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A MEASURE THEORETICAL SUBSEQUENCE CHARACTERIZATION OF STATISTICAL CONVERGENCE

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ABSTRACT. The concept of statistical convergence of a sequence was first introduced by H. Fast. Statistical convergence was generalized by R. C. Buck, and studied by other authors, using a regular nonnegative summability matrix A in place of C_1.

The main result in this paper is a theorem that gives meaning to the statement: $S = \{s_n\}$ converges to L statistically (T) if and only if "most" of the subsequences of S converge, in the ordinary sense, to L. Here T is a regular, nonnegative and triangular matrix.

Corresponding results for lacunary statistical convergence, recently defined and studied by J. A. Fridy and C. Orhan, are also presented.

INTRODUCTION

The concept of the statistical convergence of a sequence of reals $S = \{s_n\}$ was first introduced by H. Fast [9].

The sequence $S = \{s_n\}$ is said to converge statistically to L and we write

$$\lim_{n\to\infty} s_n = L \text{ (stat)} \quad \text{if for every } \varepsilon > 0,$$

$$\lim_{n\to\infty} n^{-1}|\{k \le n : |s_k - L| \ge \varepsilon\}| = 0,$$

where $|A|$ denotes the cardinality of the set A.

Properties of statistically convergent sequences were studied in [5, 6, 12, and 16]. In [13] Fridy and Miller gave a characterization of statistical convergence for bounded sequences using a family of matrix summability methods.

Statistical convergence can be generalized by using a regular nonnegative summability matrix A in place of C_1. This idea was first mentioned by R. C. Buck [3] in 1953 and has been further studied by Sember and Freedman ([10 and 11]) and Connor ([5 and 7]). Regular nonnegative summability matrices turn out to be too general for our purposes here, instead we use the concept of a mean.

A matrix $T = (a_{mn})$ will be called a mean if $a_{mn} > 0$ when $n \le m$, $a_{mn} = 0$ if $n > m$, $\sum_{n=1}^{\infty} a_{mn} = 1$ for all m and $\lim_{m\to\infty} a_{mn} = 0$ for each n.

If $T = (a_{mn})$ is a mean, following Buck, a sequence $S = \{s_n\}$ is said to be statistically T-summable to L and we write

$$s_n \to L \text{ (stat } T) \quad \text{if for every } \varepsilon > 0$$

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we have

$$\sum_{n=1}^{\infty} [a_{mn} : |s_n - L| \geq \varepsilon] \to 0 \quad \text{as } m \to \infty .$$

The main result in this paper is a theorem that gives meaning to the statement $S = \{s_n\}$ converges to L statistically (T) if and only if "most" of the subsequences of S converge, in the ordinary sense, to L.

In [14] and [15] Fridy and Orhan studied lacunary statistical convergence. We will present a measure theoretical subsequence characterization of lacunary statistical convergence.

Results. We recall that Fridy proved [12] that a sequence S is statistically convergent if and only if there exists a subset A of $\mathbb{N}$ (the natural numbers), having density zero, such that the subsequence of S obtained by removing the terms of S with indices in A is convergent in the ordinary sense. Here, A having density zero means

$$\lim_{n\to\infty} n^{-1}|\{k \leq n : k \in A\}| = 0 .$$

Our first step toward obtaining a subsequence characterization of statistical (T) convergence is the following generalization of the result of Fridy just mentioned. In the statement of our theorem we will need a definition of T-density zero.

If $T = (a_{mn})$ is a mean, then a subset A of $\mathbb{N}$ is said to have T-density zero if

$$\lim_{m\to\infty} \sum_{n\in A} a_{mn} = 0 .$$

Theorem 1. $s_n \to L$ $(\operatorname{stat} T)$ *if and only if there is a subset* A *of* $\mathbb{N}$ *such that* $s_{n_k} \to L$ *(in the usual sense) as* $k \to \infty$, *where* $\mathbb{N}\backslash A = \{n_k : k \in \mathbb{N}\}$ *and* A *has* T*-density zero.*

Proof. Suppose $s_n \to L$ $(\operatorname{stat} T)$. For each $\varepsilon_n = \frac{1}{n}$, $n = 2, 3, \ldots$, there exists a positive integer r_n (with the sequence $\{r_n\}_{n=2}^{\infty}$ strictly increasing) such that

$$(*) \qquad r \geq r_n \text{ implies } \sum_{k=1}^{\infty} \left[a_{rk} : |s_k - L| \geq \frac{1}{n}\right] < \frac{1}{n^2} .$$

Set,

$$A = \bigcup_{n=2}^{\infty} \left\{k \colon r_n \leq k < r_{n+1} \text{ and } |s_k - L| \geq \frac{1}{n}\right\} .$$

The subsequence of S obtained by removing the terms with indices in A clearly converges, in the ordinary sense, to L.

