SINGULAR LIMIT OF SOLUTIONS OF
\[ u_t = \Delta u^m - A \cdot \nabla (u^q/q) \text{ as } q \to \infty \]

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Abstract. We will show that the solutions of \( u_t = \Delta u^m - A \cdot \nabla (u^q/q) \) in \( \mathbb{R}^n \times (0, T) \), \( T > 0 \), \( m > 1 \), \( u(x, 0) = f(x) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) converge weakly in \( (L^\infty(G))^* \) for any compact subset \( G \) of \( \mathbb{R}^n \times (0, T) \) as \( q \to \infty \) to the solution of the porous medium equation \( v_t = \Delta v^m \) in \( \mathbb{R}^n \times (0, T) \) with \( v(x, 0) = g(x) \) where \( g \in L^1(\mathbb{R}^n) \), \( 0 \leq g \leq 1 \), satisfies \( g(x) + (\tilde{g}(x))_{x_1} = f(x) \) in \( \mathcal{D}'(\mathbb{R}^n) \) for some function \( \tilde{g}(x) \in L^1(\mathbb{R}^n) \), \( \tilde{g}(x) \geq 0 \) such that \( g(x) = f(x) \), \( \tilde{g}(x) = 0 \) whenever \( g(x) < 1 \) a.e. \( x \in \mathbb{R}^n \). The convergence is uniform on compact subsets of \( \mathbb{R}^n \times (0, T) \) if \( f \in C^0(\mathbb{R}^n) \).

In this paper we will study the asymptotic behaviour of nonnegative solutions \( u = u(x) \) of the equation
\[
\begin{cases}
  u_t = \Delta u^m - A \cdot \nabla (u^q/q), & (x, t) \in \mathbb{R}^n \times (0, T), \\
  u(x, 0) = f(x) \geq 0, & f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),
\end{cases}
\]
where \( 0 \neq A = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \) is a constant vector, \( T > 0 \), \( m > 1 \), as \( q \to \infty \). Recently there is a lot of research on the above equation ([A],[DiK],[G1],[G2]) The equation arises in many physical applications such as the flow of water through a homogeneous isotropic rigid porous medium [G1]. When \( A = 0 \), the above equation reduces to the well-known porous medium equation ([Ar],[P]). In the paper [CF], Caffarelli and A. Friedman studied the asymptotic behaviour of solutions of (0.1) when \( A = 0 \) and showed that the solutions of (0.1) converge as \( m \to \infty \) if \( f \) satisfies (0.1) and the following conditions:

\[
\begin{align*}
  f &\in C^1 \text{ in supp } f, \\
  f(0) > 1, &\quad f_r < 0 \text{ in } \mathbb{R}^n \setminus \{0\} \cap \text{supp } f, \\
  f_{r_{x_0}} &\leq 0 \text{ in } \mathbb{R}^n \setminus B_r(0) \cap \text{supp } f \quad \forall x_0 \in B_{r_0}(0)
\end{align*}
\]

for some \( e_0 > 0 \) where \( r_{x_0} = |x - x_0| \), \( B_r(0) = \{ x : |x| < r \} \) and \( f_{r_{x_0}} \) is the radial derivative of \( f \) with center at \( x_0 \).

This result has been extended in various directions by P. Bénilan, L. Boccardo and M. Herrero [BBH], P. E. Sacks [S2] in the case \( A = 0 \), X. Xu [X] in the case of hyperbolic equations and K. M. Hui [H1], [H2] in the case of a porous
medium equation with absorption and in the case of the generalized p-Laplacian equation.

For simplicity we will assume that $T = 1$ and $A = (1, 0, \ldots, 0)$ throughout the rest of the paper. We will show that as $q \to \infty$, the convection term in (0.1) disappears. More precisely, we will show that for fixed $m > 1$ the solutions $u = u^{(q)}$ of (0.1) converge weakly in $(L^\infty(G))^*$ for any compact subset $G$ of $R^n \times (0, 1)$ as $q \to \infty$. Moreover the limit $u^{(\infty)} = \lim_{q \to \infty} u^{(q)}$ satisfies the porous medium equation

$$
\begin{cases}
u_t = \Delta v^m, & (x, t) \in R^n \times (0, 1), \\
u(\cdot, t) \searrow g & \text{as } t \to 0 \text{ in } D'(R^n),
\end{cases}
$$

where $g \in L^1(R^n)$, $0 \leq g \leq 1$, satisfies

$$
g(x) + (\tilde{g}(x))_{x_i} = f(x) \text{ in } D'(R^n)
$$

for some function $\tilde{g}(x) \geq 0$, $\tilde{g}(x) \in L^1(R^n)$ and $g(x) = f(x)$, $\tilde{g}(x) = 0$ whenever $g(x) < 1$ a.e. $x \in R^n$. This extends the recent results obtained by M. Escobedo and E. Zuazua [EZ], who showed that the convection term was negligible compared with the other terms appearing in (0.1) for the case $m = 1$ and $q > 1 + 1/n$ as $t \to \infty$. Although we were not able to prove it, we suspect that the same result should remain valid when $A = A(x) \in L^\infty(R^n)$.

We will first start with some definitions. For any open set $Q_0 \subset R^n$, $h \in C(R)$, we say that $u$ is a solution (respectively subsolution, supersolution) of

$$
u_t = \Delta u^m - (h(u))_{x_i} \text{ in } Q_0 \times (0, 1)
$$

if $u$ is continuous and nonnegative in $Q_0 \times (0, 1)$, $u \in L^\infty([0, 1]; L^1(Q_0)) \cap L^\infty(Q_0 \times (0, 1))$ and satisfies

$$
\int_{\tau_1}^{\tau_2} \int_{Q_0} [u^m \Delta \eta + u \frac{\partial \eta}{\partial t} + h(u) \eta_{x_i}] \, dx \, dt = \int_{\tau_1}^{\tau_2} \int_{\partial Q_0} u^m \frac{\partial \eta}{\partial N} \, ds \, ds + \int_{\tau_1}^{\tau_2} \int_{Q_0} u \eta \, dx
$$

(respectively $>$, $< $) for all bounded open sets $Q \subset Q_0$ with $\partial Q \in C^2$, $0 < \tau_1 \leq \tau_2 < 1$, $\eta \in C^\infty(\Omega \times [\tau_1, \tau_2])$, $\eta \equiv 0$ on $\partial Q \times [\tau_1, \tau_2]$ where $\partial / \partial N$ is the exterior normal derivative on $\partial Q$ and $d\sigma$ is the surface measure on $\partial Q$.

If $u$ is a solution of (0.4) in $\Omega_0 \times (0, 1)$, we say that $u$ has initial trace or initial value $du$ if

$$
\lim_{t \to 0} \int u(x, t) \eta(x) \, dx = \int \eta \, du \quad \forall \eta \in C_0^\infty(\Omega_0).
$$

We let $\rho \in C_0^\infty(R^n)$, $\rho \geq 0$, $\int \rho = 1$ and for any $g$ we define

$$
g_\varepsilon = g \ast \rho_\varepsilon(x) = \int \rho_\varepsilon(x - y) g(y) \, dy, \quad \varepsilon > 0,
$$

where $\rho_\varepsilon(y) = \rho(y/\varepsilon) / \varepsilon^n$. For any $r > 0$, $x_0 \in R^n$, let $B_r(x_0) = \{x \in R^n : |x - x_0| < r\}$. For any set $A \subset R^n$, we let $\chi_A$ be the characteristic function of the set $A$. We will also assume $m > 1$, $q > m + 1$, and let $u^{(q)}$ be the solution of (0.1) for the rest of the paper.

The plan of the paper is as follows. In section 1 we will state and prove the existence of solutions of (0.1). We will also prove a comparison theorem.
for solutions of (0.1) and obtain some bounds on $u^{(q)}$ by constructing explicit supersolutions to (0.1). In section 2 we will first prove a comparison lemma for solutions of (0.3). We then prove the main theorem under the assumption $f \in C^1_c(R^n)$ (Theorem 2.9). Finally we will prove the main theorem (Theorem 2.10) by an approximation argument.

We first state and prove an uniqueness theorem for solutions of (0.1).

**Theorem 1.1.** If $u_1^{(q)}, u_2^{(q)} \in L^\infty((0, 1); L^1(R^n)) \cap L^\infty(R^n \times (0, 1)) \cap C(R^n \times (0, 1))$ are the solutions of

\[(1.1) 
 u_t = \Delta u^m - (u^{q}/q)x_1 
\]

in $R^n \times (0, 1)$ with initial values $f_1$ and $f_2 \in L^1(R^n) \cap L^\infty(R^n)$ respectively, $f_1, f_2 \geq 0$, then there exists a constant $C > 0$ such that

(i) $\int_{R^n} (u_1^{(q)} - u_2^{(q)})_+(x, t)dx \leq e^{Ct} \int_{R^n} (f_1 - f_2)_+(x)dx$, 

(ii) $\int_{R^n} |u_1^{(q)} - u_2^{(q)}|(x, t)dx \leq e^{Ct} \int_{R^n} |f_1 - f_2|(x)dx$

for all $0 < t < 1$. Hence $u_1^{(q)} \leq u_2^{(q)}$ if $f_1 \leq f_2$. In particular the solution of (1.1) in $R^n \times (0, 1)$ with initial value in $L^1(R^n) \cap L^\infty(R^n)$ is unique in the class $L^\infty((0, 1); L^1(R^n)) \cap L^\infty(R^n \times (0, 1)) \cap C(R^n \times (0, 1))$.

**Proof.** The proof of the theorem is similar to the proof of Theorem 2.3 of [A]. By subtracting the equation for $u_1^{(q)}$ and $u_2^{(q)}$, we get

\[
\int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})(x, t)\eta(x, t)dx = \int_{B_R(0)} (f_1 - f_2)(x)\eta(x, 0)dx \\
+ \int_0^t \int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})(\eta_t + A\Delta \eta + B\eta_{x_1})dxd\tau \\
- \int_0^t \int_{\partial B_R(0)} (u_1^{(q)m} - u_2^{(q)m})\frac{\partial \eta}{\partial N}d\sigma d\tau
\]

for all $0 < t < 1$, $\eta \in C^\infty(\overline{B_R(0) \times [0, t]}), R > 0$, such that $\eta \equiv 0$ on $\partial B_R(0) \times [0, t]$ where

\[
A = \begin{cases} 
\frac{u_1^{(q)m} - u_2^{(q)m}}{u_1^{(q)} - u_2^{(q)}} & \text{for } u_1^{(q)} \neq u_2^{(q)}, \\
u_1^{(q)m-1} & \text{for } u_1^{(q)} = u_2^{(q)},
\end{cases} \\
B = \begin{cases} 
\frac{1}{q} \frac{u_1^{(q)q} - u_2^{(q)q}}{u_1^{(q)} - u_2^{(q)}} & \text{for } u_1^{(q)} \neq u_2^{(q)}, \\
u_1^{(q)q-1} & \text{for } u_1^{(q)} = u_2^{(q)}.
\end{cases}
\]

Since $u_1^{(q)}, u_2^{(q)} \in L^\infty(R^n \times (0, 1))$, there exists a constant $C_1 > 0$ such that

\[
\|u_1^{(q)}\|_{L^\infty(R^n)}, \|u_2^{(q)}\|_{L^\infty(R^n)} \leq C_1 \\
\Rightarrow B^2/2A \leq \frac{1}{2m} C_1^{2q-m-1}, B/A \leq \frac{1}{m} C_1^{q-m}.
\]
By an argument similar to section 4 of [A], there exists smooth functions \( A_i, R \) and \( B_i, R \) and constant \( c_i > 0 \) such that \( c_i \leq A_i, R \leq mC_i^{m-1} + 1, 0 \leq B_i, R \leq C_i^{q-1} + 1, B_i, R / 2A_i, R \leq (C_i^{q-1} / 2m) + 1 = C_2, B_i, R / A_i, R \leq (C_i^{q-1} / m) + 1 = C_3, (A_i, R - A) / A_i^{1/2} \rightarrow 0 \) and \( B_i, R - B \rightarrow 0 \) in \( L^2(B_R(0) \times (0, 1)) \) as \( R \rightarrow 0 \) for all \( R > 0 \).

