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INTERPRETATION OF LAVRENTIEV PHENOMENON BY RELAXATION: THE HIGHER ORDER CASE

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ABSTRACT. We consider integral functionals of the calculus of variations of the form

$$F(u) = \int_0^1 f(x, u, u', \dots, u^{(n)})\, dx$$

defined for $u \in W^{n,\infty}(0, 1)$, and we show that the relaxed functional $\overline{F}$ with respect to the weak $W^{n,1}_{\mathrm{loc}}(0, 1)$ convergence can be written as

$$\overline{F}(u) = \int_0^1 f(x, u, u', \dots, u^{(n)})\, dx + L(u),$$

where the additional term $L(u)$, the Lavrentiev Gap, is explicitly identified in terms of F.

1. INTRODUCTION

In 1926 M. Lavrentiev (see [L]) first demonstrated this surprising result: given a variational integral of a two-point Lagrange problem, which is sequentially weakly lower semicontinuous on the admissible class of absolutely continuous functions, its infimum on the dense subclass of C^1 admissible functions may be *strictly greater* than its minimum value on the full admissible class. Some years later Manià (see [M]) gave an example of this phenomenon with a polynomial integrand. In recent years there have been additional works by several authors; for further bibliographical references the reader can see for instance [BuM].

In this paper we follow the Buttazzo and Mizel [BuM] approach which consists in studying the *Lavrentiev Phenomenon* from the point of view of relaxation theory. More precisely let X be a topological space, $Y \subset X$ a dense subset, $F : X \to [0, +\infty]$ a given functional, and define

$$\overline{F}_X = \sup\{G : X \to [0, +\infty] : G \text{ l.s.c., } G \le F \text{ on } X\},$$
$$\overline{F}_Y = \sup\{G : X \to [0, +\infty] : G \text{ l.s.c., } G \le F \text{ on } Y\},$$
$$L(u) = \begin{cases} \overline{F}_Y(u) - \overline{F}_X(u) & \text{if } \overline{F}_X(u) < +\infty, \\ 0 & \text{otherwise}, \end{cases} \qquad u \in X.$$

We call this nonnegative functional L (notice that $\overline{F}_X \le \overline{F}_Y$) the "*Lavrentiev Gap*" associated to F, X and Y. In their paper Buttazzo and Mizel [BuM]

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considered integral functionals of the form

$$G(u) = \int_0^1 f(x, u(x), u'(x))\,dx$$

with $X = W^{1,1}(0, 1)$ and $Y = W^{1,+\infty}(0, 1)$, and gave a characterization of L in term of the "*Value Function*" V (see §2 below). Then they obtained an explicit representation of L for a large class of integrands.

In this paper we extend the results of [BuM] to integral functionals depending on higher order derivatives, of the form

$$G(u) = \int_0^1 f(x, u(x), \dots, u^{(n)}(x))\,dx$$

with $X = W^{n,1}(0, 1)$ and $Y = W^{n,+\infty}(0, 1)$. More precisely, in §2 we obtain a characterization of L in terms of the "*Value Function*" V; in §3 we provide an explicit representation of L for some integrands which satisfy a "*homogeneity condition*", and an integrand of Manià type (see [M], [BM1], [BM2]) is analyzed in detail by following this approach. Our results deal with regular integrands (in a sense to be specified), but we want to point out an interesting result involving autonomous second order integrands (see Cheng [C], Cheng and Mizel [CM]) showing the nonoccurrence of the gap phenomenon when the integrand satisfies some continuity assumptions, with an example of a nonvanishing gap when a constraint of the form $\{u \geq 0\}$ is added.

2. The representation theorem

Let Ω be the interval $(0, 1)$; we consider the following spaces:

$W^{n,1}(0, 1)$	the space of all functions $u : \Omega \to \mathbf{R}$ which are absolutely continuous together with their $(n-1)$ derivatives;
$W^{n,\infty}[0, 1]$	the space of all functions $u : \Omega \to \mathbf{R}$ which are Lipschitz continuous together with their $(n-1)$ derivatives;
$W^{n,\infty}_{\text{loc}}]0, 1]$	the space of all functions $u : \Omega \to \mathbf{R}$ which are Lipschitz continuous together with their $(n-1)$ derivatives on every interval $[\delta, 1]$, with $\delta > 0$;
$\mathscr{A}_\infty$	the space of all function $u \in W^{n,1}(0, 1) \cap W^{n,\infty}_{\text{loc}}]0, 1]$ such that $u^{(i)}(0) = 0$ for $i = 0, \dots, (n-1)$.

