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SIMULTANEOUS TRIANGULARIZABILITY, NEAR COMMUTATIVITY AND ROTA'S THEOREM

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ABSTRACT. In this paper we consider simultaneously triangularizable collections of compact operators and show that similarities of any finite subcollection can be made arbitrarily close to commuting normal operators. As a consequence, we obtain a variant of a theorem of G.-C. Rota. Also, we give some sufficient conditions for simultaneous triangularization of collections of compact operators. Finally, several counterexamples are given.

1. INTRODUCTION

A collection $\mathscr{C}$ of bounded linear operators on a Hilbert space $\mathscr{H}$ is said to be *simultaneously triangularizable* if there is a maximal subspace chain (i.e., a maximal totally ordered set of closed subspaces) whose elements are invariant under every member of $\mathscr{C}$. If $\mathscr{H}$ is finite dimensional, then it is easy to see that this is equivalent to the more familiar concept of triangularizability, that is, the existence of an orthonormal basis with respect to which every element of $\mathscr{C}$ has an upper triangular matrix. The problem of simultaneous triangularizability has been studied by several authors. (See e.g. [2, 5–10].)

In this paper we consider simultaneously triangularizable collections of compact operators and show that similarities of any finite subcollection can be made arbitrarily close to commuting normal operators. As a consequence, we obtain an extension of Rota's theorem [14] which asserts that the spectral radius of an operator is the infimum of the norms of the operators similar to it. Also, we give some sufficient conditions for simultaneous triangularization of collections of compact operators. Finally, several counterexamples are given.

2. RESULTS

Throughout, $\mathscr{H}$ will denote a complex Hilbert space and $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. For an operator A in $\mathscr{B}(\mathscr{H})$, $\sigma(A)$ and $r(A)$ will denote the spectrum and the spectral radius of A, respectively. For a complete chain $\mathscr{N}$ of subspaces in $\mathscr{H}$ and $N \in \mathscr{N}$, let N_- denote the smallest member of $\mathscr{N}$ containing all $M \in \mathscr{N}$ such that $M \subsetneq N$. It is obvious that $\mathscr{N}$ is maximal if and only if $\dim(N/N_-) \leq 1$, for all $N \in \mathscr{N}$. A well-known

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result of Ringrose [12] (see [3, p. 31]) asserts that if $A \in \mathscr{B}(\mathscr{H})$ is compact and leaves every member of a maximal chain $\mathscr{N}$ invariant, then the eigenvalues of A are, with the possible exception of 0, its "diagonal coefficients"; i.e., the numbers $\lambda_N(A)$, $N \in \mathscr{N}$, where $\lambda_N(A)$ is the scalar operator induced by A on N/N_- for $N_- \neq N$. Thus A can be written as $D+Q$, where D and Q are the "diagonal" and the "strictly upper triangular" parts of A (D is $\sum \lambda_N(A)(P_N - P_{N_-})$, where P_N and P_{N_-} are the orthogonal projections onto N and N_- respectively.)

We consider the following two versions of "near commutativity" of a collection $\mathscr{C}$ of compact operators:

(I) For every $A_1, \dots, A_m \in \mathscr{C}$, there exist commuting compact normal operators $B_1, \dots, B_m$ such that for every $\varepsilon > 0$, there is an invertible operator S for which $\|S^{-1}A_kS - B_k\| < \varepsilon$, for every k.

(II) For every $A_1, \dots, A_m \in \mathscr{C}$ and every $\varepsilon > 0$, there exist commuting compact normal operators $B_1, \dots, B_m$ (possibly depending on ε) and an invertible operator S satisfying $\|S^{-1}A_kS - B_k\| < \varepsilon$, for every k.

We show that a triangularizable collection of compact operators satisfies condition (I) and hence also (II). We also prove a partial converse, showing that condition (I) implies simultaneous triangularizability and that the formally weaker condition (II) implies simultaneous triangularizability under additional conditions.

