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# SINGULAR LIMIT OF SOLUTIONS OF $u_t = \Delta u^m - A \cdot \nabla(u^q/q)$ AS $q \to \infty$

KIN MING HUI

ABSTRACT. We will show that the solutions of $u_t = \Delta u^m - A \cdot \nabla(u^q/q)$ in $R^n \times (0, T)$, $T > 0$, $m > 1$, $u(x, 0) = f(x) \in L^1(R^n) \cap L^\infty(R^n)$ converge weakly in $(L^\infty(G))^*$ for any compact subset $G$ of $R^n \times (0, T)$ as $q \to \infty$ to the solution of the porous medium equation $v_t = \Delta v^m$ in $R^n \times (0, T)$ with $v(x, 0) = g(x)$ where $g \in L^1(R^n)$, $0 \le g \le 1$, satisfies $g(x) + (\tilde{g}(x))_{x_1} = f(x)$ in $\mathscr{D}'(R^n)$ for some function $\tilde{g}(x) \in L^1(R^n)$, $\tilde{g}(x) \ge 0$ such that $g(x) = f(x)$, $\tilde{g}(x) = 0$ whenever $g(x) < 1$ a.e. $x \in R^n$. The convergence is uniform on compact subsets of $R^n \times (0, T)$ if $f \in C_0(R^n)$.

In this paper we will study the asymptotic behaviour of nonnegative solutions $u = u^{(q)}$ of the equation

$$(0.1) \qquad \begin{cases} u_t = \Delta u^m - A \cdot \nabla(u^q/q), & (x, t) \in R^n \times (0, T), \\ u(x, 0) = f(x) \ge 0, & f \in L^1(R^n) \cap L^\infty(R^n), \end{cases}$$

where $0 \ne A = (a_1, a_2, \dots, a_n) \in R^n$ is a constant vector, $T > 0$, $m > 1$, as $q \to \infty$. Recently there is a lot of research on the above equation ([A],[DiK],[G1],[G2]) The equation arises in many physical applications such as the flow of water through a homogeneous isotropic rigid porous medium [G1]. When $A = 0$, the above equation reduces to the well-known porous medium equation ([Ar],[P]). In the paper [CF], Caffarelli and A. Friedman studied the asymptotic behaviour of solutions of (0.1) when $A = 0$ and showed that the solutions of (0.1) converge as $m \to \infty$ if $f$ satisfies (0.1) and the following conditions:

$$\begin{aligned} &f \in C^1 \text{ in supp } f, \\ &f(0) > 1, \quad f_r < 0 \text{ in } R^n \setminus \{0\} \cap \text{supp } f, \\ &f_{r_{x_0}} \le 0 \text{ in } R^n \setminus B_1(0) \cap \text{supp } f \qquad \forall x_0 \in B_{\varepsilon_0}(0) \end{aligned}$$

for some $\varepsilon_0 > 0$ where $r_{x_0} = |x - x_0|$, $B_r(0) = \{x : |x| < r\}$ and $f_{r_{x_0}}$ is the radial derivative of $f$ with center at $x_0$.

This result has been extended in various directions by P. Bénilan, L. Boccardo and M. Herrero [BBH], P. E. Sacks [S2] in the case $A = 0$, X. Xu [X] in the case of hyperbolic equations and K. M. Hui [H1], [H2] in the case of a porous

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medium equation with absorption and in the case of the generalized p-Laplacian equation.

For simplicity we will assume that $T = 1$ and $A = (1, 0, \dots, 0)$ throughout the rest of the paper. We will show that as $q \to \infty$, the convection term in (0.1) disappears. More precisely, we will show that for fixed $m > 1$ the solutions $u = u^{(q)}$ of (0.1) converge weakly in $(L^\infty(G))^*$ for any compact subset $G$ of $R^n \times (0, 1)$ as $q \to \infty$. Moreover the limit $u^{(\infty)} = \lim_{q\to\infty} u^{(q)}$ satisfies the porous medium equation

$$\begin{cases} v_t = \Delta v^m, & (x, t) \in R^n \times (0, 1), \\ v(\cdot, t) \searrow g & \text{as } t \to 0 \text{ in } \mathscr{D}'(R^n), \end{cases} \tag{0.2}$$

where $g \in L^1(R^n)$, $0 \le g \le 1$, satisfies

$$g(x) + (\tilde{g}(x))_{x_1} = f(x) \text{ in } \mathscr{D}'(R^n) \tag{0.3}$$

for some function $\tilde{g}(x) \ge 0$, $\tilde{g}(x) \in L^1(R^n)$ and $g(x) = f(x)$, $\tilde{g}(x) = 0$ whenever $g(x) < 1$ a.e. $x \in R^n$. This extends the recent results obtained by M. Escobedo and E. Zuazua [EZ], who showed that the convection term was negligible compared with the other terms appearing in (0.1) for the case $m = 1$ and $q > 1 + 1/n$ as $t \to \infty$. Although we were not able to prove it, we suspect that the same result should remain valid when $A = A(x) \in L^\infty(R^n)$.

We will first start with some definitions. For any open set $\Omega_0 \subset R^n$, $h \in C(R)$, we say that $u$ is a solution (respectively subsolution, supersolution) of

$$u_t = \Delta u^m - (h(u))_{x_1} \tag{0.4}$$

in $\overline{\Omega}_0 \times (0, 1)$ if $u$ is continuous and nonnegative in $\overline{\Omega}_0 \times (0, 1)$, $u \in L^\infty([0, 1); L^1(\Omega_0)) \cap L^\infty(\Omega_0 \times (0, 1))$ and satisfies

$$\int_{\tau_1}^{\tau_2} \int_\Omega \left[ u^m \Delta\eta + u\frac{\partial \eta}{\partial t} + h(u)\eta_{x_1} \right] dx dt = \int_{\tau_1}^{\tau_2} \int_{\partial\Omega} u^m \frac{\partial \eta}{\partial N} d\sigma ds + \int_\Omega u\eta dx \Big|_{\tau_1}^{\tau_2} \tag{0.5}$$

(respectively $\ge$, $\le$ ) for all bounded open sets $\Omega \subset \Omega_0$ with $\partial\Omega \in C^2$, $0 < \tau_1 \le \tau_2 < 1$, $\eta \in C^\infty(\Omega \times [\tau_1, \tau_2])$, $\eta \equiv 0$ on $\partial\Omega \times [\tau_1, \tau_2]$ where $\partial/\partial N$ is the exterior normal derivative on $\partial\Omega$ and $d\sigma$ is the surface measure on $\partial\Omega$.

If $u$ is a solution of (0.4) in $\overline{\Omega}_0 \times (0, 1)$, we say that $u$ has initial trace or initial value $d\mu$ if

$$\lim_{t\to 0} \int u(x, t)\eta(x) dx = \int \eta d\mu \quad \forall \eta \in C_0^\infty(\overline{\Omega}_0).$$

We let $\rho \in C_0^\infty(R^n)$, $\rho \ge 0$, $\int \rho = 1$ and for any $g$ we define

$$g_\varepsilon = g * \rho_\varepsilon(x) = \int \rho_\varepsilon(x - y) g(y) dy, \quad \varepsilon > 0,$$

where $\rho_\varepsilon(y) = \rho(y/\varepsilon)/\varepsilon^n$. For any $r > 0$, $x_0 \in R^n$, let $B_r(x_0) = \{x \in R^n : |x - x_0| < r\}$. For any set $A \subset R^n$, we let $\chi_A$ be the characteristic function of the set $A$. We will also assume $m > 1$, $q > m + 1$, and let $u^{(q)}$ be the solution of (0.1) for the rest of the paper.

The plan of the paper is as follows. In section 1 we will state and prove the existence of solutions of (0.1). We will also prove a comparison theorem

for solutions of (0.1) and obtain some bounds on $u^{(q)}$ by constructing explicit supersolutions to (0.1). In section 2 we will first prove a comparison lemma for solutions of (0.3). We then prove the main theorem under the assumption $f \in C_0^1(R^n)$ (Theorem 2.9). Finally we will prove the main theorem (Theorem 2.10) by an approximation argument.

## 1

We first state and prove an uniqueness theorem for solutions of (0.1).

**Theorem 1.1.** *If* $u_1^{(q)}$, $u_2^{(q)} \in L^\infty([0,1);\ L^1(R^n)) \cap L^\infty(R^n \times (0,1)) \cap C(R^n \times (0,1))$ *are the solutions of*

$$u_t = \Delta u^m - (u^q/q)_{x_1} \tag{1.1}$$

*in* $R^n \times (0,1)$ *with initial values* $f_1$ *and* $f_2 \in L^1(R^n) \cap L^\infty(R^n)$ *respectively,* $f_1, f_2 \geq 0$, *then there exists a constant* $C > 0$ *such that*

$$\text{(i)} \qquad \int_{R^n} (u_1^{(q)} - u_2^{(q)})_+(x,t)dx \leq e^{Ct} \int_{R^n} (f_1 - f_2)_+(x)dx,$$

$$\text{(ii)} \qquad \int_{R^n} |u_1^{(q)} - u_2^{(q)}|(x,t)dx \leq e^{Ct} \int_{R^n} |f_1 - f_2|(x)dx$$

*for all* $0 < t < 1$. *Hence* $u_1^{(q)} \leq u_2^{(q)}$ *if* $f_1 \leq f_2$. *In particular the solution of* (1.1) *in* $R^n \times (0,1)$ *with initial value in* $L^1(R^n) \cap L^\infty(R^n)$ *is unique in the class* $L^\infty([0,1); L^1(R^n)) \cap L^\infty(R^n \times (0,1)) \cap C(R^n \times (0,1))$.

*Proof.* The proof of the theorem is similar to the proof of Theorem 2.3 of [A]. By subtracting the equation for $u_1^{(q)}$ and $u_2^{(q)}$, we get

$$\begin{aligned} \int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})(x,t)\eta(x,t)dx &= \int_{B_R(0)} (f_1 - f_2)(x)\eta(x,0)dx \\ &+ \int_0^t \int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})(\eta_t + A\Delta\eta + B\eta_{x_1})dxd\tau \\ &- \int_0^t \int_{\partial B_R(0)} (u_1^{(q)m} - u_2^{(q)m})\frac{\partial \eta}{\partial N} d\sigma d\tau \end{aligned} \tag{1.2}$$

for all $0 < t < 1$, $\eta \in C^\infty(\overline{B_R(0)} \times [0,t])$, $R > 0$, such that $\eta \equiv 0$ on $\partial B_R(0) \times [0,t]$ where

$$A = \begin{cases} \dfrac{u_1^{(q)m} - u_2^{(q)m}}{u_1^{(q)} - u_2^{(q)}} & \text{for } u_1^{(q)} \neq u_2^{(q)}, \\ m u_1^{(q)m-1} & \text{for } u_1^{(q)} = u_2^{(q)}, \end{cases}$$

$$B = \begin{cases} \dfrac{1}{q} \cdot \dfrac{u_1^{(q)q} - u_2^{(q)q}}{u_1^{(q)} - u_2^{(q)}} & \text{for } u_1^{(q)} \neq u_2^{(q)}, \\ u_1^{(q)q-1} & \text{for } u_1^{(q)} = u_2^{(q)}. \end{cases}$$

Since $u_1^{(q)}$, $u_2^{(q)} \in L^\infty(R^n \times (0,1))$, there exists a constant $C_1 > 0$ such that

$$\|u_1^{(q)}\|_{L^\infty(R^n)}, \|u_2^{(q)}\|_{L^\infty(R^n)} \leq C_1$$
$$\Rightarrow B^2/2A \leq \frac{1}{2m} C_1^{2q-m-1}, \ B/A \leq \frac{1}{m} C_1^{q-m}.$$

By an argument similar to section 4 of [A], there exists smooth functions $A_{i,R}$ and $B_{i,R}$ and constant $c_i > 0$ such that $c_i \le A_{i,R} \le mC_1^{m-1} + 1$, $0 \le B_{i,R} \le C_1^{q-1}+1$, $B_{i,R}^2/2A_{i,R} \le (C_1^{2q-m-1}/2m)+1 = C_2$, $B_{i,R}/A_{i,R} \le (C_1^{q-m}/m)+1 = C_3$, $(A_{i,R} - A)/A_{i,R}^{1/2} \to 0$ and $B_{i,R} - B \to 0$ in $L^2(B_R(0) \times (0, 1))$ as $i \to \infty$ for all $R > 0$.