We will now show that A has T-density zero. Let $\varepsilon > 0$. There exists an $n(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n=n(\varepsilon)}^{\infty} \frac{1}{n^2} < \frac{\varepsilon}{2} .$$

The regularity of T implies there exists an $R_{n(\varepsilon)}$, a term in the sequence $\{r_n\}$ with index larger than $n(\varepsilon)$, i.e., $R_{n(\varepsilon)} = r_{m(\varepsilon)}$, $m(\varepsilon) > n(\varepsilon)$, such that

$$\sum_{i=1}^{r_{n(\varepsilon)}-1} a_{ri} < \frac{\varepsilon}{2} \quad \text{for all } r \geq R_{n(\varepsilon)} \; (= r_{m(\varepsilon)}) .$$

Now suppose $r \geq r_{m(\varepsilon)}$. We have

$$(**) \qquad \sum_{i \in A} a_{ri} \leq \sum_{i < r_{n(\varepsilon)}} a_{ri} + \sum_{\substack{i \in A \\ i \geq r_{n(\varepsilon)}}} a_{ri},$$

with $r \geq r_{m(\varepsilon)} > r_{n(\varepsilon)}$, $r_j \leq r < r_{j+1}$, so

$$\sum_{\substack{i \in A \\ i \geq r_{n(\varepsilon)}}} a_{ri} = \sum_{\substack{i \in A \\ r_n(\varepsilon) \leq i \leq r_{n(\varepsilon)+1}}} a_{ri} + \sum_{\substack{i \in A \\ r_{n(\varepsilon)+1} \leq i < r_{n(\varepsilon)+2}}} a_{ri} + \cdots + \sum_{\substack{i \in A \\ r_j \leq i < r_{j+1}}} a_{ri}$$
$$< \frac{1}{n^2(\varepsilon)} + \frac{1}{(n(\varepsilon)+1)^2} + \cdots + \frac{1}{j^2} < \sum_{n=n(\varepsilon)}^{\infty} \frac{1}{n^2} < \frac{\varepsilon}{2}$$

by $(*)$ and the definition of A. So, by $(**)$

$$\sum_{i \in A} a_{ri} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{if } r \geq r_{m(\varepsilon)},$$

or A has T-density zero.

Now we look at the converse. Suppose that A has T-density zero, $\{n_k\} = \mathbb{N} \backslash A$ and $s_{n_k} \to L$ (in the ordinary sense). We must show that $s_n \to L$ (stat T). Let $\varepsilon > 0$. Then there exists a $k(\varepsilon)$ such that $k \geq k(\varepsilon)$ implies $|s_{n_k} - L| < \varepsilon$ and therefore

$$\begin{aligned} &\sum_k [a_{rk} : |s_k - L| \geq \varepsilon] \\ &\quad \leq \sum_{k < n_{k(\varepsilon)}} a_{rk} + \sum_{k \geq n_{k(\varepsilon)}} [a_{rk} : |s_k - L| \geq \varepsilon] \\ &\quad \leq \sum_{k < n_{k(\varepsilon)}} a_{rk} + \sum [a_{rk} : k \in A]. \end{aligned}$$

$\lim_{r\to\infty} \sum_{k<n_k(\varepsilon)} a_{rk} = 0$ since T is regular and $\lim_{r\to\infty} \sum [a_{rk} : k \in A] = 0$ because A has T-density 0. □

We now observe that there is a one-to-one onto correspondence between the interval $(0, 1]$ and the collection of all subsequences of the sequence $S = \{s_n\}$. Namely, if $x \in (0, 1]$, then x has a unique binary expansion $x = \sum_{n=1}^{\infty} e_n(x) 2^{-n}$, $e_n(x) \in \{0, 1\}$, with infinitely many ones. For each $x \in (0, 1]$, let $S(x)$ denote the subsequence of S obtained by the following rule: s_n is in the subsequence $S(x)$ if and only if $e_n(x) = 1$. Clearly the mapping $x \mapsto S(x)$ is a one-to-one onto mapping between $(0, 1]$ and the collection of all subsequences of S.