Theorem 1. *Suppose that $\mathscr{C}$ is a collection of simultaneously triangularizable compact operators on $\mathscr{H}$. Then for every $A_1, \dots, A_m$ in $\mathscr{C}$, there exist commuting compact normal operators $D_1, \dots, D_m$ such that for every $\varepsilon > 0$ there is an invertible operator S for which*

$$\|S^{-1}A_iS - D_i\| < \varepsilon, \qquad 1 \leq i \leq m. \tag{1}$$

Proof. Since $A_1, \dots, A_m$ are simultaneously triangularizable operators, the above-described result of Ringrose provides a maximal subspace chain $\mathscr{N}$ of common invariant subspaces with respect to which $A_i = D_i + Q_i$, for $1 \leq i \leq m$, where the D_i's are diagonal (and hence normal), the Q_i's are strictly upper triangular, and D_i commutes with D_j for all $1 \leq i, j \leq m$. Because $A_1, \dots, A_m$ are compact, so are the D_i's and hence the Q_i's. Since each Q_i is quasinilpotent, the repeated application of a result of Ringrose ([11], [3, p. 32]) yields, for every $\varepsilon > 0$, a finite sequence of projections $P_0 = 0 < P_1 < \cdots < P_n = I$ in $\mathscr{N}$ such that for every $1 \leq i \leq m$ and $1 \leq k \leq n$:

$$\|(P_k - P_{k-1})Q_i(P_k - P_{k-1})\| < \frac{\varepsilon}{2}. \tag{2}$$

Now let $S = \sum_{k=1}^n \eta^k (P_k - P_{k-1})$, where $0 < \eta < 1$ is a constant to be specified. The operator S commutes with D_i for every i, and

$$S^{-1}A_iS - D_i = S^{-1}(A_i - D_i)S = S^{-1}Q_iS.$$

For an operator T whose matrix with respect to $\mathscr{N}$ is block upper triangular, let $T^{(k)}$ denote the kth superdiagonal of T; i.e. the operator obtained from T by replacing all block entries of T, except those which are located on the $(r, r+k)$ positions, by zero. In other words let

$$T^{(k)} = \sum_{j=1}^{n-k} (P_j - P_{j-1})T(P_{j+k} - P_{j+k-1}). \tag{3}$$

Now since $S^{-1}Q_iS$ is block upper triangular, using the notation just mentioned, it is easy to see that

$$S^{-1}Q_iS = Q_i^{(0)} + \eta Q_i^{(1)} + \eta^2 Q_i^{(2)} + \cdots + \eta^{n-1}Q_i^{(n-1)}. \tag{4}$$

By inequality (2), $\|Q_i^{(0)}\| < \varepsilon/2$. It is not hard to verify that

$$\|T^{(k)}\| \leq \|T\| \tag{5}$$

for every T and hence

$$\|S^{-1}Q_iS\| < \frac{\varepsilon}{2} + \frac{\eta}{1-\eta}\|Q_i\|.$$

Now choose η small enough to get the desired conclusion. □

Remark 1. The above proof shows that S commutes with D_i, $1 \leq i \leq m$, and hence (1) can be written in the form

$$\|S^{-1}(A_i - D_i)S\| < \varepsilon. \tag{6}$$

Since S is positive, it defines an equivalent Hilbert space norm $\|\cdot\|_1$ on $\mathscr{H}$, with respect to which (6) can be written as

$$\|A_i - D_i\|_1 < \varepsilon.$$

Remark 2. In the above, by a suitable approximation, we can choose the D_k's to be of finite rank as well.

