

## HOLOMORPHIC MARTINGALES AND INTERPOLATION BETWEEN HARDY SPACES: THE COMPLEX METHOD

P. F. X. MÜLLER

**ABSTRACT.** A probabilistic proof is given to identify the complex interpolation space of  $H^1(\mathbb{T})$  and  $H^\infty(\mathbb{T})$  as  $H^p(\mathbb{T})$ .

### INTRODUCTION

In this note, a soft probabilistic proof of P.W. Jones's theorem on the complex interpolation space between  $H^1$  and  $H^\infty$  is given. We shall work with N.Th. Varopoulos's space of holomorphic martingales.

The observation presented in this article is that the use of a stopping time decomposition simplifies constructions of Serguei V. Kislyakov and Quanhua Xu to obtain the following

**Theorem.** *The complex interpolation space*

$$[H^1(\mathbb{T}), H^\infty(\mathbb{T})]_\Theta, \quad 0 < \Theta < 1,$$

*coincides with  $H^p(\mathbb{T})$  provided that  $\frac{1}{p} = 1 - \Theta$ .*

As is well known this result has been obtained by P.W. Jones using  $L^\infty$  estimates to the  $\bar{\partial}$  problem (see [J]). His work also contains the description of the real interpolation spaces for the couple  $(H^1, H^\infty)$ . At about the same time Jean Bourgain obtained a Marcinkiewicz type decomposition, using completely different techniques (see [B1]). Recently in a series of papers Jean Bourgain [B2], Serguei Kislyakov [K1], [K2], [K-X], Gilles Pisier [P] and Quanhua Xu [X1], [X2] obtained deep results concerning real and complex interpolation methods between vector-valued, weighted Hardy spaces of analytic functions. S. Kislyakov's paper [K1] contains the following idea to approximate the characteristic function  $\mathbf{1}_{\{|f|<\lambda\}}$  by analytic functions on the circle  $\mathbb{T}$ : He starts

$$\alpha = \max\left\{1, \frac{|f|}{\lambda}\right\}$$

and then considers

$$\varphi = \frac{1}{\alpha + iH\alpha}$$

where  $H$  denotes the Hilbert transform on  $\mathbb{T}$ .

Received by the editors August 11, 1992 and, in revised form, February 1, 1994; originally communicated to the *Proceedings of the AMS* by Dale Alspach.

1991 *Mathematics Subject Classification.* Primary 60G46, 42B30.

*Key words and phrases.* Holomorphic martingales, complex interpolation method, Hardy spaces.

In [K2] S. Kislyakov uses these approximations and constructs an “analytic partition of unity”, thereby proving J. Bourgain’s theorem on absolutely summing operators on the disc algebra.

Then Q. Xu, using this analytic partition of unity, was able to give an elementary proof of P.W. Jones’s complex interpolation theorem (see [X1]).

The use of probability allows us to simplify Q. Xu’s proof further: We start with a stopping-time argument which gives a decomposition of a given element  $f \in H^p(\Omega)$  into a sum of functions  $d_i \in H^\infty(\Omega)$  in such a way that we have good control over the supports and  $H^\infty(\Omega)$  norms (of the functions  $d_i$ ). Then one defines the vector-valued analytic functions on the strip  $S = \{\xi : 0 < \operatorname{Re} \xi < 1\}$ , which is required to conclude that

$$H^p(\Omega) \subseteq [H^1(\Omega), H^\infty(\Omega)]_\theta$$

(for a definition of the complex interpolation method see [B-L] or [Ja-Jo]).

These notes are also a continuation of [M] where a probabilistic argument was given to identify the real interpolation spaces between  $H^\infty(\mathbb{T})$  and  $H^1(\mathbb{T})$ . For sake of completeness let us recall the concept of holomorphic martingales:  $(z_t)_{t \geq 0}$  denotes the complex Brownian motion on the Wiener space  $(\Omega, (\mathcal{F}_t), \mathcal{F}, P)$ .

**Definition.** A random variable  $X : \Omega \rightarrow \mathbb{C}$  is called holomorphic if and only if the conditional expectations

$$X_t = \mathbf{E}(X | \mathcal{F}_t)$$

admit a stochastic integral representation of the form

$$X_t = X_0 + \int_0^t f_s dz_s$$

where  $f_s : \Omega \rightarrow \mathbb{C}$  is measurable with respect to  $\mathcal{F}_s$ .