For any $R_0 > 2$, $R > R_0 + 1$, $\lambda > C_2$, $\theta \in C_0^\infty(B_{R_0}(0))$, $0 \le \theta \le 1$, let $\eta_{i,R}$ be the solution of

$$\begin{cases} \eta_s + A_{i,R}\Delta\eta + B_{i,R}\eta_{x_1} - \lambda\eta = 0 & \text{for } (x, s) \in B_R(0) \times (0, t), \\ \eta(x, s) = 0 & \text{for } (x, s) \in \partial B_R(0) \times (0, t], \\ \eta(x, t) = \theta(x) & \text{for } x \in B_R(0). \end{cases}$$

Since $0 \le \theta \le 1$, by the maximum principle $0 \le \eta_{i,R} \le 1$. By Lemma 4.1 of [A], we have

$$\begin{aligned} &\int_0^t \int_{B_R(0)} A_{i,R}(\Delta\eta_{i,R})^2 dx d\tau + 2(\lambda - C_2) \int_0^t \int_{B_R(0)} |\nabla\eta_{i,R}|^2 dx d\tau \\ &\quad \le \int_{B_R(0)} |\nabla\theta|^2 dx. \end{aligned} \tag{1.3}$$

By the same argument as the proof of Theorem 2.1 (ii) of [PV], we see that for any $\beta > 0$, the function

$$g(x, s) = e^{h(s)} \left( \frac{1 + R_0^2}{1 + |x|^2} \right)^\beta$$

where $h(s) = C'(t - s)$, $C' = 4\beta(\beta + 1)(mC_1^{m-1} + 1) + \beta(C_1^{q-1} + 1)$, satisfies

$$\begin{cases} g_s + A_{i,R}\Delta g + B_{i,R} g_{x_1} - \lambda g < 0, & \text{for } (x, s) \in B_R(0) \times (0, t), \\ g(x, s) \ge \eta_{i,R}(x, s), & \text{for } (x, s) \in B_R(0) \times \{t\} \cup \partial B_R(0) \times (0, t]. \end{cases}$$

Hence by the maximum principle [LSU], $g \ge \eta_{i,R}$ in $B_R(0) \times (0, t)$. We next consider the function

$$g^*(x, s) = ae^{h(s)}\Gamma(|x|), \quad R - \alpha \le r \le R, 0 \le s \le t,$$

where $\alpha = 1/2(C_3 + n - 1)$, $\Gamma(r) = (R - r) - C_3(R - r)^2$ and

$$a = (1 + R_0^2)^\beta / \{\Gamma(R - \alpha)(1 + (R - \alpha)^2)^\beta\}.$$

Then $g^* \ge 0$, $g_s^* = h'(s)g^* \le 0$ and

$$\begin{aligned} &\Delta g^* + (B_{i,R}/A_{i,R}) g_{x_1} \\ &\quad \le ae^{h(s)} \left( \Gamma''(r) + \frac{n-1}{r}\Gamma'(r) + C_3|\Gamma'(r)| \right) \\ &\quad \le ae^{h(s)} \left( -2C_3 + \frac{n-1}{r}(-1 + 2C_3(R - r)) + C_3(1 + 2C_3(R - r)) \right) \\ &\quad \le aC_3 e^{h(s)}(-1 + 2(C_3 + n - 1)(R - r)) \\ &\quad \le 0 \end{aligned}$$

for all $R - \alpha < r < R$, $0 < s < t$ since $R - \alpha \ge R_0 \ge 2$. Hence $g^*$ satisfies

$$g_s^* + A_{i,R}\Delta g^* + B_{i,R} g_{x_1}^* - \lambda g^* < 0, \quad \text{for } (x, s) \in B_R(0) \setminus B_{R-\alpha}(0) \times (0, t)$$

with $g^*(x, s) \geq \eta_{i,R}(x, s)$ for all

$$(x, s) \in B_R(0) \setminus B_{R-\alpha}(0) \times \{t\} \cup (\partial B_R(0) \cup \partial B_{R-\alpha}(0)) \times (0, t].$$

By the maximum principle, $0 \leq \eta_{i,R} \leq g^*$ in $B_R(0) \setminus B_{R-\alpha}(0) \times (0, t)$. Since $g^* \equiv \eta_{i,R} \equiv 0$ on $\partial B_R(0) \times [0, t]$,

$$\|\partial \eta_{i,R}/\partial N\|_{L^\infty(\partial B_R(0)\times(0,t))} \leq \|\partial g^*/\partial N\|_{L^\infty(\partial B_R(0)\times(0,t))} \leq CR^{-2\beta}. \tag{1.4}$$

Putting $\eta = \eta_{i,R}$ in (1.2), we get by (1.3) and (1.4),

(1.5)

$$\begin{aligned}
&\int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})(x, t)\theta(x)dx \\
&= \int_{B_R(0)} (f_1 - f_2)(x)\eta_{i,R}(x, 0)dx + \int_0^t \int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})[(A - A_{i,R})\Delta\eta_{i,R} \\
&\quad + (B - B_{i,R})(\eta_{i,R})_{x_1} + \lambda\eta_{i,R}]dxd\tau \\
&\quad - \int_0^t \int_{\partial B_R(0)} (u_1^{(q)m} - u_2^{(q)m})\frac{\partial \eta_{i,R}}{\partial N} d\sigma d\tau \\
&\leq \int_{R^n} (f_1 - f_2)_+ dx + 2C_1\|(A_{i,R} - A)/A_{i,R}^{1/2}\|_{L^2(B_R(0)\times(0,t))}\|\nabla\theta\|_{L^2(B_R(0))} \\
&\quad + (2C_1/(2(\lambda - C_2))^{1/2})\|B_{i,R} - B\|_{L^2(B_R(0)\times(0,t))}\|\nabla\theta\|_{L^2(B_R(0))} \\
&\quad + \lambda \int_0^t \int_{R^n} (u_1^{(q)} - u_2^{(q)})_+ dxd\tau + CR^{n-1-2\beta}t
\end{aligned}$$

for all $\theta \in C_0^\infty(B_{R_0}(0))$, $0 \leq \theta \leq 1$, $0 < t < 1$. Choose now $\beta = n/2$ and let first $i \to \infty$ and then $R \to \infty$, $\lambda \to C_2$ in (1.5), we get

$$\int_{R^n} (u_1^{(q)} - u_2^{(q)})(x, t)\theta(x)dx \leq \int_{R^n} (f_1 - f_2)_+ dx + C_2 \int_0^t \int_{R^n} (u_1^{(q)} - u_2^{(q)})_+ dxd\tau$$

for all $\theta \in C_0^\infty(B_{R_0}(0))$, $0 \leq \theta \leq 1$, $R_0 > 2$. Putting $\theta = \chi_{\{u_1^{(q)} \geq u_2^{(q)}\} \cap B_{R_0-1}(0)} * \rho_\varepsilon$ into the above inequality and letting first $\varepsilon \to 0$ and then $R_0 \to \infty$, we get

$$\begin{aligned}
&\int_{R^n} (u_1^{(q)} - u_2^{(q)})_+(x, t)dx \\
&\qquad \leq \int_{R^n} (f_1 - f_2)_+ dx + C_2 \int_0^t \int_{R^n} (u_1^{(q)} - u_2^{(q)})_+ dxd\tau \quad \forall 0 < t < 1.
\end{aligned}$$

(i) then follows from the Gronwall's inequality. Similarly,

$$\int_{R^n} (u_1^{(q)} - u_2^{(q)})_-(x, t)dx \leq e^{C_2 t} \int_{R^n} (f_1 - f_2)_- dx \quad \forall 0 < t < 1.$$

By combining the above inequality with (i), we get (ii).

**Corollary 1.2.** *If $u_1^{(q)}$ is a subsolution and $u_2^{(q)}$ is a supersolution of* (1.1) *in $Q = D \times (0, 1)$ where $D = (-\infty, R_0] \times R^{n-1}$ for some $R_0 \in R$ (or $D = [R_0, R_1] \times R^{n-1}$ for some $R_0$, $R_1 \in R$, $R_0 < R_1$) with $u_1^{(q)}$, $u_2^{(q)} \in L^\infty([0, 1); L^1(D)) \cap L^\infty(D \times (0, 1)) \cap C(D \times (0, 1))$ with initial values $u_1^{(q)}(x, 0)$, $u_2^{(q)}(x, 0)$ and boundary values satisfying*

$$u_1^{(q)}(x, t) \leq u_2^{(q)}(x, t) \quad \forall (x, t) \in \partial_p Q$$

*where* $\partial_p Q = \{R_0\} \times R^{n-1} \times (0, 1) \cup (-\infty, R_0] \times R^{n-1} \times \{0\}$ (*respectively* $\partial_p Q = \{R_0, R_1\} \times R^{n-1} \times (0, 1) \cup [R_0, R_1] \times R^{n-1} \times \{0\}$), *then*

$$u_1^{(q)}(x, t) \le u_2^{(q)}(x, t) \quad \forall (x, t) \in Q$$

*Proof*. The proof is the same as the proof of Theorem 1.1.

**Theorem 1.3.** *The equation*

$$\begin{cases} u_t = \Delta u^m - (u^q/q)_{x_1}, \ u \ge 0, & (x, t) \in R^n \times (0, 1), \\ u(x, 0) = f(x) \ge 0, & f \in L^1(R^n) \cap L^\infty(R^n), \end{cases} \tag{1.6}$$

*has a unique solution* $u^{(q)} \in L^\infty([0, 1); L^1(R^n)) \cap L^\infty(R^n \times (0, 1)) \cap C(R^n \times (0, 1))$ *with*

$$\text{(i)} \qquad \int u^{(q)}(x, t)dx = \int f dx \quad \forall 0 < t < 1, \tag{1.7}$$

$$\text{(ii)} \qquad \|u^{(q)}\|_{L^\infty(R^n \times (0,1))} \le \|f\|_{L^\infty(R^n)}. \tag{1.8}$$

*Proof*. The proof is similar to that of [ERV] and [DK]. Let $\psi \in C_0^\infty(R^n)$, $0 \le \psi \le 1$, be such that $\psi(x) \equiv 1$ for all $|x| \le 1/2$ and $\psi \equiv 0$ for all $|x| \ge 1$. For any $\varepsilon > 0$, $0 < \varepsilon < 1$, $R > 0$, let $f_{\varepsilon, R}(x) = f * \rho_\varepsilon(x) \cdot \psi(x/R) + \varepsilon$ and let $a_\varepsilon(s)$, $b_\varepsilon(s) \in C^\infty(R)$ be such that $a_\varepsilon'(s)$, $b_\varepsilon'(s) \ge 0$,

$$a_\varepsilon(s) = \begin{cases} m(\|f\|_{L^\infty(R^n)} + 2)^{m-1} & \text{for } s \ge \|f\|_{L^\infty(R^n)} + 2, \\ ms^{m-1} & \text{for } \varepsilon \le s \le \|f\|_{L^\infty(R^n)} + 1, \\ m(\varepsilon/2)^{m-1} & \text{for } s \le \varepsilon/2, \end{cases}$$

$$b_\varepsilon(s) = \begin{cases} (\|f\|_{L^\infty(R^n)} + 2)^{q-1} & \text{for } s \ge \|f\|_{L^\infty(R^n)} + 2, \\ s^{q-1} & \text{for } \varepsilon \le s \le \|f\|_{L^\infty(R^n)} + 1, \\ (\varepsilon/2)^{q-1} & \text{for } s \le \varepsilon/2. \end{cases}$$

By standard parabolic theory [LSU], there exists a unique solution $u_{\varepsilon, R}^{(q)}$ to the equation

$$\begin{cases} u_t = \operatorname{div}(a_\varepsilon(u)\nabla u) - b_\varepsilon(u)u_{x_1}, & \text{for } (x, t) \in B_R(0) \times (0, 1), \\ u(x, t) = \varepsilon & \text{for } (x, t) \in \partial B_R(0) \times (0, 1), \\ u(x, 0) = f_{\varepsilon, R}(x), & \text{for } x \in B_R(0). \end{cases} \tag{1.9}$$

Since $\varepsilon \le f_{\varepsilon, R} \le \|f\|_{L^\infty(R^n)} + \varepsilon$, by the maximum principle,

$$\varepsilon \le u_{\varepsilon, R}^{(q)} \le \|f\|_{L^\infty(R^n)} + \varepsilon. \tag{1.10}$$

Hence $a_\varepsilon(u_{\varepsilon, R}^{(q)}) = m u_{\varepsilon, R}^{(q)m-1}$, $b_\varepsilon(u_{\varepsilon, R}^{(q)}) = u_{\varepsilon, R}^{(q)q-1}$. Since (1.3) is a nondegenerate parabolic equation, by Schauder's estimate [LSU], $u_{\varepsilon, R}^{(q)} \in C^\infty(B_R(0) \times [0, 1))$. Thus $u_{\varepsilon, R}^{(q)}$ satisfies (1.1) in $B_R(0) \times (0, 1)$. Since $u_{\varepsilon, R}^{(q)}$ is uniformly bounded by $\|f\|_{L^\infty(R^n)} + 1$, by the result of P. Sacks [S1], $\{u_{\varepsilon, R}^{(q)}\}_{R>0}$ has a convergent subsequent $\{u_{\varepsilon, R_j}^{(q)}\}_{j=1}^\infty$, $R_j \to \infty$ as $j \to 0$, such that $\{u_{\varepsilon, R_j}^{(q)}\}_{j=1}^\infty$ converges uniformly on compact subsets of $R^n \times (0, 1)$. Let $u_\varepsilon^{(q)} = \lim_{j \to \infty} u_{\varepsilon, R_j}^{(q)}$. Then $u_\varepsilon^{(q)} \in C(R^n \times (0, 1))$ and

$$\varepsilon \le u_\varepsilon^{(q)} \le \|f\|_{L^\infty(R^n)} + \varepsilon. \tag{1.11}$$

Putting $u = u^{(q)}_{\varepsilon, R_j}$ in (1.1) and letting $j \to 0$, we see that $u^{(q)}_\varepsilon$ satisfies (1.1) in $R^n \times (0, 1)$ with $u^{(q)}_\varepsilon(x, 0) = f * \rho_\varepsilon(x) + \varepsilon$. Thus $u^{(q)}_\varepsilon \in C^\infty(R^n \times (0, 1))$ by (1.11) and Schauder's estimates. Since $\|u^{(q)}_\varepsilon\|_{L^\infty(R^n \times (0,1))} \le \|f\|_{L^\infty(R^n)} + \varepsilon$, by [S1], $\{u^{(q)}_\varepsilon\}_{\varepsilon > 0}$ has a convergent subsequent $\{u^{(q)}_{\varepsilon_i}\}_{i=1}^\infty$, $\varepsilon_i \to 0$ as $i \to 0$, such that $\{u^{(q)}_{\varepsilon_i}\}_{i=1}^\infty$ converges uniformly on compact subsets of $R^n \times (0, 1)$. Let $u^{(q)} = \lim_{i \to \infty} u^{(q)}_{\varepsilon_i}$. Then $u^{(q)} \in C(R^n \times (0, 1))$.

