SOME INEQUALITIES OF ALGEBRAIC POLYNOMIALS WITH NONNEGATIVE COEFFICIENTS

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Abstract. Let $S_n$ be the collection of all algebraic polynomials of degree $\leq n$ with nonnegative coefficients. In this paper we discuss the extremal problem

$$
\sup_{p_n(x) \in S_n} \frac{\int_{a}^{b} (p_n'(x))^2 \omega(x) \, dx}{\int_{a}^{b} p_n^2(x) \omega(x) \, dx}
$$

where $\omega(x)$ is a positive and integrable function. This problem is solved completely in the cases

(i) $[a, b] = [-1, 1], \, \omega(x) = (1 - x^2)^\alpha, \, \alpha > -1$;
(ii) $[a, b] = [0, \infty), \, \omega(x) = x^\alpha e^{-x}, \, \alpha > -1$;
(iii) $[a, b] = (-\infty, \infty), \, \omega(x) = e^{-ax^2}, \, \alpha > 0$.

The second case was solved by Varma for some values of $\alpha$ and by Milovanović completely. We provide a new proof here in this case.

1. Introduction

In this paper we investigate the following extremal problem

$$
\sup_{p_n(x) \in S_n} \frac{\int_{a}^{b} (p_n'(x))^2 \omega(x) \, dx}{\int_{a}^{b} p_n^2(x) \omega(x) \, dx}
$$

where

$$
S_n = \left\{ p_n(x): p_n(x) = \sum_{i=0}^{n} a_i x^i, \, a_i \geq 0, \, 0 \leq i \leq n \right\},
$$

and $\omega(x): (a, b) \to \mathbb{R}$ is a positive and integrable function.

In the case $[a, b] = [0, \infty), \, \omega(x) = x^\alpha e^{-x}, \, \alpha > -1$, the extremal problem (1) was initiated and solved by Varma [10] in the cases $0 \leq \alpha \leq 1/2$ and $(\sqrt{5} - 1)/2 \leq \alpha < \infty$. Later, it was solved completely by Milovanović [4] for $-1 < \alpha < \infty$.

In this note we consider the above extremal problem (1) for different weight functions on different intervals. Throughout this paper, we denote $S_n$ the collection of all algebraic polynomials of degree $\leq n$ with nonnegative coefficients. In Section 2, we provide the complete answer to the case $[a, b] = [-1, 1], \, \omega(x) = (1 - x^2)^\alpha, \, \alpha > -1$. In the case $\alpha = 0$, this result is an analogue of a
theorem of Lorentz [3] in the \( L_\infty \) norm. Indeed, that theorem holds for a wider class (Lorentz class) of polynomials, which was studied extensively by Scheick [7]. For some subsets of Lorentz class of polynomials, the extremal problem (1) was discussed by Milovanović and Petković [5] for the Jacobi weight.

In Section 3, we give a new proof of Milovanović's Theorem [4]. In our last section, Section 4, we consider the weight function \( \omega(x) = e^{-ax^2} \), \( a > 0 \), on the interval \((-\infty, \infty)\).

The corresponding extremal problem for the unrestricted polynomials was discussed in Dörfler [1], [2], Mirsky [6] and Turan [8], which are Markov type inequalities in \( L_2 \) norm.

2. The weight \( \omega(x) = (1 - x^2)^\alpha \)

In this section, we discuss the extremal problem in the \( L_2 \) norm under the weight function \( \omega(x) = (1 - x^2)^\alpha \), \( \alpha > -1 \), on \([-1, 1]\). For some special values of \( \alpha \), we obtain several corollaries corresponding to some classic weight functions. The main result in this section is the following theorem.

**Theorem 2.1.** Let \( p_n(x) \in S_n \), \( \alpha > -1 \); then

\[
\int_{-1}^{1} (p'_n(x))^2(1 - x^2)^\alpha \, dx \leq \frac{2n + 2\alpha + 1}{2n - 1} n^2 \int_{-1}^{1} p_n^2(x)(1 - x^2)^\alpha \, dx
\]

with equality when \( p_n(x) = x^n \).