Suppose T is a mean and $s_n \to L$ (stat T). It is natural to consider the set

$$C_L := \{x \in (0, 1] : S(x) \text{ converges to } L\}.$$

This set may well have Lebesgue measure zero as the following example shows and hence "most" of the subsequences of $S = \{s_n\}$, in the sense of Lebesgue measure, need not converge to L.

Example 1. Let T denote the $(C, 1)$ matrix and hence (stat T) convergence is statistical convergence. Let A be any infinite subset of $\mathbb{N}$ having density zero. Define $S = \{s_n\}$ as follows. $s_n = 1$ if $n \notin A$ and $s_n = 0$ if $n \in A$. Two easy applications of Borel's normal number theorem [2, p. 9] shows that

$$m(\{x \in (0, 1] : S(x) \text{ has infinitely many zero terms and infinitely many one terms}\}) = 1$$

where m is Lebesgue measure. Also, $\lim_{n\to\infty} s_n = 1$ (stat).

This example shows that to get the theorem mentioned in the introduction it will be necessary to use a measure different from Lebesgue measure.

In the following, if $A = \{k_n\}$ is any subset of $\mathbb{N}$, m_A will denote the unique probability measure defined on the Borel subsets of $(0, 1]$ having the following property:

$$m_A(\{x \in (0, 1] : e_j(x) = 1\}) = \begin{cases} \frac{1}{2} & \text{if } j \notin A, \\ \frac{1}{2^n} & \text{if } j = k_n \end{cases}$$

and $\{e_n(x)\}_{n=1}^{\infty}$ is a sequence of independent random variables with respect to m_A. See [1].

To get a little better feel for m_A, consider the following inductive process. Suppose m_A has been defined for the

$$\begin{array}{llllllll} 2^1 & \text{half} & \text{closed} & \text{intervals} & \text{of} & \text{length} & \frac{1}{2}, \\ 2^2 & \text{half} & \text{closed} & \text{intervals} & \text{of} & \text{length} & \frac{1}{2^2}, \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot, \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot, \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot, \\ 2^{j-1} & \text{half} & \text{closed} & \text{intervals} & \text{of} & \text{length} & \frac{1}{2^{j-1}}. \end{array}$$

Each of the last-mentioned 2^{j-1} intervals I is divided into two abutting half-closed intervals of length $\frac{1}{2^j}$, call them I (left) and I (right), the domain of m_A is extended as follows:

$$\left.\begin{aligned} m_A(I(\text{left})) &= \tfrac{1}{2} m_A(I) \\ m_A(I(\text{right})) &= \tfrac{1}{2} m_A(I) \end{aligned}\right\} \quad (\text{if } j \notin A),$$

$$\left.\begin{aligned} m_A(I(\text{left})) &= \left(1 - \tfrac{1}{2^n}\right) m_A(I) \\ m_A(I(\text{right})) &= \tfrac{1}{2^n} m_A(I). \end{aligned}\right\} \quad (\text{if } j = k_n).$$

m_A is the unique probability measure on the Borel subset of $(0, 1]$ whose values on half-closed dyadic subintervals are given above.

The purpose of using $\frac{1}{2^n}$ instead of $\frac{1}{2}$ when $j = k_n$ is to avoid "picking" elements of the "bad" set A.

We are now ready to prove the main result in this paper.

Theorem 2. *Suppose $T = (a_{mn})$ is a mean. The sequence $S = \{s_n\}$ converges (stat T) to L (i.e., $s_n \to L$ (stat T)) if and only if there exists a subset A of $\mathbb{N}$ having T-density zero such that*

$$m_A(C_L) = m_A\left(\left\{x \in (0, 1] : \lim_{n\to\infty} (S(x))_n = L\right\}\right) = 1.$$

Proof. Suppose $s_n \to L$ (stat T). Then, by Theorem 1, there exists a subset A of $\mathbb{N}$, having T-density zero such that $\{x_{n_k}\}$ converges, in the ordinary sense, to L, where $\{n_k : k \in \mathbb{N}\} = \mathbb{N}\backslash A$.

Notice that $S(x)$ converges to L if $\{n \in A : e_n(x) = 1\}$ is a finite set. However, by the first part of the Borel-Cantelli Lemma (see [2, p. 46])

$$m_A(\{x \in (0, 1] : \{n \in A : e_n(x) = 1\} \text{ is infinite}\}) = 0$$

since $\sum_{n\in A} m_A\{x \in (0, 1] : e_n(x) = 1\} = \sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent. Therefore $m_A(C_L) = 1$.