Putting $u = u^{(q)}_{\varepsilon_i}$ in (1.1) and letting $i \to 0$, we see that $u^{(q)}$ satisfies (1.1) in $R^n \times (0, 1)$. Moreover,

$$
\begin{aligned}
&\left| \int_{R^n} u^{(q)}_{\varepsilon, R_j}(x, t)\eta(x)dx - \int_{R^n} f_{\varepsilon, R_j}\eta dx \right| \\
&\quad = \left| \int_0^t \int_{R^n} (u^{(q)}_{\varepsilon, R_j})_t(x, \tau)\eta(x)dxd\tau \right| \\
&\quad = \left| \int_0^t \int_{R^n} \left[ \Delta u^{(q)m}_{\varepsilon, R_j} - \left( \frac{u^{(q)q}_{\varepsilon, R_j}}{q} \right)_{x_1} \right] \eta dxd\tau \right| \qquad (1.12) \\
&\quad = \left| \int_0^t \int_{R^n} \left( u^{(q)m}_{\varepsilon, R_j} \Delta\eta + \frac{u^{(q)q}_{\varepsilon, R_j}}{q} \eta_{x_1} \right) dxd\tau \right| \\
&\quad \le (\|f\|_{L^\infty(R^n)} + 1)^m \|\Delta\eta\|_{L^1(R^n)} t + \frac{(\|f\|_{L^\infty(R^n)} + 1)^q}{q} \|\eta_{x_1}\|_{L^1(R^n)} t
\end{aligned}
$$

for all $\eta \in C_0^\infty(R^n)$. Letting first $j \to 0$ and then $\varepsilon = \varepsilon_i \to 0$, $t \to 0$, we get

$$
\lim_{t \to 0} \int_{R^n} u^{(q)}(x, t)\eta(x)dx = \int_{R^n} f\eta dx \quad \forall \eta \in C_0^\infty(R^n).
$$

Hence $u^{(q)}$ has initial trace $f$ and $\|u^{(q)}\|_{L^\infty(R^n \times (0,1))} \le \|f\|_{L^\infty(R^n)}$ by (1.11). On the other hand, since $u^{(q)}_\varepsilon$ satisfies (1.1) in $R^n \times (0, 1)$,

$$
\begin{aligned}
\int_{B_R(0)} u^{(q)}_\varepsilon(x, t)\eta(x, t)dx &= \int_{B_R(0)} (f * \rho_\varepsilon(x) + \varepsilon)\eta(x, 0)dx \\
&\quad + \int_0^t \int_{B_R(0)} u^{(q)}_\varepsilon(\eta_t + A_\varepsilon \Delta\eta + B_\varepsilon \eta_{x_1})dxd\tau \qquad (1.13) \\
&\quad - \int_0^t \int_{\partial B_R(0)} u^{(q)m}_\varepsilon \frac{\partial \eta}{\partial N} d\sigma d\tau
\end{aligned}
$$

for all $0 < t < 1$, $\eta \in C^\infty(\overline{B_R(0)} \times [0, t])$, $R > 0$ such that $\eta \equiv 0$ on $\partial B_R(0) \times [0, t]$ where $A_\varepsilon = u^{(q)m-1}_\varepsilon$, $B_\varepsilon = u^{(q)q-1}_\varepsilon/q$.

For any $R_0 > 2$, $R > R_0 + 1$, $\theta \in C_0^\infty(B_{R_0}(0))$, $0 \le \theta \le 1$, $\theta \equiv 1$ for $|x| \le R_0 - 1$, let $\eta_{\varepsilon, R}$ be the solution of

$$
\begin{cases}
\eta_s + A_\varepsilon \Delta\eta + B_\varepsilon \eta_{x_1} = 0 & \text{for } (x, s) \in B_R(0) \times (0, t), \\
\eta(x, s) = 0 & \text{for } (x, s) \in \partial B_R(0) \times (0, t], \\
\eta(x, t) = \theta(x) & \text{for } x \in B_R(0).
\end{cases}
$$

By an argument similar to the proof of Theorem 1.1, we have $0 \le \eta_{\varepsilon, R} \le 1$,

$$
\eta_{\varepsilon, R}(x, s) \le e^{h(s)} \left( \frac{1 + R_0^2}{1 + |x|^2} \right)^n \quad \forall 0 \le s \le t,
$$

where $h(s) = C'(t-s)$, $C' = 4n(n+1)(b_1^{m-1}+1)+n(b_1^{q-1}+1)$, $b_1 = \|f\|_{L^\infty(R^n)}+1$, and

$$\|\partial\eta_{\varepsilon,R}/\partial N\|_{L^\infty(\partial B_R(0)\times(0,t))} \le CR^{-2n}$$

for some constant $C > 0$ depending only on $R_0$ and $b_1$. Putting $\eta = \eta_{\varepsilon,R}$ into (1.13), we get

$$\int_{B_R(0)} u_\varepsilon^{(q)}\theta(x)dx \le \int f dx + C'_{R_0}R^{-n-1} + \varepsilon C_{R_0} \tag{1.14}$$

for some constant $C_{R_0}$, $C'_{R_0} > 0$ depending only on $R_0$ and $b_1$. Letting $R \to \infty$, $\varepsilon = \varepsilon_i \to 0$,

$$\int_{|x|\le R_0-1} u^{(q)}(x,t)dx \le \int_{R^n} u_\varepsilon^{(q)}(x,t)\theta(x)dx \le \int f dx$$

for all $0 < t < 1$. Letting $R_0 \to \infty$,

$$\int_{R^n} u^{(q)}(x,t)dx \le \int f dx \quad \forall 0 < t < 1. \tag{1.15}$$

Hence $u^{(q)} \in L^\infty([0,1); L^1(R^n)) \cap L^\infty(R^n\times(0,1)) \cap C(R^n\times(0,1))$ and satisfies (1.8). It remains to show (1.7). Since

$$\begin{aligned}(1.15) \Rightarrow \quad & \int_0^1\int_{R^n} u^{(q)}(x,\tau)dxd\tau \le \int_{R^n} f dx \\ \Rightarrow \quad & \int_0^1\int_{R/2\le|x|\le R} u^{(q)}(x,\tau)dxd\tau \to 0 \quad \text{as } R\to\infty,\end{aligned}$$

putting $\eta(x) = \psi(x/R)$, $R > 0$, in (1.12), we have

$$\begin{aligned}&\left|\int_{R^n} u_{\varepsilon,R_j}^{(q)}(x,t)\psi(x/R)dx - \int_{R^n} f_{\varepsilon,R_j}(x)\psi(x/R)dx\right| \\ &\le \frac{(\|f\|_{L^\infty(R^n)}+1)^{m-1}}{R^2}\|\Delta\psi\|_{L^\infty(R^n)}\int_0^t\int_{R/2\le|x|\le R} u_{\varepsilon,R_j}^{(q)}(x,\tau)dxd\tau \\ &\quad + \frac{(\|f\|_{L^\infty(R^n)}+1)^{q-1}}{qR}\|\psi_{x_1}\|_{L^\infty(R^n)}\int_0^t\int_{R/2\le|x|\le R} u_{\varepsilon,R_j}^{(q)}(x,\tau)dxd\tau.\end{aligned}$$

By letting first $j \to \infty$ and then $\varepsilon = \varepsilon_i \to 0$, $R \to \infty$, in the above inequality, we get (1.7). Since uniqueness of solution of (1.6) follows from Theorem 1.1. This completes the proof of the theorem.

**Theorem 1.4.** *Let* $u_1^{(q)}$, $u_2^{(q)}$, $f_1$, $f_2$ *be as in Theorem* 1.1. *Then*

$$\int_{R^n} |u_1^{(q)} - u_2^{(q)}|(x,t)dx \le \int_{R^n} |f_1 - f_2|dx \quad \forall 0 < t < 1.$$

*Proof.* By Theorem 1.1 and the proof of Theorem 1.3, there exist solutions $u_{1,\varepsilon}^{(q)}$, $u_{2,\varepsilon}^{(q)} \in C^\infty(R^n\times(0,1)) \cap L^\infty(R^n\times(0,1))$, $0 < \varepsilon < 1$, of (1.6) with initial values $u_{1,\varepsilon}^{(q)}(x,0) = f_1 * \rho_\varepsilon + \varepsilon$, $u_{2,\varepsilon}^{(q)}(x,0) = f_2 * \rho_\varepsilon + \varepsilon$ respectively such that $u_{1,\varepsilon}^{(q)}$ and $u_{2,\varepsilon}^{(q)}$ converges uniformly to $u_1^{(q)}$ and $u_2^{(q)}$ respectively on compact subsets of $R^n\times(0,1)$ as $\varepsilon \to 0$.

By a proof similar to the proof of (1.14), we have

$$\int_{B_R(0)} (u_{1,\varepsilon}^{(q)} - u_{2,\varepsilon}^{(q)})(x, t)\theta(x)dx \le \int_{R^n} (f_1 * \rho_\varepsilon - f_2 * \rho_\varepsilon)_+ dx + C'_{R_0} R^{-1-n} + \varepsilon C_{R_0}$$

for all $\theta \in C_0^\infty(B_{R_0}(0))$, $R_0 > 2$, $R > R_0 + 1$, $0 < t < 1$ where $C_{R_0}$ and $C'_{R_0} > 0$ are constants depending only on $R_0$, $\|u_1^{(q)}\|_{L^\infty(R^n)}$ and $\|u_2^{(q)}\|_{L^\infty(R^n)}$. Letting $R \to \infty$, $\varepsilon \to 0$, we get

$$\int_{R^n} (u_1^{(q)} - u_2^{(q)})(x, t)\theta(x)dx \le \int_{R^n} (f_1 - f_2)_+ dx$$

for all $\theta \in C_0^\infty(B_{R_0}(0))$, $R_0 > 2$, $0 < t < 1$. Putting $\theta = \chi_{\{u_1^{(q)} \ge u_2^{(q)}\}} * \rho_\varepsilon$ and letting first $\varepsilon \to 0$ and then $R_0 \to \infty$,

$$\int_{R^n} (u_1^{(q)} - u_2^{(q)})_+(x, t)dx \le \int_{R^n} (f_1 - f_2)_+ dx \quad \forall 0 < t < 1.$$

Similarly,

$$\int_{R^n} (u_1^{(q)} - u_2^{(q)})_-(x, t)dx \le \int_{R^n} (f_1 - f_2)_- dx \quad \forall 0 < t < 1.$$

Combining the above two inequalities the theorem follows.

**Lemma 1.5.** *If $f \in C_0^1(R^n)$ and $f_\varepsilon = f + \varepsilon$, $0 < \varepsilon < 1$, then (1.1) has a unique solution $u_\varepsilon^{(q)} \in C^\infty(R^n \times (0, 1)) \cap C^1(R^n \times [0, 1))$ in $R^n \times (0, 1)$ with $u_\varepsilon^{(q)}(x, 0) = f_\varepsilon(x)$ such that $u_\varepsilon^{(q)}$ converges uniformly on compact subsets of $R^n \times (0, 1)$ to the solution $u^{(q)}$ of (1.6) with $u^{(q)}(x, 0) = f(x)$ as $\varepsilon \to 0$. Moreover*

$$\|u_{\varepsilon, x_k}^{(q)}\|_{L^\infty(R^n)} \le \|f_{x_k}\|_{L^\infty(R^n)} \quad \forall k = 1, 2, \dots, n.$$

*Proof.* By Theorem 1.4 and an argument similar to the proof of Theorem 1.3, for any $0 < \varepsilon < 1$ there exists a unique solution $u_\varepsilon^{(q)} \in C^\infty(R^n \times (0, 1)) \cap C^1(R^n \times [0, 1))$ to (1.1) in $R^n \times (0, 1)$ with $u_\varepsilon^{(q)}(x, 0) = f(x) + \varepsilon$ and

$$\varepsilon \le u_\varepsilon^{(q)} \le \|f\|_{L^\infty(R^n)} + \varepsilon \tag{1.16}$$

such that $u_\varepsilon^{(q)}$ converges uniformly on compact subsets of $R^n \times (0, 1)$ to the solution $u^{(q)}$ of (1.6) with $u^{(q)}(x, 0) = f(x)$ as $\varepsilon \to 0$.

Since $u_\varepsilon^{(q)} \in C^\infty(R^n \times (0, 1)) \cap C^1(R^n \times [0, 1))$, differentiating (1.1) with respect to $x_k$ and writing $z = u_{\varepsilon, x_k}^{(q)}$, we get

$$\begin{cases} z_t = \Delta(m u_\varepsilon^{(q)m-1} z) + (u_\varepsilon^{(q)q-1} z)_{x_1}, & (x, t) \in R^n \times (0, 1), \\ z(x, 0) = f_{x_k}(x), & x \in R^n, \end{cases}$$

for all $k = 1, 2, \dots, n$. Since the above equation is nondegenerate by (1.16), by the maximum principle,

$$\|z\|_{L^\infty(R^n)} \le \|f_{x_k}\|_{L^\infty(R^n)} \quad \forall k = 1, 2, \dots, n$$

and the lemma follows.

**Lemma 1.6.** *Let $0 \le f \le M$ with $\operatorname{supp} f \subset B_{R_1}(0)$ for some $R_1 > 0$. Then there exists $R' > 0$ depending only on $m$, $R_1$, $M$ and is independent of $q > m + 1$ such that*

$$u^{(q)}(x, t) = 0 \quad \forall x = (x_1, \dots, x_n) \in R^n, x_1 \le -R', 0 \le t < 1, q > m + 1,$$

*and*

$$0 \le u^{(q)}(x, t) \le \left(\frac{x_1 + R' + 1}{t + (1/M^{q-1})}\right)^{1/q-1} \le \left(\frac{x_1 + R' + 1}{t}\right)^{1/q-1}$$

*for all* $x = (x_1, \dots, x_n) \in R^n$, $x_1 \ge -R'$, $0 < t < 1$, $q > m + 1$.