For any \( R_0 > 2, R > R_0 + 1, \lambda > C_2, \theta \in C_0^\infty(B_{R_0}(0)), 0 \leq \theta \leq 1 \), let \( \eta_i, R \) be the solution of

\[
\begin{aligned}
\eta_t + A_i, R \Delta \eta + B_i, R \eta_x, - \lambda \eta &= 0 & &\text{for } (x, s) \in B_R(0) \times (0, t), \\
\eta(x, s) &= 0 & &\text{for } (x, s) \in \partial B_R(0) \times (0, t), \\
\eta(x, t) &= \theta(x) & &\text{for } x \in B_R(0).
\end{aligned}
\]

Since \( 0 \leq \theta \leq 1 \), by the maximum principle \( 0 \leq \eta_i, R \leq 1 \). By Lemma 4.1 of [A], we have

\[
\int_0^t \int_{B_R(0)} A_i, R(\Delta \eta_i, R)^2 \, dx \, dt + 2(\lambda - C_2) \int_0^t \int_{B_R(0)} |\nabla \eta_i, R|^2 \, dx \, dt \\
\leq \int_{B_R(0)} |\nabla \theta|^2 \, dx.
\]

By the same argument as the proof of Theorem 2.1 (ii) of [PV], we see that for any \( \beta > 0 \), the function

\[
g(x, s) = e^{h(s)} \left( \frac{1 + R_0^2}{1 + |x|^2} \right)^\beta
\]

where \( h(s) = C'(t - s), C' = 4\beta(\beta + 1)(mC_i^{m-1} + 1) + \beta(C_i^{q-1} + 1) \), satisfies

\[
\begin{aligned}
g_s + A_i, R \Delta g + B_i, R g_x, - \lambda g &< 0, & &\text{for } (x, s) \in B_R(0) \times (0, t), \\
g(x, s) \geq \eta_i, R(x, s), & &\text{for } (x, s) \in B_R(0) \times \{t\} \cup \partial B_R(0) \times (0, t).
\end{aligned}
\]

Hence by the maximum principle [LSU], \( g \geq \eta_i, R \) in \( B_R(0) \times (0, t) \). We next consider the function

\[
g^*(x, s) = a e^{h(s)} \Gamma(|x|), \quad R - \alpha \leq r \leq R, 0 \leq s \leq t,
\]

where \( \alpha = 1/2(C_3 + n - 1), \Gamma(r) = (R - r) - C_3(R - r)^2 \) and

\[
a = (1 + R_0^2)^\beta / \{\Gamma(R - \alpha)(1 + (R - \alpha)^2)^\beta\}.
\]

Then \( g^* \geq 0, g^*_s = h'(s)g^* \leq 0 \) and

\[
\begin{aligned}
\Delta g^* + (B_i, R / A_i, R) g^*_x, & \leq a e^{h(s)} \left( \Gamma''(r) + \frac{n - 1}{r} \Gamma'(r) + C_3 |\Gamma'(r)| \right) \\
& \leq a e^{h(s)} \left( -2C_3 + \frac{n - 1}{r} (-1 + 2C_3(R - r)) + C_3(1 + 2C_3(R - r)) \right) \\
& \leq aC_3 e^{h(s)} (-1 + 2(C_3 + n - 1)(R - r)) \\
& \leq 0
\end{aligned}
\]

for all \( R - \alpha < r < R, 0 < s < t \) since \( R - \alpha \geq R_0 \geq 2 \). Hence \( g^* \) satisfies

\[
g^*_s + A_i, R \Delta g^* + B_i, R g^*_x, - \lambda g^* < 0, \quad \text{for } (x, s) \in B_R(0) \setminus B_{R-\alpha}(0) \times (0, t)
\]
with \( g^*(x, s) \geq \eta_i, R(x, s) \) for all
\[(x, s) \in B_R(0) \setminus B_{R-\alpha}(0) \times \{t\} \cup (\partial B_R(0) \cup \partial B_{R-\alpha}(0)) \times (0, t]. \]

By the maximum principle, \( 0 \leq \eta_i, R \leq g^* \) in \( B_R(0) \setminus B_{R-\alpha}(0) \times (0, t) \). Since \( g^* \equiv \eta_i, R \equiv 0 \) on \( \partial B_R(0) \times [0, t] \),
\[
(1.4) \quad \|\partial \eta_i, R / \partial N\|_{L^\infty(\partial B_R(0) \times (0, t))} \leq \|\partial g^*/\partial N\|_{L^\infty(\partial B_R(0) \times (0, t))} \leq CR^{-2\beta}.
\]
Putting \( \eta = \eta_i, R \) in (1.2), we get by (1.3) and (1.4),
\[
(1.5) \quad \int_{B_R(0)} (u_1^{(q)} - u_2^{(q)})(x, t) \theta(x) dx
\]
for all \( \theta \in C_0^\infty(B_R(0)) \), \( 0 < \theta < 1 \), \( 0 < t < 1 \). Choose now \( \beta = n/2 \) and let first \( i \to \infty \) and then \( R \to \infty \), \( R \to C_2 \) in (1.5), we get
\[
\int_{R^n} (u_1^{(q)} - u_2^{(q)})(x, t) \theta(x) dx \leq \int_{R^n} (f_1 - f_2)_+ dx + C_2 \int_{R^n} (u_1^{(q)} - u_2^{(q)})_+ dx \quad \forall 0 < t < 1.
\]
for all \( \theta \in C_0^\infty(B_R(0)) \), \( 0 \leq \theta \leq 1 \), \( R_0 > 2 \). Putting \( \theta = \chi_{\{u_1^{(q)} \geq u_2^{(q)}\} \cap B_R(0)-0} * \rho_\epsilon \) into the above inequality and letting first \( \epsilon \to 0 \) and then \( R_0 \to \infty \), we get
\[
\int_{R^n} (u_1^{(q)} - u_2^{(q)})_+(x, t) dx \leq \int_{R^n} (f_1 - f_2)_+ dx + C_2 \int_{R^n} (u_1^{(q)} - u_2^{(q)})_+ dx \quad \forall 0 < t < 1.
\]
(i) then follows from the Gronwall's inequality. Similarly,
\[
\int_{R^n} (u_1^{(q)} - u_2^{(q)})_-(x, t) dx \leq e^{C_2 t} \int_{R^n} (f_1 - f_2)_- dx \quad \forall 0 < t < 1.
\]
By combining the above inequality with (i), we get (ii).

**Corollary 1.2.** If \( u_1^{(q)} \) is a subsolution and \( u_2^{(q)} \) is a supersolution of (1.1) in \( Q = D \times (0, 1) \) where \( D = (-\infty, R_0) \times R^{n-1} \) for some \( R_0 \in R \) (or \( D = [R_0, R_1] \times R^{n-1} \) for some \( R_0, R_1 \in R \), \( R_0 < R_1 \)) with \( u_1^{(q)} \), \( u_2^{(q)} \) in \( L^\infty([0, 1); L^1(D)) \cap L^\infty(D \times (0, 1)) \cap C(D \times (0, 1)) \) with initial values \( u_1^{(q)}(x, 0), u_2^{(q)}(x, 0) \) and boundary values satisfying
\[
u^{(q)}(x, t) \leq u^{(q)}(x, t) \quad \forall (x, t) \in \partial \rho Q
\]
where $\partial_p Q = \{ R_0 \} \times R^{n-1} \times (0, 1) \cup (-\infty, R_0) \times R^{n-1} \times \{ 0 \} \) (respectively $\partial_p Q = \{ R_0, R_1 \} \times R^{n-1} \times (0, 1) \cup [R_0, R_1] \times R^{n-1} \times \{ 0 \} \), then
\[
\begin{align*}
u_p(q)(x, t) &\leq u_2(q)(x, t) \quad \forall (x, t) \in Q
\end{align*}
\]
Proof. The proof is the same as the proof of Theorem 1.1.

**Theorem 1.3.** The equation
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + (u^a/q) x_i, \quad u \geq 0, \quad (x, t) \in R^n \times (0, 1), \\
u(x, 0) &= f(x) \geq 0, \quad f \in L^1(R^n) \cap L^\infty(R^n),
\end{align*}
\]
has a unique solution $u(q) \in L^\infty([0, 1); L^1(R^n)) \cap L^\infty(R^n \times (0, 1)) \cap C(R^n \times (0, 1))$ with
\[
\begin{align*}
\int u(q)(x, t)dx &= \int f dx \quad \forall 0 < t < 1, \\
\|u(q)\|_{L^\infty(R^n \times (0, 1))} &\leq \|f\|_{L^\infty(R^n)}.
\end{align*}
\]
Proof. The proof is similar to that of [ERV] and [DK]. Let $\psi \in C_0^\infty(R^n)$, $0 \leq \psi \leq 1$, be such that $\psi(x) \equiv 1$ for all $|x| \leq 1/2$ and $\psi \equiv 0$ for all $|x| \geq 1$. For any $\epsilon > 0, 0 < \epsilon < 1, R > 0$, let $f_{\epsilon,R}(x) = f \ast \rho(x) \cdot \psi(x/R) + \epsilon$ and let $a_\epsilon(s), b_\epsilon(s) \in C^\infty(R)$ be such that $a_\epsilon(s), b_\epsilon(s) \geq 0,$
\[
\begin{align*}
a_\epsilon(s) &= \begin{cases} 
ms & \text{for } s \leq 0, \\
m(\|f\|_{L^\infty(R^n)} + 2)^{m-1} & \text{for } s \geq 0,
\end{cases} \\
b_\epsilon(s) &= \begin{cases} 
s^{q-1} & \text{for } s \leq 0, \\
(e/2)^{q-1} & \text{for } s \geq 0.
\end{cases}
\end{align*}
\]
By standard parabolic theory [LSU], there exists a unique solution $u_{\epsilon,R}$ to the equation
\[
\begin{align*}
\frac{\partial u}{\partial t} &= div(a_\epsilon(u) \nabla u) - b_\epsilon(u) u x_i, \quad \forall (x, t) \in B_R(0) \times (0, 1), \\
u(x, t) &= \epsilon \quad \forall (x, t) \in \partial B_R(0) \times (0, 1), \\
u(x, 0) &= f_{\epsilon,R}(x), \quad \forall x \in B_R(0).
\end{align*}
\]
Since $\epsilon \leq f_{\epsilon,R} \leq \|f\|_{L^\infty(R^n)} + \epsilon$, by the maximum principle,
\[
\epsilon \leq u_{\epsilon,R} \leq \|f\|_{L^\infty(R^n)} + \epsilon.
\]
Hence $a_\epsilon(u_{\epsilon,R}) = m u_{\epsilon,R}^{m-1}$, $b_\epsilon(u_{\epsilon,R}) = u_{\epsilon,R}^{q-1}$. Since (1.3) is a nondegenerate parabolic equation, by Schauder's estimate [LSU], $u_{\epsilon,R} \in C^\infty(B_R(0) \times (0, 1))$. Thus $u_{\epsilon,R}$ satisfies (1.1) in $B_R(0) \times (0, 1)$. Since $u_{\epsilon,R}$ is uniformly bounded by $\|f\|_{L^\infty(R^n)} + 1$, by the result of P. Sacks [S1], $\{u_{\epsilon,R}\}_{R>0}$ has a convergent subsequent $\{u_{\epsilon,R}\}_{j=1}^{\infty}$, $R_j \rightarrow \infty$ as $j \rightarrow 0$, such that $\{u_{\epsilon,R}\}_{j=1}^{\infty}$ converges uniformly on compact subsets of $R^n \times (0, 1)$. Let $u_{\epsilon,R} = \lim_{j \rightarrow \infty} u_{\epsilon,R}$. Then $u_{\epsilon,R} \in C(R^n \times (0, 1))$ and
\[
\epsilon \leq u_{\epsilon,R} \leq \|f\|_{L^\infty(R^n)} + \epsilon.
\]
Putting \( u = u^{(q)}_{e,R_j} \) in (1.1) and letting \( j \to 0 \), we see that \( u^{(q)}_e \) satisfies (1.1) in \( \mathbb{R}^n \times (0, 1) \) with \( u^{(q)}_e(x, 0) = f \ast \rho(x) + \varepsilon \). Thus \( u^{(q)}_e \in C^\infty(\mathbb{R}^n \times (0, 1)) \) by (1.11) and Schauder’s estimates. Since \( \|u^{(q)}_e\|_{L^\infty(\mathbb{R}^n \times (0, 1))} \leq \|f\|_{L^\infty(\mathbb{R}^n)} + \varepsilon \), by [S1], \( \{u^{(q)}_e\}_{e>0} \) has a convergent subsequence \( \{u^{(q)}_{e_i}\}_{i=1}^\infty \), \( e_i \to 0 \) as \( i \to 0 \), such that \( \{u^{(q)}_{e_i}\}_{i=1}^\infty \) converges uniformly on compact subsets of \( \mathbb{R}^n \times (0, 1) \).

Let \( u^{(q)} = \lim_{i \to \infty} u^{(q)}_{e_i} \). Then \( u^{(q)} \in C(\mathbb{R}^n \times (0, 1)) \).

Putting \( u = u^{(q)}_e \) in (1.1) and letting \( i \to 0 \), we see that \( u^{(q)} \) satisfies (1.1) in \( \mathbb{R}^n \times (0, 1) \). Moreover,

\[
\int_{\mathbb{R}^n} u^{(q)}_{e,R_j}(x, t)\eta(x)dx - \int_{\mathbb{R}^R} \eta(x)dx
\]

\[
= \left| \int_{0}^{t} \int_{\mathbb{R}^n} \left( \Delta u^{(q)}_{e,R_j} \frac{u^{(q)}_{e,R_j}}{q} \right) \eta(x)dxdt \right|
\]

\[
= \left| \int_{0}^{t} \int_{\mathbb{R}^n} \left( \Delta u^{(q)}_{e,R_j} \frac{u^{(q)}_{e,R_j}}{q} \right) \eta(x)dxdt \right|
\]

\[
\leq \left( \|f\|_{L^\infty(\mathbb{R}^n)} + 1 \right)^m \|\Delta \eta\|_{L^1(\mathbb{R}^n)} t + \frac{\left( \|f\|_{L^\infty(\mathbb{R}^n)} + 1 \right)^q}{q} \|\eta_i\|_{L^1(\mathbb{R}^n)} t
\]

for all \( \eta \in C^\infty(\mathbb{R}^n) \). Letting first \( j \to 0 \) and then \( e = e_i \to 0 \), \( t \to 0 \), we get

\[
\lim_{t \to 0} \int_{\mathbb{R}^n} u^{(q)}(x, t)\eta(x)dx = \int_{\mathbb{R}^n} f\eta(x)dx \quad \forall \eta \in C^\infty(\mathbb{R}^n).
\]

Hence \( u^{(q)} \) has initial trace \( f \) and \( \|u^{(q)}\|_{L^\infty(\mathbb{R}^n \times (0, 1))} \leq \|f\|_{L^\infty(\mathbb{R}^n)} \) by (1.11).