Suppose now that $S = \{s_n\}$ is not statistically (T) convergent and A is any subset of $\mathbb{N}$ having T-density zero. Then, by Theorem 1 $\{s_{n_k}\}$, where $\{n_k\} = \mathbb{N}\backslash A$, does not converge. Then we have either

$$\lim_{j\to\infty} s_{n_{k_j}} = +\infty \quad \text{for some subsequence } \{n_{k_j}\} \text{ of } \{n_k\}$$

or

$$\lim_{j\to\infty} s_{n_{k_j}} = -\infty \quad \text{for some subsequence } \{n_{k_j}\} \text{ of } \{n_k\}$$

or there exist $\lambda < \mu$ and two infinite subsets B and C of $\mathbb{N}$ such that $A \cap B = A \cap C = B \cap C = \varnothing$ and $s_n < \mu$ if $n \in B$ and $s_n > \mu$ if $n \in C$.

Now, since $m_A(\{x \in (0, 1] : e_n(x) = 1\}) = \frac{1}{2}$ if $n \notin A$, we have

$$\text{In Case 1,} \quad \sum_{j=1}^{\infty} m_A(\{x \in (0, 1] : e_{n_{k_j}}(x) = 1\}) = \sum \frac{1}{2} = \infty.$$

$$\text{In Case 2,} \quad \sum_{j=1}^{\infty} m_A(\{x : (0, 1] : e_{n_{k_j}}(x) = 1\}) = \sum \frac{1}{2} = \infty.$$

$$\text{In Case 3,} \quad \sum_{n\in B} \frac{1}{2} = \infty = \sum_{n\in C} \frac{1}{2} = \infty.$$

So, by the second part of the Borel-Cantelli Lemma [2, p. 48] we have:

In Case 1, $m_A(\{x \in (0, 1] : e_{n_{k_j}}(x) = 1 \text{ for infinitely many } j\}) = 1$.

In Case 2, $m_A(\{x \in (0, 1] : e_{n_{k_j}}(x) = 1 \text{ for infinitely many } j\}) = 1$.

In Case 3, $m_A(\{x \in (0, 1] : e_n(x) = 1 \text{ for infinitely many } n \in B$ and also for infinitely many $n \in C\}) = 1$.

Therefore, in each of the above three cases we have

$$m_A(\{x \in (0, 1] : S(x) \text{ is convergent}\}) = 0. \quad \square$$

Example 1 shows that $s_n \to L$ (stat T), where $S = \{s_n\}$ is a sequence and T is a mean, does not imply that

$$m(\{x \in (0, 1] : S(x) \text{ is convergent}\}) = 1.$$

It is natural to ask if

$$(*) \qquad s_n \to L \text{ (stat } T) \text{ implies } m(\{x \in (0, 1] : (S(x))_n \to L \text{ (stat } T)\}) = 1.$$

It is easy to construct examples to show that $(*)$ does not hold in general.

A matrix summability method is said to have the Borel property if it sums "almost all" sequences of 0's and 1's to the value $\frac{1}{2}$; see [8, p. 207]. It is more involved to find a mean T that has the Borel property and a sequence $S = \{s_n\}$ such that $s_n \to L$ (stat T) but

$$m(\{x \in (0, 1] : (S(x))_n \to L \text{ (stat } T)\}) = 0.$$

Example 2. Let $S = \{s_n\} = \{1, 0, 1, 0, 1, \dots\}$. Let T be the mean defined in the following way. $a_{11} = 1$ and for each row m, $m \geq 2$, spread the total weight $1 - \frac{1}{m}$ equally in the odd columns and spread the total weight $\frac{1}{m}$ equally in the even columns. Of course, $a_{mn} = 0$ if $n > m$. Then T is a mean satisfying the Borel property (see [6, p. 211]) and

$$s_n \to 1 \ (\operatorname{stat} T).$$

To show that $m(\{x \in (0, 1] : S(x) \text{ is } (\operatorname{stat} T) \text{ convergent}\}) = 0$, we consider the sequences $\{X_n\}_{n=1}^{\infty}$ of random variables on the probability space $((0, 1], \mathscr{B}, m)$, where $\mathscr{B}$ are the Borel subsets of $(0, 1]$, that are defined in the following manner. For each x in $(0, 1]$:

$$X_1(x) = \begin{cases} 1 & \text{if } (S(x))_1 \neq (S(x))_3, \\ 0 & \text{if } (S(x))_1 = (S(x))_3, \end{cases}$$
$$X_2(x) = \begin{cases} 1 & \text{if } (S(x))_5 \neq (S(x))_7, \\ 0 & \text{if } (S(x))_5 = (S(x))_7, \end{cases}$$
$$\vdots \qquad\qquad \cdots \qquad\qquad .$$

