TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 5, May 1995

INEQUALITIES FOR ZERO-BALANCED HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. The authors study certain monotoneity and convexity properties of the Gaussian hypergeometric function and those of the Euler gamma function.

1. INTRODUCTION

The Gaussian *hypergeometric function* is defined by

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!}, \tag{1.1}$$

for $x \in (-1, 1)$, where (a, n) denotes the shifted factorial function $(a, n) = a(a+1)\cdots(a+n-1)$, $n = 1, 2, \ldots$, and $(a, 0) = 1$ for $a \neq 0$. The sum is well defined at least when $(c, n) \neq 0$, i.e., when $c \neq 0, -1, -2, \ldots$. This function has found frequent applications in various fields of the mathematical and natural sciences [Ask2]. Many elementary transcendental functions are special cases or limiting cases of $F(a, b; c; x)$; for an extensive list see [AS, pp. 556–566], [PBM, pp. 430–615]. Two important special cases are the complete elliptic integrals

$$\mathscr{K}(x) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right), \qquad \mathscr{E}(x) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; x^2\right). \tag{1.2}$$

As a function of its parameters a, b, c the function $F(a, b; c; x)$ is smooth, and it is thus natural to expect that the properties of $\mathscr{K}(x)$ extend also to $F(a, b; c; x^2)$ for (a, b, c) close to $(\frac{1}{2}, \frac{1}{2}, 1)$. Recall that $F(a, b; c; x)$ is called *zero-balanced* if $c = a + b$. We obtain several results for the zero-balanced $F(a, b; a+b; x)$, $a, b > 0$, extending well-known properties of $\mathscr{K}(x)$.

1.3. **Theorem.** (1) *For* $a, b \in (0, \infty)$ *the function*

$$f(x) \equiv \frac{1 - F(a, b; a+b; x)}{\log(1-x)}$$

Received by the editors September 17, 1993; originally communicated to the *Proceedings of the AMS* by Hal L. Smith.

1991 *Mathematics Subject Classification.* Primary 33C05, 33B15; Secondary 26D07.

Key words and phrases. Hypergeometric function, gamma function, elliptic integral.

is strictly increasing from $(0, 1)$ *onto* $(ab/(a+b), 1/B)$, *where* $B = B(a, b)$ *is the Euler beta function.*

(2) *For* $a, b \in (0, \infty)$ *the function*

$$g(x) \equiv BF(a, b; a+b; x) + \log(1-x)$$

is strictly decreasing from $(0, 1)$ *onto* (R, B), *where*

$$R = -\Psi(a) - \Psi(b) - 2\gamma.$$

Here $\Psi(a) = \Gamma'(a)/\Gamma(a)$, *and* γ *is the Euler-Mascheroni constant.*

1.4. **Theorem.** *For* $a, b \in (0, \infty)$, *let*

$$f(x) = xF(a, b; a+b; x)/\log(1/(1-x))$$

on $(0, 1)$ *and let* B, R *be as in Theorem* 1.3.

(1) *If* $a, b \in (0, 1)$, *then* f *is decreasing with range* $(1/B, 1)$.

(2) *If* $a, b \in (1, \infty)$, *then* f *is increasing with range* $(1, 1/B)$.

(3) *If* $a = b = 1$, *then* $f(x) = 1$ *for all* $x \in (0, 1)$.

(4) *If* $a, b \in (0, 1)$ *the function* $g_1(x) \equiv BF(a, b; a+b; x) + (1/x)\log(1-x)$ *is increasing from* $(0, 1)$ *onto* $(B-1, R)$.

(5) *If* $a, b \in (1, \infty)$, *then* g_1 *is decreasing from* $(0, 1)$ *onto* $(R, B-1)$.