On the other hand, since \( u^{(q)}_e \) satisfies (1.1) in \( \mathbb{R}^n \times (0, 1) \),

\[
\int_{\mathbb{R}^n} u^{(q)}_{e,R_j}(x, t)\eta(x, t)dx = \int_{\mathbb{R}^n} \left( f \ast \rho(x) + \varepsilon \right)\eta(x, 0)dx
\]

\[
+ \int_{0}^{t} \int_{\partial B_R(0)} u^{(q)}_e(\eta_t + A_\varepsilon \Delta \eta + B_\varepsilon \eta_{x_1})dxd\tau
\]

\[
- \int_{0}^{t} \int_{\partial B_R(0)} u^{(q)}_e \frac{\partial \eta}{\partial N} d\sigma d\tau
\]

for all \( 0 < t < 1 \), \( \eta \in C^\infty(\overline{B_R(0)} \times [0, t]) \), \( R > 0 \) such that \( \eta \equiv 0 \) on \( \partial B_R(0) \times [0, t] \) where \( A_\varepsilon = u^{(q)}_e m^{m-1} \), \( B_\varepsilon = u^{(q)}_e m^{1/q} \).

For any \( R_0 > 2 \), \( R > R_0 + 1 \), \( \theta \in C^\infty(B_{R_0}(0)) \), \( 0 \leq \theta \leq 1 \), \( \varepsilon \equiv 1 \) for \( |x| \leq R_0 - 1 \), let \( \eta_{e,R} \) be the solution of

\[
\begin{cases}
\eta_\varepsilon + A_\varepsilon \Delta \eta + B_\varepsilon \eta_{x_1} = 0 & \text{for } (x, s) \in B_R(0) \times (0, t), \\
\eta(x, s) = 0 & \text{for } (x, s) \in \partial B_R(0) \times (0, t), \\
\eta(x, t) = \theta(x) & \text{for } x \in B_R(0).
\end{cases}
\]

By an argument similar to the proof of Theorem 1.1, we have \( 0 \leq \eta_{e,R} \leq 1 \),

\[
\eta_{e,R}(x, s) \leq e^{h(s)} \left( \frac{1 + R_0^2}{1 + |x|^2} \right)^n \forall 0 \leq s \leq t,
\]
where \( h(s) = C'(t-s), \) \( C' = 4n(n+1)(b^m_1+1)+n(b^q_1+1), \) \( b_1 = \|f\|_{L^\infty(R^n)} + 1, \) and
\[
\|\partial \eta_{e,R}/\partial N\|_{L^\infty(\partial B_R(0) \times (0,t))} \leq CR^{-2n}
\]
for some constant \( C > 0 \) depending only on \( R_0 \) and \( b_1 \). Putting \( \eta = \eta_{e,R} \) into (1.13), we get
\[
\int_{B_R(0)} u^{(e)}_e \theta(x) dx \leq \int f dx + C'R_0 R^{-n-1} + \varepsilon C_R_0
\]
for some constant \( C_{R_0}, C'_R > 0 \) depending only on \( R_0 \) and \( b_1 \). Letting \( R \to \infty, \varepsilon = \varepsilon_i \to 0, \)
\[
\int_{|x| \leq R_0-1} u^{(e)}_e(x, t) dx \leq \int_{R^n} u^{(e)}_e(x, t) \theta(x) dx \leq \int f dx
\]
for all \( 0 < t < 1. \) Letting \( R_0 \to \infty, \)
\[
\int_{R^n} u^{(e)}_e(x, t) dx \leq \int f dx \forall 0 < t < 1.
\]
Hence \( u^{(e)}_e \in L^\infty((0, 1); L^1(R^n)) \cap L^\infty(R^n \times (0, 1)) \cap C(R^n \times (0, 1)) \) and satisfies (1.8). It remains to show (1.7). Since
\[
(1.15) \Rightarrow \int_0^1 \int_{R^n} u^{(e)}_e(x, \tau) dx d\tau \leq \int_{R^n} f dx
\]
\[
\Rightarrow \int_0^1 \int_{R/2 \leq |x| \leq R} u^{(e)}_e(x, \tau) dx d\tau \to 0 \quad \text{as} \quad R \to \infty,
\]
putting \( \eta(x) = \psi(x/R), \) \( R > 0, \) in (1.12), we have
\[
\left| \int_{R^n} u^{(e)}_{e,R}(x, t) \psi(x/R) dx - \int_{R^n} f_{e,R}(x) \psi(x/R) dx \right| \\
\leq \frac{(\|f\|_{L^\infty(R^n)} + 1)^m-1}{R^2} \|\Delta \psi\|_{L^\infty(R^n)} \int_0^t \int_{R/2 \leq |x| \leq R} u^{(e)}_{e,R}(x, \tau) dx d\tau \\
+ \frac{(\|f\|_{L^\infty(R^n)} + 1)^q-1}{qR} \|\psi_{x_1}\|_{L^\infty(R^n)} \int_0^t \int_{R/2 \leq |x| \leq R} u^{(e)}_{e,R}(x, \tau) dx d\tau.
\]
By letting first \( j \to \infty \) and then \( \varepsilon = \varepsilon_i \to 0, \) \( R \to \infty, \) in the above inequality, we get (1.7). Since uniqueness of solution of (1.6) follows from Theorem 1.1. This completes the proof of the theorem.

Theorem 1.4. Let \( u^{(1)}_e, u^{(2)}_e, f_1, f_2 \) be as in Theorem 1.1. Then
\[
\int_{R^n} |u^{(1)}_e - u^{(2)}_e|(x, t) dx \leq \int_{R^n} |f_1 - f_2| dx \forall 0 < t < 1.
\]

Proof. By Theorem 1.1 and the proof of Theorem 1.3, there exist solutions \( u^{(1)}_{e, \varepsilon}, u^{(2)}_{e, \varepsilon} \in C^\infty(R^n \times (0, 1)) \cap L^\infty(R^n \times (0, 1)), \) \( 0 < \varepsilon < 1, \) of (1.6) with initial values \( u^{(1)}_{e, \varepsilon}(x, 0) = f_1 * \rho_\varepsilon + \varepsilon, \) \( u^{(2)}_{e, \varepsilon}(x, 0) = f_2 * \rho_\varepsilon + \varepsilon \) respectively such that \( u^{(1)}_{e, \varepsilon} \) and \( u^{(2)}_{e, \varepsilon} \) converges uniformly to \( u^{(1)}_e \) and \( u^{(2)}_e \) respectively on compact subsets of \( R^n \times (0, 1) \) as \( \varepsilon \to 0. \)
By a proof similar to the proof of (1.14), we have
\[
\int_{B_R(0)} (u_1^{(q)} - u_2^{(q)}) (x, t) \theta(x) dx \leq \int_{R^n} (f_1 - f_2)_+ dx + C_{R_0} R^{-n} + \varepsilon C_{R_0}
\]
for all \( \theta \in C_0^\infty(B_{R_0}(0)) \), \( R_0 > 2 \), \( R > R_0 + 1 \), \( 0 < t < 1 \) where \( C_{R_0} \) and \( C_{R_0}^\varepsilon > 0 \) are constants depending only on \( R_0 \), \( \|u_1^{(q)}\|_{L^\infty(R^n)} \) and \( \|u_2^{(q)}\|_{L^\infty(R^n)} \).

Letting \( R \to \infty \), \( \varepsilon \to 0 \), we get
\[
\int_{R^n} (u_1^{(q)} - u_2^{(q)})(x, t) \theta(x) dx \leq \int_{R^n} (f_1 - f_2)_+ dx
\]
for all \( \theta \in C_0^\infty(B_{R_0}(0)) \), \( R_0 > 2 \), \( 0 < t < 1 \). Putting \( \theta = \chi_{\{u_1^{(q)} > u_2^{(q)}\}} \) \( \rho_\varepsilon \) and letting first \( \varepsilon \to 0 \) and then \( R_0 \to \infty \),
\[
\int_{R^n} (u_1^{(q)} - u_2^{(q)})_+(x, t) dx \leq \int_{R^n} (f_1 - f_2)_+ dx \quad \forall 0 < t < 1.
\]

Similarly,
\[
\int_{R^n} (u_1^{(q)} - u_2^{(q)})_-(x, t) dx \leq \int_{R^n} (f_1 - f_2)_- dx \quad \forall 0 < t < 1.
\]

Combining the above two inequalities the theorem follows.

**Lemma 1.5.** If \( f \in C_0^1(R^n) \) and \( f_\varepsilon = f + \varepsilon, \ 0 < \varepsilon < 1 \), then (1.1) has a unique solution \( u_\varepsilon^{(q)} \in C_0^\infty(R^n \times (0, 1)) \cap C^1(R^n \times [0, 1)) \) in \( R^n \times (0, 1) \) with \( u_\varepsilon^{(q)}(x, 0) = f_\varepsilon(x) \) such that \( u_\varepsilon^{(q)} \) converges uniformly on compact subsets of \( R^n \times (0, 1) \) to the solution \( u^{(q)} \) of (1.6) with \( u^{(q)}(x, 0) = f(x) \) as \( \varepsilon \to 0 \).

Moreover
\[
\|u_\varepsilon^{(q)}\|_{L^\infty(R^n)} \leq \|f_\varepsilon\|_{L^\infty(R^n)} \quad \forall k = 1, 2, \ldots, n.
\]

**Proof.** By Theorem 1.4 and an argument similar to the proof of Theorem 1.3, for any \( 0 < \varepsilon < 1 \) there exists a unique solution \( u_\varepsilon^{(q)} \in C_0^\infty(R^n \times (0, 1)) \cap C^1(R^n \times [0, 1)) \) to (1.1) in \( R^n \times (0, 1) \) with \( u_\varepsilon^{(q)}(x, 0) = f(x) + \varepsilon \) and
\[
(1.16) \quad \varepsilon \leq u_\varepsilon^{(q)} \leq \|f\|_{L^\infty(R^n)} + \varepsilon
\]
such that \( u_\varepsilon^{(q)} \) converges uniformly on compact subsets of \( R^n \times (0, 1) \) to the solution \( u^{(q)} \) of (1.6) with \( u^{(q)}(x, 0) = f(x) \) as \( \varepsilon \to 0 \).

Since \( u_\varepsilon^{(q)} \in C_0^\infty(R^n \times (0, 1)) \cap C^1(R^n \times [0, 1)) \), differentiating (1.1) with respect to \( x_k \) and writing \( z = u_\varepsilon^{(q)}(x_k) \), we get
\[
\begin{cases}
z_t = \Delta(mu_\varepsilon^{(q)m-1}z) + (u_\varepsilon^{(q)q-1}z)_{x_1}, & (x, t) \in R^n \times (0, 1), \\
z(x, 0) = f_\varepsilon(x), & x \in R^n,
\end{cases}
\]
for all \( k = 1, 2, \ldots, n \). Since the above equation is nondegenerate by (1.16), by the maximum principle,
\[
\|z\|_{L^\infty(R^n)} \leq \|f_\varepsilon\|_{L^\infty(R^n)} \quad \forall k = 1, 2, \ldots, n
\]
and the lemma follows.

**Lemma 1.6.** Let \( 0 \leq f \leq M \) with \( \text{supp} f \subset B_{R_1}(0) \) for some \( R_1 > 0 \). Then there exists \( R' > 0 \) depending only on \( m, R_1, M \) and is independent of \( q > m + 1 \) such that
\[
u^{(q)}(x, t) = 0 \quad \forall x = (x_1, \ldots, x_n) \in R^n, \ x_i \leq -R', \ 0 \leq t < 1, \ q > m + 1,
\]
and

\[ 0 \leq u^{(q)}(x, t) \leq \left( \frac{x_1 + R' + 1}{t + (1/M^{q-1})} \right)^{1/q-1} \leq \left( \frac{x_1 + R' + 1}{t} \right)^{1/q-1} \]

for all \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ x_1 \geq -R', \ 0 < t < 1, \ q > m + 1. \)

**Proof.** Let

\[ w(x_1, t) = \frac{1}{(t + t_0)^{1/m+1}} \left( a^2 - C_1 \left( \frac{x_1}{(t + t_0)^{1/m+1}} \right)^2 \right)^{1/m-1}, \quad x_1 \in \mathbb{R}, \ t \geq 0, \]

be the Barenblatt solution for the porous medium equation \( \frac{w_t}{(\alpha x_1)^q} = \left( \frac{\partial w}{\partial x_1} \right)^q \) ([B], [HP]) where \( C_1 = \frac{m-1}{2m} \left( \frac{1}{(m+1)} \right), \ t_0 = \min \left( 1, \left( \frac{4CR_1^2}{2m-1/M} \right)^{(m+1)/2} \right) \) and

\[ a = \left( C_1 (2R_1/t_0)^{1/m+1} \right)^{1/2} + \left( M t_0^{1/m+1} \right)^{m-1}. \]

