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HOLOMORPHIC MARTINGALES AND INTERPOLATION BETWEEN HARDY SPACES: THE COMPLEX METHOD

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ABSTRACT. A probabilistic proof is given to identify the complex interpolation space of $H^1(\mathbb{T})$ and $H^\infty(\mathbb{T})$ as $H^p(\mathbb{T})$.

INTRODUCTION

In this note, a soft propabilistic proof of P.W. Jones's theorem on the *complex* interpolation space between H^1 and H^∞ is given. We shall work with N.Th. Varopoulos's space of holomorhpic martingales.

The observation presented in this article is that the use of a stopping time decomposition simplifies constructions of Serguei V. Kislyakov and Quanhua Xu to obtain the following

Theorem. *The complex interpolation space*

$$[H^1(\mathbb{T}), H^\infty(\mathbb{T})]_\Theta, \qquad 0 < \Theta < 1,$$

coincides with $H^p(\mathbb{T})$ *provided that* $\frac{1}{p} = 1 - \Theta$.

As is well known this result has been obtained by P.W. Jones using L^∞ estimates to the $\bar\partial$ problem (see [J]). His work also contains the description of the real interpolation spaces for the couple (H^1, H^∞). At about the same time Jean Bourgain obtained a Marcinkiewicz type decomposition, using completely different techniques (see [B1]). Recently in a series of papers Jean Bourgain [B2], Serguei Kislyakov [K1], [K2], [K-X], Gilles Pisier [P] and Quanhua Xu [X1], [X2] obtained deep results concerning real and complex interpolation methods between vector-valued, weighted Hardy spaces of analytic functions. S. Kislyakov's paper [K1] contains the following idea to approximate the characteristic function $\mathbf{1}_{\{|f|<\lambda\}}$ by analytic functions on the circle $\mathbb{T}$: He starts

$$\alpha = \max\left\{1, \frac{|f|}{\lambda}\right\}$$

and then considers

$$\varphi = \frac{1}{\alpha + iH\alpha}$$

where H denotes the Hilbert transform on $\mathbb{T}$.

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In [K2] S. Kislyakov uses these approximations and constructs an "analytic partition of unity", thereby proving J. Bourgain's theorem on absolutely summing operators on the disc algebra.

Then Q. Xu, using this analytic partition of unity, was able to give an elementary proof of P.W. Jones's complex interpolation theorem (see [X1]).

The use of probability allows us to simplify Q. Xu's proof further: We start with a stopping-time argument which gives a decomposition of a given element $f \in H^p(\Omega)$ into a sum of functions $d_i \in H^\infty(\Omega)$ in such a way that we have good control over the supports and $H^\infty(\Omega)$ norms (of the functions d_i). Then one defines the vector-valued analytic functions on the strip $S = \{\xi : 0 < \operatorname{Re}\xi < 1\}$, which is required to conclude that

$$H^p(\Omega) \subseteq [H^1(\Omega), H^\infty(\Omega)]_\Theta$$

(for a definition of the complex interpolation method see [B-L] or [Ja-Jo]).

These notes are also a continuation of [M] where a probabilistic argument was given to identify the real interpolation spaces between $H^\infty(\mathbb{T})$ and $H^1(\mathbb{T})$. For sake of completeness let us recall the concept of holomorphic martingales: $(z_t)_{t\geq 0}$ denotes the complex Brownian motion on the Wiener space $(\Omega, (\mathscr{F}_t), \mathscr{F}, P)$.

Definition. A random variable $X : \Omega \to \mathbf{C}$ is called holomorphic if and only if the conditional expectations

$$X_t = \mathbf{E}(X|\mathscr{F}_t)$$

admit a stochastic integral representation of the form

$$X_t = X_0 + \int_0^t f_s \, dz_s$$

where $f_s : \Omega \to \mathbf{C}$ is measurable with respect to $\mathscr{F}_s$.

Accordingly $H^p(\Omega)$ denotes the subspace of $L^p(\Omega)$ which consists of holomorphic random variables. On a general $\mathscr{F}_t$ martingale with stochastic integral representation

$$Y_t = Y_0 + \int_0^t f_s \, dx_s + \int_0^t g_s \, dy_s$$

we define

$$(\mathscr{H} Y)_t = -\int_0^t g_s \, dx_s + \int_0^t f_s \, dy_s.$$

This martingale transform is called the stochastic Hilbert transform, because for $Y \in L^2(\Omega)$ one obtains $Y + i\mathscr{H} Y \in H^2(\Omega)$.

