TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 6, June 1995

BOUNDED POINT EVALUATION IN $\mathbf{C}^n$

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ABSTRACT. A positive Borel measure μ on a domain $\Omega \subset \mathbf{C}^n$ is said to be in $\mathscr{R}(\Omega)$, if point evaluations at every $p \in \Omega$ are locally uniformly bounded in $L^2(\mu)$-norm. It is proved that the multiplication of a measure in $\mathscr{R}(\Omega)$ by a function decreasing no faster than a power of a holomorphic function produces a measure in $\mathscr{R}(\Omega)$. Some applications to classical Hardy and Bergman spaces are given.

0. INTRODUCTION

Let Ω be a bounded domain in $\mathbf{C}^n$ and let μ be a finite positive Borel measure on $\overline{\Omega}$. Following standard notation we denote by $\mathscr{O}(\overline{\Omega})$ the set of functions holomorphic in the closed domain $\overline{\Omega}$. In this paper we focus on the Hilbert space $L_a^2(\Omega, \mu)$ which is the closure of $\mathscr{O}(\overline{\Omega})$ in $L^2(\Omega, \mu)$. (Note that the inclusion $\mathscr{O}(\overline{\Omega}) \subset L^2(\Omega, \mu)$ holds because μ is finite.) A point $w \in \Omega$ is called a bounded point evaluation for μ if the linear functional $f \mapsto f(w)$ is bounded on $L_a^2(\Omega, \mu)$. The Riesz-Fischer Theorem then implies that there is an element $k_w \in L_a^2(\Omega, \mu)$ such that

$$f(w) = \int_{\overline{\Omega}} f(z)\overline{k_w}(z)\, d\mu(z)$$

for $f \in L_a^2(\Omega, \mu)$. Let $bpe(\mu)$ denote the set of bounded point evaluations for μ. If $w_0 \in \operatorname{int}(bpe(\mu))$, and if for all $z \in \Omega$, $k_w(z)$ depends on w antianalytically in a neighborhood of w_0, then w_0 is called analytic bounded point evaluation. The set of all such points is denoted by $abpe(\mu)$.

The following proposition may be found in [2, p. 63]. It was proved there in the case $n = 1$, but the proof is valid for an arbitrary n if we replace rational functions in [1] by functions from $\mathscr{O}(\overline{\Omega})$.

Proposition A. *If $w \in bpe(\mu)$, then $w \in abpe(\mu)$ if and only if there is a number $r > 0$ such that the ball of radius r centered at w is in $bpe(\mu)$ and*

$$\sup\{\|k_z\|_{L_a^2(\Omega, \mu)} : |z - w| < r\} < \infty.$$

In the last two decades questions related to bounded point evaluations were intensively investigated in the case $n = 1$, that is, in the case when the measure lives on a subset of the complex plane. The monograph [2] of John B.

Received by the editors September 1, 1993 and, in revised form, November 24, 1993; originally communicated to the *Proceedings of the AMS* by Theodore Gamelin.

1991 *Mathematics Subject Classification.* Primary 32A99, 46E20; Secondary 46E22.

Key words and phrases. Bounded point evaluation, reproducing kernel, Bergman spaces.

Conway is an excellent reference on the subject. It contains both elementary facts and the most recent achievements in the field. It also contains a detailed bibliography. The corresponding theory for $n > 1$ is still in its infancy. Concepts and problems generalize naturally to arbitrary dimension, though proofs will generally rely on techniques from multidimensional complex analysis, and thus will not just involve routine generalizations. Moreover, new phenomena are bound to appear in higher dimensions. For example, characterizations of $abpe(\mu)$, where $\mu = dV|_\Omega$, should be related to information about the envelope of holomorphy of Ω. Operator-theoretic interpretations, analogous to known results in the case $n = 1$, will lead to new classes of operators with interesting properties, which, in turn, may stimulate further questions and investigations.

One of the basic and still unanswered questions in the theory of analytic bounded point evaluations is the following: Is it true that $\operatorname{int}(bpe(\mu)) = abpe(\mu)$?

In many cases the answer is positive. That is the case, for instance, if Ω is a polynomially convex subset of $\mathbf{C}$. This and some other results can be found in [8] (see also [2]).