Then \( w \) is a supersolution of (1.1) in \( (-\infty, 0] \times \mathbb{R}^{n-1} \times (0, 1) \) with

\[ u^{(q)}(x + x_0) = f(x + x_0) \leq M \leq w(x_1, 0) \]

for all \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ x_1 \leq 0 \) where \( x_0 = (R_1, 0, \ldots, 0) \) and

\[ w(0, t) \geq \frac{1}{(1 + t_0)^{1/m+1}} a^{2/m-1} \]

\[ \geq \frac{1}{2^{1/m+1}} C_1 \left( \frac{2R_1}{t_0} \right)^{2/m-1} \]

\[ \geq \frac{1}{2^{1/m+1}} \left( \frac{4R_1^2C_1}{(2^{1/m+1} M)^{m-1}} \right)^{1/m-1} \]

\[ = M \geq u(x_0, t) \]

for all \( 0 < t < 1 \) by (1.8). Hence by applying the maximum principle (Corollary 1.2) to the functions \( u^{(q)}(x + x_0, \cdot) \) and \( w \) in the region \( (-\infty, 0] \times \mathbb{R}^{n-1} \times (0, 1) \), we get

\[ u^{(q)}(x + x_0, t) \leq w(x_1, t) \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ x_1 \leq 0, \ 0 \leq t < 1. \]

Now for each \( 0 < t < 1, \ \text{supp} \ w(x_1, t) \subset B_{R_t}(0) \) where

\[ R_t = \frac{a}{C_1^{1/2}} (t + t_0)^{1/m+1} \leq \frac{2a}{C_1^{1/2}} \quad (= R_2 \text{ say}). \]

Hence

\[ u^{(q)}(x + x_0, t) \leq w(x_1, t) = 0 \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ x_1 \leq -R_2, \ 0 \leq t < 1, \]

\[ \Rightarrow u^{(q)}(x, t) = 0 \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ x_1 \leq -R', \ 0 \leq t < 1, \]

\[ q > m + 1, \]

(1.17)

where \( R' = \max(R_2 - R_1, 0) \geq 0. \)
We next observe that

\[
\tilde{w}(x_1, t) = \left( \frac{x_1 + R' + 1}{t + (1/M^{q-1})} \right)^{1/q-1}, \quad q > m + 1,
\]
is a supersolution of (1.1) in \([-R', R_3] \times \mathbb{R}^{n-1} \times (0, \infty)\) with

\[
\begin{align*}
&u(q)(x, 0) \leq M \leq \tilde{w}(x_1, 0) \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad -R' \leq x_1 \leq R_3, \\
u(q)(x, t) \leq M \leq \tilde{w}(x_1, t) \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \\
x_1 = -R' \text{ or } x_1 = R_3, \quad 0 \leq t < 1,
\end{align*}
\]
for all \(R_3 > \max(2M^{q-1} - R' + 1, 0)\) by (1.17). Hence by applying Corollary 1.2 to the function \(u(q)\) and \(\tilde{w}\) in the region \([-R', R_3] \times \mathbb{R}^{n-1} \times (0, 1)\), we get

\[
u(q)(x, t) \leq \tilde{w}(x_1, t)
\]
for all \(x = (x_1, \ldots, x_n) \in [-R', R_3] \times \mathbb{R}^{n-1}, \quad 0 \leq t < 1, \quad q > m + 1, \quad R_3 > \max(2M^{q-1} - R' + 1, 0)\). By letting \(R_3 \to \infty\), the lemma follows.

**Lemma 1.7.** Suppose \(f\) is as in Lemma 1.6. Let \(\Omega \subset \mathbb{R}^n\) be a bounded open set with \(\partial \Omega \in C^2\) and \(\eta \in C^\infty(\mathbb{R}^n \times (0, 1))\). Then

\[
\int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{u(q)}{q} \cdot \eta \, dx \, dt \to 0 \quad \text{as } q \to \infty
\]
for any \(0 < \tau_1 \leq \tau_2 < 1\).

**Proof.** By Lemma 1.6, there exists a constant \(R' > 0\) such that

\[
u(q)(x, t) \leq \left( \frac{|x_1| + R' + 1}{t} \right)^{1/q-1} \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad 0 < t < 1,
\]
and by Theorem 1.3 \(\|u(q)\|_{L^\infty} \leq \|f\|_{L^\infty}\) for all \(q > m + 1\). Hence

\[
\begin{align*}
&\int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{u(q)}{q} \cdot \eta \, dx \, dt = \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{u(q)^2}{q} \cdot \eta \, dx \, dt \\
&\leq \|\eta\|_{L^\infty} \|f\|_{L^\infty}^2 \int_{\tau_1}^{\tau_2} \int_{\Omega} \left( \frac{|x_1| + R' + 1}{t} \right)^{q-2/q-1} \, dx \, dt \\
&\leq \|\eta\|_{L^\infty} \|f\|_{L^\infty}^2 \left( \frac{R'' + R' + 1}{\tau_1} \right)^{q-2/q-1} \to 0
\end{align*}
\]
as \(q \to \infty\) where \(R'' = \sup\{|x_1| : x = (x_1, \ldots, x_n) \in \Omega| < \infty\)).

**Lemma 1.8.** Let \(f \in C_0(\mathbb{R}^n)\) and let \(p(q)(x, t) = \int_0^t \frac{u(q)(x, \tau)}{q} \, d\tau\). Then \(\{p(q)\}_{q>m+1}\) is uniformly bounded on compact subsets of \(\mathbb{R}^n \times [0, 1)\). For any sequence \(\{p(i)\}_{i=1}^\infty, \quad q_i \to \infty\) as \(i \to \infty\), of \(\{p(q)\}_{q>m+1}\), there exists a subsequence \(\{p(q)\}_{i=1}^\infty, \quad q_i \to \infty\) of \(\{p(q)\}_{q=1}^\infty\), a sequence of functions \(\{p_j\}_{j=1}^\infty \subset L^\infty_{\text{loc}}(\mathbb{R}^n)\), \(\tilde{g} \in L^\infty_{\text{loc}}(\mathbb{R}^n)\), \(p_j, \tilde{g} \geq 0\), and a sequence \(\{\varepsilon_j\}_{j=1}^\infty \subset \mathcal{R}, \quad \varepsilon_j \to 0\) as \(j \to \infty\), such that

\[
(1.18)
\]
\[
\begin{align*}
&p^{(q)}(\cdot, \varepsilon_j) \to p_j(\cdot) \quad \text{weakly in } (L^\infty(K))^* \quad \text{as } i \to \infty, \quad \forall j = 1, 2, \ldots, \\
p_j(\cdot) \to \tilde{g}(\cdot) \quad \text{weakly in } (L^\infty(K))^* \quad \text{as } j \to \infty
\end{align*}
\]
for any compact subset \(K \subset \mathbb{R}^n\).
Proof. By Theorem 1.3, \( \|u^{(q)}\|_{L^\infty(R^n)} \leq \|f\|_{L^\infty(R^n)} \) for all \( q > m + 1 \) and by Lemma 1.6 there exists \( R' > 0 \) such that

\[
0 \leq u^{(q)}(x, \tau) \leq \left( \frac{|x_1| + R' + 1}{\tau} \right)^{1/(q-1)} \quad \forall x = (x_1, x_2, \ldots, x_n) \in R^n, \quad 0 < \tau < 1, \ q > m + 1.
\]

Hence

\[
0 \leq p^{(q)}(x, t) = \int_0^t \frac{u^{(q)}(x, t) - u^{(q)}(x, t)}{q} \, dt \leq \frac{\|f\|_{L^\infty(R^n)}^2}{q} \int_0^t \left( \frac{|x_1| + R' + 1}{\tau} \right)^{q-2/q-1} \, dt \leq \frac{q-1}{q} \|f\|_{L^\infty(R^n)}^2 (|x_1| + R' + 1)^{q-2/q-1} t^{1/q-1}
\]

for all \( x = (x_1, x_2, \ldots, x_n) \in R^n, \ 0 < t < 1, \ q > m + 1 \). Thus \( \{p^{(q)}\}_{q>m+1} \) is uniformly bounded on compact subsets of \( R^n \times [0, 1) \). So any sequence \( \{p^{(q_i)}\}_{i=1}^\infty \) of \( \{p^{(q)}\}_{q>m+1} \) will have a subsequence \( \{p^{(q_{i,i})}\}_{i=1}^\infty \) such that \( \{p^{(q_{i,i})}\}_{i=1}^\infty \) converges weakly in \( (L^\infty(K))^* \) for any compact subset \( K \subset R^n \).

Let \( p_1(\cdot) = \lim_{i \to \infty} p^{(q_{i,i})}(\cdot, 1/2) \). Then \( \{p^{(q_{i,i})}\}_{i=1}^\infty \) has a subsequence \( \{p^{(q_{i,i})}\}_{i=1}^\infty \) such that \( p^{(q_{i,i})}(x, 1/2) \to p_1(x) \) a.e. \( x \in R^n \) as \( i \to \infty \). Without loss of generality we may assume \( p^{(q_{i,i})}(x, 1/2) \to p_1(x) \) a.e. \( x \in R^n \) as \( i \to \infty \). We may also assume that \( q_1 < q_{1,1} \). Since \( \{p^{(q_{i,i})}\}_{i=1}^\infty \) is uniformly bounded on compact subsets of \( R^n \), \( \{p^{(q_{i,i})}\}_{i=1}^\infty \) has a subsequence \( \{p^{(q_{i,i})}\}_{i=1}^\infty \) converging weakly in \( (L^\infty(K))^* \) for any compact set \( K \subset R^n \). Let \( p_2(\cdot) = \lim_{i \to \infty} p^{(q_{2,i})}(\cdot, 1/3) \). We may assume without loss of generality that \( p^{(q_{2,i})}(x, 1/3) \to p_2(x) \) a.e. \( x \in R^n \) as \( i \to \infty \) and \( q_1,1 < q_{2,1} \).

Repeating the argument, for each \( j = 2, 3, \ldots \), we can find a subsequence \( \{p^{(q_{j,i})}(x, 1/(j+1))\}_{i=1}^\infty \) of \( \{p^{(q_{j-1,i})}(x, 1/(j+1))\}_{i=1}^\infty \) with \( q_{j,1} > q_{j-1,1} \) and a function \( p_j \in L^\infty_{\text{loc}}(R^n) \) such that \( p^{(q_{j,i})}(x, 1/(j+1)) \to p_j(x) \) weakly in \( (L^\infty(K))^* \) for every compact set \( K \subset R^n \) as \( i \to \infty \) and \( p^{(q_{j,i})}(x, 1/(j+1)) \to p_j(x) \) a.e. \( x \in R^n \) as \( i \to \infty \).

Let \( q'_j = q_{j,i} \). Then for each \( j = 1, 2, \ldots, \) \( \{p^{(q'_j)}(\cdot, 1/(j+1))\}_{i=1}^\infty \) is a subsequence of \( \{p^{(q_{j,i})}(x, 1/(j+1))\}_{i=1}^\infty \). Hence \( p^{(q'_j)}(x, 1/(j+1)) \to p_j(x) \) weakly in \( (L^\infty(K))^* \) for every compact set \( K \subset R^n \) as \( i \to \infty \) and \( p^{(q'_j)}(x, 1/(j+1)) \to p_j(x) \) a.e. \( x \in R^n \) as \( i \to \infty \). Thus \( \{p_j\}_{j=1}^\infty \) is also uniformly bounded on every compact subset of \( R^n \). So there exists a subsequence \( \{p_{j_k}\}_{k=1}^\infty \) of \( \{p_j\}_{j=1}^\infty \) and a function \( \tilde{g} \in L^\infty_{\text{loc}}(R^n) \) such that \( p_{j_k} \to \tilde{g} \) weakly in \( (L^\infty(K))^* \) for any compact subset \( K \subset R^n \). Letting \( \epsilon_k = 1/(j_k + 1) \), the lemma follows.

In this section we will first establish some technical lemmas and prove the main theorem (Theorem 2.10) under the assumption that \( f \in C^1_0(R^n) \) (Theorem 2.9). The main theorem will then follow by an approximation argument.
Theorem 2.1. Suppose \( f \in C_0(\mathbb{R}^n) \). For any sequence \( \{u^{(q)}\}_{q=1}^{\infty} \), \( q_i \to \infty \) as \( i \to \infty \), of \( \{u^{(q)}\}_{q>m+1} \), there exists a subsequence \( \{u^{(q')}\}_{q=1}^{\infty} \) of \( \{u^{(q)}\}_{q=1}^{\infty} \) and a \( u^{(\infty)} \in C(\mathbb{R}^n \times (0,1)) \), \( 0 \leq u^{(\infty)} \leq 1 \), such that \( u^{(q')} \to u^{(\infty)} \) uniformly on compact subsets \( R \times (0,1) \) as \( i \to \infty \). Moreover \( u^{(\infty)} \) satisfies (0.2) with initial trace \( g \in L_1(\mathbb{R}^n) \), \( 0 \leq g \leq 1 \), satisfying (0.3) for some function \( \tilde{g} \in L_\infty(\mathbb{R}^n) \), \( \tilde{g} \geq 0 \).

