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TATE COHOMOLOGY OF PERIODIC K-THEORY WITH REALITY IS TRIVIAL

LISBETH FAJSTRUP

ABSTRACT. We calculate the $RO(\mathbb{Z}/2)$-graded spectrum for Atiyah's periodic K-theory with reality and the Tate cohomology associated to it. The latter is shown to be trivial.

1. INTRODUCTION

Let KR be the periodic $\mathbb{Z}/2$-spectrum representing Atiyah's Real K-theory or K-theory with reality [A].

In this note we prove that the generalized Tate spectrum associated to it (in the sense of Greenlees and May [GM]) is trivial. This is quite surprising, since the Tate spectrum for ordinary equivariant K-theory is not contractible for any group G.

The proof is quite simple however; it just relies on the fact that the generator η of $KR(\mathbb{R}P^1)$ is nilpotent. We give a more general statement to emphasize this in 3.1.

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2. G-SPECTRA AND G-COHOMOLOGY THEORIES

In this paper we will be working in the equivariant stable category of [LMS], and we begin by recalling some definitions. We let G be a finite group, since we do not need the Lie group setup of [LMS].

A complete universe U is an infinite dimensional real inner product space with G acting through isometries such that U contains a countably infinite direct sum of regular representations of G as a subspace.

A G-spectrum k_G indexed on a given U associates a based G-space $k_G(V)$ to each finite dimensional G-subspace $V \subset U$ such that for any two G-subspaces V and W of U with $V \subset W$ the usual transitive system of structure maps $k_G(V) \to \Sigma^{V-W} k_G(W)$ are G-homeomorphisms.

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Let $U = \bigoplus(V_i)^\infty$ for a set of distinct irreducible representations V_i. Then $RO(G, U)$ is the free abelian group generated by the V_i.

Given a G-spectrum k_G indexed on U, we define the associated $RO(G, U)$-graded homology and cohomology theory:

For any virtual representation $a = V - W$ with V and W in $RO(G, U)$ there are sphere G-spectra $S^a = \sum^{-W} S^V$, and we let

$$k_G^a(X) = [X \wedge S^{-a}, k_G]^G \quad \text{and} \quad k_a^G(X) = [S^a, X \wedge k_G]^G$$

for any G-spectrum X. For a G-space Y, let $k_G^a(Y) = k_G^a(\sum^\infty Y)$ and similarly for homology; here $\sum^\infty Y$ is the $RO(G, U)$-graded suspension spectrum of Y as in [LMS, p. 14]. These theories have suspension isomorphisms $k_G^V(X) \cong k_G^{V\oplus W}(S^W \wedge X)$ and similarly for homology, induced by the structure maps in the spectrum.

On the other hand, given an $RO(G, U)$-graded cohomology theory on G-spectra indexed over U (for a definition of this, see [May, Chapter 19], or [LMS, p. 34]), there is a spectrum which classifies this theory as above. Furthermore if the cohomology theory is only defined on G-spaces, it has an extension to G-spectra, which is unique up to nonunique isomorphism, and hence it gives rise to a classifying G-spectrum.

3. The Tate spectrum for some $\mathbb{Z}/2$-spectra

In [GM] a generalized Tate cohomology theory for a G-spectrum k_G is introduced for any compact Lie group G. We repeat this construction for the convenience of the reader. Let X_+ be the disjoint union of the G-space X with a fixed base point and let EG be a contractible free G-space. Let $\widetilde{EG}$ be the unreduced suspension of EG. Then there is a cofibering

$$EG_+ \to S^0 \to \widetilde{EG}. \tag{1}$$

Let $F(EG_+, k_G)$ be the function G-spectrum of maps from EG_+ to k_G. The projection $EG_+ \to S^0$ induces a map of G-spectra

$$\varepsilon\colon k_G = F(S^0, k_G) \to F(EG_+, k_G).$$

Smashing ε with the cofibering (1) yields the following map of cofiberings of G-spectra:

$$\begin{tikzcd}
k_G \wedge EG_+ \arrow[r] \arrow[d, "\varepsilon\wedge 1"] & k_G \arrow[r] \arrow[d, "\varepsilon"] & k_G \wedge \widetilde{EG} \arrow[d, "\varepsilon\wedge 1"] \\
F(EG_+, k_G) \wedge EG_+ \arrow[r] & F(EG_+, k_G) \arrow[r] & F(EG_+, k_G) \wedge \widetilde{EG}
\end{tikzcd} \tag{2}$$

The Tate G-spectrum associated to k_G is then the spectrum in the lower right hand corner of this diagram: $t(k_G) = F(EG_+, k_G) \wedge \widetilde{EG}$. The associated homology and cohomology theories are the Tate homology and cohomology of k_G.