Let $f : \Omega \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ be a function such that:

(i) $f(x, s, z)$ is of Carathéodory type (i.e. measurable in X and continuous in (s, z));
(ii) $f(x, s, \cdot)$ is convex on $\mathbf{R}$ for every $(x, s) \in \Omega \times \mathbf{R}^n$;
(iii) there exists a function $\omega : \Omega \times \mathbf{R} \times \mathbf{R} \to [0, +\infty[$, with $\omega(x, t, \tau)$ integrable in x and increasing in t, τ, such that

$$0 \leq f(x, s, z) \leq \omega(x, |s|, |z|) \quad \forall (x, s, z) \in \Omega \times \mathbf{R}^n \times \mathbf{R}.$$

For every $u \in \mathscr{A}_\infty$, define

$$F(u) = \int_0^1 f(x, u, \dots, u^{(n)})\, dx,$$
$$G(u) = \begin{cases} F(u) & \text{if } u \in W^{n,\infty}[0, 1], \\ +\infty & \text{otherwise}, \end{cases}$$

and denote by $\overline{G}$ the functional

$$\overline{G} = \max\left\{H : \mathscr{A}_\infty \to [0, +\infty] : H \text{ seq. w-}W^{n,1}_{\mathrm{loc}}\text{-l.s.c.}, H \leq G\right\}.$$

Our goal is to give a representation formula for $\overline{G}$ over $\mathscr{A}_\infty$.

Since F is sequentially weakly lower semicontinuous on $W^{n,1}_{\mathrm{loc}}(0, 1)$ (briefly seq. w-$W^{n,1}_{\mathrm{loc}}(0, 1)$-l.s.c.) (see [B]), we have

$$\overline{G}(u) \geq F(u) \quad \forall u \in \mathscr{A}_\infty,$$

and then

$$\overline{G}(u) = F(u) + L(u) \quad \forall u \in \mathscr{A}_\infty$$

for a suitable functional $L \geq 0$. We call the functional L the "*Lavrentiev Gap*" relative to G over the space $\mathscr{A}_\infty$. Obviously we have that $\overline{G} \leq G$. Then

$$\overline{G}(u) = F(u) \quad \forall u \in W^{n,\infty}[0, 1];$$

i.e. $L(u) = 0$ for every $u \in W^{n,\infty}[0, 1]$. In order to identify the functional L we introduce the "*Value Function*" $V(x, s)$ defined for every $(x, s) \in \mathbf{\Omega} \times \mathbf{R}^n$ by:

$$V(x, s) = \inf\left\{\int_0^x f(t, u, \dots, u^{(n)})\, dt : u \in W^{n,\infty}[0, 1], u^{(i)}(0) = 0, \right.$$
$$\left. u^{(i)}(x) = s_i,\ i = 0, \dots, (n-1)\right\}$$

and its lower semicontinuous envelope with respect to $s = (s_0, \dots, s_{(n-1)})$, given by

$$W(x, s) = \liminf_{\xi \to s} V(x, \xi).$$

We now state a representation result for the Lavrentiev Gap L.

Theorem 2.1. *If the integrand $f(x, s, z)$ satisfies the hypotheses above, then*

$$L(u) = \liminf_{x \to 0^+} W(x, u(x), \dots, u^{(n-1)}(x)) \quad \textit{for every } u \in \mathscr{A}_\infty.$$

In order to achieve the proof of Theorem 2.1 we need some lemmas. For the sake of simplicity in the following we set

$$M(u) = \liminf_{x \to 0^+} W(x, u(x), \dots, u^{(n-1)}(x))$$

and, when no confusion is possible, we use the notation $\overline{u}(x)$ to indicate the vector $(u^{(i)}(x))_{i=0}^{n-1}$.