Here Theorem 1.3(1) generalizes the fact that $((2/\pi)\mathscr{K}(x) - 1)/\log(1/x')$ is increasing from $(0, 1)$ onto $(1/2, 2/\pi)$; 1.3(2) generalizes the well-known fact that $\mathscr{K}(x) + \log x'$ is decreasing from $(0, 1)$ onto $(\log 4, \pi/2)$ (cf. [AVV1, Theorem 2.2(2)]); and 1.4(1) generalizes the result that $x^2\mathscr{K}(x)/\log(1/x')$ is decreasing from $(0, 1)$ onto $(1, \pi)$ [AVV3, Theorem 2(19)].

The asymptotic relation

$$F(a, b; a+b; x) \sim -\frac{1}{B(a, b)}\log(1-x) \tag{1.5}$$

as $x \to 1$ is due to Gauss, and its refined form

$$B(a, b)F(a, b; a+b; x) + \log(1-x) = R + O((1-x)\log(1-x)) \tag{1.6}$$

as $x \to 1$, with $R = -\Psi(a) - \Psi(b) - 2\gamma$, is due to S. Ramanujan [Ev, p. 553], [Be, p. 71]. Theorems 1.3 and 1.4 are refinements of these classical relations. Ramanujan also gave extensions of (1.6) to the generalized hypergeometric function ${}_pF_q$ for certain values of p and q [Ev, pp. 553–558], [Be, pp. 12, 71]. Formulas (1.5) and (1.6) follow from the identity in [AS, 15.3.10], which has been generalized recently in [B, p. 152].

1.7. **Theorem.** *For each* $a, b \in (0, \infty)$ *the function*

$$f(x) \equiv (1-x)^{1/4}F(a, b; a+b; x)$$

is a strictly decreasing automorphism of $(0, 1)$ *if and only if* $4ab \le a+b$.

While studying relationships between the arithmetic-geometric mean and some other means, J. and P. Borwein [BB2, (2.9)] recently proved that

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x^c\right) < F\left(\frac{1}{2}-\delta, \frac{1}{2}+\delta; 1; 1-x^d\right) \tag{1.8}$$

for all $x \in (0, 1)$, with $c = 2$, $d = 3$, $\delta = \frac{1}{6}$.

We obtain the following generalization.

1.9. **Theorem.** *For* $c, d \in (0, \infty), 4c < \pi d$, *inequality* (1.8) *holds for all* $x \in (0, 1)$ *and for all* $\delta \in (0, \delta_o)$, *where* $\delta_o = ((d\pi - 4c)/(4\pi d))^{1/2}$.

We conclude the paper by proving some inequalities for the gamma function. Our notation is mostly standard. For $x \in [0, 1]$, we often denote $x' = \sqrt{1 - x^2}$.

2. Proofs

For many important properties of the functions $F(a, b; c; x)$, $\mathscr{K}(x)$, and $\mathscr{E}(x)$ see [AS, pp. 556–566, 589–626], [C, Chapters 2, 9], [PBM, pp. 430–615]. Some algorithms for computing these functions are given in [Ba, pp. 345–372, 416–424], [M, Chapter 7], [PNB, p. 249], and [PT, pp. 404–426]. Various aspects of the theory of special functions are surveyed in [Ask1], [Ask2], [AVV3], [LO, 5.3, 5.5].

We shall give here a proof of the Gauss asymptotic formula (1.5), since we have been unable to find a proof in the recent literature or in the standard texts. Various generalizations of (1.5) appear in [Ev, pp. 553–558], [B, p. 152].

2.1. **Lemma.** (1) *For* $a, b \in (0, \infty)$, *the sequence*

$$f(n) \equiv \frac{(a, n)(b, n)}{(a + b, n)(n - 1)!}$$

is increasing to the limit $1/B(a, b)$, *as* $n \to \infty$.

(2) *For* $a, b \in (0, 1)$, *the sequence*

$$g(n) \equiv \frac{(a, n)(b, n)(n + 1)}{(a + b, n)n!}$$

is decreasing to the limit $1/B(a, b)$, *as* $n \to \infty$.

(3) *For* $a, b \in (1, \infty)$, *the sequence* $g(n)$ *is increasing to the limit* $1/B(a, b)$, *as* $n \to \infty$.