Proof. The proof is a modification of the proof of Theorem 4 of [H1]. We first observe that \( u^{(q)} \) is uniformly bounded by \( \|f\|_{L_\infty} \) by Theorem 1.3 and there exists \( R' > 0 \) such that
\[
0 < u^{(q)}(x,t) < M^1 + R' \quad \forall x = (x_1, x') \in \mathbb{R}^n, \quad 0 < t < 1, \quad q > m + 1,
\]
by Lemma 1.6. If \( \gamma(s) = s^{q/m} q \), then \( \gamma(u^{(q)}m) = \frac{u^{(q)}m}{q} \) and
\[
\gamma'(u^{(q)}m(x,t)) = \frac{1}{q} u^{(q)}m(x,t)^{q-m} \leq \frac{1}{q} \left( \frac{|x_1| + R' + 1}{t} \right)^{q-m/q-1} \leq \frac{1}{q} \left( \frac{|x_1| + R' + 1}{t} \right)^{q-m/q-1} \forall x = (x_1, x') \in \mathbb{R}^n, \quad 0 < t < 1, \quad q > m + 1,
\]
by (2.1). Hence both \( u^{(q)}m \) and \( \gamma'(u^{(q)}m) \) are uniformly bounded on compact subsets of \( \mathbb{R}^n \times (0,1) \) for \( q > m + 1 \). By the result of P. Sacks [S1], \( \{u^{(q)}m\}_{q=m+1}^{\infty} \) is uniformly Hölder continuous on every compact subset of \( \mathbb{R}^n \times (0,1) \). Hence \( \{u^{(q)}m\}_{q=m+1}^{\infty} \) has a convergent subsequence \( \{u^{(q')}\}_{q=1}^{\infty} \) such that \( \{u^{(q')}\}_{q=1}^{\infty} \) converges uniformly on every compact subset of \( \mathbb{R}^n \times (0,1) \). Without loss of generality we may assume that \( \{u^{(q)}\}_{q=1}^{\infty} \) converges uniformly on every compact subset of \( \mathbb{R}^n \times (0,1) \). Let \( u^{(\infty)} = \lim_{i \to \infty} u^{(q)} \). Then \( u^{(\infty)} \in C(\mathbb{R}^n \times (0,1)) \).

Putting \( q = q_i \) and letting \( i \to \infty \) in (2.1), we get \( 0 \leq u^{(\infty)} \leq 1 \). Putting \( h(u) = u^{q_i/q_i} \), \( u = u^{(q)} \) in (0.5) and letting \( i \to \infty \) we see that, by Lemma 1.6, \( u^{(\infty)} \) satisfies
\[
\int_{t_1}^{t_2} \int_{\Omega} \left[ u^{m} \Delta \eta + u \frac{\partial \eta}{\partial t} \right] dx dt = \int_{t_1}^{t_2} \int_{\partial \Omega} u^{m} \frac{\partial \eta}{\partial N} d\sigma ds + \int_{\Omega} u \eta dx \bigg|_{t_1}^{t_2}
\]
for all bounded open sets \( \Omega \subset \mathbb{R}^n \) with \( \partial \Omega \in C^2 \), \( 0 < \tau_1 \leq \tau_2 < 1 \), \( \eta \in C^\infty(\Omega \times [\tau_1, \tau_2]) \), \( \eta = 0 \) on \( \partial \Omega \times [\tau_1, \tau_2] \). Hence \( u^{(\infty)} \) is a solution of the equation \( u_t = \Delta u^m \) in \( \mathbb{R}^n \times (0,1) \). Since \( \|u^{(\infty)}\|_{L_\infty} \leq \|f\|_{L_\infty} \), \( u^{(\infty)} \) has an initial trace \( d\mu \) by [DK] and \( d\mu \) is absolutely continuous with respect to the Lebesgue measure. Hence \( d\mu = g(x)dx \) for some function \( g \geq 0 \). Since \( 0 \leq u^{(\infty)} \leq 1 \) and
\[
\lim_{i \to 0} u^{(\infty)}(x,t) = g(x) \quad \text{a.e.} \quad x \in \mathbb{R}^n
\]
by the result of [DFK], \( 0 \leq g \leq 1 \). Since
\[
\int_{\mathbb{R}^n} u^{(q)}(x,t)dx = \int_{\mathbb{R}^n} f(x)dx, \quad \forall 0 < t \leq 1, \quad i = 1, 2, \ldots
\]
Letting $i \to \infty$, we get by Fatou's lemma,
\[
\int_{R^n} u^{(\infty)}(x, t) dx \leq \int_{R^n} f(x) dx, \quad \forall 0 < t \leq 1.
\]

Letting $t \to 0$, we get by Fatou's lemma and (2.3),
\[
\int_{R^n} g(x) dx \leq \int_{R^n} f(x) dx.
\]

Hence $g \in L^1(R^n)$. Let $p^{(q_i)}$ be as in Lemma 1.8 and $\Omega$ be a bounded open subset of $R^n$ with $\partial \Omega \in C^2$. Then by Lemma 1.8 there exists a constant $C_1 > 0$ such that $\|p^{(q_i)}\|_{L^{\infty}(\Omega \times [0,1])} \leq C_1$ for all $q > m + 1$ and there exists a subsequence $\{p^{(q'_i)}\}_{i=1}^{\infty}$, a sequence of functions $\{p_j\}_{j=1}^{\infty} \subset L^{\infty}_{\text{loc}}(R^n)$, $\bar{g} \in L^{\infty}_{\text{loc}}(R^n)$, $p_j \geq 0$, and a sequence $\{e_j\}_{j=1}^{\infty} \subset R$, $e_j \to 0$ as $j \to \infty$, such that (1.18) holds. Hence for any $0 < \tau_2 < 1$, $\eta \in C_0^\infty(R^n)$,
\[
\begin{align*}
\int_0^{\tau_2} \int_{\Omega} \frac{u^{(q'_i)}q_i}{q_i} \eta_{x_i} dx d\tau \leq \int_{\Omega} \int_0^{\tau_2} \frac{u^{(q'_i)}q_i}{q_i} \eta_{x_i} dx d\tau \\
+ \int_{\Omega} \left( \int_0^{e_j} \frac{u^{(q'_i)}q_i}{q_i}(x, \tau) d\tau \right) \eta_{x_i}(x) dx - \int_{R^n} \bar{g} \eta_{x_i} dx \\
\leq \|\eta_{x_i}\|_{L^{\infty}(R^n)} \int_{\Omega} \int_0^{\tau} \frac{u^{(q'_i)}q_i}{q_i} \eta_{x_i} dx d\tau \\
+ \int_{\Omega} p^{(q'_i)}(x, e_j) \eta_{x_i}(x) dx - \int_{\Omega} p_j(x) \eta_{x_i}(x) dx \\
+ \int_{\Omega} p_j(x) \eta_{x_i}(x) dx - \int_{\Omega} \bar{g}(x) \eta_{x_i}(x) dx.
\end{align*}
\]
Letting first $i \to \infty$ and then $j \to \infty$, we get by Lemma 1.7 and Lemma 1.8,
\[
\limsup_{i \to \infty} \left| \int_0^{\tau_2} \int_{\Omega} \frac{u^{(q'_i)}q_i}{q_i} \eta_{x_i} dx d\tau - \int_{\Omega} \bar{g} \eta_{x_i} dx \right| = 0
\]
(2.4)

Putting $h(u) = \frac{u^{(q'_i)}}{q_i}$, $u = u^{(q_i)}$, in (0.5) and letting $\tau_1 \to 0$, we have
\[
\begin{align*}
\int_0^{\tau_2} \int_{R^n} u^{(q'_i)}q_i \Delta \eta dx d\tau + \int_0^{\tau_2} \int_{R^n} \frac{u^{(q'_i)}q_i}{q_i} \eta_{x_i} dx d\tau \\
= \int_{R^n} u^{(q'_i)}(x, \tau_2) \eta(x) dx - \int_{R^n} f \eta dx
\end{align*}
\]
for all $\eta \in C_0^\infty(R^n)$, $0 < \tau_2 < 1$. Letting $i \to \infty$, we get by (2.4) and Lebesgue dominated convergence theorem,
\[
\begin{align*}
\int_0^{\tau_2} \int_{R^n} u^{(\infty)}q_i \Delta \eta dx d\tau + \int_{R^n} \bar{g} \eta_{x_i} dx d\tau = \int_{R^n} u^{(\infty)}(x, \tau_2) \eta(x) dx - \int_{R^n} f \eta dx
\end{align*}
\]
for all $\eta \in C_0^\infty(R^n)$. Letting $\tau_2 \to 0$,
\[
\int \bar{g} \eta_{x_i} dx = \int g \eta dx - \int f \eta dx \quad \forall \eta \in C_0^\infty(R^n)
\]
\[
\Rightarrow g + \bar{g} \eta_{x_i} = f \quad \text{in } \mathcal{D}'(R^n).
\]
This completes the proof of Theorem 2.1.

We will now let
\[ S(g) = \left\{ x_0 \in \mathbb{R}^n : \lim_{h \to 0} \frac{1}{|B_h(0)|} \int_{B_h(x_0)} |g(x) - g(x_0)| dx = 0 \right\}, \]
\[ G(u^{(\infty)}, g) = \left\{ x \in \mathbb{R}^n : \lim_{i \to 0} u^{(\infty)}(x, t) = g(x) \right\}. \]

**Lemma 2.2.** Let \( f, u^{(\infty)}, u^{(q_i)}, g \) be as in Theorem 2.1 and let \( S^* = S(g) \cap G(u^{(\infty)}, g) \cap \{ g < 1 \} \). If \( x_0 \in S^* \) is such that \( g(x_0) \leq \theta < 1 \), then for any \( \theta_1 \in (\theta, 1) \) and \( \delta > 0 \), there exists \( q_0 > m + 1 \), \( e_0 > 0 \), \( 0 < e_0 < 1/2 \), such that
\[
\inf_{|x - x_0| \leq \delta} u^{(q_i)}(x, t) < \theta_1 \quad \forall 0 < t \leq e_0, q_i > q_0.
\]

**Proof.** The proof is similar to the proof of Theorem 3.3 of [CF]. Suppose the lemma is not true. Then there exists \( \theta_1 \in (\theta, 1) \), \( \delta > 0 \), and \( \{ e_i \}_{i=1}^{\infty}, 0 < e_i < 1/2, i = 1, 2, \ldots, e_i \to 0 \) as \( i \to \infty \) and a subsequence \( \{ u^{(q_i')} \}_{i=1}^{\infty} \) of \( \{ u^{(q_i')} \}_{i=1}^{\infty} \) such that
\[
\inf_{|x - x_0| \leq \delta} u^{(q_i')} (x, e_i) > \theta_1.
\]

Let \( \tilde{u}^{(q_i''')} \) be the solution of (1.1) in \( \mathbb{R}^n \times (0, 1) \) with initial value \( \tilde{u}^{(q_i''')} (x, 0) = \theta_1 \chi_{B_\delta(x_0)} \) where \( \chi_{B_\delta(x_0)} \) is the characteristic function of the set \( B_\delta(x_0) \). By Theorem 1.1,
\[
\tilde{u}^{(q_i''')} (x, t) \leq u^{(q_i''')} (x, t + e_i) \quad \forall x \in \mathbb{R}^n, 0 < t \leq 1/2
\]
\[
\Rightarrow \int \int \tilde{u}^{(q_i''')} (x, t) \eta(x, t) dx dt \leq \int \int u^{(q_i''')} (x, t + e_i) \eta(x, t) dx dt
\]
\[
= \int \int u^{(q_i''')} (x, t) \eta(x, t - e_i) dx dt
\]
(2.5)
for all \( \eta \in C_0^\infty (\mathbb{R}^n \times (0, 1/2)) \) and \( e_i \) sufficiently small. By Theorem 2.1, \( \{ \tilde{u}^{(q_i''')} \}_{i=1}^{\infty} \) has a convergent subsequence converging uniformly on compact subsets of \( \mathbb{R}^n \times (0, 1) \). Without loss of generality, we may assume that \( \{ \tilde{u}^{(q_i''')} \}_{i=1}^{\infty} \) converges uniformly on compact subsets of \( \mathbb{R}^n \times (0, 1) \). Let \( \tilde{u}^{(\infty)} = \lim_{i \to \infty} \tilde{u}^{(q_i''')} \). Since \( 0 \leq u^{(q_i'')} \leq \theta_1 < 1 \), letting \( i \to \infty \) in (2.5), we get by Lebesgue dominated convergence theorem
\[
\int \int \tilde{u}^{(\infty)} (x, t) \eta(x, t) dx dt \leq \int \int u^{(\infty)} (x, t) \eta(x, t) dx dt
\]
\[
\Rightarrow \tilde{u}^{(\infty)}(x, t) \leq u^{(\infty)}(x, t) \quad \forall x \in \mathbb{R}^n, 0 < t < 1/2
\]
\[
\text{since } \tilde{u}^{(\infty)}, u^{(\infty)} \in C(\mathbb{R}^n \times (0, 1/2))
\]
\[
\Rightarrow \int_{\mathbb{R}^n} \tilde{u}^{(\infty)} (x, t) \eta(x) dx \leq \int_{\mathbb{R}^n} u^{(\infty)} (x, t) \eta(x) dx \quad \forall \eta \in C_0^\infty (\mathbb{R}^n)
\]
\[
\Rightarrow \int_{\mathbb{R}^n} \theta_1 \chi_{B_\delta(x_0)} (x) dx \leq \int_{\mathbb{R}^n} g(x) \eta(x) dx \quad \text{as } t \to 0 \quad \forall \eta \in C_0^\infty (\mathbb{R}^n)
\]
\[
\Rightarrow \theta < \theta_1 \leq g(x_0)
\]
since \( x_0 \in S(g) \). Thus contradiction arise and the lemma follows.
Lemma 2.3. Suppose \( f \in C^1_0(R^n) \). Let \( u^{(\infty)}, u^{(q_i)} \), \( g \) be as in Theorem 2.1 and let \( S^* \) be as in Lemma 2.2. If \( x_0 \in S^* \) is such that \( g(x_0) \leq \theta < 1 \), then for any \( \theta_1 \in (0, 1) \), there exists \( \theta_0 \) depending only on \( \theta \), \( \theta_1 \) and \( \|f_{x_k}\|_{L^\infty(R^n)}, \ k = 1, 2, \ldots, n \), such that

\[
u^{(q_i)}(x, t) \leq \theta_1 \quad \forall x \in B_\delta(x_0), \ 0 < t \leq \theta_0, \ q_i' \geq q_0,
\]

where \( \delta = (\theta_1 - \theta)/4(\sqrt{n} \max_{1 \leq k \leq n} \|f_{x_k}\|_{L^\infty(R^n)} + 1) \).