Lemma 2.2. *Take* $u, u_h \in \mathscr{A}_\infty$ *with* $u_h \in W^{n,\infty}[0, 1]$ *and assume that* $u_h \to u$ *weakly in* $W^{n,1}_{\mathrm{loc}}(0, 1)$. *Then*

$$F(u) + M(u) \leq \liminf_{h\to+\infty} G(u_h).$$

Proof. Take $\delta > 0$; for every $h \in \mathbf{N}$, by the definitions of $V(x, s)$ and $W(x, s)$ we get

$$\begin{aligned} G(u_h) &= \int_0^\delta f(x, u_h, \dots, u_h^{(n)})\,dx + \int_\delta^1 f(x, u_h, \dots, u_h^{(n)})\,dx \\ &\geq V(\delta, \overline{u}_h(\delta)) + \int_\delta^1 f(x, u_h, \dots, u_h^{n})\,dx \\ &\geq W(\delta, \overline{u}_h(\delta)) + \int_\delta^1 f(x, u_h, \dots, u_h^{(n)})\,dx. \end{aligned}$$

As $h \to +\infty$, taking into account that W is seq. w-$W^{n,1}_{\mathrm{loc}}$-l.s.c. and that the assumptions on the integrand f provide the seq. w-$W^{n,1}_{\mathrm{loc}}$-l.s.c. of the integral term, we get

$$\begin{aligned} \liminf_{h\to+\infty} G(u_h) &\geq \liminf_{h\to+\infty} \left[W(\delta, \overline{u}_h(\delta)) + \int_\delta^1 f(x, u_h, \dots, u_h^{(n)})\,dx \right] \\ &\geq W(\delta, \overline{u}(\delta)) + \int_\delta^1 f(x, u, \dots, u^{(n)})\,dx. \end{aligned}$$

Finally, as $\delta \to 0$ we obtain

$$\begin{aligned} \liminf_{h\to+\infty} G(u_h) &\geq \liminf_{\delta\to 0} \left[W(\delta, \overline{u}(\delta)) + \int_\delta^1 f(x, u, \dots, u^{(n)})\,dx \right] \\ &\geq M(u) + \int_0^1 f(x, u, \dots, u^{(n)})\,dx \\ &= M(u) + F(u), \end{aligned}$$

and the lemma is proved. □

Lemma 2.3. *The functional* $F + M$ *is seq.* w-$W^{n,1}_{\mathrm{loc}}$-*l.s.c. on* $\mathscr{A}_\infty$.

Proof. Taking $u, u_h \in \mathscr{A}_\infty$ with $u_h \to u$ weakly in $W^{n,1}_{\mathrm{loc}}$, we have to show that

$$F(u) + M(u) \leq \liminf_{h\to+\infty} [F(u_h) + M(u_h)].$$

Assume that the right-hand side is finite (otherwise there is nothing to prove), and consider a sequence (x_h) in Ω with $x_h \to 0$ such that

$$W(x_h, \overline{u}_h(x_h)) \geq M(u_h) + \frac{1}{h} \quad \forall h \in \mathbf{N}. \tag{2.1}$$

It is now possible to find a sequence (s_h) in $\mathbf{R}^n$, with $s_h \to 0$ such that

$$|s_h - \overline{u}_h(x_h)| \leq \frac{1}{h}; \tag{2.2}$$

$$V(x_h, s_h) \leq W(x_h, \overline{u}_h(x_h)) + \frac{1}{h}. \tag{2.3}$$

Moreover, denoting by P_{n-1} the polynomial of degree $n-1$ such that $\overline{P}_{n-1}(x_h) = s_h - \overline{u}_h(x_h)$, it is easy to see that, since f is of Carathéodory type, the sequence (s_h) can be taken such that

$$\int_{x_h}^1 f(x, \overline{u}_h + \overline{P}_{n-1}, u_h^{(n)})\,dx \le \int_{x_h}^1 f(x, \overline{u}_h, u_h^{(n)})\,dx + \frac{1}{h}. \tag{2.4}$$

Finally, let $v_h \in W^{n,\infty}[0, x_h]$ be such that $\overline{v}_h(0) = 0$, $\overline{v}_h(x_h) = s_h$ and

$$\int_0^{x_h} f(x, \overline{v}_h, v_h^{(n)})\,dx \le V(x_h, s_h) + \frac{1}{h}. \tag{2.5}$$

Setting

$$w_h(x) = \begin{cases} u_h(x) + P_{n-1}(x) & \text{if } x > x_h, \\ v_h(x) & \text{if } 0 \le x \le x_h, \end{cases}$$

we have $w_h \in W^{n,\infty}[0, 1]$, $\overline{w}_h(0) = 0$, and

$$w_h \xrightarrow[h\to+\infty]{} u \text{ w-}W^{n-1}_{\text{loc}}(0, 1). \tag{2.6}$$