*Proof*. Let

$$w(x_1, t) = \frac{1}{(t + t_0)^{1/m+1}} \left(a^2 - C_1 \left(\frac{x_1}{(t + t_0)^{1/m+1}}\right)^2\right)_+^{1/m-1}, \qquad x_1 \in R, t \ge 0,$$

be the Barenblatt solution for the porous medium equation $w_t = (w^m)_{x_1 x_1}$ ([B], [HP]) where $C_1 = \frac{m-1}{2m}\left(\frac{1}{(m+1)}\right)$, $t_0 = \min\left(1, \left(\frac{4C_1 R_1^2}{2^{m-1/m+1} M^{m-1}}\right)^{(m+1)/2}\right)$ and

$$a = (C_1(2R_1/t_0^{1/m+1})^2 + (M t_0^{1/m+1})^{m-1})^{1/2}.$$

Then $w$ is a supersolution of (1.1) in $(-\infty, 0] \times R^{n-1} \times (0, 1)$ with

$$u^{(q)}(x + x_0) = f(x + x_0) \le M \le w(x_1, 0)$$

for all $x = (x_1, \dots, x_n) \in R^n$, $x_1 \le 0$ where $x_0 = (R_1, 0, \dots, 0)$ and

$$\begin{aligned} w(0, t) &\ge \frac{1}{(1 + t_0)^{1/m+1}} a^{2/m-1} \\ &\ge \frac{1}{2^{1/m+1}} \left(C_1 \left(\frac{2R_1}{t_0^{1/m+1}}\right)^2\right)^{1/m-1} \\ &\ge \frac{1}{2^{1/m+1}} \left(\frac{4R_1^2 C_1}{4C_1 R_1^2/(2^{1/m+1} M)^{m-1}}\right)^{1/m-1} \\ &= M \ge u(x_0, t) \end{aligned}$$

for all $0 < t < 1$ by (1.8). Hence by applying the maximum principle (Corollary 1.2) to the functions $u^{(q)}(\cdot + x_0, \cdot)$ and $w$ in the region $(-\infty, 0] \times R^{n-1} \times (0, 1)$, we get

$$u^{(q)}(x + x_0, t) \le w(x_1, t) \quad \forall x = (x_1, \dots, x_n) \in R^n, x_1 \le 0, 0 \le t < 1.$$

Now for each $0 < t < 1$, supp $w(x_1, t) \subset B_{R_t}(0)$ where

$$R_t = \frac{a}{C_1^{1/2}}(t + t_0)^{1/m+1} \le \frac{2a}{C_1^{1/2}} \quad (= R_2 \text{ say }).$$

Hence

$$\begin{aligned} u^{(q)}(x + x_0, t) \le w(x_1, t) = 0 \quad \forall x = (x_1, \dots, x_n) \in R^n, \\ x_1 \le -R_2, 0 \le t < 1, \end{aligned}$$

$$\begin{aligned} \Rightarrow u^{(q)}(x, t) = 0 \quad \forall x = (x_1, \dots, x_n) \in R^n, x_1 \le -R', 0 \le t < 1, \\ q > m + 1, \end{aligned}$$

(1.17)

where $R' = \max(R_2 - R_1, 0) \ge 0$.

We next observe that

$$\widetilde{w}(x_1, t) = \left(\frac{x_1 + R' + 1}{t + (1/M^{q-1})}\right)^{1/q-1}, \quad q > m+1,$$

is a supersolution of (1.1) in $[-R', R_3] \times R^{n-1} \times (0, \infty)$ with

$$\begin{cases} u^{(q)}(x, 0) \le M \le \widetilde{w}(x_1, 0) & \text{for } x = (x_1, \dots, x_n) \in R^n, -R' \le x_1 \le R_3, \\ u^{(q)}(x, t) \le M \le \widetilde{w}(x_1, t) & \text{for } x = (x_1, \dots, x_n) \in R^n, \\ & x_1 = -R' \text{ or } x_1 = R_3, 0 \le t < 1, \end{cases}$$

for all $R_3 > \max(2M^{q-1} - R' + 1, 0)$ by (1.17). Hence by applying Corollary 1.2 to the function $u^{(q)}$ and $\widetilde{w}$ in the region $[-R', R_3] \times R^{n-1} \times (0, 1)$, we get

$$u^{(q)}(x, t) \le \widetilde{w}(x_1, t)$$

for all $x = (x_1, \dots, x_n) \in [-R', R_3] \times R^{n-1}$, $0 \le t < 1$, $q > m+1$, $R_3 > \max(2M^{q-1} - R' + 1, 0)$. By letting $R_3 \to \infty$, the lemma follows.

**Lemma 1.7.** *Suppose $f$ is as in Lemma 1.6. Let $\Omega \subset R^n$ be a bounded open set with $\partial\Omega \in C^2$ and $\eta \in C^\infty(R^n \times (0, 1))$. Then*

$$\int_{\tau_1}^{\tau_2} \int_\Omega \frac{u^{(q)q}}{q} \cdot \eta dx dt \to 0 \quad \text{as } q \to \infty$$

*for any $0 < \tau_1 \le \tau_2 < 1$.*

*Proof.* By Lemma 1.6, there exists a constant $R' > 0$ such that

$$u^{(q)}(x, t) \le \left(\frac{|x_1| + R' + 1}{t}\right)^{1/q-1} \qquad \forall x = (x_1, \dots, x_n) \in R^n, 0 < t < 1, \quad q > m+1,$$

and by Theorem 1.3 $\|u^{(q)}\|_{L^\infty} \le \|f\|_{L^\infty}$ for all $q > m+1$. Hence

$$\begin{aligned} \int_{\tau_1}^{\tau_2} \int_\Omega \frac{u^{(q)q}}{q} \cdot \eta dx dt &= \int_{\tau_1}^{\tau_2} \int_\Omega \frac{u^{(q)q-2} u^{(q)2}}{q} \cdot \eta dx dt \\ &\le \|\eta\|_{L^\infty} \|f\|_{L^\infty}^2 \int_{\tau_1}^{\tau_2} \int_\Omega \frac{1}{q} \left(\frac{|x_1| + R' + 1}{t}\right)^{q-2/q-1} dx dt \\ &\le \frac{\|\eta\|_{L^\infty} \|f\|_{L^\infty}^2 |\Omega|}{q} \left(\frac{R'' + R' + 1}{\tau_1}\right)^{q-2/q-1} (\tau_2 - \tau_1) \to 0 \end{aligned}$$

as $q \to \infty$ where $R'' = \sup\{|x_1| : x = (x_1, \dots, x_n) \in \Omega\} < \infty$.

**Lemma 1.8.** *Let $f \in C_0(R^n)$ and let $p^{(q)}(x, t) = \int_0^t \frac{u^{(q)q}(x, \tau)}{q} d\tau$. Then $\{p^{(q)}\}_{q>m+1}$ is uniformly bounded on compact subsets of $R^n \times [0, 1)$. For any sequence $\{p^{(q_i)}\}_{i=1}^\infty$, $q_i \to \infty$ as $i \to \infty$, of $\{p^{(q)}\}_{q>m+1}$, there exists a subsequence $\{p^{(q_i')}\}_{i=1}^\infty$ of $\{p^{(q_i)}\}_{i=1}^\infty$, a sequence of functions $\{p_j\}_{j=1}^\infty \subset L^\infty_{\text{loc}}(R^n)$, $\widetilde{g} \in L^\infty_{\text{loc}}(R^n)$, $p_j, \widetilde{g} \ge 0$, and a sequence $\{\varepsilon_j\}_{j=1}^\infty \subset R$, $\varepsilon_j \to 0$ as $j \to \infty$, such that*

(1.18)

$$\begin{cases} p^{(q_i')}(\cdot, \varepsilon_j) \to p_j(\cdot) & \text{weakly in } (L^\infty(K))^* \text{ as } i \to \infty, \quad \forall j = 1, 2, \dots, \\ p_j(\cdot) \to \widetilde{g}(\cdot) & \text{weakly in } (L^\infty(K))^* \text{ as } j \to \infty \end{cases}$$

*for any compact subset $K \subset R^n$.*

*Proof*. By Theorem 1.3, $\|u^{(q)}\|_{L^\infty(R^n)} \le \|f\|_{L^\infty(R^n)}$ for all $q > m+1$ and by Lemma 1.6 there exists $R' > 0$ such that

$$0 \le u^{(q)}(x, \tau) \le \left(\frac{|x_1| + R' + 1}{\tau}\right)^{1/(q-1)} \qquad \forall x = (x_1, x_2, \dots, x_n) \in R^n, \quad 0 < \tau < 1, \, q > m+1.$$

Hence

$$\begin{aligned} 0 \le p^{(q)}(x, t) &= \int_0^t \frac{u^{(q)q-2}u^{(q)2}}{q} d\tau \\ &\le \frac{\|f\|^2_{L^\infty(R^n)}}{q} \int_0^t \left(\frac{|x_1| + R' + 1}{\tau}\right)^{q-2/q-1} d\tau \\ &\le \frac{q-1}{q} \|f\|^2_{L^\infty(R^n)} (|x_1| + R' + 1)^{q-2/q-1} t^{1/q-1} \\ &\le \|f\|^2_{L^\infty(R^n)} (|x_1| + R' + 1) \end{aligned}$$

for all $x = (x_1, x_2, \dots, x_n) \in R^n$, $0 < t < 1$, $q > m+1$. Thus $\{p^{(q)}\}_{q>m+1}$ is uniformly bounded on compact subsets of $R^n \times [0, 1)$. So any sequence $\{p^{(q_i)}\}_{i=1}^\infty$ of $\{p^{(q)}\}_{q>m+1}$ will have a subsequence $\{p^{(q_{1,i})}\}_{i=1}^\infty$ such that $\{p^{(q_{1,i})}(\cdot, 1/2)\}_{i=1}^\infty$ converges weakly in $(L^\infty(K))^*$ for any compact subset $K \subset R^n$.

Let $p_1(\cdot) = \lim_{i\to\infty} p^{(q_{1,i})}(\cdot, 1/2)$. Then $\{p^{(q_{1,i})}(\cdot, 1/2)\}$ has a subsequence $\{p^{(q'_{1,i})}(\cdot, 1/2)\}$ such that $p^{(q'_{1,i})}(x, 1/2) \to p_1(x)$ a.e. $x \in R^n$ as $i \to \infty$. Without loss of generality we may assume $p^{(q_{1,i})}(x, 1/2) \to p_1(x)$ a.e. $x \in R^n$ as $i \to \infty$. We may also assume that $q_1 < q_{1,1}$. Since $\{p^{(q_{1,i})}(\cdot, 1/3)\}_{i=1}^\infty$ is uniformly bounded on compact subsets of $R^n$, $\{p^{(q_{1,i})}(\cdot, 1/3)\}_{i=1}^\infty$ has a subsequence $\{p^{(q_{2,i})}(\cdot, 1/3)\}_{i=1}^\infty$ converging weakly in $(L^\infty(K))^*$ for any compact set $K \subset R^n$. Let $p_2(\cdot) = \lim_{i\to\infty} p^{(q_{2,i})}(\cdot, 1/3)$. We may assume without loss of generality that $p^{(q_{2,i})}(x, 1/3) \to p_2(x)$ a.e. $x \in R^n$ as $i \to \infty$ and $q_{1,1} < q_{2,1}$.

Repeating the argument, for each $j = 2, 3, \dots$, we can find a subsequence $\{p^{(q_{j,i})}(x, 1/(j+1))\}_{i=1}^\infty$ of $\{p^{(q_{j-1,i})}(x, 1/(j+1))\}_{i=1}^\infty$ with $q_{j,1} > q_{j-1,1}$ and a function $p_j \in L^\infty_{\mathrm{loc}}(R^n)$ such that $p^{(q_{j,i})}(x, 1/(j+1)) \to p_j(x)$ weakly in $(L^\infty(K))^*$ for every compact set $K \subset R^n$ as $i \to \infty$ and $p^{(q_{j,i})}(x, 1/(j+1)) \to p_j(x)$ a.e. $x \in R^n$ as $i \to \infty$.

Let $q'_i = q_{i,i}$. Then for each $j = 1, 2, \dots$, $\{p^{(q'_i)}(\cdot, 1/(j+1))\}_{i=1}^\infty$ is a subsequence of $\{p^{(q_{j,i})}(x, 1/(j+1))\}_{i=1}^\infty$. Hence $p^{(q'_i)}(x, 1/(j+1)) \to p_j(x)$ weakly in $(L^\infty(K))^*$ for every compact set $K \subset R^n$ as $i \to \infty$ and $p^{(q'_i)}(x, 1/(j+1)) \to p_j(x)$ a.e. $x \in R^n$ as $i \to \infty$. Thus $\{p_j\}_{j=1}^\infty$ is also uniformly bounded on every compact subset of $R^n$. So there exists a subsequence $\{p_{j_k}\}_{k=1}^\infty$ of $\{p_j\}_{j=1}^\infty$ and a function $\tilde{g} \in L^\infty_{\mathrm{loc}}(R^n)$ such that $p_{j_k} \to \tilde{g}$ weakly in $(L^\infty(K))^*$ for any compact subset $K \subset R^n$. Letting $\varepsilon_k = 1/(j_k + 1)$, the lemma follows.

## 2

In this section we will first establish some technical lemmas and prove the main theorem (Theorem 2.10) under the assumption that $f \in C^1_0(R^n)$ (Theorem 2.9) The main theorem will then follow by an approximation argument.