In particular, for each positive integer n,

$$\frac{ab}{a + b} \le f(n) < \frac{1}{B(a, b)} \quad \text{for all } a, b \in (0, \infty), \tag{4}$$

$$\frac{1}{B(a, b)} < g(n) \le \frac{2ab}{a + b} \quad \text{for all } a, b \in (0, 1), \tag{5}$$

$$\frac{2ab}{a + b} \le g(n) < \frac{1}{B(a, b)} \quad \text{for all } a, b \in (1, \infty), \tag{6}$$

where the weak inequalities reduce to equality if and only if $n = 1$.

Proof. Part (1) is proved in [AVV3, Theorem 6(4)]. For (2) and (3),

$$\frac{g(n + 1)}{g(n)} = \frac{(a + n)(b + n)(n + 2)}{(a + b + n)(n + 1)^2} < 1$$

iff $w_1 \equiv ((a + b + n)n + ab)(n + 2) < w_2 \equiv (a + b + n)(n + 1)^2$, which is true since $w_2 - w_1 = (a + b - 2ab) + n(1 - ab)$ is positive if $a, b \in (0, 1)$, and negative if $a, b \in (1, \infty)$. The limiting values follow form Stirling's formula [C, p. 24], [Mi, p. 184]. □

2.2. *Remark.* The following inequalities due to Wallis appear in [Mi, p. 192]:

$$\frac{1}{\sqrt{\pi(n+\frac{1}{2})}} < \frac{(\frac{1}{2}, n)}{n!} < \frac{1}{\sqrt{\pi n}}.$$

Since $B(\frac{1}{2}, \frac{1}{2}) = \pi$, the result in Lemma 2.1(4) generalizes the second Wallis inequality here.

2.3. **Lemma** (Gauss). *For* $a, b \in (0, \infty)$, *the relation* (1.5) *holds.*

Proof. Let $B = B(a, b)$. From Lemma 2.1, we have

$$F(a, b; a+b; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(a+b, n)n!} x^n \leq 1 + \sum_{n=1}^{\infty} \frac{1}{B}\frac{x^n}{n} = 1 - \frac{1}{B}\log(1-x).$$

Hence

$$\limsup_{x\to 1} \frac{F(a, b; a+b; x)}{\log(1/(1-x))} \leq \frac{1}{B}.$$

Next, fix $\varepsilon \in (0, 1/B)$. Then by Lemma 2.1 there exists a positive integer n_o such that $(a, n)(b, n)/((a+b, n)(n-1)!) > (1/B) - \varepsilon$, for all $n \geq n_o$. Hence,

$$F(a, b; a+b; x) \geq F_{n_o} + \left(\frac{1}{B} - \varepsilon\right) \sum_{n=n_o+1}^{\infty} \frac{x^n}{n} = F_{n_o} - \left(\frac{1}{B} - \varepsilon\right)(\log(1-x) + S_{n_o}),$$

where

$$F_{n_o} = \sum_{n=1}^{n_o} \frac{(a, n)(b, n)}{(a+b, n)n!} x^n, \quad S_{n_o} = \sum_{n=1}^{n_o} \frac{x^n}{n}.$$

Dividing by $\log(1/(1-x))$ and letting $x \to 1$, since $\varepsilon \in (0, 1/B)$ is arbitrary, we get

$$\liminf_{x\to 1} \frac{F(a, b; a+b; x)}{\log(1/(1-x))} \geq \frac{1}{B},$$

so that the limit is $1/B$. □

2.4. *Proof of Theorem* 1.3. (1) The limiting value as $x \to 0$ follows from series expansions, while the one as $x \to 1$ follows from Lemma 2.3. Next, let $g(x) = F(a, b; a+b; x) - 1$ and $h(x) = \log(1/(1-x))$. Then $g(0) = h(0) = 0$, and by the monotone l'Hôpital rule [AVV4, Lemma 2.2] it is enough to show that $g'(x)/h'(x)$ is strictly increasing. Now [WW, p. 281]

