \}^{-1}(1)} \}$ for some positive, finite number of $\alpha \in \omega_1$, where $\{ 0, 1 \}$ is the two-point discrete space. Note that $f \in \Pi_{\{ \alpha \}^{-1}(1)}$ if and only if $f(\{ \alpha \}) = 1$. For $\alpha \in \omega_1$, define $f_\alpha$ by

$$f_\alpha(q) = \begin{cases} 1 & \text{if } \alpha \in q \\ 0 & \text{if } \alpha \notin q \end{cases}.$$ 

Define $K_1 = \{ f_\alpha | \alpha \in \omega_1 \}$ so that $K_1 \subseteq \tilde{G}_1$. Let $G_1 = K_1 \cup \{ f \in \tilde{G}_1 - K_1 | f(q) = 1 \}$ for only finitely many $q \in Q_1 \}$. Topologize $G_1$ by adding to the induced Tychonoff product topology all singleton sets $\{ g \}$ where $g \in G_1 - K_1$. Let $K_2 = G_1$ with the topology described above. For each $\alpha < \omega_1$, let $U_\alpha^2 = \pi_{\{ \alpha \}^{-1}(1)}$. We call such a set $U_\alpha^2$ a standard subbasic open funnel. Since $\mathcal{U}^2 = \{ U_\alpha^2 | \alpha < \omega_1 \}$ covers $K_2^*$, we call $\mathcal{U}^2$ the standard open cover of $K_2^*$.

Let $Q_2 = P(G_1 - K_1)$. Using $Q_2$, construct $K_2$ and the Bing space $G_2$ as in the construction of the Bing space $G_1$ using $Q_1$. Let $K_3^* = G_1 \cup G_2$, identifying $G_1 - K_1$ with $K_2$ by the bijection $\varphi : (G_1 - K_1) \rightarrow K_2$ defined by the following:
\( \varphi(f) = g_f \in K_2 \), where
\[
    g_f(q) = \begin{cases} 
    1 & \text{if } f \in q \\
    0 & \text{if } f \notin q 
\end{cases}
\text{ for all } f \in G_1 - K_1.
\]

Topologize \( K_3^* \) as follows. If \( A \) is an open subset of \( K_2^* \), then \( A \cup \{ f | f \in K_2 \cap A \} \), where each \( A_f \) is a set containing \( f \) that is open in \( G_2 \), is a basic open set in \( K_3^* \). In addition, any open set \( B \subseteq G_2 \) is open in \( K_3^* \). For each \( \alpha < \omega_1 \), let \( U_3^\alpha = U_3^2 \cup \{ f | f \in \pi_{(k)}^{-1}(1) \} \) for some \( k \in K_2 \cup U_3^2 \). Note that \( U_3^3 = \{ U_3^\alpha | \alpha < \omega_1 \} \) is the standard open cover of \( K_3^* \). We now construct by transfinite recursion the space \( K_{\omega_1}^* \).

Let \( \gamma < \omega_1 \).

Case (1). If \( \gamma = \beta + 1 \), where \( \beta \) is a successor ordinal, construct \( K_{\gamma}^* \) from \( K_3^* \) as \( K_3^* \) was constructed from \( K_2^* \) above. For each \( \alpha < \omega_1 \), let \( U_3^\alpha = U_3^\beta \cup \{ f | f \in \pi_{(k)}^{-1}(1) \} \) for some \( k \in K_\beta \cap U_3^\beta \). Note that \( \mathcal{U}_\gamma = \{ U_3^\alpha | \alpha < \omega_1 \} \) is the standard open cover of \( K_\gamma^* \).

Case (2). If \( \gamma \) is a limit ordinal, let \( K_{\gamma}^* = \bigcup \{ K_\delta^* | \delta < \gamma \} \) with the natural identification of levels and topology.