Proof. The proof is similar to the proof of Theorem 2.4 of [H2]. Let

\[
\delta = (\theta_1 - \theta)/4(\sqrt{n} \max_{1 \leq k \leq n} \|f_{x_k}\|_{L^\infty(R^n)} + 1).
\]

Then by Lemma 2.2, there exists \( q_0 > m + 1, \ e_0 > 0, \ 0 < e_0 < 1/2 \), such that

\[
\inf_{|x-x_0| \leq \delta} \nu^{(q_i)}(x, t) \leq \frac{\theta_1 + \theta}{2} \quad \forall 0 < t \leq \theta_0, \ q_i' \geq q_0.
\]

Hence for each \( q_i' \geq q_0 \) and \( 0 < t \leq \theta_0 \), there exists an \( x_t \in B_\delta(x_0) \) such that

\[
u^{(q_i)}(x_t, t) \leq \frac{\theta_1 + \theta}{2} \quad \forall 0 < t \leq \theta_0.
\]

For any \( 0 < \epsilon < 1 \), let \( f_\epsilon = f + \epsilon \) and let \( u^{(q_i)}_\epsilon \) be the solution of (1.1) in \( R^n \times (0, 1) \) with \( u^{(q_i)}_\epsilon(x, 0) = f_\epsilon(x) \) given by Lemma 1.5. Then by Lemma 1.5,

\[
|u^{(q_i)}_\epsilon(x, t) - u^{(q_i)}_\epsilon(x_t, t)|
\]

\[
= \left| \int_0^1 \frac{d}{ds} u^{(q_i)}_\epsilon(sx + (1 - s)x_t, t) ds \right|
\]

\[
\leq \int_0^1 |\nabla u^{(q_i)}_\epsilon(sx + (1 - s)x_t, t) \cdot (x - x_t)| ds
\]

\[
\leq \sqrt{n} \max_{1 \leq k \leq n} \|f_{x_k}\|_{L^\infty(R^n)} |x - x_t|
\]

\[
\leq 2\delta \sqrt{n} \max_{1 \leq k \leq n} \|f_{x_k}\|_{L^\infty(R^n)} \leq (\theta_1 - \theta)/2
\]

\[
\Rightarrow u^{(q_i)}_\epsilon(x, t) \leq u^{(q_i)}_\epsilon(x_t, t) + (\theta_1 - \theta)/2 \leq (\theta_1 + \theta)/2 + (\theta_1 - \theta)/2 \leq \theta_1
\]

for all \( x \in B_\delta(x_0), \ 0 < t \leq \theta_0, \ q_i' \geq q_0 \). Since \( u^{(q_i)}_\epsilon \to u^{(q_i)} \) uniformly on compact subsets of \( R^n \times (0, 1) \) as \( \epsilon \to 0 \) by Lemma 1.5, letting \( \epsilon \to 0 \) we get

\[
u^{(q_i)}(x, t) \leq \theta_1 \quad \forall x \in B_\delta(x_0), \ 0 < t \leq \theta_0, \ q_i' \geq q_0.
\]

Lemma 2.4. Suppose \( f \in C^1_0(R^n) \). Let \( g, \tilde{g} \) be as in Theorem 2.1 and let \( S^* \) be as in Lemma 2.2. Then \( g(x) = f(x) \), \( \tilde{g}(x) = 0 \) for all \( x \in S^* \cap S(\tilde{g}) \).

Proof. Let \( u^{(\infty)}, u^{(q_i)} \) be as in Theorem 2.1. By Theorem 2.1 we may assume without loss of generality that \( u^{(q_i)} \) converges uniformly to \( u^{(\infty)} \) on compact subsets of \( R^n \times (0, 1) \) as \( i \to \infty \). We also let \( \rho^{(q_i)}, p_j, \varepsilon_j \) be as in Lemma 1.8. Suppose \( x_0 \in S^* \cap S(\tilde{g}) \). Then there exists \( \theta, \theta_1 > 0 \) such that
\[ g(x_0) \leq \theta < \theta_1 < 1. \] By Lemma 2.3, there exists \( q_0 > m + 1, \delta > 0, \varepsilon_0 > 0, 0 < \varepsilon_0 < 1/2 \) such that
\[ u(q_i')(x, t) \leq \theta_1 \quad \forall x \in B_\delta(x_0), 0 < t \leq \varepsilon_0, q_i' \geq q_0. \]

Hence
\[
\begin{align*}
&\left| \int_{R^n} u(q_i')(x, t) \eta(x) dx - \int_{R^n} f(x) \eta(x) dx \right| \\
&= \left| \int_0^t \int_{R^n} \left[ \frac{\theta^{q_i'}}{q_i'} \eta_{x_1} \right] dx dt \right|
\end{align*}
\]
\[
\leq \theta_i^n \|\Delta \eta\|_{L^1(R^n)} t + \frac{\theta^{q_i'}}{q_i'} \|\eta_{x_1}\|_{L^1(R^n)} t \quad \forall q_i' \geq q_0, 0 < t \leq \varepsilon_0, \eta \in C_0^\infty(B_\delta(x_0)).
\]

Letting \( i \to \infty \),
\[
\begin{align*}
&\left| \int_{R^n} u(\infty)(x, t) \eta(x) dx - \int_{R^n} f(x) \eta(x) dx \right| \\
&\leq \theta_i^n \|\Delta \eta\|_{L^1(R^n)} t + \frac{\theta^{q_i'}}{q_i'} \|\eta_{x_1}\|_{L^1(R^n)} t \quad \forall q_i' \geq q_0, 0 < t \leq \varepsilon_0, \eta \in C_0^\infty(B_\delta(x_0)).
\end{align*}
\]

Letting \( t \to 0 \),
\[
\begin{align*}
&\int_{R^n} \eta dx = \int_{R^n} f \eta dx \quad \forall \eta \in C_0^\infty(B_\delta(x_0)) \Rightarrow g(x_0) = f(x_0)
\end{align*}
\]
since \( x_0 \in S(g) \). Similarly
\[
\begin{align*}
&\int_{B_\delta(x_0)} p(q_i')(x, e_j) dx \\
&= \int_{B_\delta(x_0)} \int_0^{e_j} u(q_i')q_i' dx \leq \frac{\theta^{q_i'}}{q_i'} |B_\delta(x_0)| e_j \to 0 \text{ as } i \to 0 \quad \forall j = 1, 2, \ldots
\end{align*}
\]
\[ \Rightarrow \int_{B_\delta(x_0)} p_j(x) dx = 0 \quad \text{by Fatou's lemma since } p_j \geq 0 \]
\[ \Rightarrow \int_{B_\delta(x_0)} \tilde{g}(x) dx = 0 \quad \text{by Fatou's lemma since } \tilde{g} \geq 0 \]
\[ \Rightarrow \tilde{g} \equiv 0 \text{ on } B_\delta(x_0) \]
\[ \Rightarrow \tilde{g}(x_0) = 0 \text{ since } x_0 \in S(\tilde{g}). \]

**Lemma 2.5.** Suppose \( f \in C_0^1(R^n) \) and let \( g, \tilde{g} \) be as in Theorem 2.1. Then there exists \( r' > 0 \) such that
\[
\begin{align*}
g(x) &= f(x), \\
\tilde{g}(x) &= 0
\end{align*}
\]
a.e. \( x \in R^n \setminus B_r(0) \)

**Proof.** Let \( u(\infty), u(q_i') \) be as in Theorem 2.1, \( S^* \) be as in Lemma 2.2 and let \( S_1 = S(g) \cap S(\tilde{g}) \cap G(u(\infty), g) \). \( S_2 = S(g) \cap G(u(\infty), g) \). For any \( 0 < \theta < 1, r > 0 \), let \( A_{\theta,r} = \{ x \in R^n \setminus B_r(0) : g(x) \geq \theta \} \). We now fix \( \theta, \theta_1 \in (0, 1) \) such that \( \theta < \theta_1 \). Choose a constant \( \theta' > 0 \) such that \( \theta < \theta' < \theta_1 \) and let
\[
\delta = \min((\theta' - \theta)/4(\sqrt{n} \max_{1 \leq k \leq n} \| f_{x_k} \|_{L^\infty(R^n)} + 1), 1). \]
Since \( g \in L^1(\mathbb{R}^n) \),
\[
\int_{|x| \geq r} g\,dx \to 0 \text{ as } r \to 0.
\]
Thus there exists \( r_0 > 0 \) such that
\[
\int_{|x| \geq r_0} g\,dx \leq \frac{1}{2} \theta |B_\delta(0)|
\]
\[
\Rightarrow \theta |A_{\theta, r_0}| \leq \frac{1}{2} \theta |B_\delta(0)|
\]
\[
\Rightarrow |A_{\theta, r_0}| \leq \frac{1}{2} |B_\delta(0)|.
\]

Let \( r' = r_0 + 1 \). Since \( |\mathbb{R}^n \setminus S_1| = 0 \) by the result of [DFK] and Chapter 1 of [S], (2.6) holds for a.e. \( x \in A_{\theta, r'} \) by Lemma 2.4. Hence in order to prove the lemma, it suffices to show that (2.6) holds for a.e. \( x \in A_{\theta, r} \cap S_1 \). Let \( y_0 \in A_{\theta, r} \cap S_1 \). If \( |B_\delta(y_0) \cap A_{\theta, r}| = 0 \), then
\[
g(z) \geq \theta \text{ a.e. } z \in B_\delta(y_0) \Rightarrow |A_{\theta, r_0}| \geq |B_\delta(y_0)|
\]
since \( B_\delta(y_0) \subset \mathbb{R}^n \setminus B_\delta(0) \). This contradicts (2.7). Thus \( |B_\delta(y_0) \cap A_{\theta, r'}| \neq 0 \). Since \( |(B_\delta(y_0) \cap A_{\theta, r'}) \setminus (B_\delta(y_0) \cap A_{\theta, r'} \cap S_2)| = 0 \), \( B_\delta(y_0) \cap A_{\theta, r'} \cap S_2 \neq \emptyset \) and there exists \( x_0 \in B_\delta(y_0) \cap A_{\theta, r'} \cap S_2 \subset S^* \). By Lemma 2.3, there exists \( q_0 > m + 1 \) and \( \varepsilon_0 > 0 \), \( 0 < \varepsilon_0 < 1/2 \), such that
\[
u(q')(x, t) \leq \theta' \quad \forall x \in B_\delta(x_0), \quad 0 < t \leq \varepsilon_0, \quad q' \geq q_0.
\]

Letting \( i \to \infty \),
\[
u^{(\infty)}(x, t) \leq \theta' \quad \forall x \in B_\delta(x_0), \quad 0 < t \leq \varepsilon_0
\]
\[
\Rightarrow \int_{\mathbb{R}^n} \nu^{(\infty)}(x, t)\eta(x)\,dx \leq \theta' \int_{\mathbb{R}^n} \eta\,dx \quad \forall \eta \in C_0(B_\delta(x_0))
\]
\[
\Rightarrow \int_{\mathbb{R}^n} g\eta\,dx \leq \theta' \int_{\mathbb{R}^n} \eta\,dx \quad \forall \eta \in C_0(B_\delta(x_0)) \quad \text{as } t \to 0
\]
\[
\Rightarrow g(y_0) \leq \theta' < 1
\]
since \( y_0 \in S(g) \cap B_\delta(x_0) \). Hence \( y_0 \in S^* \cap S(\bar{g}) \). Thus (2.6) holds for \( x = y_0 \) by Lemma 2.4 and the lemma follows.

**Corollary 2.6.** Suppose \( f \in C^1_0(\mathbb{R}^n) \) and \( \bar{g} \) is as in Theorem 2.1. Then \( \bar{g} \in L^1(\mathbb{R}^n) \).

**Proof.** The lemma follows directly from Lemma 2.5 and the fact that \( \bar{g} \in L^\infty_{\text{loc}}(\mathbb{R}^n) \).