**Theorem 2.1.** *Suppose* $f \in C_0(R^n)$. *For any sequence* $\{u^{(q_i)}\}_{i=1}^{\infty}$, $q_i \to \infty$ *as* $i \to \infty$, *of* $\{u^{(q)}\}_{q>m+1}$, *there exists a subsequence* $\{u^{(q_i')}\}_{i=1}^{\infty}$ *of* $\{u^{(q_i)}\}_{i=1}^{\infty}$ *and a* $u^{(\infty)} \in C(R^n \times (0, 1))$, $0 \le u^{(\infty)} \le 1$, *such that* $u^{(q_i')} \to u^{(\infty)}$ *uniformly on compact subsets* $R \times (0, 1)$ *as* $i \to \infty$. *Moreover* $u^{(\infty)}$ *satisfies* (0.2) *with initial trace* $g \in L^1(R^n)$, $0 \le g \le 1$, *satisfying* (0.3) *for some function* $\tilde{g} \in L^{\infty}_{\mathrm{loc}}(R^n)$, $\tilde{g} \ge 0$.

*Proof*. The proof is a modification of the proof of Theorem 4 of [H1]. We first observe that $u^{(q)}$ is uniformly bounded by $\|f\|_{L^\infty}$ by Theorem 1.3 and there exists $R' > 0$ such that

(2.1)
$$0 \le u^{(q)}(x, t) \le \left(\frac{|x_1| + R' + 1}{t}\right)^{1/q-1} \quad \forall x = (x_1, x') \in R^n, \quad 0 < t < 1, q > m+1,$$

by Lemma 1.6. If $\gamma(s) = s^{q/m}/q$, then $\gamma(u^{(q)m}) = \frac{u^{(q)q}}{q}$ and

$$\begin{aligned}\gamma'(u^{(q)m}(x, t)) &= \frac{1}{m}(u^{(q)}(x, t))^{q-m} \\ &\le \frac{1}{m}\left(\frac{|x_1| + R' + 1}{t}\right)^{q-m/q-1} \\ &\le \frac{1}{m}\left(\frac{|x_1| + R' + 1}{t}\right) \quad \forall x = (x_1, x') \in R^n, 0 < t < 1, q > m+1,\end{aligned}$$

by (2.1). Hence both $u^{(q)m}$ and $\gamma'(u^{(q)m})$ are uniformly bounded on compact subsets of $R^n \times (0, 1)$ for $q > m+1$. By the result of P. Sacks [S1], $\{u^{(q)}\}_{q>m+1}^{\infty}$ is uniformly Hölder continuous on every compact subset of $R^n \times (0, 1)$. Hence $\{u^{(q_i)}\}_{i=1}^{\infty}$ has a convergent subsequence $\{u^{(q_i')}\}_{i=1}^{\infty}$ such that $\{u^{(q_i')}\}_{i=1}^{\infty}$ converges uniformly on every compact subset of $R^n \times (0, 1)$. Without loss of generality we may assume that $\{u^{(q_i)}\}_{i=1}^{\infty}$ converges uniformly on every compact subset of $R^n \times (0, 1)$. Let $u^{(\infty)} = \lim_{i\to\infty} u^{(q_i)}$. Then $u^{(\infty)} \in C(R^n \times (0, 1))$. Putting $q = q_i$ and letting $i \to \infty$ in (2.1), we get $0 \le u^{(\infty)} \le 1$. Putting $h(u) = u^{q_i}/q_i$, $u = u^{(q_i)}$ in (0.5) and letting $i \to \infty$ we see that, by Lemma 1.6, $u^{(\infty)}$ satisfies

$$\int_{\tau_1}^{\tau_2}\int_{\Omega}\left[u^m \Delta\eta + u\frac{\partial\eta}{\partial t}\right]dxdt = \int_{\tau_1}^{\tau_2}\int_{\partial\Omega} u^m \frac{\partial\eta}{\partial N} d\sigma ds + \int_{\Omega} u\eta dx\Big|_{\tau_1}^{\tau_2} \tag{2.2}$$

for all bounded open sets $\Omega \subset R^n$ with $\partial\Omega \in C^2$, $0 < \tau_1 \le \tau_2 < 1$, $\eta \in C^\infty(\Omega \times [\tau_1, \tau_2])$, $\eta \equiv 0$ on $\partial\Omega \times [\tau_1, \tau_2]$. Hence $u^{(\infty)}$ is a solution of the equation $u_t = \Delta u^m$ in $R^n \times (0, 1)$. Since $\|u^{(\infty)}\|_{L^\infty} \le \|f\|_{L^\infty}$, $u^{(\infty)}$ has an initial trace $d\mu$ by [DK] and $d\mu$ is absolutely continuous with respect to the Lebesgue measure. Hence $d\mu = g(x)dx$ for some function $g \ge 0$. Since $0 \le u^{(\infty)} \le 1$ and

$$\lim_{t\to 0} u^{(\infty)}(x, t) = g(x) \quad \text{a.e. } x \in R^n \tag{2.3}$$

by the result of [DFK], $0 \le g \le 1$. Since

$$\int_{R^n} u^{(q_i)}(x, t)dx = \int_{R^n} f(x)dx, \quad \forall 0 < t \le 1, i = 1, 2, \ldots.$$

Letting $i \to \infty$, we get by Fatou's lemma,

$$\int_{R^n} u^{(\infty)}(x, t)dx \le \int_{R^n} f(x)dx, \quad \forall 0 < t \le 1.$$

Letting $t \to 0$, we get by Fatou's lemma and (2.3),

$$\int_{R^n} g(x)dx \le \int_{R^n} f(x)dx.$$

Hence $g \in L^1(R^n)$. Let $p^{(q)}$ be as in Lemma 1.8 and $\Omega$ be a bounded open subset of $R^n$ with $\partial\Omega \in C^2$. Then by Lemma 1.8 there exists a constant $C_1 > 0$ such that $\|p^{(q)}\|_{L^\infty(\overline{\Omega}\times[0,1))} \le C_1$ for all $q > m+1$ and there exists a subsequence $\{p^{(q_i')}\}_{i=1}^\infty$ of $\{p^{(q_i)}\}_{i=1}^\infty$, a sequence of functions $\{p_j\}_{j=1}^\infty \subset L^\infty_{\text{loc}}(R^n)$, $\widetilde{g} \in L^\infty_{\text{loc}}(R^n)$, $p_j, \widetilde{g} \ge 0$, and a sequence $\{\varepsilon_j\}_{j=1}^\infty \subset R$, $\varepsilon_j \to 0$ as $j \to \infty$, such that (1.18) holds. Hence for any $0 < \tau_2 < 1$, $\eta \in C_0^\infty(R^n)$,

$$\begin{aligned}
\left|\int_0^{\tau_2}\int_\Omega \frac{u^{(q_i')q_i'}}{q_i'}\eta_{x_1}dxd\tau - \int_\Omega \widetilde{g}\eta_{x_1}dx\right| &\le \left|\int_\Omega\int_{\varepsilon_j}^{\tau_2} \frac{u^{(q_i')q_i'}}{q_i'}\eta_{x_1}dxd\tau\right| \\
&\quad + \left|\int_\Omega\left(\int_0^{\varepsilon_j} \frac{u^{(q_i')q_i'}(x,\tau)}{q_i'}d\tau\right)\eta_{x_1}(x)dx - \int_{R^n}\widetilde{g}\eta_{x_1}dx\right| \\
&\le \|\eta_{x_1}\|_{L^\infty(R^n)}\int_\Omega\int_{\varepsilon_j}^{\tau} \frac{u^{(q_i')q_i'}}{q_i'}dxd\tau \\
&\quad + \left|\int_\Omega p^{(q_i')}(x,\varepsilon_j)\eta_{x_1}(x)dx - \int_\Omega p_j(x)\eta_{x_1}(x)dx\right| \\
&\quad + \left|\int_\Omega p_j(x)\eta_{x_1}(x)dx - \int_\Omega \widetilde{g}(x)\eta_{x_1}(x)dx\right|.
\end{aligned}$$

Letting first $i \to \infty$ and then $j \to \infty$, we get by Lemma 1.7 and Lemma 1.8,

$$\limsup_{i\to\infty}\left|\int_0^{\tau_2}\int_\Omega \frac{u^{(q_i')q_i'}}{q_i'}\eta_{x_1}dxd\tau - \int_\Omega \widetilde{g}\eta_{x_1}dx\right| = 0$$

$$\Rightarrow \quad \lim_{i\to\infty}\int_0^{\tau_2}\int_\Omega \frac{u^{(q_i')q_i'}}{q_i'}\eta_{x_1}dxd\tau = \int_\Omega \widetilde{g}\eta_{x_1}dx. \tag{2.4}$$

Putting $h(u) = u^{q_i'}/q_i'$, $u = u^{(q_i')}$, in (0.5) and letting $\tau_1 \to 0$, we have

$$\begin{aligned}
&\int_0^{\tau_2}\int_{R^n} u^{(q_i')m}\Delta\eta dxd\tau + \int_0^{\tau_2}\int_{R^n} \frac{u^{(q_i')q_i'}}{q_i'}\eta_{x_1}dxd\tau \\
&\quad = \int_{R^n} u^{(q_i')}(x,\tau_2)\eta(x)dx - \int_{R^n} f\eta dx
\end{aligned}$$

for all $\eta \in C_0^\infty(R^n)$, $0 < \tau_2 < 1$. Letting $i \to \infty$, we get by (2.4) and Lebesgue dominated convergence theorem,

$$\int_0^{\tau_2}\int_{R^n} u^{(\infty)m}\Delta\eta dxd\tau + \int_{R^n}\widetilde{g}\eta_{x_1}dxd\tau = \int_{R^n} u^{(\infty)}(x,\tau_2)\eta(x)dx - \int_{R^n} f\eta dx$$

for all $\eta \in C_0^\infty(R^n)$. Letting $\tau_2 \to 0$,

$$\begin{aligned}
&\int \widetilde{g}\eta_{x_1}dx = \int g\eta dx - \int f\eta dx \quad \forall \eta \in C_0^\infty(R^n) \\
&\quad \Rightarrow g + \widetilde{g}_{x_1} = f \quad \text{in } \mathscr{D}'(R^n).
\end{aligned}$$

This completes the proof of Theorem 2.1.

We will now let

$$S(g) = \left\{ x_0 \in R^n : \lim_{h \to 0} \frac{1}{|B_h(0)|} \int_{B_h(x_0)} |g(x) - g(x_0)| dx = 0 \right\},$$

$$G(u^{(\infty)}, g) = \left\{ x \in R^n : \lim_{t \to 0} u^{(\infty)}(x, t) = g(x) \right\}.$$

**Lemma 2.2.** *Let $f$, $u^{(\infty)}$, $u^{(q_i')}$, $g$ be as in Theorem 2.1 and let $S^* = S(g) \cap G(u^{(\infty)}, g) \cap \{g < 1\}$. If $x_0 \in S^*$ is such that $g(x_0) \le \theta < 1$, then for any $\theta_1 \in (\theta, 1)$ and $\delta > 0$, there exists $q_0 > m + 1$, $\varepsilon_0 > 0$, $0 < \varepsilon_0 < 1/2$, such that*

$$\inf_{|x - x_0| \le \delta} u^{(q_i')}(x, t) \le \theta_1 \quad \forall 0 < t \le \varepsilon_0, \, q_i' \ge q_0.$$

*Proof.* The proof is similar to the proof of Theorem 3.3 of [CF]. Suppose the lemma is not true. Then there exists $\theta_1 \in (\theta, 1)$, $\delta > 0$, and $\{\varepsilon_i\}_{i=1}^{\infty}$, $0 < \varepsilon_i < 1/2$, $i = 1, 2, \ldots$, $\varepsilon_i \to 0$ as $i \to \infty$ and a subsequence $\{u^{(q_i'')}\}_{i=1}^{\infty}$ of $\{u^{(q_i')}\}_{i=1}^{\infty}$ such that

$$\inf_{|x - x_0| \le \delta} u^{(q_i'')}(x, \varepsilon_i) > \theta_1.$$

Let $\tilde{u}^{(q_i'')}$ be the solution of (1.1) in $R^n \times (0, 1)$ with initial value $\tilde{u}^{(q_i'')}(x, 0) = \theta_1 \chi_{B_\delta(x_0)}$ where $\chi_{B_\delta(x_0)}$ is the characteristic function of the set $B_\delta(x_0)$. By Theorem 1.1,

$$\begin{aligned} &\tilde{u}^{(q_i'')}(x, t) \le u^{(q_i'')}(x, t + \varepsilon_i) \quad \forall x \in R^n, \, 0 < t \le 1/2 \\ &\quad \Rightarrow \iint \tilde{u}^{(q_i'')}(x, t) \eta(x, t) dx dt \le \iint u^{(q_i'')}(x, t + \varepsilon_i) \eta(x, t) dx dt \\ &\quad = \iint u^{(q_i'')}(x, t) \eta(x, t - \varepsilon_i) dx dt \qquad (2.5) \end{aligned}$$

for all $\eta \in C_0^\infty(R^n \times (0, 1/2))$ and $\varepsilon_i$ sufficiently small. By Theorem 2.1, $\{\tilde{u}^{(q_i'')}\}_{i=1}^{\infty}$ has a convergent subsequence converging uniformly on compact subsets of $R^n \times (0, 1)$. Without loss of generality, we may assume that $\{\tilde{u}^{(q_i'')}\}_{i=1}^{\infty}$ converges uniformly on compact subsets of $R^n \times (0, 1)$. Let $\tilde{u}^{(\infty)} = \lim_{i \to \infty} \tilde{u}^{(q_i'')}$. Since $0 \le u^{(q_i'')} \le \theta_1 < 1$, letting $i \to \infty$ in (2.5), we get by Lebesgue dominated convergence theorem

$$\begin{aligned} &\iint \tilde{u}^{(\infty)}(x, t) \eta(x, t) dx dt \le \iint u^{(\infty)}(x, t) \eta(x, t) dx dt \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \eta \in C_0^\infty(R^n \times (0, 1/2)) \\ &\Rightarrow \tilde{u}^{(\infty)}(x, t) \le u^{(\infty)}(x, t) \quad \forall x \in R^n, \, 0 < t < 1/2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{since } u^{(\infty)}, \tilde{u}^{(\infty)} \in C(R^n \times (0, 1/2)) \\ &\Rightarrow \int_{R^n} \tilde{u}^{(\infty)}(x, t) \eta(x) dx \le \int_{R^n} u^{(\infty)}(x, t) \eta(x) dx \qquad \forall \eta \in C_0^\infty(R^n) \\ &\Rightarrow \int_{R^n} \theta_1 \chi_{B_\delta(x_0)}(x) dx \le \int_{R^n} g(x) \eta(x) dx \qquad \text{as } t \to 0 \quad \forall \eta \in C_0^\infty(R^n) \\ &\Rightarrow \theta < \theta_1 \le g(x_0) \end{aligned}$$

since $x_0 \in S(g)$. Thus contradiction arise and the lemma follows.