Therefore, by using Lemma 2.2 and (2.1)–(2.6), we obtain

$$\begin{aligned}
F(u) + M(u) &\le \liminf_{h\to+\infty} F(w_h) \\
&= \liminf_{h\to+\infty} \left[\int_0^{x_h} f(x, \overline{v}_h, v_h^{(n)})\,dx + \int_{x_h}^1 f(x, \overline{u}_h + \overline{P}_{n-1}(x_h), u_h^{(n)})\,dx\right] \\
&\le \liminf_{h\to+\infty} \left[\left(V(x_h, s_h) + \frac{1}{h}\right) + \left(\int_{x_h}^1 f(x, \overline{u}_h, u_h^{(n)})\,dx + \frac{1}{h}\right)\right] \\
&\le \liminf_{h\to+\infty} \left[\left(W(x_h, \overline{u}_h(x_h)) + \frac{1}{h}\right) + F(u_h) + \frac{2}{h}\right] \\
&\le \liminf_{h\to+\infty} \left[\left(M(u_h) + \frac{1}{h}\right) + F(u_h) + \frac{3}{h}\right] \\
&= \liminf_{h\to+\infty} [M(u_h) + F(u_h)],
\end{aligned}$$

and the lemma is proved. □

Proof of Theorem 2.1. It is easy to see that

$$M(u) = 0 \quad \text{for every } u \in \mathscr{A}_\infty \cap W^{n,\infty}[0, 1];$$

hence we have $F + M \le G$ on $\mathscr{A}_\infty$. By Lemma 2.3 we have $F + M \le \overline{G}$ on $\mathscr{A}_\infty$, so it remains to prove that

$$\overline{G} \le F(u) + M(u) \quad \text{for every } u \in \mathscr{A}_\infty.$$

To this aim, fix $u \in \mathscr{A}_\infty$ and take a sequence (x_h) in Ω, $x_h \to 0$, such that

$$M(u) = \lim_{h\to+\infty} W(x_h, \overline{u}(x_h)). \tag{2.7}$$

By definition of W and the assumptions on the integrand f we may find a sequence (s_h) in $\mathbf{R}^n$, $s_h \to 0$, such that for every $h \in \mathbf{N}$

$$|s_h - \overline{u}(x_h)| \leq \frac{1}{h}, \tag{2.8}$$

$$V(x_h, s_h) \leq W(x_h, \overline{u}(x_h)) + \frac{1}{h}, \tag{2.9}$$

$$\int_{x_h}^1 f(x, \overline{u} + \overline{P}_{n-1}, u^{(n)})\, dx \leq \int_{x_h}^1 f(x, \overline{u}, u^{(n)})\, dx + \frac{1}{h}, \tag{2.10}$$

where P_{n-1} is as in the proof of Lemma 2.3. Finally, let $v_h \in W^{n,\infty}[0, x_h]$ be a sequence such that $\overline{v}_h(0) = 0$, $\overline{v}_h(x_h) = s_h$ and

$$\int_0^{x_h} f(x, \overline{v}_h, v_h^{(n)})\, dx \leq V(x_h, s_h) + \frac{1}{h}. \tag{2.11}$$

As in the proof of Lemma 2.3, we define the sequence

$$w_h(x) = \begin{cases} u_h(x) + P_{n-1}(x) & \text{if } x > x_h, \\ v_h(x) & \text{if } 0 \leq x \leq x_h. \end{cases}$$

Then $w_h \in W^{n,\infty}[0, 1]$, $\overline{w}_h(0) = 0$ and

$$w_h \xrightarrow[h\to+\infty]{} u \text{ strongly } W^{n,1}_{\text{loc}}(0, 1).$$

Hence, by using (2.7)–(2.11), we obtain

$$\begin{aligned}
\overline{G}(u) &\leq \liminf_{h\to+\infty} G(w_h) \\
&= \liminf_{h\to+\infty} \left[\int_0^{x_h} f(x, \overline{v}_h, v_h^{(n)})\, dx + \int_{x_h}^1 f(x, \overline{u} + \overline{P}_{n-1}(x_h), u^{(n)})\, dx \right] \\
&\leq \liminf_{h\to+\infty} \left[\left(V(x_h, s_h) + \frac{1}{h} \right) + \left(\int_{x_h}^1 f(x, \overline{u}, u^{(n)})\, dx + \frac{1}{h} \right) \right] \\
&\leq \liminf_{h\to+\infty} \left[\left(W(x_h, \overline{u}(x_h)) + \frac{1}{h} \right) + F(u) + \frac{2}{h} \right] \\
&= M(u) + F(u);
\end{aligned}$$