In this paper we consider a related question. We say that the measure μ belongs to the class $\mathscr{R}(\Omega)$ if $\Omega \subset abpe(\mu)$. The natural question is: given a measure μ from $\mathscr{R}(\Omega)$, describe perturbations of this measure which leave the resulting measure in the same class. We prove that multiplication by a function decreasing no faster than analytically (more precisely than the absolute value of a function analytic in $\overline{\Omega}$) satisfies this condition. The same method allows to prove an analogous local result. Namely, if μ satisfies some additional local conditions (see section 1 below), then multiplying μ by some function which locally does not decrease faster than a power of the absolute value of a germ of an analytic function leaves the resulting measure in the same class $\mathscr{R}(\Omega)$.

The last section contains some applications to the case when μ is the Lebesgue measure on Ω or surface measure on $\partial\Omega$.

1. Statement of results

Let $\mu \in \mathscr{R}(\Omega)$ and let ν be a finite nonnegative Borel measure on Ω. We denote by ν^μ the absolute continuous part of ν with respect to μ, so that $\nu^\mu = \nu^\mu(w)\,d\mu(w)$, where $\nu^\mu(w) \in L^1(\Omega, \mu)$.

Theorem 1. *If $\mu \in \mathscr{R}(\Omega)$ and a finite nonnegative Borel measure ν satisfies*

$$\operatorname*{ess\,inf}_{w\in\overline{\Omega}} \frac{\nu^\mu(w)}{|\psi(w)|^\alpha} > 0 \tag{1}$$

for some holomorphic function ψ in $\overline{\Omega}$ and some $\alpha > 0$, then $\nu \in \mathscr{R}(\Omega)$.

As a direct corollary we obtain the following result.

Corollary. *For every $\varphi \in \mathscr{O}(\overline{\Omega})$ and $\alpha > 0$ there is a symmetric reproducing kernel $K^\alpha_\varphi(z, w)$ with the weight $|\varphi(z)|^\alpha$, that is, $K_\varphi(z, w)$ is holomorphic in z and antiholomorphic in w and*

$$\int_{\Omega} \overline{K^\alpha_\varphi(z, w)} f(z) |\varphi(z)|^\alpha \, d\mu(z) = f(w)$$

for every $f \in L^2_a(\Omega, \mu)$.

To formulate the local version of Theorem 1 we introduce some additional conditions.

For a compact set $K \subset \Omega$ we use the traditional notation $\widehat{K}_{\mathcal{O}(\Omega)}$ for the $\mathcal{O}(\Omega)$-convex hull of K:

$$\widehat{K}_{\mathcal{O}(\Omega)} = \{z \in \Omega\colon |f(z)| \leq \max_{w\in K} |f(w)| \text{ for all } f \in \mathcal{O}(\Omega)\}.$$

We denote by $abpe_{loc}(\mu)$ the subset of $abpe(\mu)$ consisting of all points $w \in abpe(\mu)$ for which there is a basis of neighborhoods of w, $\{U_k(w)\}_{k=1}^{\infty}$, such that $\mu|_{U_k(w)} \in \mathcal{R}(U_k(w))$ for $k = 1, 2, \ldots$.

Note that $abpe_{loc}(\mu)$ may be empty even if $\mu \in \mathcal{R}(\Omega)$. For example, if $\partial\Omega$ is smooth and μ is a $(2n-1)$-Hausdorff measure on $\partial\Omega$, then $L_a^2(\Omega, \mu) = H^2(\Omega)$ and $\mu \in \mathcal{R}(\Omega)$ but $abpe_{loc}(\mu) = \varnothing$.

We introduce the following condition on a measure μ:

(A) For every point $w \in \Omega$ there are a neighborhood U_w and a compact set $M_w \subset abpe_{loc}(\mu)$ such that

$$U_w \subset (\widehat{M_w})_{\mathcal{O}(\Omega)}.$$

It follows from Proposition A that if a measure μ satisfies the condition (A), then μ is in $\mathcal{R}(\Omega)$.

Now we state the local variant of Theorem 1.