**Lemma 2.3.** *Suppose* $f \in C_0^1(R^n)$. *Let* $u^{(\infty)}$, $u^{(q_i')}$, $g$ *be as in Theorem* 2.1 *and let* $S^*$ *be as in Lemma* 2.2. *If* $x_0 \in S^*$ *is such that* $g(x_0) \le \theta < 1$, *then for any* $\theta_1 \in (\theta, 1)$, *there exists* $q_0 > m+1$, $\varepsilon_0 \in (0, 1/2)$ *depending only on* $\theta$, $\theta_1$ *and* $\|f_{x_k}\|_{L^\infty(R^n)}$, $k = 1, 2, \dots, n$, *such that*

$$u^{(q_i')}(x, t) \le \theta_1 \quad \forall x \in B_\delta(x_0), 0 < t \le \varepsilon_0, q_i' \ge q_0,$$

*where* $\delta = (\theta_1 - \theta)/4(\sqrt{n}\max_{1\le k\le n}\|f_{x_k}\|_{L^\infty(R^n)} + 1)$.

*Proof.* The proof is similar to the proof of Theorem 2.4 of [H2]. Let

$$\delta = (\theta_1 - \theta)/4(\sqrt{n}\max_{1\le k\le n}\|f_{x_k}\|_{L^\infty(R^n)} + 1).$$

Then by Lemma 2.2, there exists $q_0 > m+1$, $\varepsilon_0 > 0$, $0 < \varepsilon_0 < 1/2$, such that

$$\inf_{|x-x_0|\le\delta} u^{(q_i')}(x, t) \le \frac{\theta_1 + \theta}{2} \quad \forall 0 < t \le \varepsilon_0, q_i' \ge q_0.$$

Hence for each $q_i' \ge q_0$ and $0 < t \le \varepsilon_0$, there exists an $x_t \in \overline{B_\delta(x_0)}$ such that

$$u^{(q_i')}(x_t, t) \le \frac{\theta_1 + \theta}{2} \quad \forall 0 < t \le \varepsilon_0.$$

For any $0 < \varepsilon < 1$, let $f_\varepsilon = f + \varepsilon$ and let $u_\varepsilon^{(q)}$ be the solution of (1.1) in $R^n \times (0, 1)$ with $u_\varepsilon^{(q)}(x, 0) = f_\varepsilon(x)$ given by Lemma 1.5. Then by Lemma 1.5,

$$\begin{aligned}
&|u_\varepsilon^{(q_i')}(x, t) - u_\varepsilon^{(q_i')}(x_t, t)| \\
&\quad = \left|\int_0^1 \frac{d}{ds} u_\varepsilon^{(q_i')}(sx + (1-s)x_t, t)ds\right| \\
&\quad \le \int_0^1 |\nabla u_\varepsilon^{(q_i)}(sx + (1-s)x_t, t)\cdot(x - x_t)|ds \\
&\quad \le \sqrt{n}\max_{1\le k\le n}\|f_{x_k}\|_{L^\infty(R^n)}|x - x_t| \\
&\quad \le 2\delta\sqrt{n}\max_{1\le k\le n}\|f_{x_k}\|_{L^\infty(R^n)} \le (\theta_1 - \theta)/2 \\
&\quad \Rightarrow u_\varepsilon^{(q_i')}(x, t) \le u_\varepsilon^{(q_i')}(x_t, t) + (\theta_1 - \theta)/2 \le (\theta_1 + \theta)/2 + (\theta_1 - \theta)/2 \le \theta_1
\end{aligned}$$

for all $x \in B_\delta(x_0)$, $0 < t \le \varepsilon_0$, $q_i' \ge q_0$. Since $u_\varepsilon^{(q_i')} \to u^{(q_i')}$ uniformly on compact subsets of $R^n \times (0, 1)$ as $\varepsilon \to 0$ by Lemma 1.5, letting $\varepsilon \to 0$ we get

$$u^{(q_i')}(x, t) \le \theta_1 \quad \forall x \in B_\delta(x_0), 0 < t \le \varepsilon_0, q_i' \ge q_0.$$

**Lemma 2.4.** *Suppose* $f \in C_0^1(R^n)$. *Let* $g$, $\widetilde{g}$ *be as in Theorem* 2.1 *and let* $S^*$ *be as in Lemma* 2.2. *Then*

$$\begin{cases} g(x) = f(x), \\ \widetilde{g}(x) = 0 \end{cases}$$

*for all* $x \in S^* \cap S(\widetilde{g})$.

*Proof.* Let $u^{(\infty)}$, $u^{(q_i)}$ be as in Theorem 2.1. By Theorem 2.1 we may assume without loss of generality that $u^{(q_i)}$ converges uniformly to $u^{(\infty)}$ on compact subsets of $R^n \times (0, 1)$ as $i \to \infty$. We also let $p^{(q_i)}$, $p^{(q_i')}$, $p_j$, $\varepsilon_j$ be as in Lemma 1.8. Suppose $x_0 \in S^* \cap S(\widetilde{g})$. Then there exists $\theta$, $\theta_1 > 0$ such that

$g(x_0) \le \theta < \theta_1 < 1$. By Lemma 2.3, there exists $q_0 > m+1$, $\delta > 0$, $\varepsilon_0 > 0$, $0 < \varepsilon_0 < 1/2$ such that

$$u^{(q_i')}(x, t) \le \theta_1 \quad \forall x \in B_\delta(x_0),\ 0 < t \le \varepsilon_0,\ q_i' \ge q_0.$$

Hence

$$\begin{aligned}
&\left| \int_{R^n} u^{(q_i')}(x, t)\eta(x)dx - \int_{R^n} f(x)\eta(x)dx \right| \\
&= \left| \int_0^t \int_{R^n} [u^{(q_i')m}\Delta\eta + \frac{u^{(q_i')q_i'}}{q_i'}\eta_{x_1}]dxd\tau \right| \\
&\le \theta_1^m \|\Delta\eta\|_{L^1(R^n)} t + \frac{\theta_1^{q_i'}}{q_i'} \|\eta_{x_1}\|_{L^1(R^n)} t \quad \forall q_i' \ge q_0,\ 0 < t \le \varepsilon_0,\ \eta \in C_0^\infty(B_\delta(x_0)).
\end{aligned}$$

Letting $i \to \infty$,

$$\begin{aligned}
&\left| \int_{R^n} u^{(\infty)}(x, t)\eta(x)dx - \int_{R^n} f(x)\eta(x)dx \right| \\
&\le \theta_1^m \|\Delta\eta\|_{L^1(R^n)} t + \frac{\theta_1^{q_i'}}{q_i'} \|\eta_{x_1}\|_{L^1(R^n)} t \quad \forall q_i' \ge q_0,\ 0 < t \le \varepsilon_0,\ \eta \in C_0^\infty(B_\delta(x_0)).
\end{aligned}$$

Letting $t \to 0$,

$$\int_{R^n} g\eta dx = \int_{R^n} f\eta dx \quad \forall \eta \in C_0^\infty(B_\delta(x_0)) \Rightarrow g(x_0) = f(x_0)$$

since $x_0 \in S(g)$. Similarly

$$\begin{aligned}
&\int_{B_\delta(x_0)} p^{(q_i')}(x, \varepsilon_j)dx \\
&\quad = \int_{B_\delta(x_0)} \int_0^{\varepsilon_j} \frac{u^{(q_i')q_i'}}{q_i'} d\tau dx \le \frac{\theta_1^{q_i'}}{q_i'} |B_\delta(x_0)|\varepsilon_j \to 0 \text{ as } i \to 0 \quad \forall j = 1, 2, \ldots \\
&\quad \Rightarrow \int_{B_\delta(x_0)} p_j(x)dx = 0 \qquad \text{by Fatou's lemma since } p_j \ge 0 \\
&\quad \Rightarrow \int_{B_\delta(x_0)} \widetilde{g}(x)dx = 0 \qquad \text{by Fatou's lemma since } \widetilde{g} \ge 0 \\
&\quad \Rightarrow \widetilde{g} \equiv 0 \text{ on } B_\delta(x_0) \\
&\quad \Rightarrow \widetilde{g}(x_0) = 0 \text{ since } x_0 \in S(\widetilde{g}).
\end{aligned}$$

**Lemma 2.5.** *Suppose $f \in C_0^1(R^n)$ and let $g$, $\widetilde{g}$ be as in Theorem 2.1. Then there exists $r' > 0$ such that*

$$\begin{cases} g(x) = f(x), \\ \widetilde{g}(x) = 0 \end{cases} \tag{2.6}$$

*a.e.* $x \in R^n \setminus B_{r'}(0)$

*Proof.* Let $u^{(\infty)}$, $u^{(q_i')}$ be as in Theorem 2.1, $S^*$ be as in Lemma 2.2 and let $S_1 = S(g) \cap S(\widetilde{g}) \cap G(u^{(\infty)}, g)$. $S_2 = S(g) \cap G(u^{(\infty)}, g)$. For any $0 < \theta < 1$, $r > 0$, let $A_{\theta,r} = \{x \in R^n \setminus B_r(0) : g(x) \ge \theta\}$. We now fix $\theta$, $\theta_1 \in (0, 1)$ such that $\theta < \theta_1$. Choose a constant $\theta' > 0$ such that $\theta < \theta' < \theta_1$ and let

$$\delta = \min((\theta' - \theta)/4(\sqrt{n} \max_{1 \le k \le n} \|f_{x_k}\|_{L^\infty(R^n)} + 1), 1).$$

Since $g \in L^1(R^n)$,

$$\int_{|x|\geq r} g dx \to 0 \text{ as } r \to 0.$$

Thus there exists $r_0 > 0$ such that

$$\begin{aligned} &\int_{|x|\geq r_0} g dx \leq \frac{1}{2}\theta |B_\delta(0)| \\ &\Rightarrow \theta |A_{\theta, r_0}| \leq \frac{1}{2}\theta |B_\delta(0)| \\ &\Rightarrow |A_{\theta, r_0}| \leq \frac{1}{2}|B_\delta(0)|. \end{aligned} \tag{2.7}$$

Let $r' = r_0 + 1$. Since $|R^n \setminus S_1| = 0$ by the result of [DFK] and Chapter 1 of [S], (2.6) holds for a.e. $x \in A^c_{\theta, r'}$ by Lemma 2.4. Hence in order to prove the lemma, it suffices to show that (2.6) holds for a.e. $x \in A_{\theta, r'} \cap S_1$. Let $y_0 \in A_{\theta, r'} \cap S_1$. If $|B_\delta(y_0) \cap A^c_{\theta, r'}| = 0$, then

$$g(z) \geq \theta \text{ a.e. } z \in B_\delta(y_0) \Rightarrow |A_{\theta, r_0}| \geq |B_\delta(y_0)|$$

since $B_\delta(y_0) \subset R^n \setminus B_{r_0}(0)$. This contradicts (2.7). Thus $|B_\delta(y_0) \cap A^c_{\theta, r'}| \neq 0$. Since $|(B_\delta(y_0) \cap A^c_{\theta, r'}) \setminus (B_\delta(y_0) \cap A^c_{\theta, r'} \cap S_2)| = 0$, $B_\delta(y_0) \cap A^c_{\theta, r'} \cap S_2 \neq \varnothing$ and there exists $x_0 \in B_\delta(y_0) \cap A^c_{\theta, r'} \cap S_2 \subset S^*$. By Lemma 2.3, there exists $q_0 > m + 1$ and $\varepsilon_0 > 0$, $0 < \varepsilon_0 < 1/2$, such that

$$u^{(q_i')}(x, t) \leq \theta' \quad \forall x \in B_\delta(x_0), 0 < t \leq \varepsilon_0, q_i' \geq q_0.$$

Letting $i \to \infty$,

$$\begin{aligned} &u^{(\infty)}(x, t) \leq \theta' \quad \forall x \in B_\delta(x_0), 0 < t \leq \varepsilon_0 \\ &\Rightarrow \int_{R^n} u^{(\infty)}(x, t)\eta(x) dx \leq \theta' \int_{R^n} \eta dx \quad \forall \eta \in C_0(B_\delta(x_0)) \\ &\Rightarrow \int_{R^n} g\eta dx \leq \theta' \int_{R^n} \eta dx \quad \forall \eta \in C_0(B_\delta(x_0)) \qquad \text{as } t \to 0 \\ &\Rightarrow g(y_0) \leq \theta' < 1 \end{aligned}$$

since $y_0 \in S(g) \cap B_\delta(x_0)$. Hence $y_0 \in S^* \cap S(\widetilde{g})$. Thus (2.6) holds for $x = y_0$ by Lemma 2.4 and the lemma follows.

**Corollary 2.6.** *Suppose $f \in C_0^1(R^n)$ and $\widetilde{g}$ is as in Theorem 2.1. Then $\widetilde{g} \in L^1(R^n)$.*

*Proof.* The lemma follows directly from Lemma 2.5 and the fact that $\widetilde{g} \in L^\infty_{\text{loc}}(R^n)$.