Remark 3. If $\mathscr{C}$ lies in the von Neumann-Schatten class $\mathbf{C}_p$, the conclusion of Theorem 1 is valid with the norm $\|\cdot\|_p$ of $\mathbf{C}_p$ in place of the operator norm. To see this, first note that the diagonals $D_i \in \mathbf{C}_p$ (see e.g. [3, Problem 1.4]). It follows that $Q_i \in \mathbf{C}_p$. With the same notation as in the proof of Theorem 1, we then observe that $\|Q_i^{(0)}\|_p < \varepsilon/2$, as can be seen from [3, Proposition 1.18] along with the proof of [3, Lemma 3.5]. Now using the norm inequalities $\|BT\|_p \leq \|B\|\,\|T\|_p$ and $\|TB\|_p \leq \|T\|_p\|B\|$ for $T \in \mathbf{C}_p$ and B in $\mathscr{B}(\mathscr{H})$ and equation (3), one can see that for every $T \in \mathbf{C}_p$

$$\|T^{(k)}\|_p \leq (n-k)\|T\|_p \leq n\|T\|_p.$$

Finally using these and (4) of Theorem 1 we obtain

$$\|S^{-1}Q_iS\| < \varepsilon/2 + \frac{n\eta}{1-\eta}\|Q_i\|,$$

which implies the desired result.

Remark 4. It follows from Theorem 1 that for every $\varepsilon > 0$ there exists a positive invertible operator T such that

$$\|T^{-1}(A_iA_j - A_jA_i)T\| < \varepsilon$$

or

$$\|T^{-1}A_iTT^{-1}A_jT - T^{-1}A_jTT^{-1}A_iT\| < \varepsilon,$$

which implies that certain similarities of A_i, $1 \leq i \leq m$, "approximately commute".

As an application of Theorem 1 we get the following extension of Rota's theorem in the case of compact operators [14] to finite collections of simultaneously triangularizable compact operators. In [1], Fong and Sourour proved the multivariable Rota's Theorem for any finite family of commuting operators. Also in [15], Shih and Tan obtained a similar result for a finite family of commuting operators by renorming the Hilbert space. Using the next corollary, one can extend results of section 3 of [15] from commutative to simultaneously triangularizable families, in the case of compact operators.

Corollary 1. *If $\mathscr{C}$ is a collection of compact operators which are simultaneously triangularizable, then for every $A_1, \ldots, A_m$ in $\mathscr{C}$ and every $\varepsilon > 0$ there exists a positive invertible operator S such that*

$$\|S^{-1}A_kS\| < r(A_k) + \varepsilon,$$

for every $1 \le k \le m$.

Proof. By Theorem 1, we have $\|S^{-1}A_kS\| < \|D_k\| + \varepsilon$, for all $1 \le k \le m$. Since the D_k's are normal, $r(D_k) = \|D_k\|$. The proof of Theorem 1 shows that $\sigma(A_k) = \sigma(D_k)$, $1 \le k \le m$, which finishes the proof. □

We now prove that condition (II) with an additional boundedness condition implies simultaneous triangularizability. We note that commuting "approximants" are not assumed to be normal.

Theorem 2. *Suppose that $\mathscr{C}$ is a collection of compact operators with the property that for all $A_1, \ldots, A_m \in \mathscr{C}$ there exists a $K > 0$ such that for every $\varepsilon > 0$ there exist commuting compact operators $D_1, \ldots, D_m$ with $\|D_k\| \le K$ and an invertible S such that*

$$\|S^{-1}A_kS - D_k\| < \varepsilon,$$

for every k. Then $\mathscr{C}$ is simultaneously triangularizable.

Proof. In view of [5], to show that $\mathscr{C}$ is simultaneously triangularizable it suffices to show that, in the algebra $\mathscr{A}$ generated by $\mathscr{C}$, every commutator is quasinilpotent. Given $P, Q \in \mathscr{A}$, there are noncommutative polynomials p, q in m variables and $A_1, \ldots, A_m \in \mathscr{C}$ such that $P = p(A_1, \ldots, A_m)$ and $Q = q(A_1, \ldots, A_m)$. It thus suffices to show that

$$[P, Q] = p(A_1, \ldots, A_m)q(A_1, \ldots, A_m) - q(A_1, \ldots, A_m)p(A_1, \ldots, A_m)$$

is quasinilpotent. We shall show that for an arbitrary $\rho > 0$, there is an invertible operator S such that $\|S^{-1}[P, Q]S\| < \rho$, so that the spectral radius of $[P, Q]$ is arbitrary small.