Case (3). If \( \gamma \) is the successor of a limit ordinal, say \( \gamma = \beta + 1 \), let \( \psi : \omega \to \beta \) be an increasing cofinal map where each \( \psi(n) \) is a successor ordinal. Let \( K_{\gamma} = \{ \varphi : \omega \to K_\beta^* | \varphi(n) \in K_{\psi(n)} \} \) for every \( n \in \omega \) and a tail of the image of \( \varphi \) is contained in some member of the standard open cover \( \mathcal{U}_\beta \) of \( K_\beta^* \). Define \( K_{\gamma}^* = K_\beta^* \cup K_{\gamma} \) topologized as follows: If set \( B \) is open in \( K_\beta^* \), then \( B \cup \{ \varphi \in K_{\gamma} | \text{ a tail of the image of } \varphi \text{ is in } B \} \) is an open set in \( K_{\gamma}^* \). In addition, all singleton sets \( \{ \varphi \} \) for \( \varphi \in K_{\gamma} \) are open.

Define the space \( K_{\omega_1}^* = \bigcup \{ K_\gamma^* | \gamma < \omega_1 \} \) with the natural identification of the levels and topology. Note that \( \mathcal{U}^* = \{ U_3^\delta | \delta < \omega_1 \} \) is an open cover of \( K_{\omega_1}^* \), which we call the standard open cover, that each \( K_\gamma^* \) is a closed subspace of \( K_{\omega_1}^* \), that each \( K_{\gamma} \) is a relatively closed, discrete subspace of \( K_{\omega_1}^* \cup \{ K_\beta^* | \beta < \gamma \} \), and that \( K_{\omega_1}^* \) is T4. To see that \( K_{\omega_1}^* \) is normal, suppose that \( A \) and \( B \) are disjoint nonempty subsets of \( K_{\omega_1}^* \). Then \( A \cap (K_{\gamma} \cup K_{\gamma+1}) \) and \( B \cap (K_{\gamma} \cap K_{\gamma+1}) \) are disjoint nonempty subsets of \( K_{\gamma} \cup K_{\gamma+1} \), which is a normal Bing space, and can be separated by disjoint open sets, say \( A^* \) and \( B^* \). Since \( A^* \cap K_{\gamma+1} \) and \( B^* \cap K_{\gamma+1} \) are disjoint closed subsets of \( K_{\gamma+1} \cup K_{\gamma+2} \), a straightforward induction can be used to construct disjoint open subsets of \( K_{\omega_1}^* \) separating \( A \) and \( B \).

We now state our main result.

**Theorem 1.** For every countable ordinal \( \alpha \), there exists a T4, \( B(D, \alpha) \)-refinable space which is not \( B(D, \beta) \)-refinable for any \( \beta < \alpha \).

**Proof.** Since each \( K_\alpha \) is a relatively closed discrete subspace of \( K_{\omega_1}^* - \{ K_\beta^* | \beta < \gamma \} \) is a successor ordinal), it is clear that \( K_\alpha^* \) is \( B(D, \alpha) \)-refinable.

To prove that \( K_\alpha^* \) is not \( B(D, \beta) \)-refinable for any \( \beta < \alpha \), our inductive proof proceeds in the following way. Let \( \{ \mathcal{B}_\mu | \mu < \alpha \} \) be a collection of partial refinements of \( \mathcal{U} \) such that the following conditions hold:

1. \( \bigcup_{\gamma<\mu}(\bigcup \mathcal{B}_\gamma) \) is closed in \( K_{\omega_1}^* \) for every \( \mu < \alpha \);
$\mathcal{B}_\mu$ is a relatively discrete, closed collection in $K^{\omega_1}_* - (\bigcup_{\gamma < \mu} (\bigcup \mathcal{B}_\gamma))$ for each $\mu < \alpha$.