so $M = L$, and the theorem is completely proved. □

Remark 2.4. Fix a subset β of $\{0, 1, \ldots, n-1\}$ and consider the class $\mathscr{A}_\infty^\beta$ of all functions $u \in W^{n,1}(0, 1) \cap W^{n,\infty}_{\text{loc}}]0, 1]$ such that $u^{(i)}(0) = 0$ for $i \in \beta$. We denote by $\overline{G}_\beta$ the functional

$$\overline{G}_\beta = \max\{H : \mathscr{A}_\infty^\beta \to [0, +\infty] : H \text{ seq. w-}W^{n,1}_{\text{loc}}\text{-l.s.c.}, H \leq G\}.$$

As in the previous case, we have

$$\overline{G}_\beta(u) = F(u) + L_\beta(u) \quad \forall u \in \mathscr{A}_\infty$$

for a suitable functional $L_\beta \geq 0$, the "*Lavrentiev Gap*" relative to G over the space $\mathscr{A}_\infty^\beta$. In order to identify the functional L_β we introduce the Value

Function $V_\beta(x, s)$ defined for every $(x, s) \in \Omega \times \mathbf{R}^k$ by:

$$V_\beta(x, s) = \inf\left\{ \int_0^x f(t, u, \dots, u^{(n)})\, dt : u \in W^{n,\infty}[0, 1],\, u^{(i_j)}(0) = 0, \right.$$
$$\left. u^{(i_j)}(x) = s_{i_j},\ j = 0, 1, \dots, k-1 \right\}$$

and its lower semicontinuous envelope with respect to $s = (s_{i_0}, \dots, s_{i_{k-1}})$, given by

$$W_\beta(x, s) = \liminf_{\xi \to s} V_\beta(x, \xi).$$

By repeating step by step the proof of Theorem 2.1, we obtain the following result:

Theorem 2.5. *If the integrand $f(x, s, z)$ satisfies the assumptions of Theorem 2.1, then*

$$L_\beta(u) = \liminf_{x \to 0^+} W_\beta(x, u^{(i_0)}(x), \dots, u^{(i_{k-1})}(x)) \quad \textit{for every } u \in \mathscr{A}_\infty^\beta.$$

Note that the polynomial P_{n-1} may be chosen, in this case, such that

$$P_{n-1}^{(r)} = \begin{cases} u_h^{(r)}(x_h) - (s_h)_r & \text{if } r \in \beta, \\ 0 & \text{otherwise}. \end{cases}$$

3. Some examples

In this section we give an explicit representation formula for a class of second order integrands f (we mean that f is a function depending on x, u, u', u''). We introduce the so-called "*invariance property*" for second order integrands (analogous to the one introduced in [HM1] for first order integrands, and to the one of [CM] for second order autonomous integrands):

there exists $\gamma \in]1, 2[$ such that for every $t > 0$ and $(x, s, z, w) \in \Omega \times \mathbf{R}^3 t f(tx, t^\gamma s, t^{\gamma-1} z, t^{\gamma-2} w) = f(x, s, z, w)$.

We want to analyze a class of second order integrands $f(x, s, z, w)$ that satisfies this invariance property only in an asymptotic sense near the relevant singular abscissa. Let us take $\delta > 1$, $\tau \in [1, \delta[$; we suppose the integrand $f : \Omega \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ has the form

$$f(x, s, z, w) = x^{\tau-1} a(x, s) b(x, z) |w|^\delta,$$

with $a(x, s)$, $b(x, z)$ nonnegative, continuous functions such that, setting $\gamma = 2 - \frac{\tau}{\delta}$, for every $y \in \Omega$ the functions $m_y, n_y, M_y, N_y : \Omega \to \mathbf{R}$ defined by

$$m_y(s) = \inf\{a(x, x^\gamma s) : x \le y\}, \quad n_y(s) = \inf\{b(x, x^{\gamma-1} s) : x \le y\},$$
$$M_y(s) = \sup\{a(x, x^\gamma s) : x \le y\}, \quad N_y(s) = \sup\{b(x, x^{\gamma-1} s) : x \le y\}$$

are locally bounded. Take now $x, y \in \Omega$, $x \le y$, and consider the following functionals:

$$F_x(u) = \int_0^x f(t, u, u', u'')\, dt,$$
$$F_{*,x,y}(u) = \int_0^x t^{\tau-1} m_y(t^{-\gamma} u) n_y(t^{1-\gamma} u') |u''|^\delta\, dt,$$
$$F_{x,y}^*(u) = \int_0^x t^{\tau-1} M_y(t^{-\gamma} u) N_y(t^{1-\gamma} u') |u''|^\delta\, dt.$$