Observe that every noncommutative polynomial h in m operators is a uniformly continuous function of its arguments on any bounded set in $\mathscr{B}(\mathscr{H})$. Thus for $h = [P, Q]$, $M = K + 1$, and any $\rho > 0$ there exists a $\delta \in (0, 1)$ such that $\|B_i - C_i\| < \delta$, $i = 1, \ldots, m$, implies

$$\|h(B_1, \ldots, B_m) - h(C_1, \ldots, C_m)\| < \rho \tag{1}$$

for all operators B_i and C_i of norm not exceeding M. By the hypothesis of the theorem for $\varepsilon = \delta$, there exist commuting compact operators $D_1, \ldots, D_m$ with $\|D_k\| \le K$ and an invertible operator S such that

$$\|S^{-1}A_iS - D_i\| < \delta, \qquad i = 1, \ldots, m. \tag{2}$$

Since $h(D_1, \ldots, D_m) = 0$, by the commutativity of D_i, it follows from (1) and (2) that

$$\begin{aligned}\|S^{-1}h(A_1, \ldots, A_m)S\| &= \|h(S^{-1}A_1S, \ldots, S^{-1}A_mS)\| \\ &= \|h(S^{-1}A_1S, \ldots, S^{-1}A_mS) - h(D_1, \ldots, D_m)\| < \rho. \quad \square\end{aligned}$$

We can now get the converse of Theorem 1, showing that condition (I) implies simultaneous triangularizability.

Theorem 3. *A collection $\mathscr{C}$ of compact operators on $\mathscr{H}$ is simultaneously triangularizable if and only if for all $A_1, \ldots, A_m \in \mathscr{C}$ there exist commuting compact normal operators $D_1, \ldots, D_m$ with the property that for every $\varepsilon > 0$ there exists an invertible operator S satisfying*

$$\|S^{-1}A_kS - D_k\| < \varepsilon,$$

for every k.

Proof. Let $K = \max\{\|D_k\| : 1 \leq k \leq m\}$ and apply Theorem 2. $\square$

Remark. In view of Theorem 2, we can drop the normality condition in Theorem 3.

In the statements of the equivalence results above, restricting the "nearly commuting" operators to a finite number is essential if the underlying space is infinite dimensional. Even if the collection $\mathscr{C}$ is assumed bounded this restriction is necessary. (See counterexample 4.) However, in finite dimensions, the following result holds and its proof is similar to the argument given after (2) in the proof of Theorem 1.

Proposition 1. *In a finite-dimensional Hilbert space, a bounded collection $\{A_\alpha\}$ of operators is simultaneously triangularizable if and only if there exists a commuting set $\{D_\alpha\}$ of normal operators such that to every $\varepsilon > 0$ there corresponds an invertible operator S with $\|S^{-1}A_\alpha S - D_\alpha\| < \varepsilon$, for every α.*

The next result will be used to get a version of Theorem 3, using the weaker condition (II), for finite-dimensional spaces.

Proposition 2. *Let $\dim \mathscr{H} < \infty$, $\{D_n\}$ be a sequence of normal operators, and $\{S_n\}$ a sequence of invertible operators such that*

$$\lim_{n\to\infty} \|S_n^{-1}AS_n - D_n\| = 0,$$

for some operator A. Then $\{D_n\}$ is bounded.

Proof. Suppose the contrary. Then, with no loss of generality, we may assume that $\|D_n\| > n$ and

$$\|S_n^{-1}AS_n - D_n\| < \frac{1}{n}. \tag{1}$$

Let $c_n = \|D_n\|$. Equation (1) implies that

$$\left\|S_n^{-1}\frac{A}{c_n}S_n - \frac{D_n}{c_n}\right\| < \frac{1}{n^2}. \tag{2}$$

Now

$$r\left(\frac{D_n}{c_n}\right) = 1 \quad \text{and} \quad r\left(S_n^{-1}\frac{A}{c_n}S_n\right) = \frac{r(A)}{c_n} \to 0 \quad \text{as } n \to \infty. \tag{3}$$

But it is well known that on finite-dimensional spaces the spectral radius is continuous and, hence, uniformly continuous on bounded sets of operators. This is in contradiction to (2) and (3). Thus $\{D_n\}$ must be bounded. □

This result together with Theorem 2 gives the following .