First, for every limit ordinal $\alpha < \omega_1$ and $x \in (U_\delta \cap K_{\alpha+1}) - \bigcup_{\mu < \alpha} (\bigcup \mathcal{B}_\mu)$, let $S(x, \delta)$ be an open funnel about $x$ such that

$$S(x, \delta) \subseteq U_\delta - \left( \left( \bigcup_{\mu < \alpha} K_{\mu} \right) \cup \left( \bigcup_{\mu < \alpha} (\bigcup \mathcal{B}_\mu) \right) \right).$$

For every $\alpha < \omega_1$, we will show that there exists a $\delta_\alpha < \omega_1$ such that for every $\delta \geq \delta_\alpha$ there exists an $x(\alpha, \delta) \in K_{\alpha+1} \cap U_\delta$ and an open set $V(\alpha, \delta)$ about $x(\alpha, \delta)$ such that the following conditions hold:

1. $y \geq \alpha \Rightarrow \delta_y < \delta_\alpha$;
2. $y < \alpha$ and $\delta \geq \delta_\alpha \Rightarrow V(\alpha, \delta) \subseteq V(y, \delta)$;
3. $\delta \geq \delta_\alpha \Rightarrow x(\alpha, \delta) \notin (\bigcup_{\sigma < \delta} U_\sigma) \cup (\bigcup_{\mu < \alpha} (\bigcup \mathcal{B}_\mu))$;
4. $\delta \geq \delta_\alpha \Rightarrow V(\alpha, \delta) \cap (\bigcup_{\mu < \alpha} (\bigcup \mathcal{B}_\mu)) = \emptyset$;
5. For every limit ordinal $\tau > \alpha$, $\delta \geq \delta_\alpha$ and $x \in K_{\tau+1} \cap V(\alpha, \delta) \Rightarrow S(x, \delta) \subseteq V(\alpha, \delta)$.

For $\alpha = 0$, let $\delta_0 = 0$ and $x(\alpha, \delta) = f_\delta$ for every $\delta < \omega_1$. Let $V(\alpha, \delta) = U_\delta$ for each $\delta < \omega_1$. Then conditions (1)-(5) are satisfied for $\alpha = 0$.

Suppose that $\alpha > 0$ and that conditions (1)-(5) above hold for every $\beta < \alpha$. We show that these conditions hold for $\alpha$.

I. Case 1. $\alpha$ is a limit ordinal.

Since $\alpha < \omega_1$, choose $\psi < \omega_1$ such that $\psi > \delta_\beta$ for every $\beta < \alpha$. Hence for every $\delta > \psi$, there exists for each $\beta < \alpha$, a $x(\beta, \delta) \in K_{\beta+1} \cap U_\delta$ and an open set $V(\beta, \delta)$ about $x(\beta, \delta)$ such that the inductive conditions (1)-(5) hold. Let $\delta_\alpha = \psi$.

For each $\delta > \psi$, let $x(\alpha, \delta) = \phi_\delta \in K_{\alpha+1}$, where $\phi_\delta(\beta) = x(\beta, \delta)$ for every $\beta$ in the image of $\psi_\alpha$ (see above the construction Case 3 of $K_{\alpha+1}$, where $\gamma$ is a limit ordinal). By inductive conditions (2) and (3), each $x(\alpha, \delta)$ is well defined and $x(\alpha, \delta) \notin (\bigcup_{\sigma < \delta} U_\sigma) \cup (\bigcup_{\mu < \alpha} (\bigcup \mathcal{B}_\mu))$. By the choice of $\psi$ it follows that $\gamma < \alpha \Rightarrow \delta_\gamma < \psi$. Since if $\delta \geq \psi$, we have $x(\alpha, \delta) \notin \bigcup_{\mu < \alpha} (\bigcup \mathcal{B}_\mu)$, by the definition of the sets $V(\gamma, \delta)$, we can choose an open set $V(\alpha, \delta)$ about $x(\alpha, \delta)$ such that (2), (4), and (5) hold.

Before we consider the case where $\alpha$ is not a limit ordinal, we need the lemma below.