Accordingly  $H^p(\Omega)$  denotes the subspace of  $L^p(\Omega)$  which consists of holomorphic random variables. On a general  $\mathcal{F}_t$  martingale with stochastic integral representation

$$Y_t = Y_0 + \int_0^t f_s dx_s + \int_0^t g_s dy_s$$

we define

$$(\mathcal{H}Y)_t = - \int_0^t g_s dx_s + \int_0^t f_s dy_s.$$

This martingale transform is called the stochastic Hilbert transform, because for  $Y \in L^2(\Omega)$  one obtains  $Y + i\mathcal{H}Y \in H^2(\Omega)$ .

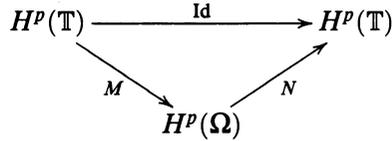
Being a martingale transform,  $\mathcal{H}$  defines a bounded operator on  $L^p(\Omega)$  for  $1 < p < \infty$ . The corresponding norms are denoted by  $N_p$ . For a function  $f \in L^p(\Omega)$  we denote by  $f^*$  its martingale maximal function, i.e.

$$f^* := \sup_s |\mathbf{E}(f | \mathcal{F}_s)|.$$

For  $L^p(\Omega)$ ,  $1 < p \leq \infty$ , we have  $\|f\|_p \leq M_p \|f^*\|_p$ . We shall write  $C_p$  for the larger of  $N_p$  and  $M_p$ . Below we shall give a proof of

**Theorem 1.** *The complex interpolation space  $[H^1(\Omega), H^\infty(\Omega)]_\Theta, 0 < \Theta < 1$ , coincides with  $H^p(\Omega)$  provided that  $\frac{1}{p} = 1 - \Theta$ .*

As shown by N. Th. Varopoulos [V], there exist operators  $M, N$  such that for  $1 \leq p \leq \infty$  the diagram



commutes. We thus derive Jones's theorem from Theorem 1.

THE BASIC DECOMPOSITION

In this section, we use a sequence of stopping times to decompose a given  $f \in H^p(\Omega)$  into a  $p$ -absolutely convergent series of functions  $d_j \in H^\infty(\Omega)$ .

Take  $f \in H^p(\Omega), 1 < p < \infty$ . Let

$$m_1 := \inf \left\{ n : P\{f^* > 2^n\} \leq \frac{1}{4} \right\}.$$

Having defined  $m_1, \dots, m_{j-1}$  we put

$$E_{j-1} = \{f^* > 2^{m_{j-1}}\}$$

and define

$$m_j := \inf \left\{ n > m_{j-1} : P\{f^* > 2^n\} \leq \frac{1}{4} P(E_{j-1}) \right\}.$$

Using this sequence of integers we define natural stopping times as

$$\tau_0(\omega) := 0, \quad \tau_j(\omega) := \inf \{t \in \mathbf{R} : |\mathbf{E}(f|\mathcal{F}_t)(\omega)| \geq 2^{m_j}\}.$$

These stopping times are used to chop off the large parts of  $f$ . We let  $\mathcal{F}_{\tau_j}$  be the  $\sigma$ -algebra generated by the stopping time  $\tau_j$ , and we put  $f_j := \mathbf{E}(f|\mathcal{F}_{\tau_j})$ . Finally we form the martingale differences  $d_j = f_{j+1} - f_j$ .

It is easily observed that  $d_j$  lies in  $H^\infty(\Omega)$ , has its support contained in  $E_j$ , and is dominated by  $2^{m_{j+1}}$  and that

$$f - \mathbf{E}(f) = \sum_{j=0}^{\infty} d_j.$$

For our purposes we need to improve this decomposition.