Theorem 2. *Let μ satisfy the condition* (A) *and let ν satisfy the following condition. There is a compact set $D \subset \Omega$ (depending on ν) such that for every point $z \in \Omega \setminus D$ there are a neighborhood U_z, an analytic function in U_z, which we denote by φ_z, and a positive number α_z satisfying* (1) *in U_z, that is,*

$$\operatorname*{ess\,sup}_{w\in U_z} \frac{\nu^{\mu}(w)}{|\varphi_z(w)|^{\alpha_z}} > 0.$$

Then ν satisfies condition (A).

2. Auxiliary proposition

From now on we denote by $B(w, r)$ the ball of radius r centered at w:

$$B(w, r) = \{z \in \mathbf{C}^n\colon |z - w| < r\}.$$

We denote by $K(z, w)$ the reproducing kernel for the measure $\mu \in \mathcal{R}(\Omega)$. It is well known that $K(z, w)$ is Hermitian symmetric in the sense that $K(z, w) = \overline{K(w, z)}$.

Proposition 1. *Let $\mu \in \mathcal{R}(\Omega)$. Then for every $w_0 \in \Omega$ we have the inclusion*

$$\frac{\partial^{|m|}}{\partial \overline{w}^m} K(z, w)|_{w=w_0} \in L_a^2(\Omega, \mu) \quad \textit{for any multi-index } m = (m_1, \ldots, m_n).$$

Proof. By Proposition A there is a ball $B(w_0, r)$ and a constant $M > 0$ such that

$$\|k_z\|_{L_a^2(\Omega, \mu)} = (K(z, z))^{1/2} < M$$

for all $|z - w_0| < r$. It means that

$$\sup\{|f(z)|\colon |z - w_0| = \tfrac{r}{2}\} \leq M\|f\|_{L_a^2(\Omega, \mu)}.$$

If r_1 satisfies the condition

$$\Delta(w_0, r_1) = \{z \in \mathbf{C}^n : |z_i - (w_0)_i| < r_1, \ i = 1, \dots, n\} \subset B\left(w_0, \frac{1}{2}r\right),$$

then the Cauchy inequality shows that

$$\left| \left. \frac{\partial^{|m|} f}{\partial z^m} \right|_{z=w_0} \right| \le \frac{M}{r_1^{|m|}} \|f\|_{L^2_a(\Omega, \mu)} \tag{2}$$

for every multi-index $m = (m_1, \dots, m_n)$ (here we use the common notation $|m| = m_1 + \cdots + m_n$, $\frac{\partial^{|m|} f}{\partial z^m} = \frac{\partial^{|m|} f}{\partial_{z_1}^{m_1} \dots \partial z_m^{m_n}}$). The relation (2) means that $f \mapsto \frac{\partial^{|m|} f}{\partial z^m}$ is a bounded linear functional on $L^2_a(\Omega, \mu)$. To prove the required assertion it suffices to prove this functional is generated by $\frac{\partial^{|m|} K(z, w)}{\partial \overline{w}^m}|_{w=w_0}$. We use induction in $|m|$. Let $w_k \to w_0$ as $k \to \infty$ be a sequence such that

(i) $(w_k)_i = (w_0)_i$, $i = 1, \dots, n-1$;

(ii) $(w_k)_n \neq (w_0)_n$.

Consider the following functionals on $L^2_a(\Omega, \mu)$

$$\langle l_k, f \rangle = \frac{1}{(w_k)_n - (w_0)_n} \left[\left. \frac{\partial^{|m|-1} f}{\partial z_1^{m_1} \cdots \partial z_n^{m_n - 1}} \right|_{z=w_k} - \left. \frac{\partial^{|m|-1} f}{\partial z_1^{m_1} \cdots \partial z_n^{m_n - 1}} \right|_{z=w_0} \right]$$

and

$$\langle l, f \rangle = \left. \frac{\partial^{|m|} f}{\partial z^m} \right|_{z=w_0}$$

(without loss of generality we may assume that $m_n \neq 0$ in all cases except $|m| = 0$ where the assertion in question follows from the definition of the kernel).