Being a martingale transform, $\mathscr{H}$ defines a bounded operator on $L^p(\Omega)$ for $1 < p < \infty$. The corresponding norms are denoted by N_p. For a function $f \in L^p(\Omega)$ we denote by f^* it's martingale maximal function, i.e.

$$f^* := \sup_s |\mathbf{E}(f|\mathscr{F}_s)|.$$

For $L^p(\Omega)$, $1 < p \leq \infty$, we have $||f||_p \leq M_p ||f^*||_p$. We shall write C_p for the larger of N_p and M_p. Below we shall give a proof of

Theorem 1. *The complex interpolation space* $[H^1(\Omega), H^\infty(\Omega)]_\Theta$, $0 < \Theta < 1$, *coincides with* $H^p(\Omega)$ *provided that* $\frac{1}{p} = 1 - \Theta$.

As shown by N. Th. Varopoulos [V], there exist operators M, N such that for $1 \leq p \leq \infty$ the diagram

$$\begin{tikzcd} H^p(\mathbb{T}) \arrow[rr, "\mathrm{Id}"] \arrow[dr, "M"'] & & H^p(\mathbb{T}) \\ & H^p(\Omega) \arrow[ur, "N"'] & \end{tikzcd}$$

commutes. We thus derive Jones's theorem from Theorem 1.

The basic decomposition

In this section, we use a sequence of stopping times to decompose a given $f \in H^p(\Omega)$ into a p-absolutely convergent series of functions $d_i \in H^\infty(\Omega)$.

Take $f \in H^p(\Omega)$, $1 < p < \infty$. Let

$$m_1 := \inf\left\{n : P\{f^* > 2^n\} \leq \frac{1}{4}\right\}.$$

Having defined $m_1, \ldots, m_{j-1}$ we put

$$E_{j-1} = \{f^* > 2^{m_{j-1}}\}$$

and define

$$m_j := \inf\left\{n > m_{j-1} : P\{f^* > 2^n\} \leq \frac{1}{4}P(E_{j-1})\right\}.$$

Using this sequence of integers we define natural stopping times as

$$\tau_0(\omega) := 0, \quad \tau_j(\omega) := \inf\{t \in \mathbf{R} : |\mathbf{E}(f|\mathscr{F}_t)(\omega)| \geq 2^{m_j}\}.$$

These stopping times are used to chop off the large parts of f. We let $\mathscr{F}_{\tau_j}$ be the σ-algebra generated by the stopping time τ_j, and we put $f_j := \mathbf{E}(f|\mathscr{F}_{\tau_j})$. Finally we form the martingale differences $d_j = f_{j+1} - f_j$.

It is easily observed that d_j lies in $H^\infty(\Omega)$, has its support contained in E_j, and is dominated by $2^{m_{j+1}}$ and that

$$f - \mathbf{E}(f) = \sum_{j=0}^{\infty} d_j.$$

For our purposes we need to improve this decomposition.

We let

$$\alpha_j(\omega) = \max\left(1, \delta\frac{f^*}{2^{m_j}}\right).$$

The function

$$\varphi_j = \frac{1}{\alpha_j + i\mathscr{H}\alpha_j}$$

satisfies the following conditions:

(i) $\varphi_j \in H^\infty(\Omega)$ with $||\varphi_j||_\infty \leq 1$.

(ii) For $k \geq j+1$ and $\omega \in E_k$ we obtain the pointwise estimate

$$|\varphi_j(\omega)| \leq \frac{1}{\delta}\left(\frac{2^{m_{j+1}}}{2^{m_k}}\right).$$

(iii)

$$\int_\Omega |1-\varphi_j|^p\,dP \le C_p^p \int_\Omega |1-\alpha_j|^p\,dP$$
$$\le \delta^p C_p^p \int_{E_{j+1}} \left(\frac{f^*}{2^{m_{j+1}}}\right)^p dP.$$

Subsequently we fix δ so that $|\log\delta|^{1/p}\delta C_p^2 \le \frac{1}{64}$.

The proof of Theorem 1

Given $1 < p < \infty$, $\Theta := 1 - 1/p$ and $f \in H^p(\Omega)$. In this section we shall construct an analytic function F on the strip $S = \{\zeta = \eta + i\xi : 0 \le \eta \le 1\}$ which satisfies

(1) $\|F(\Theta) - f\|_{H^p(\Omega)} \le \frac{1}{4}\|f\|_{H^p(\Omega)}$,
(2) $\sup_{\xi\in\mathbf{R}} \|F(i\xi)\|_{H^1(\Omega)} \le C\|f\|^p_{H^p(\Omega)}$,
(3) $\sup_{\xi\in\mathbf{R}} \|F(1+i\xi)\|_{H^\infty(\Omega)} \le C$.