When $G = \mathbb{Z}/2$ and k_G is a ring spectrum, the following special case of [GM, §16] gives a method for calculating the upper right-hand corner of (2).

Let m denote the trivial real m-dimensional representation of $\mathbb{Z}/2$ and let $k\xi$ be $\mathbb{R}^k$ with involution given by multiplication by -1. We allow m and k to be infinite and let $\mathbb{R}^\infty$ have the direct limit topology as usual.

When $G=\mathbb{Z}/2$, our universe will always be $U=\infty\xi\oplus\infty$.

A filtration of $\widetilde{E\mathbb{Z}/2}$ can be obtained from the fact that $B(\infty\xi)/S(\infty\xi)$ is a model for $\widetilde{E\mathbb{Z}/2}$, which implies that the one point compactifications $S^{p\xi}$ give a filtration. We get canonical isomorphisms in homology for any X and for any subspace V of our universe; in cohomology this works for finite X:

$$(3)\qquad (k_{\mathbb{Z}/2}\wedge\widetilde{E\mathbb{Z}/2})^V(X)\cong\operatorname{colim}(k_{\mathbb{Z}/2}\wedge S^{p\xi})^V(X)\cong\operatorname{colim}(k_{\mathbb{Z}/2}^{V\oplus p\xi}(X)),$$

$$(4)\qquad (k_{\mathbb{Z}/2}\wedge\widetilde{E\mathbb{Z}/2})_V(X)\cong\operatorname{colim}(k_{\mathbb{Z}/2}\wedge S^{p\xi})_V(X)\cong\operatorname{colim}(k^{\mathbb{Z}/2}_{V-p\xi}(X)).$$

Here the maps are induced by the inclusion $S^{p\xi}\to S^{(p+1)\xi}$. Studying these maps a little closer, we find that the maps giving the latter colimits may also be described as multiplication by a certain element of $k^{\xi}_{\mathbb{Z}/2}(S^0)$, namely the image of the identity element of $k^0_{\mathbb{Z}/2}(S^0)$ under the isomorphism $k^0_{\mathbb{Z}/2}(S^0)\to k^{\xi}_{\mathbb{Z}/2}(S^{\xi})$ composed with the map induced by the inclusion of S^0 in S^{ξ}

$$i^*\colon k^{\xi}_{\mathbb{Z}/2}(S^{\xi})\to k^{\xi}_{\mathbb{Z}/2}(S^0).$$

This element is called the Euler class of ξ and is denoted χ_ξ. This leads to the following lemma:

Lemma 3.1. *Let $k^*_{\mathbb{Z}/2}(-)$ be an $RO(\mathbb{Z}/2,U)$-graded cohomology theory such that χ_ξ is nilpotent and the classifying spectrum $k_{\mathbb{Z}/2}$ is a ring spectrum. Then the projection p of $E\mathbb{Z}/2_+$ to S^0 induces a weak $\mathbb{Z}/2$ homotopy equivalence $p^*\colon k_{\mathbb{Z}/2}\to F(E\mathbb{Z}/2_+,k_{\mathbb{Z}/2})$, and the Tate spectrum $t(k_{\mathbb{Z}/2})$ is trivial.*

Proof. The long exact sequence induced by the cofibration (1) and the fact that $k^0_{\mathbb{Z}/2}(E\mathbb{Z}/2_+\wedge X)=[X,F(E\mathbb{Z}/2_+,k_{\mathbb{Z}/2})]^{\mathbb{Z}/2}$ implies that for a proof of the first statement it suffices to see that $k^*_{\mathbb{Z}/2}(\widetilde{E\mathbb{Z}/2}\wedge X)=0$ for X compact. There is a Milnor $\lim^1$-exact sequence [M]

$$0\to\lim{}^1k^{*-1}_{\mathbb{Z}/2}(S^{p\xi}\wedge X)\to k^*_{\mathbb{Z}/2}(\widetilde{E\mathbb{Z}/2}\wedge X)\to\lim k^*_{\mathbb{Z}/2}(S^{p\xi}\wedge X)\to 0.$$

The maps in the inverse system are the same as in (3), namely

$$i^*\colon k^*_G(S^{k\xi}\wedge X)\to k^*_G(S^{(k-1)\xi}\wedge X)$$

or equivalently $i^*\colon k_G^{*-k\xi}(X)\to k_G^{*-k\xi+\xi}(X)$, and they are multiplication by χ_ξ. Since χ_ξ is nilpotent by assumption, we conclude that the inverse limit and the $\lim^1$ are both trivial.