$$\begin{aligned}\frac{g'(x)}{h'(x)} &= \frac{ab}{a+b}(1-x)F(a+1, b+1; a+b+1; x) \\ &= \frac{ab}{a+b}\left(1 + \sum_{n=0}^{\infty}\left(\frac{(a+1, n+1)(b+1, n+1)}{(a+b+1, n+1)(n+1)!}\right.\right. \\ &\qquad \left.\left. - \frac{(a+1, n)(b+1, n)}{(a+b+1, n)n!}\right) x^{n+1}\right).\end{aligned}$$

The coefficient of x^{n+1} here is positive if and only if $(a+1+n)(b+1+n) > (a+b+n+1)(n+1)$, which holds if and only if $ab + (a+b)(n+1) + (n+1)^2 > (a+b)(n+1) + (n+1)^2$, which is true.

(2) The limiting values are clear from (1.6). Next, by series expansion,

$$g(x) - B = \sum_{n=1}^{\infty} \left(B \frac{(a, n)(b, n)}{(a+b, n)(n-1)!} - 1 \right) \frac{x^n}{n},$$

so that all coefficients are negative by Lemma 2.1(4). □

2.5. *Proof of Theorem* 1.4. For (1), let $g(x) = xF(a, b; c; x)$, $c = a + b$, and $h(x) = \log(1/(1-x))$. Then $g(0) = h(0) = 0$ and

$$\begin{aligned}
\frac{g'(x)}{h'(x)} &= (1-x)\left(F(a, b; c; x) + \frac{abx}{c} F(a+1, b+1; c+1; x)\right) \\
&= (1-x)\left(\sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!} + \frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a+1, n)(b+1, n)}{(c+1, n)} \frac{x^{n+1}}{n!}\right) \\
&= (1-x)\left(\sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(a, n+1)(b, n+1)}{(c, n+1)} \frac{x^{n+1}}{n!}\right) \\
&= \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \left(\frac{(a, n+1)(b, n+1)}{(c, n+1)} - \frac{(a, n)(b, n)}{(c, n)} \right) \frac{x^{n+1}}{n!} \\
&\quad - \sum_{n=0}^{\infty} \frac{(a, n+1)(b, n+1)}{(c, n+1)} \frac{x^{n+2}}{n!}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{g'(x)}{h'(x)} - 1 = &\left(\frac{2ab}{c} - 1 \right) x + \sum_{n=0}^{\infty} \frac{(a, n+2)(b, n+2)}{(c, n+2)} \frac{x^{n+2}}{(n+2)!} \\
&+ \sum_{n=0}^{\infty} \frac{(a, n+2)(b, n+2)}{(c, n+2)} \frac{x^{n+2}}{(n+1)!} - \sum_{n=0}^{\infty} \frac{(a, n+1)(b, n+1)}{(c, n+1)} \frac{x^{n+2}}{n!} \\
&- \sum_{n=0}^{\infty} \frac{(a, n+1)(b, n+1)}{(c, n+1)} \frac{x^{n+2}}{(n+1)!} \\
= &\left(\frac{2ab}{c} - 1 \right) x + \sum_{n=0}^{\infty} \frac{(a, n+1)(b, n+1)}{(c, n+2)(n+2)!} \\
&\times [(n+1)(ab-1) + (2ab - a - b)] x^{n+2},
\end{aligned}$$

in which all coefficients are negative. Thus $g'(x)/h'(x)$ is decreasing, hence so is $g(x)/h(x)$, by the monotone l'Hôpital rule [AVV4, Lemma 2.2], and (1) follows.

The proof for (2) is similar, except that all coefficients are positive.

Part (3) follows from (1.1) and the series for $\log(1/(1-x))$.

(4) In the series expansion,

$$g_1(x) - (B-1) = \sum_{n=1}^{\infty} \left(\frac{(a, n)(b, n)}{(a+b, n)n!} B - \frac{1}{n+1} \right) x^n,$$

all coefficients are positive by Lemma 2.1(5).