These three properties of F imply that $H^p(\Omega) \subset [H^1(\Omega), H^\infty(\Omega)]_\Theta$, which is enough to identify the complex interpolation space $[H^1(\Omega), H^\infty(\Omega)]_\Theta$ with $H^p(\Omega)$ (see [Ja-Jo]).

We start with the basic decomposition

$$f = \sum_{i=0}^\infty d_i,$$

construct φ_i as in §2, and define

$$F(\xi) = \sum_{j=0}^\infty d_j\varphi_j 2^{(p(1-\xi)-1)m_{j+1}}.$$

As $F(\Theta) - f = \sum_{j=0}^\infty d_j(1-\varphi_j)$, we have to check the following three inequalities: For all $\xi \in \mathbf{R}$

(1) $\|\sum_{j=0}^\infty d_j(1-\varphi_j)\|_{L^p(\Omega)} \le \frac{1}{4}\|f\|_{L^p(\Omega)}$,
(2) $\|\sum_{j=0}^\infty d_j\varphi_j 2^{p(1-i\xi-1)m_{j+1}}\|_{L^1(\Omega)} \le C\|f\|^p_{L^p(\Omega)}$,
(3) $\|\sum_{j=0}^\infty d_j\varphi_j 2^{pi\xi}2^{-m_{j+1}}\|_{L^\infty(\Omega)} \le C\delta^{-1}$.

Verification of (1). The important idea to change the order of summation below is taken from S. Kislyakov's paper [K2].

The choice of the sequence (m_j) implies that the pth power of the left-hand side of (1) is dominated by

$$|\log\delta| \sum_{j=0}^\infty \int |d_j(1-\varphi_j)|^p\,dP.$$

Using the L^∞ estimates for d_j and the L^p boundedness of the stochastic Hilbert's transform we estimate the above sum by

$$|\log\delta| C_p \sum_{j=0}^\infty 2^{m_{j+1}p} \int_\Omega |1-\alpha_j|^p\,dP.$$

Above we obtained

$$\int_\Omega |1-\alpha_j|^p \le \delta^p 2^{-(m_{j+1})p} \sum_{k=j+1}^{\infty} 2^{(m_{k+1})p} P(E_k).$$

Using this estimate and *changing the order of summation* we can dominate the left-hand side of (1) by

$$|\log\delta|\delta^p C_p^p \sum_{k=1}^{\infty} P(E_k) 2^{m_{k+1}p} \sum_{j=0}^{k} 2^{m_{j+1}p} 2^{-m_{k+1}p} \le |\log\delta|\delta^p 16 C_p^p \sum_{k=1}^{\infty} P(E_k) 2^{m_{k+1}p}$$
$$\le |\log\delta|\delta^p 16 C_p^p \int_\Omega f^{*p}\, dP \le |\log\delta|\delta^p 16 C_p^{2p} \int_\Omega |f|^p\, dP.$$

Verification of (2). The left-hand side of (2) is clearly dominated by

$$\sum_{i=0}^{\infty} \int |d_i|\varphi_i 2^{(p-1)m_{i+1}}\, dP.$$

Using L^∞ estimates of d_i and the fact that d_i is supported in the set $\{f^* > 2^{m_i}\}$ we can dominate the above sum by

$$\sum_{j=0}^{\infty} 2^{m_{j+1}p} P\{f^* > 2^{m_j}\}.$$

The choice of (m_j) allows one to dominate this sum by

$$2\sum_{j=0}^{\infty} 2^{m_{j+1}p} P\{f^* > 2^{m_{j+1}-1}\},$$

which is bounded by $4\|f^*\|^p_{L^p(\Omega)}$.

Verification of (3). The L^∞ bounds on d_j and the fact that supp $d_j \subseteq E_j$ gives an estimate for the left-hand side of (3) by

$$\delta^{-1}\left\| \sum_{j=0}^{\infty} \mathbf{1}_{E_j}|\varphi_j| \right\|_\infty .$$

By construction the above sum can be dominated by

$$\delta^{-1}\left\| \sum_{j=0}^{\infty} \mathbf{1}_{E_j\setminus E_{j+1}} 2^{-m_j} \left(\sum_{l=0}^{j} 2^{+m_l}\right)\right\|_\infty ,$$

which is bounded independently of f.

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