**Proof.** Since \( p_n(x) \in S_n \), we can write

\[
p_n(x) = \sum_{i=0}^{n} a_i x^i
\]

with \( a_i \geq 0 \), \( 0 \leq i \leq n \). Then

\[
p'_n(x) = \sum_{i=1}^{n} ia_i x^{i-1}
\]

and

\[
\int_{-1}^{1} p_n^2(x)(1 - x^2)^\alpha \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j \int_{-1}^{1} x^i x^j (1 - x^2)^\alpha \, dx,
\]

\[
\int_{-1}^{1} (p'_n(x))^2(1 - x^2)^\alpha \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j i j \int_{-1}^{1} x^i x^j (1 - x^2)^\alpha \, dx.
\]

Let

\[
b_{ij} = \int_{-1}^{1} x^i x^j (1 - x^2)^\alpha \, dx
\]

\[
= \frac{1 - (-1)^{i+j+1}}{2} B \left( \frac{i+j+1}{2}, \alpha + 1 \right)
\]

where \( B(x, y) \) is the Beta function and

\[
c_{ij} = i j \int_{-1}^{1} x^i x^j (1 - x^2)^\alpha \, dx
\]

\[
= i j \frac{1 - (-1)^{i+j+1}}{2} B \left( \frac{i+j-1}{2}, \alpha + 1 \right)
\]
for \(1 \leq i, j \leq n\), \(c_{ij} = 0\) if \(i = 0\) or \(j = 0\). Now denote
\[
B = (b_{ij})_{0 \leq i, j \leq n}, \quad C = (c_{ij})_{0 \leq i, j \leq n},
\]
and
\[
a = (a_0, a_1, \ldots, a_n)^T;
\]
then we can derive that
\[
\int_{-1}^{1} p_n^2(x)(1 - x^2)dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j b_{ij} = a^T B a,
\]
\[
\int_{-1}^{1} (p_n'(x))^2(1 - x^2)dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j c_{ij} = a^T C a.
\]
Now it suffices to consider the following extremal problem:
\[
(3) \quad \sup_{a \in R_{n+1}^+} \frac{a^T C a}{a^T B a}
\]
where \(R_{n+1}^+ = \{a: a = (a_0, a_1, \ldots, a_n)^T, a_i \geq 0, 0 \leq i \leq n\}\). Or find the least \(\lambda\) such that
\[
a^T C a \leq \lambda, \quad \text{for all } a \in R_{n+1}^+,
\]
which is
\[
(4) \quad a^T (\lambda B - C) a \geq 0, \quad \text{for all } a \in R_{n+1}^+.
\]
Observe that \(b_{ij} \geq 0\), \(c_{ij} \geq 0\), \(0 \leq i, j \leq n\). If we can find a smallest \(\lambda\) such that all the elements of \(\lambda B - C\) are nonnegative, then we obtain (4) automatically. Notice also that the matrices \(B\) and \(C\) have the same structure; thus it suffices to find \(\lambda\) such that
\[
\lambda b_{ij} - c_{ij} \geq 0, \quad \text{when } b_{ij} \neq 0,
\]
i.e.,
\[
\lambda \geq \frac{c_{ij}}{b_{ij}} = \frac{ij(i + j + 2\alpha + 1)}{i + j - 1}, \quad 1 \leq i, j \leq n.
\]
If we consider \(c_{ij}/b_{ij}\) as a function of two continuous variables \(i\) and \(j\), then we have
\[
\frac{\partial}{\partial i} \left( \frac{ij(i + j + 2\alpha + 1)}{i + j - 1} \right) = \frac{j[i^2 + (j - 1)(2i + j + 2\alpha + 1)]}{(i + j - 1)^2} \geq 0
\]
and similarly
\[
\frac{\partial}{\partial j} \left( \frac{ij(i + j + 2\alpha + 1)}{i + j - 1} \right) = \frac{i[j^2 + (i - 1)(2j + i + 2\alpha + 1)]}{(i + j - 1)^2} \geq 0;
\]
thus this is an increasing function of \(i\) and \(j\), and we can pick up
\[
\lambda = \frac{ij(i + j + 2\alpha + 1)}{i + j - 1} \bigg|_{i=n, j=n} = \frac{2n + 2\alpha + 1}{2n - 1} n^2.
\]
To see that \(\lambda\) is the best one, we can consider \(p_n(x) = x^n\) or \(a^T = (0, 0, \ldots, 0, 1)\). This completes the proof of the theorem. \(\Box\)

For some special values of \(\alpha\), we have the following corollaries.
Corollary 2.2. Let \( p_n(x) \in S_n \); then

\[
\int_{-1}^{1} (p_n'(x))^2 \, dx \leq \frac{2n + 1}{2n - 1} n^2 \int_{-1}^{1} p_n^2(x) \, dx
\]

with equality when \( p_n(x) = x^n \).