We obviously have $l_k \to l$ weakly as $k \to \infty$. By induction, the representative $L^2_a(\Omega, \mu)$-element for l_k is

$$\varphi_k(z) = \frac{1}{(\overline{w_k})_n - \overline{w_0}_n} \left[\left. \frac{\partial^{|m|-1} K(z, w)}{\partial \overline{w}_1^{m_1} \cdots \partial \overline{w}_n^{m_n - 1}} \right|_{w=w_k} - \left. \frac{\partial^{|m|-1} K(z, w)}{\partial \overline{w}_1^{m_1} \cdots \partial w_n^{m_n - 1}} \right|_{w=w_0} \right]. \tag{3}$$

Let us denote by $\varphi(z)$ the representative $L^2_a(\Omega, \mu)$ element for l. Weak convergence of l_k as $k \to \infty$ implies that for every $\tau \in \Omega$

$$\int_\Omega K(z, \tau) \overline{\varphi_k(z)} \, d\mu(z) \to \int_\Omega K(z, \tau) \overline{\varphi(z)} \, d\mu(z) \quad \text{as } k \to \infty$$

or, equivalently,

$$\varphi_k(\tau) \to \varphi(\tau) \quad \text{as } k \to \infty$$

for all $\tau \in \Omega$. Since $\varphi_k(\tau) \to \frac{\partial^{|m|} K(\tau, w)}{\partial \overline{w}^n}|_{w=w_0}$ pointwise, we conclude that $\frac{\partial^{|m|} K(z, w)}{\partial \overline{w}^n}|_{w=w_0}$ is the $L^2_a(\Omega, \mu)$-representative element for l.

It follows directly from the proof of this proposition (relation (2)) and general properties of duality that the $L^2_a(\Omega, \mu)$-norm of $\frac{\partial^{|m|} K(z, w)}{\partial \overline{w}^n}$ is locally uniformly bounded. More precisely, the following corollary was, in fact, proved.

Corollary. *If $\mu \in \mathscr{R}(\Omega)$, then for every point $z_0 \in \Omega$ and every natural number k there is a neighborhood of z_0, W_{z_0}, and a positive constant $M(z_0, k)$ such that*

$$\left\| \left. \frac{\partial^{|m|} K(z, w)}{\partial \overline{w}^m} \right|_{w=w_0} \right\|_{L_a^2(\Omega, \mu)} \leq M(z_0, k)$$

for all $w_0 \in W_{z_0}$ and all $|m| \leq k$.

3. Proof of theorems

Proof of Theorem 1. It is easily seen that it is sufficient to prove that the measure $\nu_1 = |\psi(z)|^2 d\mu(z)$ is in $\mathscr{R}(\Omega)$. By Proposition A it is enough to prove that for every $z_0 \in \Omega$ there is a neighborhood W_{z_0} of z_0 and a constant M such that

$$|f(z)| \leq M \|f\|_{L_a^2(\Omega, \nu_1)}$$

for all $z \in W_{z_0}$. First, let us consider the case $\psi(z_0) \neq 0$. Then, in a small neighborhood W_{z_0}, $\psi(z)$ does not vanish, and we have

$$\begin{aligned} |\psi(z) f(z)| &= \left| \int_\Omega \overline{K(\tau, z)} \psi(\tau) f(\tau)\, d\mu(\tau) \right| \\ &\leq \|K(\cdot, z)\|_{L_a^2(\Omega, \mu)} \cdot \|f\|_{L_a^2(\Omega, \nu_1)}. \end{aligned}$$

Therefore,

$$|f(z)| \leq \frac{1}{\min_{z \in \overline{W}_{z_0}} |\psi(z)|} \max_{z \in W_0} \|K(\cdot, z)\|_{L_a^2(\Omega, \mu)} \cdot \|f\|_{L_a^2(\Omega, \mu)}.$$

By the Corollary to Proposition 1 of the preceding section the maximum in the right-hand side of the last relation is bounded.

Now, let $\psi(z_0) = 0$. To simplify the notation we assume that $z_0 = 0$. Without loss of generality we may also assume that $\psi(0', z_n) \neq 0$ as a function of z_n. By the Weierstrass Preparation Theorem [4, p. 4], there are a neighborhood U of the origin and a Weierstrass polynomial $Q(z) = z_n^k + a_1(z') z_n^{k-1} + \cdots + a_k(z')$ such that for $z \in U$, $\psi(z) = Q(z) \cdot \chi(z)$, where $\chi(z)$ does not vanish in U.