Lemma 1. Let $\mathcal{U}$ be the standard subbasic open cover of $K^{\omega_1}_*$. Fix $\beta < \omega_1$ where $\beta$ is a successor ordinal. Suppose that $X$ is a closed subset of $K^{\omega_1}_*$ and $\mathcal{B} = \{B_x | x_0 \in K_1\}$ is a relatively discrete closed collection in $K^{\omega_1}_* \setminus X$ which partially refines $\mathcal{U}$ such that the following conditions hold:

(A) There exists an $\alpha_1 \in \omega_1$ such that for every $\gamma > \alpha_1$ there exists some
1. $g(\beta, \gamma) \in U_\gamma \cap K_\beta \setminus (X \cup (\bigcup \{U_\tau | \tau < \gamma\}))$

and
2. for each $g(\beta, \gamma)$ in (1) we can choose a funnel $V(\beta, \gamma) \subseteq U_\gamma$ about $g(\beta, \gamma)$ such that $V(\beta, \gamma) \cap X = \emptyset$ and $V(\beta, \gamma)$ hits at most one member of $\mathcal{B}$; i.e. $V(\beta, \gamma) \cap B \neq \emptyset$ iff $g(\beta, \gamma) \in B$ for each $B \in \mathcal{B}$. (*)

(B) Then there exists an $\alpha_2 \in \omega_1$ such that for every $\rho > \alpha_2$ we have

$$[V(\beta, \rho) \setminus ((\bigcup \mathcal{B}) \cup (\bigcup \{U_\tau | \tau < \rho\}))] \cap K_{\beta+1} \neq \emptyset.$$
Remark. Note that in (A), if (1) holds, then (2) follows.

Proof. Assume (A) and suppose (B) is false; that is, no such $\alpha_2$ exists. Choose $\gamma_0 > \alpha_1$ and a funnel $V(\beta, \gamma_0)$ such that
$$\left[ V(\beta, \gamma_0) \setminus \left( \bigcup_{(\tau, \gamma_0)} \cup \left( \{U_f, \tau < \gamma_0\} \right) \right) \right] \cap K_{\beta+1} = \emptyset.$$ 

By our supposition we can choose $\gamma_1 > \gamma_0$ such that $\gamma_1 > \tau$ if $V(\beta, \gamma_0) \cap B_f, \neq \emptyset$ and
$$\left[ V(\beta, \gamma_1) \setminus \left( \bigcup_{(\tau, \gamma_0)} \cup \left( \{U_f, \tau < \gamma_1\} \right) \right) \right] \cap K_{\beta+1} = \emptyset.$$ 

Assume that for $\rho < \Gamma$, $\gamma_\rho$ has been chosen such that the following conditions hold:
(i) $\gamma_\rho > \gamma_\delta$ if $\delta < \rho$;
(ii) $\gamma_\rho > \tau$ if $V(\beta, \gamma_\delta) \cap B_f, \neq \emptyset$ for any $\delta < \rho$;
(iii) $\left[ V(\beta, \gamma_\rho) \setminus \left( \bigcup_{(\tau, \gamma_\delta)} \cup \left( \{U_f, \tau < \gamma_\rho\} \right) \right) \right] \cap K_{\beta+1} = \emptyset.$

By our supposition there exists
$$\gamma_T > \sup(\{\gamma_\rho | \rho < \Gamma\} \cup \{\tau \in \omega_1 | B_f \cap V(\beta, \gamma_\rho) = \emptyset \text{ for some } \rho < \Gamma\})$$
such that (i), (ii), and (iii) hold. Thus we can continue the induction on $\omega_1$.