Corollary 2.3. Let \( p_n(x) \in S_n \); then

\[
\int_{-1}^{1} (p_n'(x))^2 (1 - x^2)^{-1/2} \, dx \leq \frac{2n + 2}{2n - 1} n^2 \int_{-1}^{1} p_n^2(x)(1 - x^2)^{-1/2} \, dx
\]

with equality when \( p_n(x) = x^n \).

Corollary 2.4. Let \( p_n(x) \in S_n \); then

\[
\int_{-1}^{1} (p_n'(x))^2 (1 - x^2)^{-1/2} \, dx \leq \frac{2n + 2}{2n - 1} n^2 \int_{-1}^{1} p_n^2(x)(1 - x^2)^{-1/2} \, dx
\]

with equality when \( p_n(x) = x^n \).

In the case \( \alpha = 1 \), a similar result was proved by Varma [9] for polynomials having real roots.

3. THE WEIGHT \( \omega(x) = x^\alpha e^{-x} \)

We give a new proof of Milovanović’s Theorem [4] in this section. Indeed we use the same argument as was used in the proof of Theorem 2.1. This time, we consider the weight function \( \omega(x) = x^\alpha e^{-x} \), \( \alpha > -1 \), on the interval \((0, \infty)\).

Theorem 3.1. Let \( p_n(x) \in S_n \), \( \alpha > -1 \); then

\[
\int_0^\infty (p_n'(x))^2 x^\alpha e^{-x} \, dx \leq C_n(\alpha) \int_0^\infty p_n^2(x)x^\alpha e^{-x} \, dx
\]

where

\[
C_n(\alpha) = \begin{cases} 
\frac{1}{(2 \alpha + 1)(1 + \alpha)}, & -1 < \alpha \leq \alpha_n, \\
\frac{n^2}{(2n + \alpha)(2n + \alpha - 1)}, & \alpha_n \leq \alpha < \infty,
\end{cases}
\]

and

\[
\alpha_n = \frac{1}{2}(n + 1)^{-1}[(17n^2 + 2n + 1)^{1/2} - 3n + 1].
\]

Moreover, \( C_n(\alpha) \) is the best possible constant.

Proof. Let \( p_n(x) = \sum_{i=0}^{n} a_i x^i \), \( a_i \geq 0 \), \( 0 \leq i \leq n \), then

\[
\int_0^\infty p_n^2(x)x^\alpha e^{-x} \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j \int_0^\infty x^{i+j+\alpha} e^{-x} \, dx
\]

\[
= \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j b_{ij} = a^T B a
\]

where

\[
b_{ij} = \int_0^\infty x^{i+j+\alpha} e^{-x} \, dx = \Gamma(i + j + \alpha + 1),
\]

\[
B = (b_{ij})_{0 \leq i, j \leq n}.
\]
And similarly, we have
\[
\int_0^\infty (p'_n(x))^2 x^\alpha e^{-x} \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j c_{ij} = a^T Ca
\]
where
\[
c_{ij} = \begin{cases} 
  ij \Gamma(i + j + \alpha - 1), & 1 \leq i, j \leq n, \\
  0, & i = 0 \text{ or } j = 0,
\end{cases}
C = (c_{ij})_{0 \leq i, j \leq n}.
\]

Therefore, we need to find the least \( \lambda \) such that
\[
\lambda b_{ij} - c_{ij} \geq 0, \quad \text{for } 1 \leq i, j \leq n.
\]
That is, the maximum value of the function
\[
f(i, j) := \frac{c_{ij}}{b_{ij}} = \frac{ij}{(i + j + \alpha)(i + j + \alpha - 1)}.
\]

Let \( k = i + j \); then
\[
f(i, j) = \frac{ij}{(i + j + \alpha)(i + j + \alpha - 1)} = \frac{i(k - i)}{(k + \alpha)(k + \alpha - 1)} =: g(i, k).
\]

If we consider \( g \) as a function of two continuous variables \( i \) and \( k \), then we have
\[
\frac{\partial g(i, k)}{\partial i} = \frac{k - 2i}{(k + \alpha)(k + \alpha - 1)}.
\]