We suppose that there exists $\overline{y} \in \Omega$ such that, for every $x \in \{x \in \Omega : x \leq \overline{y}\}$, we have

$$F^*_{x,\overline{y}}(u) < +\infty \quad \text{whenever } F_x(u) < +\infty . \tag{3.1}$$

Obviously for every $x, y \in \Omega$ with $x \leq y$

$$F_{*,x,y}(u) \leq F_x(u) \leq F^*_{x,y}(u) \quad \forall u \in \mathscr{A}^1_\infty ;$$

then for every $x \in \Omega$

$$\sup_{\substack{y\in\Omega \\ y\leq\overline{y}}} F_{*,x,y}(u) \leq F_x(u) \leq \inf_{\substack{y\in\Omega \\ y\leq\overline{y}}} F^*_{x,y}(u) \quad \forall u \in \mathscr{A}^1_\infty ,$$

say

$$\lim_{y\to 0^+} F_{*,x,y}(u) \leq F_x(u) \leq \lim_{y\to 0^+} F^*_{x,y}(u) \quad \forall u \in \mathscr{A}^1_\infty . \tag{3.2}$$

Define, for $y \leq \overline{y}$,

$$\lim_{y\to 0^+} m_y(s) = m_0(s), \qquad \lim_{y\to 0^+} M_y(s) = M_0(s),$$
$$\lim_{y\to 0^+} n_y(s) = n_0(s), \qquad \lim_{y\to 0^+} N_y(s) = N_0(s);$$

by the assumptions (3.1) we apply the Monotone and Lebesgue Convergence Theorems to (3.2) obtaining

$$F_{0,x}(u) \leq F_x(u) \leq F^0_x(u) \quad \forall u \in \mathscr{A}^1_\infty \tag{3.3}$$

where

$$F_{0,x}(u) = \int_0^x t^{\tau-1} m_0(t^{-\gamma}u) n_0(t^{1-\gamma}u')|u''|^\delta \, dt,$$
$$F^0_x(u) = \int_0^x t^{\tau-1} M_0(t^{-\gamma}u) N_0(t^{1-\gamma}u')|u''|^\delta \, dt.$$

We suppose also that $m_0(s) = P \leq Q = M_0(s)$, with $P, Q \in [0, +\infty[$.

Theorem 3.1. *Under the previous assumptions, for every*

$$u \in \mathscr{A}^1_\infty = \{u \in W^{2,1}(0, 1) \cap W^{2,\infty}_{\text{loc}}]0, 1] : u'(0) = 0\}$$

we have

$$\begin{aligned} &P\delta k^{\delta-1} \left| \int_0^{\liminf_{x\to 0} u'(x)x^{1-\gamma}} n_0(\xi)|\xi|^{\delta-1} d\xi \right| \\ &\quad \leq L(u) \leq Q\delta k^{\delta-1} \left| \int_0^{\liminf_{x\to 0} u'(x)x^{1-\gamma}} N_0(\xi)|\xi|^{\delta-1} \, d\xi \right| , \end{aligned} \tag{3.4}$$

where $k = \frac{\delta(\gamma-1)}{\delta-1}$.

In order to achieve the proof of Theorem 3.1, we need a lemma.

Lemma 3.2. *Let $h(Z)$ be the solution of the minimum problem*

$$\inf\{G(u) : u \in W^{2,\infty}(0, 1), u'(0) = 0, u'(1) = Z\},$$

where

$$G(u) = \int_0^1 x^{\tau-1} n(x^{1-\gamma} u'(x)) |u''(x)|^\delta \, dx.$$

We have

$$h(Z) = \delta k^{\delta-1} \left| \int_0^Z n(\xi) |\xi|^{\delta-1} \, d\xi \right|,$$

where $k = \frac{\delta(\gamma-1)}{\delta-1}$ *and the function* $h(Z)$ *is the solution of the equation*

$$\begin{cases} (\gamma-1) Z h'(Z) = \sup\{Q h'(Z) - n(Z)|Q|^\delta : Q \in \mathbf{R}\}, \\ h(0) = 0. \end{cases} \tag{3.5}$$