Since the singletons $\{g(\beta, \gamma_\delta) | \delta \in \omega_1\}$ cannot be separated by pairwise disjoint open sets in $G_\beta = K_\beta \cup K_{\beta+1}$, and since $V(\beta, \gamma_\delta) \cap G_\beta$ is open in $G_\beta$ for every $\delta \in \omega_1$, there exists $\delta_1, \delta_2 \in \omega_1$ ($\delta_1 < \delta_2$), such that $V(\beta, \gamma_\delta) \cap V(\beta, \gamma_{\delta_1}) \cap K_{\beta+1} \neq \emptyset$. Now $G_\beta \cap V(\beta, \gamma_\delta) \cap V(\beta, \gamma_{\delta_1}) = \bigcap_{j=1}^{k} \pi_{j-1}(G_\beta)$ for some $q_1, q_2, \ldots, q_k \in Q_\beta$, and each $t_j$ has the value 1 or 0. Since $g(\beta, \gamma_\delta) \in V(\beta, \gamma_\delta)$ and $g(\beta, \gamma_{\delta_1}) \in V(\beta, \gamma_{\delta_1})$, by (A) if any $q_j = \{h\}$, where $h \in \bigcup\{U_f, \tau < \gamma_\delta\} \cap K_\beta$, it follows that $t_j = 0$. Hence $V(\beta, \gamma_\delta) \cap V(\beta, \gamma_{\delta_1}) \cap V(\beta, \gamma_{\delta_2}) \cap \bigcup\{U_f, \tau < \gamma_\delta\} \cap K_{\beta+1} \neq \emptyset$. Choose $x \in V(\beta, \gamma_\delta) \cap V(\beta, \gamma_{\delta_1}) \cap V(\beta, \gamma_{\delta_2}) \cap \bigcup\{U_f, \tau < \gamma_\delta\} \cap K_{\beta+1}$. By (iii) above, we must have that $x \in \bigcup B_x$. Choose $B_x \subset B_x$ with $x \in B_x$. Then
$$x \in V(\beta, \gamma_\delta) \cap V(\beta, \gamma_{\delta_1}) \cap B_x.$$ 

By (A-2) above, $g(\beta, \gamma_\delta) \in B_x$. Thus it follows from (ii) and the assumption (A) that $g(\beta, \gamma_{\delta_1}) \notin B_x$. However, by (A-2) we have that $V(\beta, \gamma_{\delta_1}) \cap B_x = \emptyset$, contradicting (**) Therefore, the lemma is proved.

We now consider the case where $\alpha$ is a successor ordinal.

II. Case 2. $\alpha = \beta + 1$.

Since $\beta$ satisfies the inductive hypothesis (1)–(5), condition (A) of Lemma 1 is satisfied. To see this, let $X = \bigcup_{\mu \in \alpha} (\bigcup U_\mu)$ and let $\mathcal{B} = \mathcal{B}_\alpha$. Then the $\alpha_1$ of Lemma 1, condition (A) is fulfilled by $\delta_\beta$, since for each $\delta \geq \delta_\beta$ the element $x(\beta, \delta) \in (U_f \cap K_\beta) \setminus \left( \bigcup_{\mu \in \alpha} (\bigcup U_\mu) \right)$ by inductive hypothesis (3).

Hence by Lemma 1, there exists $\delta_\alpha < \omega_1$ satisfying condition (3). Clearly, we can choose $\delta_\alpha > \delta_\beta$, so (1) is satisfied. Since $\mathcal{B}_\alpha$ is a relatively closed, discrete collection in $K_{\omega_1} - (\bigcup_{\mu < \alpha} (\bigcup U_\mu))$, we can choose an open set $V(\alpha, \delta)$ about each $x(\alpha, \delta)$ such that conditions (2), (4), and (5) are satisfied. The proof of Theorem 1 is now complete.

Theorem 2. $G_n$ is metacompact for each $n \in N$.

Proof. Let $\mathcal{U}$ be any open cover of $G_n$. For each $f \in K_n$, choose a member $U_f$ of $\mathcal{U}$ that contains $f$. For each $f \in K_n$, let $V_f$ represent the standard
funnel about \( f \). Then \( \{ U_f \cap V_f \mid f \in K_1 \} \cup \{ \{ g \} \mid g \in G_n \setminus K_n \} \) is a point-finite refinement of \( \mathcal{V} \).