Corollary 2. *Let $\mathscr{C}$ be a collection of operators on a finite-dimensional Hilbert space. Then $\mathscr{C}$ is simultaneously triangularizable if and only if for every $A_1, \ldots, A_m \in \mathscr{C}$ and every $\varepsilon > 0$ there exist commuting normal operators $D_1, \ldots, D_m$ and an invertible operator S such that*

$$\|S^{-1}A_iS - D_i\| < \varepsilon, \qquad 1 \le i \le m.$$

For the next corollary, which extends part of a result of [13], we need the following. Recall that an operator A on $\mathscr{H}$ is called *unicellular* if its lattice of invariant subspaces is totally ordered. For an operator A on $\mathscr{H}$, its lattice of invariant subspaces is denoted by $\operatorname{Lat} A$.

Corollary 3. *Let C and U be compact operators on $\mathscr{H}$ with U unicellular. If for every $\varepsilon > 0$ there exists an invertible operator S such that $\|S^{-1}US\| < \varepsilon$ and $\|S^{-1}CS\| < \varepsilon$, then* $\operatorname{Lat} U \subseteq \operatorname{Lat} C$.

Proof. By Theorem 2, with $A_1 = U$, $A_2 = C$ and $D_1 = D_2 = 0$, $\{U, C\}$ is simultaneously triangularizable. Thus $\operatorname{Lat} U \subseteq \operatorname{Lat} C$. □

3. Counterexamples

Here we give four counterexamples. The first one shows that Corollary 1 may not be true if the operators are not compact.

Counterexample 1. Consider the following infinite matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Then A and B are noncompact nilpotent operators. Though norms of similarities of each one can be made arbitrarily small, this can't be done simultaneously.

For if it could be simultaneous, the norms of similarities of $A+B$, which is the unilateral backward shift, would be made arbitrarily small as well, which is impossible because $r(A+B)=1$. □

Is Proposition 2 true in infinite-dimensional spaces if A is compact? As we will see below, the answer is no.

Counterexample 2. Let M be the Hilbert matrix; i.e., $M=[a_{ij}]$, where $a_{ij}=\frac{1}{i-j}$ if $1\le i\ne j<\infty$ and $a_{ij}=0$ if $i=j$. Let U denote the "upper triangular part" of M, i.e., the matrix obtained from M by replacing the below-diagonal entries with 0. Similarly, let L be the "lower triangular part" of M. It is known that both U and L are unbounded, but M is bounded by π [4, Section 8.12]. Also, let M_n be the $n\times n$ matrix obtained by truncating M in the north west corner, U_n and L_n the "upper triangular" and the "lower triangular" parts of M_n respectively, and $H_n=L_n-U_n$. For every n, H_n is Hermitian and U_n and L_n are nilpotent. Since $\|L_n\|\to\infty$ and $\|U_n\|\to\infty$, for a subsequence we have $\|L_{m_n}\|>n$ and $\|U_{m_n}\|>n$ and hence, $\|H_{m_n}\|>n$. Let $T_n=L^t_{m_n}-U_{m_n}=-2U_{m_n}$, which is a nilpotent of size $m_n\times m_n$; and let J_n be the Jordan cell of size $m_n\times m_n$. Now consider the compact operator

$$A=J_1\oplus\frac{1}{\sqrt{2}}J_2\oplus\frac{1}{\sqrt{3}}J_3\oplus\cdots.$$

Let n be a positive integer. For $1\le i<n$ there exists an invertible operator S_i such that

$$\left\|S_i^{-1}\left(\frac{1}{\sqrt{i}}J_i\right)S_i\right\|<\frac{1}{\sqrt{n}}.$$