Now let us consider the following difference operator. Given $(m+1)$ points $w_1, \ldots, w_{m+1}$ satisfying

$$\text{(4)} \qquad \begin{aligned} &\text{(i)} \quad (w_i)_n \neq (w_j)_n,\ i \neq j; \\ &\text{(ii)} \quad (w_i)_l = (w_j)_l \text{ for all } 1 \leq i, j \leq m+1,\ l = 1, \ldots, n-1, \end{aligned}$$

we define the operator $\mathscr{D}^m(w_1, \ldots, w_{m+1})$ by induction

(5)

$$\begin{aligned} &\mathscr{D}^1(w_1, w_2)(f) = \frac{1}{(w_1)_n - (w_2)_n} [f(w_1) - f(w_2)], \\ &\mathscr{D}^m(w_1, \ldots, w_{m+1}) \\ &\quad = \frac{1}{(w_1)_n - (w_{m+1})_n} [\mathscr{D}^{m-1}(w_1, \ldots, w_m)(f) - \mathscr{D}^{m-1}(w_2, \ldots, w_{m+1})]. \end{aligned}$$

This operator approximates $\frac{1}{m!} \frac{\partial^m}{\partial z_n^m}$ in the following sense.

For every $w_0 \in \Omega$ and every $\varepsilon > 0$ there is a neighborhood V_ε of w_0 such that for every $(m+1)$-tuple $w_1, \dots, w_{m+1} \in V_\varepsilon$ satisfying (4) and every $f \in L^2_a(\Omega, \mu)$

$$\left| \left.\frac{\partial^m f}{\partial z_n^m}\right|_{z=w_1} - \mathscr{D}^m(w_1, \dots, w_{m+1})(f) \right| \le \varepsilon \|f\|_{L^2_a(\Omega, \mu)}. \tag{6}$$

It follows directly from the easily checked equality

$$\mathscr{D}^m(w_1, \dots, w_{m+1})(z_n^k) = \begin{cases} 0 & \text{if } k < m, \\ \sum_{\substack{l_1 \ge 0, \dots, l_{m+1} \ge 0 \\ l_1 + \dots + l_{m+1} = k-m}} (w_1)_n^{l_1} \cdots (w_{m+1})_n^{l_{m+1}} & \text{if } k \ge m. \end{cases}$$

Passing, if necessary, to a smaller neighborhood, we assume that the neighborhood U satisfies

$$\left| \frac{\partial^k \psi}{\partial z_n^k} \right| \ge a > 0 \quad \text{for all } z \in U \tag{7}$$

and, moreover, that (6) holds in U with $\varepsilon = \frac{a}{4\|\psi\|_{L^2_a(\Omega, \mu)}}$. For each $w_1 \in U$ let us denote $\tau_2, \dots, \tau_{k+1}$ the roots of

$$Q(w_1', z_n) = z_n^k + a_1(w_1') z_n^{k-1} + \dots + a_k(w_1') = 0 \tag{8}$$

and define

$$w_l = (w_1', \tau_l), \qquad l = 2, \dots, k+1. \tag{9}$$

It is well known [4, p. 36] that the set of $w' \in \mathbf{C}^{n-1}$ such that (8) has a multiple root is a set of zero Lebesgue measure. Let w_1' not belong to this discriminant set. Note that the operator $\mathscr{D}^k(w_1, \dots, w_{k+1})$ may be written in the form

$$\mathscr{D}^k(w_1, \dots, w_{k+1})(f) = \sum_{l=1}^{k+1} d_l(w_1, \dots, w_{k+1}) f(w_l)$$

where $d_l(w_1, \dots, w_{k+1})$ are functions of $(w_1)_n, \dots, (w_{k+1})_n$. The relation (6) implies that

$$\left\| \sum_{l=1}^{k+1} \overline{d_l(w_1, \dots, w_{k+1})} K(z, w_l) \right\|_{L^2_a(\Omega, \mu)} \le \left\| \left. \frac{\partial^k K(z, w)}{\partial \overline{w}_n^k} \right|_{w=w_1} \right\|_{L^2_a(\Omega, \mu)} + \frac{a}{4}. \tag{10}$$