**Lemma 2.7.** For any \( 0 \leq f_1, f_2, g_1, g_2, \bar{g}_1, \bar{g}_2 \in L^1(\mathbb{R}^n) \), \( 0 \leq g_1, g_2 \leq 1 \), \( \bar{g}_1, \bar{g}_2 \geq 0 \), if
\[
g_i + (\bar{g}_i)_{x_1} = f_i \quad \text{in } \mathcal{D}'(\mathbb{R}^n)
\]
and
\[
g_i(x) = f_i(x), \quad \bar{g}_i(x) = 0 \quad \text{whenever } g_i(x) < 1 \text{ a.e. } x \in \mathbb{R}^n
\]
for \( i = 1, 2 \), then
\[
\int_{|x_1| \leq R'} \int_{\mathbb{R}^{n-1}} |\bar{g}_1 - \bar{g}_2|(x_1, x')\,dx'\,dx_1 \leq 2R'\|f_1 - f_2\|_{L^1(\mathbb{R}^n)} \quad \forall R' > 0.
\]
Proof. We will use a modification of an argument of [SX]. By (2.8),
\[(g_1 - g_2) + (\tilde{g}_1 - \tilde{g}_2)_{x_1} = f_1 - f_2 \quad \text{in} \mathcal{D}'(\mathbb{R}^n) \]
(2.10)
\[
\Rightarrow \int_{\mathbb{R}^n} [(g_1 - g_2)\eta - (\tilde{g}_1 - \tilde{g}_2)\eta_{x_1}] \, dx
\]
\[
= \int_{\mathbb{R}^n} (f_1 - f_2)\eta \, dx \quad \forall \eta \in C_0^\infty(\mathbb{R}^n).
\]
Putting \(\eta(x) = \rho_{\epsilon}(\xi - x)\) in (2.10), we get
(2.11)
\[
(\tilde{g}_1,\epsilon - \tilde{g}_2,\epsilon)_{\xi,\epsilon}(\xi) = (f_1,\epsilon - f_2,\epsilon)(\xi) - (g_1,\epsilon - g_2,\epsilon)(\xi) \quad \forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n.
\]
For any \(k = 1, 2, \ldots\), we let \(p_k(\cdot) \in C_0^\infty(\mathbb{R})\), \(0 \leq p_k \leq 1\), be such that \(p_k(x) \equiv 1\) for \(x \geq 1/k\), \(p_k(x) \equiv 0\) for \(x \leq 1/2k\) and \(\|p_k, x\|_{L^\infty} \leq 5k\). Then for all \(z_1, y_1 \in \mathbb{R}\),
\[
\int_{\mathbb{R}^n} (\tilde{g}_1,\epsilon - \tilde{g}_2,\epsilon)(z_1, x')p_k(\tilde{g}_1,\epsilon - \tilde{g}_2,\epsilon)(z_1, x') \, dx'
\]
\[- \int_{\mathbb{R}^n} (\tilde{g}_1,\epsilon - \tilde{g}_2,\epsilon)(y_1, x')p_k(\tilde{g}_1,\epsilon - \tilde{g}_2,\epsilon)(y_1, x') \, dx'
\]
(2.12)
\[
\Rightarrow \int_{\mathbb{R}^n} (\tilde{g}_1,\epsilon - \tilde{g}_2,\epsilon)(z_1, x')p_k(\tilde{g}_1,\epsilon - \tilde{g}_2,\epsilon)(z_1, x') \, dx'
\]
\[- \int_{\mathbb{R}^n} (\tilde{g}_1,\epsilon - \tilde{g}_2,\epsilon)(y_1, x')p_k(\tilde{g}_1,\epsilon - \tilde{g}_2,\epsilon)(y_1, x') \, dx'
\]
by (2.11). Since \(\tilde{g}_1, \tilde{g}_2 \in L^1(\mathbb{R}^n)\),
\[
\int_{\mathbb{R}^n} |\tilde{g}_1,\epsilon - \tilde{g}_2,\epsilon| \cdot |p_k(\tilde{g}_1,\epsilon - \tilde{g}_2,\epsilon)| \, dx \leq \int_{\mathbb{R}^n} (\tilde{g}_1 + \tilde{g}_2) \, dx < \infty.
\]
Hence there exists a sequence \(\{y_1^j\}_{j=1}^\infty \subset \mathbb{R}\), \(y_1^j \to -\infty\) as \(j \to \infty\) such that
\[
\int_{\mathbb{R}^n} (\tilde{g}_1,\epsilon - \tilde{g}_2,\epsilon)(y_1^j, x')p_k(\tilde{g}_1,\epsilon - \tilde{g}_2,\epsilon)(y_1^j, x') \, dx' \to 0 \quad \text{as} \quad j \to \infty.
\]
Putting $y_1 = y_1^j$ in (2.12) and letting $j \to \infty$, we get

\[
\int_{R^{n-1}} (\tilde{g}_1, e - \tilde{g}_2, e)(z_1, x')p_k(\tilde{g}_1, e - \tilde{g}_2, e)(z_1, x') \, dx' \\
+ \int_{R^{n-1}} \int_{-\infty}^{z_1} (g_1, e - g_2, e)(x_1, x')p_k(\tilde{g}_1, e - \tilde{g}_2, e)(x_1, x') \, dx_1 \, dx' \\
= \int_{R^{n-1}} \int_{-\infty}^{z_1} (\tilde{g}_1, e - \tilde{g}_2, e)p_k'(\tilde{g}_1, e - \tilde{g}_2, e) \\
\cdot [(f_1, e - f_2, e) - (g_1, e - g_2, e)](x_1, x') \, dx_1 \, dx' \\
(2.13) + \int_{R^{n-1}} \int_{-\infty}^{z_1} (f_1, e - f_2, e)(x_1, x')p_k(\tilde{g}_1, e - \tilde{g}_2, e)(x_1, x') \, dx_1 \, dx'.
\]

Since $\tilde{g}_1, \tilde{g}_2 \in L^1(R^n)$,

\[
\int_R \left| \int_{R^{n-1}} (\tilde{g}_1, e - \tilde{g}_2, e)(z_1, x')p_k(\tilde{g}_1, e - \tilde{g}_2, e)(z_1, x') \, dx' \right| \, dz_1 \\
\leq \int_{R^n} |(\tilde{g}_1, e - \tilde{g}_2, e) - (\tilde{g}_1 - \tilde{g}_2)|p_k(\tilde{g}_1, e - \tilde{g}_2, e) \, dx \\
+ \int_{R^n} (\tilde{g}_1 - \tilde{g}_2) \cdot |p_k(\tilde{g}_1, e - \tilde{g}_2, e) - p_k(\tilde{g}_1 - \tilde{g}_2)| \, dx \\
\leq \int_{R^n} |\tilde{g}_1, e - \tilde{g}_1| \, dx + \int_{R^n} |\tilde{g}_2, e - \tilde{g}_2| \, dx \\
+ \int_{R^n} (\tilde{g}_1 + \tilde{g}_2) \cdot |p_k(\tilde{g}_1, e - \tilde{g}_2, e) - p_k(\tilde{g}_1 - \tilde{g}_2)| \, dx \\
\to 0 \quad \text{as } \varepsilon \to 0
\]

by the Lebesgue dominated convergence theorem and Theorem 2 in Chapter 3 of [S]. Hence there exists a sequence $\{\varepsilon_j\}_{j=1}^\infty \subset R$, $\varepsilon_j \to 0$ as $j \to \infty$, such that

\[
\int_{R^{n-1}} (\tilde{g}_1, \varepsilon_j - \tilde{g}_2, \varepsilon_j)(z_1, x')p_k(\tilde{g}_1, \varepsilon_j - \tilde{g}_2, \varepsilon_j)(z_1, x') \, dx' \\
\to \int_{R^{n-1}} (\tilde{g}_1 - \tilde{g}_2)(z_1, x')p_k(\tilde{g}_1 - \tilde{g}_2)(z_1, x') \, dx'.
\]
a.e. \( z_1 \in R \) as \( j \to \infty \). On the other hand,

\[
\left| \int_{R^n-1} \int_{-\infty}^{z_1} (\bar{g}_1, e - \bar{g}_2, e) p'_k (\bar{g}_1, e - \bar{g}_2, e) \cdot [(g_1, e - g_2, e) - (f_1, e - f_2, e)] (x_1, x') \mathrm{d}x_1 \mathrm{d}x' \right|
\]

\[
- \int_{R^n-1} \int_{-\infty}^{z_1} (\bar{g}_1 - \bar{g}_2) p'_k (\bar{g}_1 - \bar{g}_2) [(g_1 - g_2) - (f_1 - f_2)] (x_1, x') \mathrm{d}x_1 \mathrm{d}x'
\]

\[
\leq \int_{R^n} \left| (\bar{g}_1, e - \bar{g}_2, e) p'_k (\bar{g}_1, e - \bar{g}_2, e) [(g_1, e - g_2, e) - (g_1 - g_2)] \right| \mathrm{d}x
\]

\[
+ \int_{R^n} \left| (\bar{g}_1, e - \bar{g}_2, e) p'_k (\bar{g}_1, e - \bar{g}_2, e) [(f_1, e - f_2, e) - (f_1 - f_2)] \right| \mathrm{d}x
\]

\[
+ \int_{R^n} \left| (\bar{g}_1, e - \bar{g}_2, e) p'_k (\bar{g}_1, e - \bar{g}_2, e) - (\bar{g}_1 - \bar{g}_2) p'_k (\bar{g}_1 - \bar{g}_2) \right| (g_1 - g_2) \mathrm{d}x
\]

\[
+ \int_{R^n} \left| (\bar{g}_1, e - \bar{g}_2, e) p'_k (\bar{g}_1, e - \bar{g}_2, e) - (\bar{g}_1 - \bar{g}_2) p'_k (\bar{g}_1 - \bar{g}_2) \right| (f_1 - f_2) \mathrm{d}x
\]

\[
\leq 5 \int_{R^n} \left| (g_1, e - g_1) + |g_2, e - g_2| + |f_1, e - f_1| + |f_2, e - f_2| \right| \mathrm{d}x
\]

\[
+ \int_{R^n} \left| (\bar{g}_1, e - \bar{g}_2, e) p'_k (\bar{g}_1, e - \bar{g}_2, e) - (\bar{g}_1 - \bar{g}_2) p'_k (\bar{g}_1 - \bar{g}_2) \right| (g_1 + g_2 + f_1 + f_2) \mathrm{d}x
\]

\[
\to 0 \quad \text{as } \varepsilon \to 0
\]

by the Lebesgue dominated convergence theorem since the integrand of the last integral above is bounded by \( 5(g_1 + g_2 + f_1 + f_2) \in L^1(R^n) \) and tends to 0 as \( k \to \infty \). Similarly

\[
\int_{R^n-1} \int_{-\infty}^{z_1} (g_1, e - g_2, e) (x_1, x') p_k (\bar{g}_1, e - \bar{g}_2, e) (x_1, x') \mathrm{d}x_1 \mathrm{d}x'
\]

\[
\to \int_{R^n-1} \int_{-\infty}^{z_1} (g_1 - g_2) (x_1, x') p_k (\bar{g}_1 - \bar{g}_2) (x_1, x') \mathrm{d}x_1 \mathrm{d}x' \quad \text{as } \varepsilon \to 0
\]

and

\[
\int_{R^n-1} \int_{-\infty}^{z_1} (f_1, e - f_2, e) (x_1, x') p_k (\bar{g}_1, e - \bar{g}_2, e) (x_1, x') \mathrm{d}x_1 \mathrm{d}x'
\]

\[
\to \int_{R^n-1} \int_{-\infty}^{z_1} (f_1 - f_2) (x_1, x') p_k (\bar{g}_1 - \bar{g}_2) (x_1, x') \mathrm{d}x_1 \mathrm{d}x' \quad \text{as } \varepsilon \to 0.
\]
Putting $\varepsilon = \varepsilon_j$ in (2.13) and letting $j \to \infty$, we get

\[ \int_{\mathbb{R}^n} (\overline{\varepsilon}_1 - \overline{\varepsilon}_2)(z_1, x')p_k(\overline{g}_1 - \overline{g}_2)(z_1, x')dx' \]

\[ + \int_{\mathbb{R}^n} \int_{-\infty}^{z_1} (g_1 - g_2)(x_1, x')p_k(\overline{g}_1 - \overline{g}_2)(x_1, x')dx_1dx' \]

\[ = \int_{\mathbb{R}^n} \int_{-\infty}^{z_1} (\overline{g}_1 - \overline{g}_2)p_k(\overline{g}_1 - \overline{g}_2)[(f_1 - f_2) - (g_1 - g_2)](x_1, x')dx_1dx' \]

\[ + \int_{\mathbb{R}^n} \int_{-\infty}^{z_1} (f_1 - f_2)(x_1, x')p_k(\overline{g}_1 - \overline{g}_2)(x_1, x')dx_1dx' \]

\[ \leq I_1 + \int_{\mathbb{R}^n} (f_1 - f_2)_+dx \quad \text{a.e. } z_1 \in \mathbb{R} \]

Since $p'_k(s) = 0$ for $s \leq 1/2k$ or $s \geq 1/k$, $I_1$ is bounded by

\[ \int_{\mathbb{R}^n} |(\overline{g}_1 - \overline{g}_2)(x)|p'_k \|L^\infty \cdot (g_1 + g_2 + f_1 + f_2)(x) \cdot \chi_{A_k}(x_1, x')dx \]

\[ \leq \int_{\mathbb{R}^n} \frac{1}{k} \cdot 5k \cdot (g_1 + g_2 + f_1 + f_2)(x) \cdot \chi_{A_k}(x)dx \]

\[ \leq 5 \int_{\mathbb{R}^n} (g_1 + g_2 + f_1 + f_2)(x) \cdot \chi_{A_k}(x)dx \]

\[ \to 0 \quad \text{as } k \to \infty \]

by the Lebesgue dominated convergence theorem since $g_1, g_2, f_1, f_2 \in L^1(\mathbb{R}^n)$ and

\[ (g_1 + g_2 + f_1 + f_2)(x)\chi_{A_k}(x) \to 0 \quad \text{as } k \to \infty \text{ a.e. } x \in \mathbb{R}^n \]

where $A_k = \{ x \in \mathbb{R}^n : 1/2k \leq (g_1 - g_2)(x) \leq 1/k \}$. Hence by letting $k \to \infty$ in (2.14), we get

\[ \int_{\mathbb{R}^n} (\overline{g}_1 - \overline{g}_2)_+(z_1, x')(z_1, x')dx' \]

\[ + \int_{\mathbb{R}^n} \int_{-\infty}^{z_1} (g_1 - g_2)(x_1, x')\text{sign}_+(\overline{g}_1 - \overline{g}_2)(x_1, x')dx_1dx' \]

\[ \leq \int_{\mathbb{R}^n} (f_1 - f_2)_+dx \quad \text{a.e. } z_1 \in \mathbb{R} \]

a.e. $z_1 \in \mathbb{R}$. Since $(g_1 - g_2)(x)\text{sign}_+(\overline{g}_1(x) - \overline{g}_2(x)) \geq 0$ a.e. $x \in \mathbb{R}^n$ by (2.9),

\[ \int_{|x_1| \leq R'} \int_{\mathbb{R}^n} (\overline{g}_1 - \overline{g}_2)_+(z_1, x')(z_1, x')dx' \leq \int_{\mathbb{R}^n} (f_1 - f_2)_+dx \quad \text{a.e. } z_1 \in \mathbb{R} \]

\[ \Rightarrow \int_{|x_1| \leq R'} \int_{\mathbb{R}^n} (\overline{g}_1 - \overline{g}_2)_+(x_1, x')dx'dx_1 \leq 2R' \int_{\mathbb{R}^n} (f_1 - f_2)_+dx \quad \forall R' > 0. \]