Proof. By explicitly carrying out the maximization, the equation (3.5) becomes

$$h'(Z) = \delta k^{\delta-1} n(Z) |Z|^{\delta-2}$$

and by direct integration

$$h(Z) = \delta k^{\delta-1} \left| \int_0^Z n(\xi) |\xi|^{\delta-1} \, d\xi \right|.$$

Let us take $u \in \mathscr{A}(x, z) = \{u \in W^{2,\infty} : u'(0) = 0, u'(x) = z\}$; from (3.5), setting $Z(t) = t^{1-\gamma} u'(t)$ and $Q(t) = t^{2-\gamma} u''(t)$ we obtain

$$(\gamma-1) Z(t) h'(Z(t)) \geq |Q(t)| h'(Z(t)) - n(Z(t)) |Q(t)|^\delta.$$

Then

$$\begin{aligned} t^{-1} n(Z(t)) |Q(t)|^\delta &\geq t^{-1} h'(Z(t)) [Q(t) + (1-\gamma) Z(t)] \\ &= h'(Z(t)) Z'(t) = (h \circ Z)'(t) \end{aligned}$$

(for the last equality see [MM]). Integrating on $]0, x[$ yields

$$\begin{aligned} I(u) &= \int_0^x t^{\tau-1} n(t^{1-\gamma} u'(t)) |u''(t)|^\delta \, dt \\ &= \int_0^x t^{-1} n(t^{1-\gamma} u'(t)) |t^{\tau/\delta} u''(t)|^\delta \, dt \\ &= \int_0^x t^{-1} n(Z(t)) |Q(t)|^\delta \, dt \\ &\geq \int_0^x (h \circ Z)'(t) \, dt \\ &= h(Z(x)) - \lim_{t \to 0^+} h(Z(t)) \\ &= h(Z(x)) \end{aligned}$$

(in fact $u \in W^{2,\infty}[0, x]$ with $u'(0) = 0$ implies

$$\lim_{t \to 0^+} t^{1-\gamma} u'(t) = 0 \quad \forall \gamma \in [1, 2[,$$

and hence $\lim_{t \to 0^+} h(Z(t)) = 0$). It follows that

$$W(x, z) = \inf\{I(u) : u \in \mathscr{A}(x, z)\} \geq h(x^{1-\gamma} z) = h(Z). \tag{3.6}$$

Consider now the sequence $(u_\varepsilon) \subset \mathscr{A}(x, z)$ defined by

$$u_\varepsilon(0) = 0, \quad u'_\varepsilon(t) = \begin{cases} (\frac{t}{x})^k z & \text{if } t \geq \varepsilon, \\ t\frac{\varepsilon^{k-1}}{x^k} z & \text{if } t < \varepsilon. \end{cases}$$

Taking ε sufficiently small we have

$$W(x, z) \leq I(u_\varepsilon) = \int_0^\varepsilon t^{\tau-1} n(t^{1-\gamma} u'_\varepsilon)|u''_\varepsilon|^\delta \, dt + \int_\varepsilon^x t^{\tau-1} n(t^{1-\gamma} u'_0)|u''_0|^\delta \, dt,$$

where $u'_0(t) = (\frac{t}{x})^k$, $u_0(0) = 0$; passing to the limit for $\varepsilon \to 0$ the first integral tends to 0, and hence

$$W(x, s) \leq I(u_0).$$

At this point we can easily verify that $I(u_0) = h(x^{1-\gamma} z)$, and the proof of the lemma is then complete. □

Proof of Theorem 3.1. We fix $u \in \mathscr{A}^1_\infty$; by Theorem 2.5

$$L_1(u) = \liminf_{x \to 0} W_1(x, u'(x))$$

where $W_1(x, z) = \liminf_{q \to z} V_1(x, q)$ and

$$\begin{aligned} V_1(x, z) &= \inf\{F_x(u) : u \in W^{2,\infty}(0, 1), \, u'(0) = 0, \, u'(x) = z\} \\ &= \inf\{F_x(u) : u \in \mathscr{A}(x, z)\}, \end{aligned}$$

where $\mathscr{A}(x, z) = \{u \in W^{2,\infty}(0, x) : u'(0) = 0, \, u'(x) = z\}$.