Set

$$Y_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

A little reflection shows that there exists an $a > 0$ such that:

$$m([X_1 = 0]) > a, \quad m([X_1 = 1]) > a$$

and for each $n \geq 2$ and $i_1, i_2, \dots, i_{n+1} \in \{0, 1\}$

$$m([X_n = i_{n+1} | X_1 = i_1, \dots, X_n = i_n]) > a.$$

This implies that (see [3])

$$m\left(\left\{x \in (0, 1] : \liminf_{n\to\infty} Y_n(x) > 0\right\}\right) = 1.$$

Moreover, if $\liminf_{n\to\infty} Y_n(x) > 0$, $S(x)$ is not convergent $(\operatorname{stat} T)$ and hence $m(\{x \in (0, 1] : S(x) \text{ is } (\operatorname{stat} T) \text{ convergent}\}) = 0$.

Despite the above example we do have a characterization of statistical convergence.

Theorem 3. *The sequence* $S = \{s_n\}$ *converges statistically to* L (*i.e.*, $\lim_{n\to\infty} s_n = L$ (stat)) *if and only if*

$$m\left(\left\{x \in (0, 1] : \lim_{n\to\infty} (S(x))_n = L \text{ (stat)}\right\}\right) = 1.$$

Proof. Suppose $\lim_{n\to\infty} s_n = L$ (stat) and $x \in (0, 1]$ is a normal number, i.e., $\frac{1}{n}\sum_{k=1}^{n} e_k(x) \to \frac{1}{2}$ as $n \to \infty$. Then $S(x) = \{s_{n_1}, s_{n_2}, \dots\}$ where $\lim_{k\to\infty} \frac{n_k}{k} = 2$. Let $\varepsilon > 0$. Then

$$\begin{aligned} \frac{1}{k}|\{i \leq k : |s_{n_i} - L| \geq \varepsilon\}| &\leq \frac{1}{k}|\{i \leq n_k : |s_i - L| \geq \varepsilon\}| \\ &= \frac{n_k}{k}\frac{|\{i \leq n_k : |s_i - L| \geq \varepsilon\}|}{n_k} \to 2 \cdot 0 = 0 \end{aligned}$$

as $k \to \infty$. Therefore $\lim_{n\to\infty}(S(x))_n = L$ (stat) if x is a normal number. Since $M = \{x : (0, 1] : x$ is a normal$\}$ has Lebesgue measure one the proof of the forward implication is complete.

Conversely, assume

$$m\left(\left\{x \in (0, 1] : \lim_{n\to\infty}(S(x))_n = L \text{ (stat)}\right\}\right) = 1.$$

Then there exist two disjoint subsets $\{n_k : k \in \mathbb{N}\}$ and $\{n'_k : k \in \mathbb{N}\}$ of $\mathbb{N}$ such that:

(1) $\{n_k : k \in \mathbb{N}\} \cup \{n'_k : k \in \mathbb{N}\} = \mathbb{N}$,

(2) $\lim_{k\to\infty} \frac{n_k}{k} = 2 = \lim_{k\to\infty} \frac{n'_k}{k}$, and

(3) $s_{n_k} \to L$ (stat) $s_{n'_k} \to L$ (stat).

These three properties imply that

$$s_n \to L \text{ (stat)}, \text{ completing the proof. } \square$$

Remark 1. Suppose $0 < c_1 < 1 < c_2$ and $T = (a_{mn})$ is a mean satisfying

$$(**) \qquad \frac{c_1}{m} \le a_{mn} \le \frac{c_2}{m} \quad \text{for each } m \in \mathbb{N} \text{ and } n = 1, 2, 3, \dots, m.$$

Then $s_n \to L$ (stat) if and only if $s_n \to L$ (stat T). Therefore Theorem 3 can be extended to (stat T) convergence if T satisfies $(**)$.

Fridy and Orhan in [14] and [15] studied lacunary statistical convergence. By a lacunary sequence we mean an increasing sequence of positive integers $\theta = \{k_r\}$ such that $h_r : k_r - k_{r-1} \to \infty$ as $r \to \infty$. In the following we denote by $I_r := (k_{r-1}, k_r]$. Let θ be a lacunary sequence; they defined the sequence of numbers $S = \{s_n\}$ to be S_θ-convergent to L provided for every $\varepsilon > 0$,

$$\lim_{r\to\infty} h_r^{-1}|\{k \in I_r : |s_k - L| \ge \varepsilon\}| = 0$$

and we write $s_n \to L(S_\theta)$.