We let

$$\alpha_j(\omega) = \max \left( 1, \delta \frac{f^*}{2^{m_j}} \right).$$

The function

$$\varphi_j = \frac{1}{\alpha_j + i\mathcal{H}\alpha_j}$$

satisfies the following conditions:

- (i)  $\varphi_j \in H^\infty(\Omega)$  with  $\|\varphi_j\|_\infty \leq 1$ .
- (ii) For  $k \geq j + 1$  and  $\omega \in E_k$  we obtain the pointwise estimate

$$|\varphi_j(\omega)| \leq \frac{1}{\delta} \left( \frac{2^{m_{j+1}}}{2^{m_k}} \right).$$

(iii)

$$\begin{aligned} \int_{\Omega} |1 - \varphi_j|^p dP &\leq C_p^p \int_{\Omega} |1 - \alpha_j|^p dP \\ &\leq \delta^p C_p^p \int_{E_{j+1}} \left(\frac{f^*}{2^{m_{j+1}}}\right)^p dP. \end{aligned}$$

Subsequently we fix  $\delta$  so that  $|\log \delta|^{1/p} \delta C_p^2 \leq \frac{1}{64}$ .

THE PROOF OF THEOREM 1

Given  $1 < p < \infty$ ,  $\Theta := 1 - 1/p$  and  $f \in H^p(\Omega)$ . In this section we shall construct an analytic function  $F$  on the strip  $S = \{\zeta = \eta + i\xi : 0 \leq \eta \leq 1\}$  which satisfies

- (1)  $\|F(\Theta) - f\|_{H^p(\Omega)} \leq \frac{1}{4} \|f\|_{H^p(\Omega)}$ ,
- (2)  $\sup_{\xi \in \mathbf{R}} \|F(i\xi)\|_{H^1(\Omega)} \leq C \|f\|_{H^p(\Omega)}^p$ ,
- (3)  $\sup_{\xi \in \mathbf{R}} \|F(1 + i\xi)\|_{H^\infty(\Omega)} \leq C$ .

These three properties of  $F$  imply that  $H^p(\Omega) \subset [H^1(\Omega), H^\infty(\Omega)]_{\Theta}$ , which is enough to identify the complex interpolation space  $[H^1(\Omega), H^\infty(\Omega)]_{\Theta}$  with  $H^p(\Omega)$  (see [Ja-Jo]).

We start with the basic decomposition

$$f = \sum_{i=0}^{\infty} d_i,$$

construct  $\varphi_i$  as in §2, and define

$$F(\xi) = \sum_{j=0}^{\infty} d_j \varphi_j 2^{(p(1-\xi)-1)m_{j+1}}.$$

As  $F(\Theta) - f = \sum_{j=0}^{\infty} d_j(1 - \varphi_j)$ , we have to check the following three inequalities: For all  $\xi \in \mathbf{R}$

- (1)  $\|\sum_{j=0}^{\infty} d_j(1 - \varphi_j)\|_{L^p(\Omega)} \leq \frac{1}{4} \|f\|_{L^p(\Omega)}$ ,
- (2)  $\|\sum_{j=0}^{\infty} d_j \varphi_j 2^{p(1-i\xi-1)m_{j+1}}\|_{L^1(\Omega)} \leq C \|f\|_{L^p(\Omega)}^p$ ,
- (3)  $\|\sum_{j=0}^{\infty} d_j \varphi_j 2^{pi\xi} 2^{-m_{j+1}}\|_{L^\infty(\Omega)} \leq C \delta^{-1}$ .

*Verification of (1).* The important idea to change the order of summation below is taken from S. Kislyakov’s paper [K2].

The choice of the sequence  $(m_j)$  implies that the  $p$ th power of the left-hand side of (1) is dominated by

$$|\log \delta| \sum_{j=0}^{\infty} \int |d_j(1 - \varphi_j)|^p dP.$$

Using the  $L^\infty$  estimates for  $d_j$  and the  $L^p$  boundedness of the stochastic Hilbert’s transform we estimate the above sum by

$$|\log \delta| C_p \sum_{j=0}^{\infty} 2^{m_{j+1}p} \int_{\Omega} |1 - \alpha_j|^p dP.$$

Above we obtained

$$\int_{\Omega} |1 - \alpha_j|^p \leq \delta^p 2^{-(m_{j+1})p} \sum_{k=j+1}^{\infty} 2^{(m_{k+1})p} P(E_k).$$