Therefore, \( g(i, k) \) takes on its maximum value at \( i = k/2 \) if we fix \( k \) (consider it as a function of \( i \) alone). Now it suffices to consider the maximum value of the function
\[
h(k) := g \left( \frac{k}{2}, k \right) = \frac{k^2}{4(k + \alpha)(k + \alpha - 1)}.
\]

Following the exactly same argument of Milovanović [4, p. 425], we can see that the best possible value of \( \lambda \) is \( C_n(\alpha) \). We omit the details. This completes the proof. \( \square \)

Remark. The same idea also seems to work for other \( L_p \) norms when \( p \) is an integer, but they become more and more complicated as \( p \) is bigger and bigger. We will not formulate them here. However, for the \( L_1 \) norm, the result is simple.

**Theorem 3.2.** Let \( p_n(x) \in S_n, \alpha > -1; \) then
\[
(9) \quad \int_0^\infty p'_n(x)x^\alpha e^{-x} \, dx \leq \lambda_n(\alpha) \int_0^\infty p_n(x)x^\alpha e^{-x} \, dx
\]
where
\[
\lambda_n(\alpha) = \begin{cases} 
  1/(1 + \alpha), & -1 < \alpha \leq 0, \\
  n/(n + \alpha), & 0 \leq \alpha < \infty.
\end{cases}
\]

Moreover, \( \lambda_n(\alpha) \) is the best possible constant.
4. The weight $\omega(x) = e^{-\alpha x^2}$

In this section we discuss the weight function $\omega(x) = e^{-\alpha x^2}$, $\alpha > 0$, on the whole real line. The corresponding result is the following theorem.

**Theorem 4.1.** Let $p_n(x) \in S_n$, $\alpha > 0$; then

$$
\int_{-\infty}^{\infty} (p_n(x))' e^{-\alpha x^2} \, dx \leq \frac{2\alpha}{2n - 1} n^2 \int_{-\infty}^{\infty} p_n(x)e^{-\alpha x^2} \, dx
$$

with equality when $p_n(x) = x^n$.

**Proof.** Let $p_n(x) = \sum_{i=0}^{n} a_i x^i \in S_n$; then

$$
\int_{-\infty}^{\infty} p_n^2(x)e^{-\alpha x^2} \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j b_{ij} = a^T B a
$$

where

$$
b_{ij} = \int_{-\infty}^{\infty} x^{i+j} e^{-\alpha x^2} \, dx
$$

$$
= (1 - (-1)^{i+j+1})(i + j - 1)!!2^{-(i+j)/2 - 1} \alpha^{-(i+j+1)/2} \sqrt{\pi},
$$

$$
B = (b_{ij})_{0 \leq i, j \leq n},
$$

and

$$
\int_{-\infty}^{\infty} (p_n'(x))^2 e^{-\alpha x^2} \, dx = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i a_j c_{ij} = a^T C a
$$

where

$$
c_{ij} = ij \int_{-\infty}^{\infty} x^{i+j-2} e^{-\alpha x^2} \, dx
$$

$$
= (1 - (-1)^{i+j+1})ij(i + j - 3)!!2^{-(i+j)/2} \alpha^{-(i+j-1)/2} \sqrt{\pi},
$$

$$
C = (c_{ij})_{0 \leq i, j \leq n}.
$$

For $i + j$ even, let

$$
f(i, j) := \frac{c_{ij}}{b_{ij}} = 2\alpha \frac{ij}{i + j - 1}, \quad 1 \leq i, j \leq n;
$$

then considering $f$ as a function of two continuous variables $i$ and $j$, we can obtain

$$
\frac{\partial f(i, j)}{\partial i} = \frac{2\alpha j(j - 1)}{(i + j - 1)^2} \geq 0 \quad \text{for } 1 \leq i, j \leq n,
$$

and

$$
\frac{\partial f(i, j)}{\partial j} = \frac{2\alpha i(i - 1)}{(i + j - 1)^2} \geq 0, \quad \text{for } 1 \leq i, j \leq n.
$$

Therefore, $f(i, j)$ attains its maximum value at $i = n$, $j = n$, which implies the desired result. $\Box$

**Added in Proof.** After this manuscript was written, the author learned that Professor A. K. Varma [11] had written a paper on the same subject. There are some overlaps between his results and our results in §§2 and 3, but we do use different methods.
REFERENCES


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