Similarly

\[ \int_{|x_1| \leq R'} \int_{\mathbb{R}^n} (\overline{g}_1 - \overline{g}_2)_-(x_1, x')dx'dx_1 \leq 2R' \int_{\mathbb{R}^n} (f_1 - f_2)_-dx \quad \forall R' > 0. \]

Thus

\[ \int_{|x_1| \leq R'} \int_{\mathbb{R}^n} |\overline{g}_1 - \overline{g}_2|(x_1, x')(x_1, x')dx'dx_1 \leq 2R' \int_{\mathbb{R}^n} |f_1 - f_2|dx \quad \forall R' > 0. \]
Corollary 2.8. Let \( 0 \leq f \in L^1(\mathbb{R}^n) \). Then there exists at most one function \( g \), \( g \in L^1(\mathbb{R}^n), 0 \leq g \leq 1 \), and one function \( \tilde{g} \in L^1(\mathbb{R}^n), \tilde{g} \geq 0 \) satisfying
\[
\begin{cases}
g + (\tilde{g})_x = f & \text{in } D'(\mathbb{R}^n), \\
g(x) = f(x), \tilde{g}(x) = 0 & \text{whenever } g(x) < 1 \text{ a.e. } x \in \mathbb{R}^n.
\end{cases}
\]

As a consequence of Theorem 2.1, Lemmas 2.4, 2.5, Corollary 2.8 and the uniqueness theorem (Theorem 6.13) of [DK], we have

Theorem 2.9. Suppose \( f \in C^0(\mathbb{R}^n) \). Then there exists a unique function \( u^{(\infty)} \in C(\mathbb{R}^n \times (0, 1)), 0 \leq u^{(\infty)} \leq 1 \) such that \( u^{(q)} \) converges uniformly to \( u^{(\infty)} \) on compact subsets of \( \mathbb{R}^n \times (0, 1) \) as \( q \to \infty \). Moreover \( u^{(\infty)} \) satisfies (0.2) with initial value \( g \in L^1(\mathbb{R}^n), 0 \leq g \leq 1 \), satisfying (2.15) and (2.6) for some function \( \tilde{g} \in L^1(\mathbb{R}^n), \tilde{g} \geq 0 \). The convergence is uniform on every compact subsets of \( \mathbb{R}^n \times (0, 1) \) if \( f \in C^0(\mathbb{R}^n) \).

We are now ready to state and prove the main theorem.

Theorem 2.10. For any \( m > 1 \) fixed, there exists a unique function \( u^{(\infty)} \in C(\mathbb{R}^n \times (0, 1)), 0 \leq u^{(\infty)} \leq 1 \) such that \( u^{(q)} \) converges weakly to \( u^{(\infty)} \) in \( (L^\infty(G))^* \) for any compact subset \( G \) of \( \mathbb{R}^n \times (0, 1) \) as \( q \to \infty \). Moreover \( u^{(\infty)} \) satisfies (0.2) with initial value \( g \in L^1(\mathbb{R}^n), 0 \leq g \leq 1 \), satisfying (2.15) and (2.6) for some function \( \tilde{g} \in L^1(\mathbb{R}^n), \tilde{g} \geq 0 \). The convergence is uniform on every compact subsets of \( \mathbb{R}^n \times (0, 1) \) if \( f \in C_0(\mathbb{R}^n) \).

Proof. Since \( f \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \), we can choose a sequence \( \{f_j\}_{j=1}^\infty \subset C^0(\mathbb{R}^n) \) such that \( \|f_j\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^\infty(\mathbb{R}^n)} + 1 \), \( \|f_j\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} + 1 \) for all \( j = 1, 2, \ldots \) and \( \|f_j - f\|_{L^1(\mathbb{R}^n)} \to 0 \) as \( j \to \infty \).

For all \( j = 1, 2, \ldots \), let \( u^{(q)}_j \) be the solution of (1.1) in \( \mathbb{R}^n \times (0, 1) \) with initial value \( u^{(q)}_j(x, 0) = f_j(x) \). By Theorem 2.9, for each \( j = 1, 2, \ldots \), there exists an unique function \( g_j \in L^1(\mathbb{R}^n), 0 \leq g_j \leq 1 \), satisfying (2.15) and (2.6) for some function \( \tilde{g} \in L^1(\mathbb{R}^n), \tilde{g} \geq 0 \). The convergence is uniform on every compact subsets of \( \mathbb{R}^n \times (0, 1) \) if \( f \in C_0(\mathbb{R}^n) \).

We are now ready to state and prove the main theorem.
Letting \( i \to \infty \), we get by Fatou’s lemma,
\[
\int_{\tau_1}^{\tau_2} \int_{\mathbb{R}^n} |u_j^{(\infty)}(x, t) - u^{(\infty)}(x, t)| \, dx \, dt \leq (\tau_2 - \tau_1) \int_{\mathbb{R}^n} |f_j - f|(x) \, dx \to 0 \quad \text{as } j \to \infty
\]
for all \( 0 < \tau_1 \leq \tau_2 < 1 \).

Hence \( u^{(\infty)} \) is the limit of the functions \( \{u_j^{(\infty)}\}_{j=1}^{\infty} \) in \( L^1_{\text{loc}}(\mathbb{R}^n \times (0, 1)) \) as \( j \to \infty \). Thus \( u^{(\infty)} \) is unique and \( u^{(q)} \) converges weakly to \( u^{(\infty)} \) in \( (L^\infty(G))^* \) for any compact subset \( G \) of \( \mathbb{R}^n \times (0, 1) \) as \( q \to \infty \). This together with Theorem 2.1 implies that \( u^{(q)} \) converges uniformly to \( u^{(\infty)} \) on every compact subsets of \( \mathbb{R}^n \times (0, 1) \) as \( q \to \infty \) if \( f \in C_0(\mathbb{R}^n) \).

Moreover \( \{u_j^{(\infty)}\}_{j=1}^{\infty} \) has a subsequence converging a.e. \( (x, t) \in \mathbb{R}^n \times (0, 1) \) to \( u^{(\infty)} \). Without loss of generality we may assume that \( u_j^{(\infty)}(x, t) \to u^{(\infty)}(x, t) \) a.e. \( (x, t) \in \mathbb{R}^n \times (0, 1) \) as \( j \to \infty \).

On the other hand since \( u_j^{(\infty)} \) satisfies (0.2) and
\[
|u_j^{(q)}(x, t)| \leq \|f_j\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^\infty(\mathbb{R}^n)} + 1 \quad \forall (x, t) \in \mathbb{R}^n \times (0, 1),
\]
(2.17)
\[
\Rightarrow \|u_j^{(\infty)}\|_{L^\infty(\mathbb{R}^n \times (0, 1))} \leq \|f\|_{L^\infty(\mathbb{R}^n)} + 1 \quad \forall j = 1, 2, \ldots
\]
as \( q \to \infty \) by Theorem 1.3, by the result of [S1] \( \{u_j^{(\infty)}\}_{j=1}^{\infty} \) has a subsequence \( \{u_k^{(\infty)}\}_{k=1}^{\infty} \) converging uniformly on compact subsets of \( \mathbb{R}^n \times (0, 1) \). Hence we may assume without loss of generality that \( \{u_j^{(\infty)}\}_{j=1}^{\infty} \) converges uniformly on compact subsets of \( \mathbb{R}^n \times (0, 1) \) to \( u^{(\infty)} \). Thus \( u^{(\infty)} \in C(\mathbb{R}^n \times (0, 1)) \).

Putting \( h(u) = 0, \quad u = u_j^{(\infty)} \) in (0.5) and letting \( j \to \infty \), we see that \( u^{(\infty)} \) satisfies (0.2). By (2.17) and the result of [DK], \( u^{(\infty)} \) has an initial trace \( d\mu \) and \( d\mu \) is absolutely continuous with respect to the Lebesgue measure. Hence \( d\mu = g(x) \, dx \) for some \( g \geq 0, \quad g \in L^1(\mathbb{R}^n) \). By (2.16) and Lemma 2.7,
\[
\int_{|x_1| \leq R'} \int_{R^n-1} |\tilde{g}_j - \tilde{g}_j'|(x_1, x') \, dx' \, dx_1 \leq 2R'\|f_j - f_{j'}\|_{L^1(\mathbb{R}^n)} \to 0 \quad \text{as } j, j' \to \infty \quad \forall R' > 0.
\]
Hence \( \{\tilde{g}_j\}_{j=1}^{\infty} \) is a Cauchy sequence in \( L^1_{\text{loc}}(\mathbb{R}^n) \) and there exists \( \tilde{g} \in L^1_{\text{loc}}(\mathbb{R}^n) \) such that \( \tilde{g}_j \to \tilde{g} \) in \( L^1_{\text{loc}}(\mathbb{R}^n) \) as \( j \to \infty \). Without loss of generality we may assume that \( \tilde{g}_j(x) \to \tilde{g}(x) \) a.e. \( x \in \mathbb{R}^n \). By the proof of Theorem 2.1, \( u_j^{(\infty)} \) satisfies, for all \( \eta \in C_0^\infty(\mathbb{R}^n), \quad 0 < \tau_2 < 1, \)
\[
\int_{0}^{\tau_2} \int_{\mathbb{R}^n} u_j^{(\infty)m} \Delta \eta \, dx \, dt + \int_{\mathbb{R}^n} \tilde{g}_j \eta x_1 \, dx = \int_{\mathbb{R}^n} u_j^{(\infty)}(x, t) \eta(x) \, dx - \int_{\mathbb{R}^n} f_j \eta \, dx
\]
\[
\Rightarrow \int_{0}^{\tau_2} \int_{\mathbb{R}^n} u^{(\infty)m} \Delta \eta \, dx \, dt + \int_{\mathbb{R}^n} \tilde{g} \eta x_1 \, dx
\]
\[
= \int_{\mathbb{R}^n} u^{(\infty)}(x, t) \eta(x) \, dx - \int_{\mathbb{R}^n} f \eta \, dx \quad \text{as } j \to \infty
\]
\[
\Rightarrow \int_{\mathbb{R}^n} \tilde{g} \eta x_1 \, dx = \int_{\mathbb{R}^n} g(x) \eta(x) \, dx - \int_{\mathbb{R}^n} f \eta \, dx \quad \text{as } \tau_2 \to 0
\]
\[
\Rightarrow \tilde{g} + \tilde{g}_x = f \quad \text{in } D'(\mathbb{R}^n).
Thus
\[
\left| \int (g - g_j) \eta \, dx \right| = \left| \int (\tilde{g} - \tilde{g}_j) \eta \, dx + \int (f - f_j) \eta \, dx \right|
\leq \| \eta \|_{L^\infty(\mathbb{R}^n)} \int_{|x_1| \leq R'} \int_{\mathbb{R}^{n-1}} |\tilde{g} - \tilde{g}_j|(x_1, x') \, dx' \, dx_1
+ \| \eta \|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f - f_j| \, dx
\rightarrow 0
\]
as \( j \rightarrow \infty \) for all \( \eta \in C_0^\infty(\mathbb{R}^n) \) such that \( \text{supp} \ \eta \subset B_{R'}(0) \) for some \( R' > 0 \). Hence \( g_j \) converges weakly to \( g \) in \( \mathcal{D}'(\mathbb{R}^n) \) as \( j \rightarrow \infty \). We may assume without loss of generality that \( g_j(x) \rightarrow g(x) \) and \( \tilde{g}_j(x) \rightarrow \tilde{g}(x) \) a.e. \( x \in \mathbb{R}^n \). Let
\[
E = \{ x \in \mathbb{R}^n : g_j(x) \rightarrow g(x) \text{ and } \tilde{g}_j \rightarrow \tilde{g}(x) \text{ as } j \rightarrow \infty \},
E_0 = E \cap \{ g < 1 \} \cap \left( \bigcap_{j=1}^{\infty} (S(g_j) \cap S(\tilde{g}_j)) \cap G(u_j^{(\infty)}, g_j) \right).
\]
For any \( x_0 \in E_0 \), since \( g_j(x_0) \rightarrow g(x_0) \) as \( j \rightarrow \infty \), there exists \( j_0 \in \mathbb{Z}^+ \) such that \( g_j(x_0) < 1 \ \forall j \geq j_0 \). So \( g_j(x_0) = f(x_0) \) and \( \tilde{g}_j(x_0) = 0 \) for all \( j \geq j_0 \) by Lemma 2.4. Letting \( j \rightarrow \infty \), we have \( g(x_0) = f(x_0) \) and \( \tilde{g}(x_0) = 0 \). Since \( \{ g < 1 \} \setminus E_0 = 0 \), \( g(x_0) = f(x_0) \) and \( \tilde{g}(x_0) = 0 \) a.e. \( x_0 \in \{ g < 1 \} \) and the theorem follows.

References


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