(5) The proof is similar to (4), except that all coefficients are negative by Lemma 2.1(6). The limiting values are clear by (1.6). □

2.6. **Lemma.** *For* $a, b \in (0, \infty)$, $n = 1, 2, 3, \ldots$,

$$\sum_{k=0}^{n} \frac{(a, k)(b, k)}{(a+b, k)k!} > \frac{(a+1, n)(b+1, n)}{(a+b+1, n)n!} > \frac{(a, n)(b, n)}{(a+b, n)n!}.$$

Proof. The first inequality follows by induction, and the second one by the factorial property of (a, n). □

2.7. **Lemma.** *For* $a, b \in (0, \infty)$ *and* $x \in (0, 1)$ *the Maclaurin series of the functions*

$$\frac{1}{1-x}F(a, b; a+b; x) - F(a+1, b+1; a+b+1; x)$$

and

$$F(a+1, b+1; a+b+1; x) - F(a, b; a+b; x)$$

have constant term zero and all other coefficients strictly positive. In particular, these functions are increasing and convex on $(0, 1)$ *and*

$$F(a, b; a+b; x) < F(a+1, b+1; a+b+1; x) < \frac{1}{1-x}F(a, b; a+b; x)$$

for all $x \in (0, 1)$.

Proof. The results follow immediately from Lemma 2.6. □

2.8. *Proof of Theorem* 1.7. The limiting value $f(0) = 1$ is obvious, while $f(1-) = 0$ follows from Theorem 1.3(1). Next, for each $x \in (0, 1)$ we have

$$4(1-x)^{-1/4}f'(x) = \frac{4ab}{a+b}F(a+1, b+1; a+b+1; x) - \frac{1}{1-x}F(a, b; a+b; x).$$

If $4ab \le a+b$, then Lemma 2.7 implies that $f'(x) < 0$. Conversely, if f is decreasing, then $f'(x) \le 0$ for all $x \in (0, 1)$, and letting x tend to 0 we get $4ab \le a+b$. □

The special case $a = b = \frac{1}{2}$ of Theorem 1.7 is well known. This special case follows, for instance, from the infinite product formulas for $x'/\sqrt{x}$ and $(2/\pi)\sqrt{x}\mathscr{K}(x)$ in terms of the Jacobi nome $q = \exp(-\pi\mathscr{K}'/\mathscr{K})$ [WW, p. 488, Exercise 10] or from [AVV1, Theorem 2.2(3)].

2.9. *Remark.* It is easy to show that $(\sqrt{ab}, n)^2 \le (a, n)(b, n) \le ((a+b)/2, n)^2$ for all positive a, b and all $n = 1, 2, \ldots$, with equality if and only if $a = b$. Thus

$$F(\sqrt{ab}, \sqrt{ab}; c; x) \le F(a, b; c; x) \le F((a+b)/2, (a+b)/2; c; x) \tag{2.10}$$

for $a, b, c > 0$, $x \in [0, 1)$, with equality if and only if $a = b$ or $x = 0$. Furthermore, for $a, c > 0$ and for $t \in (0, a)$ we see that $(a+t, n)(a-t, n)$ is a decreasing function of t on $[0, a]$ for $n = 1, 2, \ldots$, so that

$$F(a+t_2, a-t_2; c; x) \le F(a+t_1, a-t_1; c; x) \tag{2.11}$$

for $x \in (0, 1)$, $0 \le t_1 < t_2 \le a$. The second assertion of Lemma 2.7 also follows from the fact that $(a, n)(b, n)/(a+b, n)$, $a, b > 0$, is a strictly increasing function of a.

3. The gamma function

We next study some properties of the functions

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad \text{and} \quad \Psi(x) = \frac{d}{dx}\log\Gamma(x)$$

when x is real and positive.

3.1. **Theorem.** *The function* $f(x) = x(\log x - \Psi(x))$ *is decreasing and convex from* $(0, \infty)$ *onto* $(\frac{1}{2}, 1)$.