Also, there exists an invertible S_n such that

$$S_n^{-1}\left(\frac{1}{\sqrt{n}}J_n\right)S_n=\frac{-1}{2\sqrt{n}}T_n.$$

For $i>n$, let $S_i=I_{m_i}$. The operator $S:=\sum_\oplus S_i$ is bounded and invertible, and

$$S^{-1}AS=\left(0\oplus\cdots\oplus 0\oplus\frac{-1}{2\sqrt{n}}T_n\oplus 0\oplus\cdots\right)+B_n, \tag{1}$$

where B_n is an operator of norm less than $\frac{1}{\sqrt{n}}$. We have

$$H_{m_n}-T_n=L_{m_n}-U_{m_n}+2U_{m_n}=L_{m_n}+U_{m_n}=M_{m_n}$$

and hence

$$\left\|\frac{1}{2\sqrt{n}}H_{m_n}-\frac{1}{2\sqrt{n}}T_n\right\|\le\frac{\pi}{2\sqrt{n}}. \tag{2}$$

Relations (1) and (2) imply that A is similar to C_n+D_n, where

$$D_n=0\oplus\cdots\oplus 0\oplus\frac{-1}{2\sqrt{n}}H_{m_n}\oplus 0\oplus\cdots$$

and C_n is an operator of norm less than $3/\sqrt{n}$. Now the construction is complete, for the D_n's are normal and

$$\|D_n\|=\frac{1}{2\sqrt{n}}\|H_{m_n}\|\ge\frac{1}{2\sqrt{n}}\|L_{m_n}\|>\frac{\sqrt{n}}{2}.\quad □$$

Next, we give an example to show that, unlike Theorem 3, Corollary 2 is false if approximants are not assumed normal. This also shows that the boundedness condition of Theorem 2 cannot be removed even in finite-dimensional spaces.

Counterexample 3. Let

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Given $\varepsilon > 0$, let

$$S = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & \varepsilon^{-1} \\ 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus $\|S^{-1}A_iS - D_i\| < \varepsilon$, for $i = 1, 2$. However, A_1 and A_2 are obviously not simultaneously triangularizable. □

Finally, we give the following counterexample which shows that the compact operators in the unit ball of a nest algebra are not simultaneously similar to nearly commuting operators. Thus Proposition 1 is not true in infinite dimensions.

Counterexample 4. Fix an orthonormal basis $\{e_i\}$ for a separable Hilbert space $\mathscr{H}$, and let $\mathscr{C}$ be the collection of operators on $\mathscr{H}$ represented by strictly upper triangular matrices $\{E_{ij} : i < j\}$, where E_{ij} denotes the matrix whose only nonzero entry is 1 and occurs at the (i, j) position. Clearly $\mathscr{C}$ is contained in the compact operators of the unit ball of the nest algebra of all upper triangular operators with respect to $\{e_i\}$. We show that $\mathscr{C}$ is not simultaneously similar to nearly commuting operators. Assume it were. Then it is easy to verify that for every $\varepsilon > 0$ there exists an invertible operator S such that

$$\|S^{-1}(AB - BA)S\| < \varepsilon$$

for all $A, B \in \mathscr{C}$. In particular, since every $E_{n,n+2}$ is a commutator in $\mathscr{C}$, i.e.

$$E_{n,n+2} = E_{n,n+1}E_{n+1,n+2} - E_{n+1,n+2}E_{n,n+1},$$

it follows that $\|S^{-1}E_{n,n+2}S\| < \varepsilon$ for all n. This implies that

$$\|S^{-1}E_{n,n+2}S(S^{-1}e_{n+2})\| < \varepsilon\|S^{-1}e_{n+2}\|$$

for all n or $\|S^{-1}e_n\| < \varepsilon\|S^{-1}e_{n+2}\|$. By induction

$$\|S^{-1}e_1\| < \varepsilon^k\|S^{-1}e_{2k+1}\| \leq \varepsilon^k\|S^{-1}\|$$

for all k. This implies that $S^{-1}e_1 = 0$, a contradiction.

Modifications of the arguments above yield counterexamples for more general nests. □

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