Theorem 3. \( K_{w_0}^* \) is metacompact.

Proof. Let \( \mathcal{V} = \{ V_\alpha \mid \alpha \in A \} \) be an open cover of \( K_{w_0}^* \). Then, since \( G_1 \) is metacompact, \( \mathcal{V} \) has a 1-1, open in \( K_{w_0}^* \), partial refinement \( \mathcal{T} = \{ T_\alpha \mid \alpha \in A \} \) which is point-finite on \( G_1 \) and covers \( G_1 \). For each \( x \in G_n \cap T_\alpha \), let \( C(x, n, \alpha) \) be the intersection of the standard funnel about \( x \) with \( T_\alpha \). Define the open set \( S^n_\alpha \) inductively as follows:

(1) \( S^1_\alpha \cap G_1 = T_\alpha \cap G_1 \);
(2) for \( n > 1 \), \( S^n_\alpha \cap G_n = \left( S^n_\alpha \cap K_n \right) \cup \{ y \in G_n \mid y \in C(x, n, \alpha) \text{ for some } x \in S^n_\alpha \cap K_n \} \).

Note that \( S^n_\alpha = \{ S^n_\alpha \mid \alpha \in A \} \) is an open in \( K_{w_0}^* \), point-finite, partial refinement of \( \mathcal{V} \) which covers \( G_1 \). To see this, suppose \( x \in K_3 \). Then \( x \in \Pi_q^{-1}(1) \) for only finitely many \( q \) in \( P(K_2) \). Hence \( x \) is a member of only finitely many standard funnels about elements in \( K_2 \). Since \( K_2 \subseteq G_1 \), \( S^n_\alpha \) is point-finite on \( K_2 \). Thus, by (2) it must be the case that \( S^n_\alpha \) is point-finite on \( K_3 \). Continuing in this way it follows that \( S^n_\alpha \) is point-finite on \( K_n \) for every \( n \).

Next, since \( K_1 \cup K_2 \cup \cdots \cup K_n \) is a closed subset of \( K_{w_0}^* \), and since \( G_n \) is metacompact for each \( n \), we can construct a 1-1 point-finite open partial refinement \( \mathcal{T}^n \) of \( \mathcal{V} \) that covers \( K_n \) and misses \( K_n \cap K_{n-1} \). It then folows that \( \mathcal{T} = \bigcup(B^n : n \in N) \) is a point-finite open refinement of \( \mathcal{V} \). Hence, \( K_{w_0}^* \) is metacompact.

To show that \( K_{w_0}^* \) is mesocompact, it suffices to show that every compact subset of \( K_{w_0}^* \) is finite. In [2], J. R. Boone shows that every compact subset of \( G_1 \) is finite. We now extend this result to obtain the following lemma.

Lemma 2. For each \( n \in N \), if \( C \) is a compact subset of \( K_{w_0}^* \), then \( C \cap (\bigcup(G_i \mid i \leq n)) \) is finite.

Proof. The proof is by induction on \( N \). For \( n = 1 \) observe that \( G_1 \cap C \) is closed in \( K_{w_0}^* \) and therefore compact. Suppose that we can choose distinct elements \( f_1, f_2, f_3, \ldots, f_n, \ldots \) in \( C \cap G_1 \). Since \( K_1 \) is discrete, we may assume that each \( f_i \) belongs to \( K_2 \cap C \). Since \( \| \{q \in Q_1 \mid f_i(q) \neq 0 \text{ for some } i \in N \} \| \leq \aleph_0 \), for each \( f \in K_1 \) we can choose a basic open funnel \( V_f \) about \( f \) that misses \( \{ f_1, f_2, f_3, \ldots \} \), since \( \| \{q \in Q_1 \mid f(q) \neq 0 \} \| > \aleph_0 \). Thus \( \{ V_f \mid f \in K_1 \cap C \} \cup \{ \{ f \} \mid f \in C \cap K_2 \} \) is an open cover (open in \( G_1 \)) of \( C \cap G_1 \) with no finite subcover, contradicting the compactness of \( C \cap G_1 \). Hence \( C \cap G_1 \) must be finite.