Proof. From [WW, p. 251, §12.32, Example] it follows that

$$f(x) = \frac{1}{2} + 2x\int_0^\infty \frac{t\,dt}{(t^2+x^2)(e^{2\pi t}-1)}.$$

Hence,

$$\begin{aligned} f'(x) &= 2\int_0^\infty \frac{t\,dt}{(t^2+x^2)(e^{2\pi t}-1)} - 4x^2\int_0^\infty \frac{t\,dt}{(t^2+x^2)^2(e^{2\pi t}-1)} \\ &= 2\int_0^\infty \frac{t(t^2-x^2)\,dt}{(t^2+x^2)^2(e^{2\pi t}-1)} \\ &= 2\int_0^x \frac{t}{e^{2\pi t}-1}\frac{(t^2-x^2)\,dt}{(t^2+x^2)^2} + 2\int_x^\infty \frac{t}{e^{2\pi t}-1}\frac{(t^2-x^2)\,dt}{(t^2+x^2)^2}. \end{aligned}$$

Since $t/(e^{2\pi t}-1)$ is decreasing on $(0, \infty)$, we get

$$f'(x) < \frac{2x}{e^{2\pi x}-1}\int_0^\infty \frac{(t^2-x^2)\,dt}{(t^2+x^2)^2}.$$

Substituting $t = x\tan u$, we get

$$\int_0^\infty \frac{(t^2-x^2)\,dt}{(t^2+x^2)^2} = \frac{1}{x}\int_0^{\pi/2}(\sin^2 u - \cos^2 u)\,du = 0.$$

Thus we have shown that $f(x)$ is strictly decreasing, so that $f(0+)$ exists. To obtain this limit we observe first that $t/(e^{2\pi t}-1) < 1/(2\pi)$ implies that

$$\lim_{x\to 0+} 2x\int_0^\infty \frac{t\,dt}{(t^2+x^2)(e^{2\pi t}-1)} \le \frac{1}{2}.$$

Next, fix $\varepsilon \in (0, 1/(2\pi))$. Since $\lim_{t\to 0+} t/(e^{2\pi t}-1) = 1/(2\pi)$, there exists $\delta > 0$ such that $t/(e^{2\pi t}-1) > 1/(2\pi) - \varepsilon$ for all $t \in (0, \delta)$. Thus

$$2x\int_0^\infty \frac{t\,dt}{(t^2+x^2)(e^{2\pi t}-1)} > 2x\left(\frac{1}{2\pi} - \varepsilon\right)\int_0^\delta \frac{dt}{t^2+x^2} = 2\left(\frac{1}{2\pi} - \varepsilon\right)\arctan\frac{\delta}{x}.$$

Now letting first x, then ε, tend to zero gives

$$\lim_{x\to 0+} 2x\int_0^\infty \frac{t\,dt}{(t^2+x^2)(e^{2\pi t}-1)} \ge \frac{1}{2}.$$

For $0 \le a < b \le \infty$ denote

$$I(a, b) = \int_a^b \frac{t\,dt}{(t^2+x^2)(e^{2\pi t}-1)}.$$

Then, since $t/(e^{2\pi t}-1) \leq 1/(2\pi)$,

$$2xI(0, 1) \leq \frac{1}{\pi} \arctan \frac{1}{x},$$

while

$$\begin{aligned} I(1, \infty) &\leq \frac{1}{1+x^2} \int_1^\infty \frac{t\,dt}{e^{2\pi t}-1} = \frac{1}{1+x^2} \int_1^\infty \frac{te^{-\pi t}\,dt}{2\sinh(\pi t)} \\ &\leq \frac{1}{2\pi(1+x^2)} \int_1^\infty e^{-\pi t} dt = \frac{1}{2\pi^2(1+x^2)}. \end{aligned}$$

Hence $\lim_{x\to\infty} f(x) = \frac{1}{2}$.