**Lemma 2.7.** *For any $0 \leq f_1, f_2, g_1, g_2, \widetilde{g}_1, \widetilde{g}_2 \in L^1(R^n)$, $0 \leq g_1, g_2 \leq 1$, $\widetilde{g}_1, \widetilde{g}_2 \geq 0$, if*

$$g_i + (\widetilde{g}_i)_{x_1} = f_i \qquad \text{in } \mathscr{D}'(R^n) \tag{2.8}$$

*and*

$$g_i(x) = f_i(x), \widetilde{g}_i(x) = 0 \qquad \text{whenever } g_i(x) < 1 \text{ a.e. } x \in R^n \tag{2.9}$$

*for $i = 1, 2$, then*

$$\int_{|x_1|\leq R'} \int_{R^{n-1}} |\widetilde{g}_1 - \widetilde{g}_2|(x_1, x') dx' dx_1 \leq 2R' \|f_1 - f_2\|_{L^1(R^n)} \quad \forall R' > 0.$$

*Proof*. We will use a modification of an argument of [SX]. By (2.8),

$$(g_1 - g_2) + (\tilde{g}_1 - \tilde{g}_2)_{x_1} = f_1 - f_2 \qquad \text{in } \mathscr{D}'(R^n)$$

$$\Rightarrow \int_{R^n} [(g_1 - g_2)\eta - (\tilde{g}_1 - \tilde{g}_2)\eta_{x_1}]dx \tag{2.10}$$

$$= \int_{R^n} (f_1 - f_2)\eta dx \quad \forall \eta \in C_0^\infty(R^n).$$

Putting $\eta(x) = \rho_\varepsilon(\xi - x)$ in (2.10), we get

(2.11)

$$(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})_{\xi_1}(\xi) = (f_{1,\varepsilon} - f_{2,\varepsilon})(\xi) - (g_{1,\varepsilon} - g_{2,\varepsilon})(\xi) \quad \forall \xi = (\xi_1, \dots, \xi_n) \in R^n.$$

For any $k = 1, 2, \dots$, we let $p_k(\cdot) \in C_0^\infty(R)$, $0 \le p_k \le 1$, be such that $p_k(x) \equiv 1$ for $x \ge 1/k$, $p_k(x) \equiv 0$ for $x \le 1/2k$ and $\|p_{k,x}\|_{L^\infty} \le 5k$. Then for all $z_1, y_1 \in R$,

$$\int_{R^{n-1}} (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(z_1, x')p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(z_1, x')dx'$$

$$- \int_{R^{n-1}} (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(y_1, x')p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(y_1, x')dx'$$

$$= \int_{R^{n-1}} \int_{y_1}^{z_1} \frac{\partial}{\partial x_1}\left[(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(x_1, x')p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(x_1, x')\right]dx_1 dx'$$

$$= \int_{R^{n-1}} \int_{y_1}^{z_1} \left[\left(\frac{\partial}{\partial x_1}(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})\right)p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})\right.$$

$$\left. + (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})\frac{\partial}{\partial x_1}(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})\right]dx_1 dx'$$

$$\Rightarrow \int_{R^{n-1}} (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(z_1, x')p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(z_1, x')dx' \tag{2.12}$$

$$+ \int_{R^{n-1}} \int_{y_1}^{z_1} (g_{1,\varepsilon} - g_{2,\varepsilon})(x_1, x')p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(x_1, x')dx_1 dx'$$

$$= \int_{R^{n-1}} (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(y_1, x')p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(y_1, x')dx'$$

$$+ \int_{R^{n-1}} \int_{y_1}^{z_1} (f_{1,\varepsilon} - f_{2,\varepsilon})(x_1, x')p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(x_1, x')dx_1 dx'$$

$$+ \int_{R^{n-1}} \int_{y_1}^{z_1} (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})$$

$$\cdot [(f_{1,\varepsilon} - f_{2,\varepsilon}) - (g_{1,\varepsilon} - g_{2,\varepsilon})](x_1, x')dx_1 dx'$$

by (2.11). Since $\widetilde{g_1}, \widetilde{g_2} \in L^1(R^n)$,

$$\int_{R^n} |\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}| \cdot |p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})|dx \le \int_{R^n} (\tilde{g}_1 + \tilde{g}_2)dx < \infty.$$

Hence there exists a sequence $\{y_1^j\}_{j=1}^\infty \subset R$, $y_1^j \to -\infty$ as $j \to \infty$ such that

$$\int_{R^{n-1}} (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(y_1^j, x')p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(y_1^j, x')dx' \to 0 \qquad \text{as } j \to \infty.$$

Putting $y_1 = y_1^j$ in (2.12) and letting $j \to \infty$, we get

$$\begin{aligned}
&\int_{R^{n-1}} (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(z_1, x') p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(z_1, x') dx' \\
&\qquad + \int_{R^{n-1}} \int_{-\infty}^{z_1} (g_{1,\varepsilon} - g_{2,\varepsilon})(x_1, x') p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(x_1, x') dx_1 dx' \\
&\quad = \int_{R^{n-1}} \int_{-\infty}^{z_1} (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) \\
&\qquad\qquad \cdot [(f_{1,\varepsilon} - f_{2,\varepsilon}) - (g_{1,\varepsilon} - g_{2,\varepsilon})](x_1, x') dx_1 dx' \\
&\qquad + \int_{R^{n-1}} \int_{-\infty}^{z_1} (f_{1,\varepsilon} - f_{2,\varepsilon})(x_1, x') p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(x_1, x') dx_1 dx'. \qquad (2.13)
\end{aligned}$$

Since $\tilde{g}_1, \tilde{g}_2 \in L^1(R^n)$,

$$\begin{aligned}
&\int_R \left| \int_{R^{n-1}} (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(z_1, x') p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(z_1, x') dx' \right. \\
&\qquad\qquad \left. - \int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)(z_1, x') p_k(\tilde{g}_1 - \tilde{g}_2)(z_1, x') dx' \right| dz_1 \\
&\quad \le \int_{R^n} |(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) - (\tilde{g}_1 - \tilde{g}_2)| p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) dx \\
&\qquad + \int_{R^n} (\tilde{g}_1 - \tilde{g}_2) \cdot |p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) - p_k(\tilde{g}_1 - \tilde{g}_2)| dx \\
&\quad \le \int_{R^n} |\tilde{g}_{1,\varepsilon} - \tilde{g}_1| dx + \int_{R^n} |\tilde{g}_{2,\varepsilon} - \tilde{g}_2| dx \\
&\qquad + \int_{R^n} (\tilde{g}_1 + \tilde{g}_2) \cdot |p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) - p_k(\tilde{g}_1 - \tilde{g}_2)| dx \\
&\quad \to 0 \qquad \text{as } \varepsilon \to 0
\end{aligned}$$

by the Lebesgue dominated convergence theorem and Theorem 2 in Chapter 3 of [S]. Hence there exists a sequence $\{\varepsilon_j\}_{j=1}^{\infty} \subset R$, $\varepsilon_j \to 0$ as $j \to \infty$, such that

$$\begin{aligned}
&\int_{R^{n-1}} (\tilde{g}_{1,\varepsilon_j} - \tilde{g}_{2,\varepsilon_j})(z_1, x') p_k(\tilde{g}_{1,\varepsilon_j} - \tilde{g}_{2,\varepsilon_j})(z_1, x') dx' \\
&\qquad \to \int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)(z_1, x') p_k(\tilde{g}_1 - \tilde{g}_2)(z_1, x') dx'
\end{aligned}$$

a.e. $z_1 \in R$ as $j \to \infty$. On the other hand,

$$\begin{aligned}
&\Bigg| \int_{R^{n-1}} \int_{-\infty}^{z_1} (\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) \\
&\qquad \cdot [(g_{1,\varepsilon} - g_{2,\varepsilon}) - (f_{1,\varepsilon} - f_{2,\varepsilon})](x_1, x') dx_1 dx' \\
&\quad - \int_{R^{n-1}} \int_{-\infty}^{z_1} (\tilde{g}_1 - \tilde{g}_2) p_k'(\tilde{g}_1 - \tilde{g}_2)[(g_1 - g_2) - (f_1 - f_2)](x_1, x') dx_1 dx' \Bigg| \\
&\le \int_{R^n} |(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})| |(g_{1,\varepsilon} - g_{2,\varepsilon}) - (g_1 - g_2)| dx \\
&\quad + \int_{R^n} |(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})| |(f_{1,\varepsilon} - f_{2,\varepsilon}) - (f_1 - f_2)| dx \\
&\quad + \int_{R^n} |(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) - (\tilde{g}_1 - \tilde{g}_2) p_k'(\tilde{g}_1 - \tilde{g}_2)| |g_1 - g_2| dx \\
&\quad + \int_{R^n} |(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) - (\tilde{g}_1 - \tilde{g}_2) p_k'(\tilde{g}_1 - \tilde{g}_2)| |f_1 - f_2| dx \\
&\le 5 \int_{R^n} (|g_{1,\varepsilon} - g_1| + |g_{2,\varepsilon} - g_2| + |f_{1,\varepsilon} - f_1| + |f_{2,\varepsilon} - f_2|) dx \\
&\quad + \int_{R^n} |(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) p_k'(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon}) - (\tilde{g}_1 - \tilde{g}_2) p_k'(\tilde{g}_1 - \tilde{g}_2)| \\
&\qquad \cdot (g_1 + g_2 + f_1 + f_2) dx \\
&\to 0 \qquad \text{as } \varepsilon \to 0
\end{aligned}$$

by the Lebesgue dominated convergence theorem since the integrand of the last integral above is bounded by $5(g_1 + g_2 + f_1 + f_2) \in L^1(R^n)$ and tends to $0$ as $k \to \infty$. Similarly

$$\begin{aligned}
&\int_{R^{n-1}} \int_{-\infty}^{z_1} (g_{1,\varepsilon} - g_{2,\varepsilon})(x_1, x') p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(x_1, x') dx_1 dx' \\
&\qquad \to \int_{R^{n-1}} \int_{-\infty}^{z_1} (g_1 - g_2)(x_1, x') p_k(\tilde{g}_1 - \tilde{g}_2)(x_1, x') dx_1 dx' \qquad \text{as } \varepsilon \to 0
\end{aligned}$$

and

$$\begin{aligned}
&\int_{R^{n-1}} \int_{-\infty}^{z_1} (f_{1,\varepsilon} - f_{2,\varepsilon})(x_1, x') p_k(\tilde{g}_{1,\varepsilon} - \tilde{g}_{2,\varepsilon})(x_1, x') dx_1 dx' \\
&\qquad \to \int_{R^{n-1}} \int_{-\infty}^{z_1} (f_1 - f_2)(x_1, x') p_k(\tilde{g}_1 - \tilde{g}_2)(x_1, x') dx_1 dx' \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

Putting $\varepsilon = \varepsilon_j$ in (2.13) and letting $j \to \infty$, we get

(2.14)

$$\begin{aligned}
&\int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)(z_1, x') p_k (\tilde{g}_1 - \tilde{g}_2)(z_1, x') dx' \\
&\quad + \int_{R^{n-1}} \int_{-\infty}^{z_1} (g_1 - g_2)(x_1, x') p_k (\tilde{g}_1 - \tilde{g}_2)(x_1, x') dx_1 dx' \\
&= \int_{R^{n-1}} \int_{-\infty}^{z_1} (\tilde{g}_1 - \tilde{g}_2) p_k' (\tilde{g}_1 - \tilde{g}_2)[(f_1 - f_2) - (g_1 - g_2)](x_1, x') dx_1 dx' \\
&\quad + \int_{R^{n-1}} \int_{-\infty}^{z_1} (f_1 - f_2)(x_1, x') p_k (\tilde{g}_1 - \tilde{g}_2)(x_1, x') dx_1 dx' \\
&\le I_1 + \int_{R^n} (f_1 - f_2)_+ dx \qquad \text{a.e. } z_1 \in R
\end{aligned}$$

Since $p_k'(s) = 0$ for $s \le 1/2k$ or $s \ge 1/k$, $I_1$ is bounded by

$$\begin{aligned}
&\int_{R^n} |(\tilde{g}_1 - \tilde{g}_2)(x)| \, \|p_k'\|_{L^\infty} \cdot (g_1 + g_2 + f_1 + f_2)(x) \cdot \chi_{A_k}(x_1, x') dx \\
&\quad \le \int_{R^n} \frac{1}{k} \cdot 5k \cdot (g_1 + g_2 + f_1 + f_2)(x) \cdot \chi_{A_k}(x) dx \\
&\quad \le 5 \int_{R^n} (g_1 + g_2 + f_1 + f_2)(x) \cdot \chi_{A_k}(x) dx \\
&\quad \to 0 \qquad \text{as } k \to \infty
\end{aligned}$$

by the Lebesgue dominated convergence theorem since $g_1, g_2, f_1, f_2 \in L^1(R^n)$ and

$$(g_1 + g_2 + f_1 + f_2)(x) \chi_{A_k}(x) \to 0 \quad \text{as } k \to \infty \text{ a.e. } x \in R^n$$

where $A_k = \{x \in R^n : 1/2k \le (g_1 - g_2)(x) \le 1/k\}$. Hence by letting $k \to \infty$ in (2.14), we get

$$\begin{aligned}
&\int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)_+ (z_1, x')(z_1, x') dx' \\
&\quad + \int_{R^{n-1}} \int_{-\infty}^{z_1} (g_1 - g_2)(x_1, x') \operatorname{sign}_+ (\tilde{g}_1 - \tilde{g}_2)(x_1, x') dx_1 dx' \\
&\quad \le \int_{R^n} (f_1 - f_2)_+ dx
\end{aligned}$$

a.e. $z_1 \in R$. Since $(g_1 - g_2)(x) \operatorname{sign}_+ (\tilde{g}_1(x) - \tilde{g}_2(x)) \ge 0$ a.e. $x \in R^n$ by (2.9),

$$\int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)_+ (z_1, x')(z_1, x') dx' \le \int_{R^n} (f_1 - f_2)_+ dx \quad \text{a.e. } z_1 \in R$$

$$\Rightarrow \int_{|x_1| \le R'} \int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)_+ (x_1, x') dx' dx_1 \le 2R' \int_{R^n} (f_1 - f_2)_+ dx \quad \forall R' > 0.$$