The following result is an analogue of a theorem of Fridy for statistical convergence that can be found in [8] and related to Theorem 1 in this paper.

Theorem 4. *The sequence $S = \{s_n\}$ satisfies $s_n \to L(S_\theta)$ for a lacunary sequence $\theta = \{k_r\}$ if and only if there exists a subset A of the natural numbers such that the subsequence of S obtained by removing the terms of S with indices in A converges to L in the ordinary sense and* $\lim_{r\to\infty} |A \cap I_r| \cdot h_r^{-1} = 0$.

Proof. Suppose A is a subset of $\mathbb{N}$ such that $\lim_{r\to\infty} |A \cap I_r| \cdot h_r^{-1} = 0$ and the subsequence of S obtained by removing the terms with indices in A converges in the ordinary sense to L. Then given $\varepsilon > 0$,

$$h_r^{-1}|\{k \in I_r : |s_k - L| \ge \varepsilon\}| \le h_r^{-1}|A \cap I_r|$$

for sufficiently large r and hence $s_n \to L(S_\theta)$.

Conversely, suppose $s_n \to L(S_\theta)$. Then there exists a strictly increasing sequence of positive integers $\{r_n\}$ such that

$$h_r^{-1}\left|\left\{k \in I_r : |s_k - L| \ge \frac{1}{n}\right\}\right| < \frac{1}{n}$$

for all $r \ge r_n$.

Let

$$A = \bigcup_{i=1}^{\infty} \left\{ k \in \bigcup_{j=r_i}^{r_{i+1}-1} I_j : |s_k - L| \geq \frac{1}{i} \right\}.$$

Then $\lim_{r\to\infty} |I_r \cap A| \cdot h_r^{-1} = 0$ and the subsequence of $S = \{s_n\}$ obtained by removing the terms of S with indices in A converges in the ordinary sense to L.

The last theorem can be used to obtain a S_θ-convergence analogue of the proof of Theorem 2. Namely, we have the following:

Theorem 5. *The sequence $S = \{s_n\}$ satisfies $s_n \to L(S_\theta)$ for a lacunary sequence $\theta = \{k_r\}$ if and only if there exists a subset A of $\mathbb{N}$ such that*

$$\lim_{r\to\infty} |I_r \cap A| \cdot h_r^{-1} = 0$$

and

$$m_A(C_L) = m_A\left(\left\{x \in (0, 1] : \lim_{n\to\infty} (S(x))_n = L\right\}\right) = 1.$$

Proof. If $s_n \to L(S_\theta)$, then $m_A(C_L) = 1$. This follows as in the first half of the proof of Theorem 2, using Theorem 4.

Suppose now that $S_\theta = \{s_n\}$ is not S_θ-convergent and A is a subset of $\mathbb{N}$ satisfying $\lim_{r\to\infty} |I_r \cap A| \cdot h_r^{-1} = 0$. Then, by Theorem 4, the subsequence $\{s_{n_k}\}$ of S, where $\{n_k\} = \mathbb{N}\backslash A$, does not converge. The remainder follows exactly as in the corresponding part of the proof of Theorem 2.

We conclude by giving an example to show that the S_θ-convergence analogue of Theorem 3 is false.

Example 3. Suppose θ is a lacunary sequence such that $h_2 = 10$, $h_4 = 12$, $h_6 = 14, \dots$. Define $S = \{s_n\}$ as follows: $s_n = 0$ if $n \in I_r$ and r is odd. If r is even, the terms of $\{s_n\}$ with indices in I_r are $1, 0, 1, 0, \dots, 1, 0$. The sequence $h_1, h_3, h_5, \dots$ can be taken large enough, and increasing fast enough to guarantee:

$$m(x \in (0, 1] : h_1^{-1}|\{k \in I_1 : (S(x))_k = 0\}| > .99) > .99,$$
$$m(x \in (0, 1] : h_2^{-1}|\{k \in I_2 : (S(x))_k = 0\}| > .999) > .999,$$
$$\vdots$$
$$\dots$$

By the first part of the Borel-Cantelli Lemma

$$m(\{x \in (0, 1] : (S(x))_n \to 0 \ (S_\theta)\}) = 1,$$

but $s_n \nrightarrow 0 \ (S_\theta)$.

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