To prove the convexity, let $J[g]$ denote the integral of a function g from 0 to ∞. Then, as above, $f'(x) = 2J[g(t)h(t, x)]$, where

$$g(t) = t/(e^{2\pi t}-1) \quad \text{and} \quad h(t, x) = (t^2-x^2)/(t^2+x^2)^2.$$

Hence,

$$f''(x) = 4xJ[g(t)H(t, x)], \quad \text{where } H(t, x) = \frac{\partial h(t, x)}{\partial x} = \frac{x^2-3t^2}{(t^2+x^2)^3}.$$

Since $g(t)$ is decreasing, by splitting the integral on $(0, \infty)$ into the sum of an integral on $(0, x/\sqrt{3})$ and one on $(x/\sqrt{3}, \infty)$, we get

$$f''(x) > 4xg(x/\sqrt{3})J[H(t, x)].$$

Now substituting $t = x\tan u$, we see that $J[H(t, x)] = 0$. Hence $f''(x) > 0$ on $(0, \infty)$, so that f is convex on $(0, \infty)$. □

3.2. **Theorem.** (1) *The function* $f_1(x) = x^{1/2-x}e^x\Gamma(x)$ *is decreasing and log-convex from* $(0, \infty)$ *onto* $(\sqrt{2\pi}, \infty)$.

(2) *The function* $f_2(x) = x^{1-x}e^x\Gamma(x)$ *is increasing and log-concave from* $(0, \infty)$ *onto* $(1, \infty)$.

Proof. (1) The limiting value at 0 follows from the relation $\Gamma(x+1) = x\Gamma(x)$, while the one at ∞ follows from Stirling's formula. Next,

$$-x\frac{d}{dx}\log f_1(x) = f(x) - \frac{1}{2},$$

where $f(x)$ is as in Theorem 3.1, is clearly positive and decreasing.

(2) The limiting value at 0 follows from the relation $\Gamma(x+1) = x\Gamma(x)$, while the one at ∞ follows from Stirling's formula. Next,

$$-x\frac{d}{dx}\log f_2(x) = 1 - f(x) > 0,$$

where $f(x)$ is as in Theorem 3.1. Moreover, by the monotone l'Hôpital rule [AVV4, §2], it follows that $\frac{d}{dx}\log f_2(x)$ is decreasing. □

3.3. *Remark.* A result similar to Theorem 3.2 appears in [Lu, p. 17], whereas in [Mi, 3.6.55, p. 288] a version of Theorem 3.1 is given. In [G, p. 283] it is shown that $x\Psi(x)$ is convex for $x > 0$. For some recent results on the gamma function see [A1].

4. Refinements

A problem of interest is to obtain upper estimates for the complete elliptic integral

$$\frac{2}{\pi}\mathscr{K}(x') = F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^2\right)$$

in terms of variants of the type

$$F\left(\frac{1}{2} - \delta, \frac{1}{2} + \delta; 1; 1 - x^3\right)$$

for $x \in (0, 1)$ and $\delta \in (0, \frac{1}{2})$. Observe initially that the inequality

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^2\right) < F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^3\right) \tag{4.1}$$

for $x \in (0, 1)$ follows immediately from the monotoneity of $F(\frac{1}{2}, \frac{1}{2}; 1; x)$ as a function of x. We seek a refinement of (4.1).

J. Borwein and P. Borwein [BB2, (2.9)] have shown that

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^2\right) < F\left(\frac{1}{2} - \delta, \frac{1}{2} + \delta; 1; 1 - x^3\right) \tag{4.2}$$

for $\delta = \frac{1}{6}$ and for all $x \in (0, 1)$. It has been conjectured recently [AVV3, p. 79] that (4.2) holds for all $\delta \in (0, \frac{1}{6})$ and for all $x \in (0, 1)$. We next obtain a refinement of (4.2), which also proves the statement of Theorem 1.9.