Using this estimate and *changing the order of summation* we can dominate the left-hand side of (1) by

$$\begin{aligned} |\log \delta| \delta^p C_p^p \sum_{k=1}^{\infty} P(E_k) 2^{m_{k+1}p} \sum_{j=0}^k 2^{m_{j+1}p} 2^{-m_{k+1}p} &\leq |\log \delta| \delta^p 16 C_p^p \sum_{k=1}^{\infty} P(E_k) 2^{m_{k+1}p} \\ &\leq |\log \delta| \delta^p 16 C_p^p \int_{\Omega} f^{*p} dP \leq |\log \delta| \delta^p 16 C_p^{2p} \int_{\Omega} |f|^p dP. \end{aligned}$$

*Verification of (2).* The left-hand side of (2) is clearly dominated by

$$\sum_{i=0}^{\infty} \int |d_i| \varphi_i 2^{(p-1)m_{i+1}} dP.$$

Using  $L^\infty$  estimates of  $d_i$  and the fact that  $d_i$  is supported in the set  $\{f^* > 2^{m_i}\}$  we can dominate the above sum by

$$\sum_{j=0}^{\infty} 2^{m_{j+1}p} P\{f^* > 2^{m_j}\}.$$

The choice of  $(m_j)$  allows one to dominate this sum by

$$2 \sum_{j=0}^{\infty} 2^{m_{j+1}p} P\{f^* > 2^{m_{j+1}-1}\},$$

which is bounded by  $4 \|f^*\|_{L^p(\Omega)}^p$ .

*Verification of (3).* The  $L^\infty$  bounds on  $d_j$  and the fact that  $\text{supp } d_j \subseteq E_j$  gives an estimate for the left-hand side of (3) by

$$\delta^{-1} \left\| \sum_{j=0}^{\infty} \mathbf{1}_{E_j} |\varphi_j| \right\|_{\infty}.$$

By construction the above sum can be dominated by

$$\delta^{-1} \left\| \sum_{j=0}^{\infty} \mathbf{1}_{E_j \setminus E_{j+1}} 2^{-m_j} \left( \sum_{l=0}^j 2^{+m_l} \right) \right\|_{\infty},$$

which is bounded independently of  $f$ .

REFERENCES

[B-L] J. Berg and J. Löfström, *Interpolation spaces*, Springer Verlag, New York, 1976.  
 [B1] J. Bourgain J., *New Banach space properties of the disc algebra and  $H^\infty$* , Acta Math. **152** (1984), 1-48.  
 [B2] ———, *Some consequences of Pisier's approach to interpolations*, preprint 1991.

- [D] R. Durrett, *Brownian motion and martingales in analysis*, Wadsworth, 1984.
- [J] P. W. Jones,  $L^\infty$  estimates for the  $\bar{\partial}$  problem in a half space, *Acta Math.* **150** (1983), 37–152.
- [Ja-Jo] S. Janson and P. W. Jones, *Interpolation between  $H^p$  spaces*, *J. Funct. Analysis* **48** (1982), 58–80.
- [K1] S. V. Kislyakov, *Truncating functions in weighted  $H^p$  and two theorems of J. Bourgain*, Uppsala University, Dept. of Math., Report 1989:10.
- [K2] ———, *Absolutely summing operators on the disc algebra*, *Analysis and Algebra* **3** (1991), 1–77; English transl., *St. Petersburg Math. J.* **3** (1992), 705–774..
- [K-X] S. V. Kislyakov and Q. Xu, *Interpolation of weighted and vector valued Hardy spaces*, *Publ. Inst. Recherche. Math. Lille* **25** (1991).
- [M] P. F. X. Müller, *Holomorphic Martingales and interpolation between Hardy spaces*, *J. Analyse Math.* **61** (1993), 327–337.
- [P] G. Pisier G, *Interpolation between  $H^p$  spaces and non commutative generalization*, *Pacific J. Math.* **155** (1992), 341–368.
- [V] N. Th. Varopoulos, *The Helson-Szegö theorem and  $A_p$  functions for Brownian motion and several variables*, *J. Funct. Anal.* **39** (1980), 85–121.
- [X1] Q. Xu, *Elementary proofs of two theorems of P.W. Jones on interpolation between Hardy spaces*, *Ann. Inst. Fourier (Grenoble)* **42** (1992), 875–889.
- [X2] ———, *Notes on interpolation of Hardy spaces*, preprint 1990.

INSTITUT FÜR MATHEMATIK, J. KEPLER UNIVERSITÄT LINZ, A-4040 LINZ, AUSTRIA  
E-mail address: paul.mueller@jk.uni-linz.ac.at