Since $\psi(w_2) = \dots = \psi(w_{k+1}) = 0$, we have for $f \in L^2_a(\Omega, \mu)$

$$\begin{aligned} \mathscr{D}^k(w_1, \dots, w_{k+1})(f\psi) &= \sum_{l=1}^{k+1} d_l(w_1, \dots, w_{k+1}) f(w_l) \psi(w_l) \\ &= f(w_1) \sum_{l=1}^{k+1} d_l(w_1, \dots, w_{k+1}) \psi_l(w_l) \\ &= f(w_1) \cdot \mathscr{D}^k(w_1, \dots, w_{k+1})(\psi). \end{aligned} \tag{11}$$

By (6) and (7)

$$|\mathscr{D}^k(w_1, \dots, w_{k+1})(\psi)| \ge \frac{3a}{4}. \tag{12}$$

Thus (11) implies

$$
\begin{aligned}
|f(w_1)| &\le \frac{4}{3a}|\mathscr{D}^k(w_1, \dots, w_{k+1})(f\psi)| \\
&= \frac{4}{3a}\left|\sum_{l=1}^{k+1} d_l(w_1, \dots, w_{k+1}) f(w_l)\psi(w_l)\right| \\
&= \frac{4}{3a}\left|\int_\Omega \left(\sum_{l=1}^{k+1} d_l(w_1, \dots, w_{k+1})\overline{K(z, w_l)}\right) f(z)\psi(z)\,d\mu(z)\right| \\
&\le \frac{4}{3a}\left\|\sum_{l=1}^{k+1}\overline{d_l(w_1, \dots, w_{k+1})}K(\cdot, w_l)\right\|_{L^2_a(\Omega, \mu)} \cdot \|f\|_{L^2_a(\Omega, \nu_1)} \\
&\le \frac{4}{3a}\left(\left\|\left.\frac{\partial^k K(z, w)}{\partial \overline{w}_n^k}\right|_{w=w_1}\right\|_{L^2_a(\Omega, \mu)} + \frac{a}{4}\right)\|f\|_{L^2_a(\Omega, \nu_1)}.
\end{aligned}
\tag{13}
$$

By the Corollary to Proposition 1, the right-hand side in the last relation is bounded. The estimate (13) holds for all $w \in U$ except for a set of Lebesgue measure 0. Hence it holds everywhere in U.

The proof of the corollary to this theorem is straightforward.

Proof of Theorem 2. We prove this theorem by reduction to Theorem 1. It is enough to prove that ν^μ satisfies the condition (A). To this end it suffices to prove that point evaluation is locally uniformly bounded in terms of $L^2_a(\Omega, \nu^\mu)$ on $abpe_{loc}(\mu) \setminus D$ where D is the compact set from the hypothesis of this theorem. Particularly, if we prove that $abpe_{loc}(\mu) \setminus D \subset abpe_{loc}(\nu^\mu)$, the result will follow.

Let $w \in abpe_{loc} \setminus D$ and let U_w be a neighborhood of w satisfying both the hypothesis of the theorem about the estimation of ν^μ in terms of $|\psi_w(z)|^{\alpha_w} \cdot d\mu(w)$ and local $apbe$, that is, $\mu|_{U_w} \in \mathscr{R}(U_w)$. By Theorem 1 $\nu^\mu|_{U_w} \in \mathscr{R}(U_w)$. Since the $L^2_a(\Omega, \nu^\mu)$-norm dominates the norm in $L^2_a(U_w, \nu^\mu|_{U_w})$ for every $L^2_a(\Omega, \mu)$-function, it means that $U_w \subset abpe_{loc}(\nu^\mu)$.