4.3. **Theorem.** *Let* $x \in (0, 1)$, $c, d \in (0, \infty)$, $4c < \pi d$. *Then*

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^c\right) < F\left(\frac{1}{2} - \delta_0, \frac{1}{2} + \delta_0; 1; 1 - x^d\right) \tag{4.4}$$

$$< F\left(\frac{1}{2} - \delta, \frac{1}{2} + \delta; 1; 1 - x^d\right) \tag{4.5}$$

$$< F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^d\right) \tag{4.6}$$

for all $\delta \in (0, \delta_0)$, *where* $\delta_0 = ((d\pi - 4c)/(4\pi d))^{1/2}$.

Proof. Let $f_\delta(x)$ be defined by

$$f_\delta(x) = \frac{1 - F(\frac{1}{2} - \delta, \frac{1}{2} + \delta; 1; x)}{\log(1 - x)}. \tag{4.7}$$

It follows from Theorem 1.3 that, for all $x \in (0, 1)$ and $\delta \in [0, \frac{1}{2})$,

$$\left(\frac{1}{2} - \delta\right)\left(\frac{1}{2} + \delta\right) < f_\delta(x) < \frac{1}{B(\frac{1}{2} - \delta, \frac{1}{2} + \delta)}, \tag{4.8}$$

where $B(a, b)$ is the Euler beta function. From the reflection formula $\Gamma(a)\Gamma(1 - a) = \pi/\sin(a\pi)$ [AS, p. 256], it follows that

$$\frac{1}{4} - \delta^2 < f_\delta(x) < \frac{1}{\pi}\sin\left(\left(\frac{1}{2} - \delta\right)\pi\right) \tag{4.9}$$

for all $x \in (0, 1)$ and $\delta \in [0, \frac{1}{2})$. From (4.9) with $\delta = 0$ it follows that

$$\frac{F(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^c) - 1}{\log(1/x^c)} = f_0(1 - x^c) < \frac{1}{\pi}.$$

Again applying (4.9) with $\delta \in (0, \delta_0)$ yields

$$\begin{aligned}\frac{F(\frac{1}{2} - \delta, \frac{1}{2} + \delta; 1; 1 - x^d) - 1}{\log(1/x^c)} &= \frac{d}{c}\left[\frac{F(\frac{1}{2} - \delta, \frac{1}{2} + \delta; 1; 1 - x^d) - 1}{\log(1/x^d)}\right] \\ &= \frac{d}{c} f_\delta(1 - x^d) > \frac{d}{c}\left(\frac{1}{4} - \delta^2\right) \\ &> \frac{d}{c}\left(\frac{1}{4} - \delta_0^2\right) = \frac{1}{\pi}.\end{aligned}$$

Therefore, for $0 < \delta \le \delta_0$ and $x \in (0, 1)$,

$$\begin{aligned} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - x^c\right) &= 1 + f_0(1 - x^c) \log \frac{1}{x^c} \\ &< 1 + \frac{1}{\pi} \log \frac{1}{x^c} \\ &< 1 + \frac{d}{c} f_\delta(1 - x^d) \log \frac{1}{x^c} \\ &= F\left(\frac{1}{2} - \delta, \frac{1}{2} + \delta; 1; 1 - x^d\right), \end{aligned}$$

which establishes (4.4). Inequalities (4.5) and (4.6) are immediate consequences of (2.11). □

4.10. *Conjectures.* (1) For $c = 2$, $d = 3$ the best possible value of δ_0 for which Theorem 4.3 is valid is $\delta_0 = (\pi - 2\arcsin(2/3))/(2\pi) \approx 0.268$ (see [AVV3, p. 6]).

(2) Theorem 1.4 has a counterpart for the generalized hypergeometric function ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x)$ for the case $a_i > 0$, $b_j > 0$, $p = q + 1$, when the sum is zero-balanced, i.e. when $\sum_{i=1}^p a_i = \sum_{j=1}^q b_j$. See also [B, p. 152].

Acknowledgments

The research of the last-named author was supported by the Academy of Finland. The authors acknowledge useful conversations with K. Pearce.

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