Assume that for all \( k < n \), \( C \cap (\bigcup(G_i \mid i \leq k)) \) is finite.

By inductive hypothesis, \( C \cap (\bigcup(G_i \mid i \leq n-1)) \) is compact and therefore finite. Suppose we can choose distinct elements \( f_1, f_2, f_3, \ldots \) in \( C \cap K_{n+1} \). Since \( \| \{q \in Q_n \mid f_i(q) \neq 0 \text{ for some } i \in N \} \| \leq \aleph_0 \), for each \( f \in C \cap (\bigcup(G_i \mid i \leq n-1)) \) we can choose a funnel \( V_f \) about \( f \) which misses \( \{ f_1, f_2, f_3, \ldots \} \subseteq C \). Thus \( \{ V_f \mid f \in C \cap (\bigcup(G_i \mid i \leq n-1)) \} \cup \{ \{ f \} \mid f \in C \cap K_{n+1} \} \) is an open cover of \( C \cap (\bigcup(G_i \mid i \leq n)) \) with no finite subcover, in contradiction to the compactness of \( C \cap (\bigcup(G_i \mid i \leq n)) \), so the lemma follows.

Lemma 3. If \( C \) is a compact subset of \( K_{w_0}^* \), then \( C \) is finite.

Proof. Suppose \( C \) is an infinite compact subset of \( K_{w_0}^* \). Then by Lemma 2
there must exist a sequence \( f : N \to C \) such that \( f(n) \notin \bigcup \{ K_j \mid f(i) \in K_j \text{ for some } i < n \} \). Since \( \{ f(n) \mid n \in N \} \) is a closed subset of \( C \), it is compact. It is easy to show that this sequence is also discrete and thus cannot be compact. Hence the lemma is proved.

From Theorem 3 and Lemma 3 above we now have the following.

**Theorem 4.** \( K_{\omega_0}^* \) is mesocompact.

**Theorem 5.** (i) \( K_{\omega_0}^* \) is \( B(D, \omega_0) \)-refinable but not \( B(D, n) \)-refinable for any \( n \).

(ii) \( K_{\omega_0+1}^* \) is \( B(D, \omega_0+1) \)-refinable but not \( B(D, \omega_0) \)-refinable.

(iii) \( K_{\omega_0+1}^* \) is weak \( \theta \)-refinable and \( T_4 \).

**Proof.** The proof of (i) and (ii) follow directly from Theorem 1. Let \( \mathcal{U} \) be an open cover of \( K_{\omega_0+1}^* \). Since \( K_{\omega_0}^* \) is metacompact, \( \mathcal{U} \) has a partial refinement \( \mathcal{S}_1 \), which is point-finite on \( K_{\omega_0}^* \). Next, let \( \mathcal{S}_2 \) be the collection of singleton subsets of \( K_{\omega_0+1} \setminus K_{\omega_0}^* \). Then \( \mathcal{S}_1 \cup \mathcal{S}_2 \) is a weak \( \bar{\theta} \)-refinement of \( \mathcal{U} \).

**Question.** If we modify the construction of each \( G_n \) so that

\[
G_n = \left\{ f \in \prod_{q \in Q_n} \{0, 1\} \mid f \in \Pi_{\{\alpha\}}^{-1}(1) \text{ for some } \alpha \in \omega_1 \right\},
\]

would the space \( K_{\omega_0+1}^* \) remain weak \( \bar{\theta} \)-refinable? The authors conjecture that it would not. If not, then \( K_{\omega_0+1}^* \) is a \( T_4 \), \( B(D, \omega_0+1) \)-refinable space that is not weak \( \bar{\theta} \)-refinable.

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**References**


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