Remark. Following the standard notation we denote by $A^\infty(\overline{\Omega})$ the collection of $C^\infty(\overline{\Omega})$ functions which are analytic in Ω. If we denote by $L^2_A(\Omega, d\mu)$ the closure of $A^\infty(\overline{\Omega})$ in $L^2(\Omega, \mu)$, and by $\mathscr{R}_A(\Omega)$ the corresponding space of measures (defined analogously to $\mathscr{R}(\Omega)$ in case of $L^2_a(\Omega)$), the analysis of the proof of Theorems 1 and 2 shows that both of them hold in $L^2_A(\Omega, \mu)$ if the function ψ in (1) is in $A^\infty(\overline{\Omega})$ and μ is in $\mathscr{R}_A(\Omega)$.

4. Some applications

In this section we consider the case when μ is either the Lebesgue measure on Ω, dV, or surface measure on $b\Omega$, $d\sigma$. There are three important cases.

1. If Ω is a smoothly bounded pseudoconvex domain, then it was proved in [3] that

$$L^2_A(\Omega, dV) = \mathscr{O}L^2(\Omega) = \mathscr{O}(\Omega) \cap L^2(\Omega, dV).$$

2. Suppose that Ω is smoothly bounded and strictly pseudoconvex. Then it is known that

$$L^2_a(\Omega, dV) = L^2_A(\Omega, dV) = \mathscr{O}L^2(\Omega)$$

(see [5, p. 306, Theorem 6.4]).

Thus we obtain the following result.

Corollary 2. *In the above cases, for any* $\psi \in A^\infty(\overline{\Omega})$

$$|\psi(z)|^\alpha \, dV(z) \in \mathscr{R}_A(\Omega) \quad \textit{for all } \alpha > 0.$$

3. Let Ω be a domain in $\mathbf{C}^n$ with smooth boundary $b\Omega$ and $d\sigma$ be the surface measure on $b\Omega$. It is well known that $d\sigma \in \mathscr{R}_A(\Omega)$. If Ω is convex, for example a ball, then it is trivial that $L^2_A(\Omega, d\sigma)$ agrees with the classical Hardy space $H^2(\Omega, d\sigma)$. So we obtain

Corollary 3. *Let* $\Omega \subset \mathbf{C}^n$ *be convex with smooth boundary and suppose* $\psi \in A^\infty(\overline{\Omega})$ *and* $\alpha > 0$. *Then* $|\psi|^\alpha \, d\sigma \in \mathscr{R}_A(\Omega)$.

The classical Hardy space $H^2(\Omega, d\sigma)$ can be introduced on an arbitrary smoothly bounded domain $\Omega \subset \mathbf{C}^n$ (see [7]).

As we already noticed, the proof of Theorem 1 goes through in the case of $L^2_A(\Omega, \mu)$.

Proposition 2. *Let* $\Omega \subset\subset \mathbf{C}^n$ *be a smoothly bounded pseudoconvex domain. Then* $H^2(\Omega, d\sigma)$ *is the closure in* $L^2(b\Omega, d\sigma)$ *of* $A^\infty(\overline{\Omega})$.

Corollary 4. *Corollary* 3 *holds for arbitrary smoothly bounded pseudoconvex domains.*

Proposition 2 does not seem to be stated explicitly in the literature. However its proof is readily obtained by standard techniques from the following result.

Theorem. *Let* Ω *be a smoothly bounded pseudoconvex domain on* $\mathbf{C}^n$, $n \geq 1$. *For every* $s = 0, 1, 2, \ldots$ *there exists a constant* C_s *such that for every* $(0, 1)$*-form* α *on* $b\Omega$ *with coefficients on the Sobolev space* $W^s(b\Omega)$, *which satisfies*

$$\overline{\partial}_b \alpha = 0 \quad \textit{if } n > 2$$

or, if $n = 2$,

$$\int_{b\Omega} \alpha \wedge \phi = 0$$

for every $\overline{\partial}$*-closed* $(2, 0)$*-form on* Ω *which extends continuously to* $\overline{\Omega}$, *there exists* $u \in W^s(b\Omega)$, *such that*

$$\overline{\partial}_b u = \alpha \quad \textit{and} \quad \|u\|_{W^s} \leq C_s \|\alpha\|_{W^s}.$$

For $n \geq 3$, the theorem was proved by M. C. Shaw [6] (see the remark in [1] for $n = 3$), while the case $n = 2$ was proved by H. P. Boas and M. S. Shaw [1].

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