For the second part of the lemma, observe that we have just seen that the middle arrow in (2) is a $\mathbb{Z}/2$-homotopy equivalence; the leftmost arrow is always an equivalence and hence we see that $t(k_{\mathbb{Z}/2})$ is $\mathbb{Z}/2$-equivalent to $k_{\mathbb{Z}/2}\wedge\widetilde{E\mathbb{Z}/2}$.

From (4) it now follows that the Tate homology groups $t(k^{\mathbb{Z}/2})_V(X)$ are trivial for any X, since this is a direct limit over multiplication with the nilpotent element χ_ξ, and hence the spectrum classifying this theory, $t(k_{\mathbb{Z}/2})$, is trivial. □

4. The Real K-theory spectrum

In the following we state some facts about the $RO(\mathbb{Z}/2,U)$-graded spectrum representing Atiyah's Real K-theory.

Definition 4.1 (Atiyah [A]). A Real vector bundle is a complex vector bundle $E \downarrow X$, where (E, τ) and (X, t) are $\mathbb{Z}/2$ spaces such that the involutions commute with the projection map and with τ antilinear, i.e. $\tau(ze) = \bar{z}\tau(e)$, $z \in \mathbb{C}$.

The Real K-theory of (X, t), $KR(X)$, is then the Grothendieck group of the isometry classes of Real vector bundles over X.

The kernel of the complex dimension map gives reduced KR-theory, $\widetilde{KR}(X)$, and this is the theory which we study here. The dimension map is split, so $KR(X)$ is isomorphic to $\widetilde{KR}(X) \oplus \mathbb{Z}$.

The classifying space for reduced Real K-theory is BU with $\mathbb{Z}/2$-action given by complex conjugation [tD]. A model for this is the infinite Grassmannian $GR(\mathbb{C}^\infty) = \bigcup GR_n(\mathbb{C}^\infty)$ of complex subspaces of $\mathbb{C}^\infty$ with action induced by complex conjugation on $\mathbb{C}^\infty$.

An n-dimensional Real bundle is a $(\mathbb{Z}/2 \times_\alpha U(n))$-bundle in the sense of tom Dieck [tD], where $\alpha: \mathbb{Z}/2 \to \mathrm{Aut}(U(n))$ is complex conjugation, whereas equivariant K or KO would have trivial α in this description. Another way of stating this is that a Real bundle is a $\mathbb{Z}/2$-bundle with total group $\Gamma = \mathbb{Z}/2 \times_\alpha U(n)$, structural group $U(n)$ and fiber $\mathbb{C}$ in the sense of [LMS, p. 175].

In [A], Atiyah introduces a $\mathbb{Z} \oplus \mathbb{Z}$-graded cohomology theory on $\mathbb{Z}/2$-spaces as follows:

$$KR^{p,q}(X, Y) = KR(X \times B^{p,q}, X \times S^{p,q} \cup Y \times B^{p,q}) = KR(S^{p\xi+q} \wedge X/Y)$$

where $B^{p,q}$ and $S^{p,q}$ are the ball, respectively the sphere of the representation $p\xi + q$, and $KR(X, A) = KR(X/A)$.

This theory has two kinds of periodicity, namely 1-1-periodicity: $KR^{p,q}(X) \cong KR^{p+1,q+1}(X)$, [A, Theorem 2.3], and 8-periodicity:

$$KR^{p,q}(X) \cong KR^{p+8,q}(X)$$

[A, Theorem 3.10]. Thus it can be extended to negative p and q.

As in §3, we work in the complete universe $U = \infty\xi \oplus \infty$. For KR^* to be an $RO(\mathbb{Z}/2, U)$-graded cohomology theory, we want suspension isomorphisms $KR^V(S^V \wedge X_+) \cong KR(X)$ for any subspace V in U. For this to be true we define

$$KR^V(X_+) = KR^{-p,-q}(X)$$

for $V = p\xi \oplus q$. The reader should beware of the change of signs, since this is quite confusing.

We let KR denote the $\mathbb{Z}/2$-spectrum representing this theory, and we state some of Atiyah's results in this language.