Similarly

$$\int_{|x_1| \le R'} \int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)_- (x_1, x') dx' dx_1 \le 2R' \int_{R^n} (f_1 - f_2)_- dx \quad \forall R' > 0.$$

Thus

$$\int_{|x_1| \le R'} \int_{R^{n-1}} |\tilde{g}_1 - \tilde{g}_2| (x_1, x')(x_1, x') dx' dx_1 \le 2R' \int_{R^n} |f_1 - f_2| dx \quad \forall R' > 0.$$

**Corollary 2.8.** *Let $0 \le f \in L^1(R^n)$. Then there exists at most one function $g$, $g \in L^1(R^n)$, $0 \le g \le 1$, and one function $\tilde{g} \in L^1(R^n)$, $\tilde{g} \ge 0$ satisfying*

$$\begin{cases} g + (\tilde{g})_{x_1} = f & \text{in } \mathscr{D}'(R^n), \\ g(x) = f(x),\ \tilde{g}(x) = 0 & \text{whenever } g(x) < 1 \text{ a.e. } x \in R^n. \end{cases} \tag{2.15}$$

As a consequence of Theorem 2.1, Lemmas 2.4, 2.5, Corollary 2.8 and the uniqueness theorem (Theorem 6.13) of [DK], we have

**Theorem 2.9.** *Suppose $f \in C_0^1(R^n)$. Then there exists a unique function $u^{(\infty)} \in C(R^n \times (0,1))$, $0 \le u^{(\infty)} \le 1$, such that $u^{(q)}$ converges uniformly to $u^{(\infty)}$ on compact subsets of $R^n \times (0,1)$ as $q \to \infty$. Moreover $u^{(\infty)}$ satisfies* (0.2) *with initial value $g \in L^1(R^n)$, $0 \le g \le 1$, $\|g\|_{L^1(R^n)} \le \|f\|_{L^1(R^n)}$, satisfying* (2.15) *and* (2.6) *for some function $\tilde{g} \in L^1(R^n)$, $\tilde{g} \ge 0$.*

We are now ready to state and prove the main theorem.

**Theorem 2.10.** *For any $m > 1$ fixed, there exists a unique function $u^{(\infty)} \in C(R^n \times (0,1))$, $0 \le u^{(\infty)} \le 1$ such that $u^{(q)}$ converges weakly to $u^{(\infty)}$ in $(L^\infty(G))^*$ for any compact subset $G$ of $R^n \times (0,1)$ as $q \to \infty$. Moreover $u^{(\infty)}$ satisfies (0.2) with initial value $g \in L^1(R^n)$, $0 \le g \le 1$, satisfying* (2.15) *for some function $\tilde{g} \in L^1_{\mathrm{loc}}(R^n)$, $\tilde{g} \ge 0$. The convergence is uniform on every compact subsets of $R^n \times (0,1)$ if $f \in C_0(R^n)$.*

*Proof.* Since $f \in L^\infty(R^n) \cap L^1(R^n)$, we can choose a sequence $\{f_j\}_{j=1}^\infty \subset C_0^1(R^n)$ such that $\|f_j\|_{L^\infty(R^n)} \le \|f\|_{L^\infty(R^n)} + 1$, $\|f_j\|_{L^1(R^n)} \le \|f\|_{L^1(R^n)} + 1$ for all $j = 1, 2, \dots$ and $\|f_j - f\|_{L^1(R^n)} \to 0$ as $j \to \infty$.

For all $j = 1, 2, \dots$, let $u_j^{(q)}$ be the solution of (1.1) in $R^n \times (0,1)$ with initial value $u_j^{(q)}(x, 0) = f_j(x)$. By Theorem 2.9, for each $j = 1, 2, \dots$, there exists an unique function $u_j^{(\infty)}$ such that $u_j^{(q)}$ converges uniformly on compact subsets of $R^n \times (0,1)$ to $u_j^{(\infty)}$ as $q \to \infty$. Moreover $u_j^{(\infty)}$ satisfies (0.2) with initial value $g_j \in L^1(R^n)$, $0 \le g_j \le 1$, satisfying

$$\begin{cases} \displaystyle\int_{R^n} g_j \le \int_{R^n} f_j \le \int_{R^n} f dx + 1, \\ g_j + (\tilde{g}_j)_{x_1} = f \quad \text{in } \mathscr{D}'(R^n) \text{ for some } \tilde{g}_j \in L^1(R^n),\ \tilde{g}_j \ge 0, \\ g_j(x) = f(x),\ \tilde{g}_j(x) = 0 \quad \text{whenever } g_j(x) < 1 \text{ a.e. } x \in R^n \end{cases} \tag{2.16}$$

for all $j = 1, 2, \dots$. Since $\|u^{(q)}\|_{L^\infty(R^n)} \le \|f\|_{L^\infty(R^n)}$, any sequence $\{u^{(q_i)}\}_{i=1}^\infty$, $q_i \to \infty$ as $i \to \infty$, of $\{u^{(q)}\}_{q>1}$ has a subsequence $\{u^{(q_i')}\}_{i=1}^\infty$ such that $\{u^{(q_i')}\}_{i=1}^\infty$ converges weakly in $(L^\infty(G))^*$ for any compact subset $G$ of $R^n \times (0,1)$ as $i \to \infty$. Let $u^{(\infty)} = \lim_{i\to\infty} u^{(q_i')}$. Without loss of generality we may assume that $u^{(q_i')}(x,t) \to u^\infty(x,t)$ a.e. $(x,t) \in R^n \times (0,1)$ as $i \to \infty$. By Theorem 1.4,

$$\begin{aligned} &\int_{R^n} |u_j^{(q_i')} - u^{(q_i')}|(x,t)dx \le \int_{R^n} |f_j - f|(x)dx \quad \forall i, j = 1, 2, \dots \\ &\Rightarrow \int_{\tau_1}^{\tau_2} \int_{R^n} |u_j^{(q_i')} - u^{(q_i')}|(x,t)dxdt \\ &\le (\tau_2 - \tau_1) \int_{R^n} |f_j - f|(x)dx \quad \forall 0 < \tau_1 \le \tau_2 < 1. \end{aligned}$$

Letting $i \to \infty$, we get by Fatou's lemma,

$$\int_{\tau_1}^{\tau_2} \int_{R^n} |u_j^{(\infty)} - u^{(\infty)}|(x, t)dxdt \le (\tau_2 - \tau_1) \int_{R^n} |f_j - f|(x)dx \to 0 \text{ as } j \to \infty$$

for all $0 < \tau_1 \le \tau_2 < 1$.

Hence $u^{(\infty)}$ is the limit of the functions $\{u_j^{(\infty)}\}_{j=1}^{\infty}$ in $L^1_{\text{loc}}(R^n \times (0, 1))$ as $j \to \infty$. Thus $u^{(\infty)}$ is unique and $u^{(q)}$ converges weakly to $u^{(\infty)}$ in $(L^\infty(G))^*$ for any compact subset $G$ of $R^n \times (0, 1)$ as $q \to \infty$. This together with Theorem 2.1 implies that $u^{(q)}$ converges uniformly to $u^{(\infty)}$ on every compact subsets of $R^n \times (0, 1)$ as $q \to \infty$ if $f \in C_0(R^n)$.

Moreover $\{u_j^{(\infty)}\}_{j=1}^{\infty}$ has a subsequence converging a.e. $(x, t) \in R^n \times (0, 1)$ to $u^{(\infty)}$. Without loss of generality we may assume that $u_j^{(\infty)}(x, t) \to u^{(\infty)}(x, t)$ a.e. $(x, t) \in R^n \times (0, 1)$ as $j \to \infty$.

On the other hand since $u_j^{(\infty)}$ satisfies (0.2) and

$$\begin{aligned} &|u_j^{(q_i')}(x, t)| \le \|f_j\|_{L^\infty(R^n)} \le \|f\|_{L^\infty(R^n)} + 1 \quad \forall (x, t) \in R^n \times (0, 1), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i, j = 1, 2, \dots, \\ &\Rightarrow \|u_j^{(\infty)}\|_{L^\infty(R^n \times (0,1))} \le \|f\|_{L^\infty(R^n)} + 1 \quad \forall j = 1, 2, \dots. \end{aligned} \tag{2.17}$$

as $q \to \infty$ by Theorem 1.3, by the result of [S1] $\{u_j^{(\infty)}\}_{j=1}^{\infty}$ has a subsequence $\{u_{j_k}^{(\infty)}\}_{k=1}^{\infty}$ converging uniformly on compact subsets of $R^n \times (0, 1)$. Hence we may assume without loss of generality that $\{u_j^{(\infty)}\}_{j=1}^{\infty}$ converges uniformly on compact subsets of $R^n \times (0, 1)$ to $u^{(\infty)}$. Thus $u^{(\infty)} \in C(R^n \times (0, 1))$.

Putting $h(u) = 0$, $u = u_j^{(\infty)}$ in (0.5) and letting $j \to \infty$, we see that $u^{(\infty)}$ satisfies (0.2). By (2.17) and the result of [DK], $u^{(\infty)}$ has an initial trace $d\mu$ and $d\mu$ is absolutely continuous with respect to the Lebesgue measure. Hence $d\mu = g(x)dx$ for some $g \ge 0$, $g \in L^1(R^n)$. By (2.16) and Lemma 2.7,

$$\int_{|x_1| \le R'} \int_{R^{n-1}} |\tilde{g}_j - \tilde{g}_{j'}|(x_1, x')dx'dx_1 \le 2R'\|f_j - f_{j'}\|_{L^1(R^n)} \to 0$$
$$\text{as } j, j' \to \infty \quad \forall R' > 0.$$

Hence $\{\tilde{g}_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $L^1_{\text{loc}}(R^n)$ and there exists $\tilde{g} \in L^1_{\text{loc}}(R^n)$ such that $\tilde{g}_j \to \tilde{g}$ in $L^1_{\text{loc}}(R^n)$ as $j \to \infty$. Without loss of generality we may assume that $\tilde{g}_j(x) \to \tilde{g}(x)$ a.e. $x \in R^n$. By the proof of Theorem 2.1, $u_j^{(\infty)}$ satisfies, for all $\eta \in C_0^\infty(R^n)$, $0 < \tau_2 < 1$,

$$\begin{aligned} &\int_0^{\tau_2} \int_{R^n} u_j^{(\infty)m} \Delta\eta dxd\tau + \int_{R^n} \tilde{g}_j \eta_{x_1} dx = \int_{R^n} u_j^{(\infty)}(x, t)\eta(x)dx - \int_{R^n} f_j \eta dx \\ &\Rightarrow \int_0^{\tau_2} \int_{R^n} u^{(\infty)m} \Delta\eta dxd\tau + \int_{R^n} \tilde{g}\eta_{x_1} dx \\ &= \int_{R^n} u^{(\infty)}(x, t)\eta(x)dx - \int_{R^n} f\eta dx \quad \text{as } j \to \infty \\ &\Rightarrow \int_{R^n} \tilde{g}\eta_{x_1} dx = \int_{R^n} g(x)\eta(x)dx - \int_{R^n} f\eta dx \quad \text{as } \tau_2 \to 0 \\ &\Rightarrow g + \tilde{g}_{x_1} = f \quad \text{in } \mathscr{D}'(R^n). \end{aligned}$$

Thus

$$\begin{aligned}\left|\int (g - g_j)\eta dx\right| &= \left|\int (\tilde{g} - \tilde{g}_j)\eta_{x_1} dx + \int (f - f_j)\eta dx\right| \\ &\le \|\eta_{x_1}\|_{L^\infty(R^n)} \int_{|x_1|\le R'} \int_{R^{n-1}} |\tilde{g} - \tilde{g}_j|(x_1, x')dx'dx_1 \\ &\quad + \|\eta\|_{L^\infty(R^n)} \int_{R^n} |f - f_j| dx \\ &\to 0\end{aligned}$$

as $j \to \infty$ for all $\eta \in C_0^\infty(R^n)$ such that supp $\eta \subset B_{R'}(0)$ for some $R' > 0$. Hence $g_j$ converges weakly to $g$ in $\mathscr{D}'(R^n)$ as $j \to \infty$. We may assume without loss of generality that $g_j(x) \to g(x)$ and $\tilde{g}_j(x) \to \tilde{g}(x)$ a.e. $x \in R^n$. Let

$$E = \{x \in R^n : g_j(x) \to g(x) \text{ and } \tilde{g}_j \to \tilde{g}(x) \text{ as } j \to \infty\},$$

$$E_0 = E \cap \{g < 1\} \cap \left(\bigcap_{j=1}^{\infty} (S(g_j) \cap S(\tilde{g}_j) \cap G(u_j^{(\infty)}, g_j))\right).$$

For any $x_0 \in E_0$, since $g_j(x_0) \to g(x_0)$ as $j \to \infty$, there exists $j_0 \in Z^+$ such that $g_j(x_0) < 1 \ \forall j \ge j_0$. So $g_j(x_0) = f(x_0)$ and $\tilde{g}_j(x_0) = 0$ for all $j \ge j_0$ by Lemma 2.4. Letting $j \to \infty$, we have $g(x_0) = f(x_0)$ and $\tilde{g}(x_0) = 0$. Since $|\{g < 1\} \setminus E_0| = 0$, $g(x_0) = f(x_0)$ and $\tilde{g}(x_0) = 0$ a.e. $x_0 \in \{g < 1\}$ and the theorem follows.

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INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, NANKANG, TAIPEI, 11529, TAIWAN, R. O. C.
*E-mail address*: makmhui@ccvax.sinica.edu.tw