Let H be the canonical line bundle over $\mathbb{C}P^1$ and let $b = [H] - 1 \in KR(\mathbb{C}P^1) \cong KR^{-(\xi\oplus 1)}(*)$. The 1-1 periodicity theorem states

Theorem 4.2 (Atiyah). *Multiplication by b induces an equivariant homotopy equivalence between $KR \wedge S^V$ and $KR \wedge S^{V-\xi\oplus 1}$ for $V \subset U$ and $\xi \oplus 1 \subset V$.*

Proof. By the 1-1 periodicity theorem [A, Theorem 2.3], multiplication with b gives an isomorphism between $KR^V(X)$ and $KR^{V-\xi\oplus 1}(X)$. Since this is true for any $\mathbb{Z}/2$-space, we get the desired $\mathbb{Z}/2$ homotopy equivalence. □

Now let $\alpha(\lambda)$ be the generator of $KR^{8\xi}(*)$. The 8-periodicity theorem is

Theorem 4.3 (Atiyah). *Multiplication by $\alpha(\lambda)$ induces an equivariant homotopy equivalence between KR and $KR \wedge S^{8\xi}$.*

Proof. The Real periodicity theorem [A, Theorem 3.10] says, that multiplication by $\alpha(\lambda)$ induces an isomorphism between $KR(X)$ and $KR^{8\xi}(X)$ for any X. □

We state some consequences of the periodicity theorems: For $p \geq q$

$$KR \wedge S^{p\xi \oplus q} \simeq KR \wedge S^{(p-q)\xi}$$

and for any X

$$KR^{p\xi \oplus q}(X) \cong KR^{(p-q)\xi}(X).$$

Now $\sum^{-V}(S^V \wedge KR) \simeq KR$ and $KR \wedge S^V \simeq \sum^{-(\xi \oplus 1)}(KR \wedge S^V)$, so

$$KR \wedge S^{p\xi \oplus q} \simeq \sum{}^{-(p\xi \oplus p)}(S^{p\xi \oplus q} \wedge KR) \simeq \sum{}^{-(p-q)}(KR).$$

For $p \leq q$, by the same arguments

$$KR \wedge S^{p\xi \oplus q} \simeq \sum{}^{-(q-p)\xi}(KR) \simeq KR \wedge S^{q-p}$$

and

$$KR^{p\xi \oplus q}(X) \cong KR^{q-p}(X).$$

The fixed points of KR yield periodic real K-theory, KO, since $KR(Y) = KO(Y)$ for any $\mathbb{Z}/2$-fixed space Y and nonequivariantly KR is homotopy equivalent to periodic K-theory. Thus Real K-theory provides an example of a non-split G-spectrum, since a splitting [GM, Definition 0.2] is a homotopy equivalence $KU \to KO \to KU$ mapping the naive G-spectrum KR considered nonequivariantly (KU) through the fixed point spectrum (KO) and via the inclusion to KU. This can never be a homotopy equivalence, one reason being that $\pi_6(KO)$ is trivial and $\pi_6(KU)$ is not.

5. Calculation of $t(KR)$

We will see that KR fulfills the conditions of 3.1.

Let η be the generator of $KR^{\xi}(S^0) \cong KR(\mathbb{R}P^1) \cong KO(\mathbb{R}P^1)$, which represents the reduced Hopf bundle over $\mathbb{R}P^1$.

Lemma 5.1. *Let $i\colon S^0 \to S^{p\xi}$ be the inclusion. Then the induced map $i^*\colon KR^*(S^{p\xi}) \to KR^*(S^0)$ is multiplication by $(-\eta)^p$.*

Proof. In [A, 3.2], Atiyah studied a map induced by the inclusion of the ball $(B(p\xi)_+, +)$ in $(B(p\xi)_+, S(p\xi)_+)$. The induced map in KR-theory is equivalent to the map induced by $i\colon S^0 \to S^{p\xi}$, via equivariant homotopy equivalences of the spaces, and Atiyah proved that it is multiplication by $(-\eta)^p$. □

This is all we need to prove our main theorem.

Theorem 5.2. *The Tate spectrum $t(KR)$ is trivial, and the projection of $E\mathbb{Z}/2_+$ to S^0 induces a $\mathbb{Z}/2$-homotopy equivalence between KR and $F(E\mathbb{Z}/2_+, KR)$.*

Proof. This will follow from 3.1 once we show that χ_ξ is nilpotent, since KR is a ring spectrum via tensor product of real bundles. Now Lemma 5.1 implies that $\chi_\xi = -\eta$ and $\eta^3 = 0$ by [A]. □

By the discussion in [GM, §5], this implies that the homotopy orbit spectrum for KR is equivalent to the homotopy fixed point spectrum, which by (5.2) is the fixed point spectrum KO, but we will not go into that here.

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Department of Mathematics and Computer Science, Aalborg University, Fr. Bajersvej 7E, 9220 Aalborg Øst, Denmark

E-mail address: `fajstrup@iesd.auc.dk`