Let us introduce the Value Functions relative to the functionals $F_{0,x}$, F_x^0 given by

$$\begin{aligned} V_0(x, z) &= \inf\{F_{0,x}(u) : u \in \mathscr{A}(x, z)\}, \\ V^0(x, z) &= \inf\{F_x^0(u) : u \in \mathscr{A}(x, z)\}; \end{aligned} \tag{3.7}$$

obviously, for every $x \in \Omega$ and for every $z \in \mathbf{R}$, we have by (3.3)

$$V_0(x, z) \leq V_1(x, z) \leq V^0(x, z). \tag{3.8}$$

Setting $S = sx^{-\gamma}$, $Z = zx^{1-\gamma}$ and

$$\begin{aligned} G_0(u) &= P \int_0^1 t^{\tau-1} n_0(t^{1-\gamma} u')|u''|^\delta \, dt, \\ G^0(u) &= Q \int_0^1 t^{\tau-1} N_0(t^{1-\gamma} u')|u''|^\delta \, dt, \\ \mathscr{A}(Z) &= \{u \in W^{2,\infty}(0, 1) : u'(0) = 0, \, u'(1) = Z\}, \end{aligned}$$

by the change of variable $t = xy$ we get

$$\begin{aligned} V_0(x, z) &= H_0(Z) = \inf\{G_0(u) : u \in \mathscr{A}(Z)\}, \\ V^0(x, z) &= H^0(Z) = \inf\{G^0(u) : u \in \mathscr{A}(Z)\}, \end{aligned}$$

so that inequalities (3.8) become

$$H_0(Z) \leq V_1(x, z) \leq H^0(Z) \tag{3.9}$$

for every $x \in \Omega$ and for every $z \in \mathbf{R}$.

By Lemma 3.2 we have that $H_0(Z)$ and $H^0(Z)$ are given by

$$\begin{aligned} H_0(Z) &= P\delta k^{\delta-1}\left|\int_0^Z n_0(\xi)|\xi|^{\delta-1}\,d\xi\right|, \\ H^0(Z) &= Q\delta k^{\delta-1}\left|\int_0^Z N_0(\xi)|\xi|^{\delta-1}\,d\xi\right|; \end{aligned} \tag{3.10}$$

and inserting (3.10) into (3.9) we obtain the inequality (3.4), that is the thesis. □

Example 3.3. Consider the functional

$$F(u) = \int_0^1 f(x, u(x), u'(x), u''(x))\,dx,$$

where the integrand f has the following form, with $1 < p < 2$ and $0 < q < 1$,

$$\begin{aligned} f(x, s, z, w) &= (s - x^p)^2(z - x^q)^2|w|^\delta \\ &= x^{2(p+q)}(sx^{-p} - 1)^2(zx^{-q} - 1)^2|w|^\delta \\ &= x^{2(p+q)}a(x, s)b(x, z)|w|^\delta, \end{aligned}$$

where we set

$$\begin{aligned} a(x, s) &= (sx^{-p} - 1)^2, \\ b(x, z) &= (zx^{-q} - 1)^2. \end{aligned}$$

If $\delta \leq 1$ we can easily verify that the Lavrentiev Gap $L(u)$ is identically equal to 0: for every $u \in W^{2,1}(0, 1)$ with $u'(0) = 0$ we construct the sequence in $W^{2,\infty}$

$$\begin{aligned} u'_\varepsilon(x) &= \begin{cases} u'(x), & \text{if } x_\varepsilon \leq x, \\ \frac{u'(x_\varepsilon)}{x_\varepsilon}x, & \text{if } 0 \leq x \leq x_\varepsilon, \end{cases} \\ u_\varepsilon(0) &= u(0), \end{aligned} \tag{3.11}$$

where $x_\varepsilon \in [0, 1]$ is a sequence with limit 0 as $\varepsilon \to 0$, and we verify that $F(u_\varepsilon) \to F(u)$ as $\varepsilon \to 0$. Here, for simplicity, we restrict our attention to the case $\delta > \frac{1+2(p+q)}{2-p}$. With the notation above

$$\tau = 1 + 2(p + q), \quad \gamma = 2 - \frac{\tau}{\delta} = 2 - \frac{1 + 2(p + q)}{\delta}.$$

This integrand f has as "*zero cost curves*" the functions $z_1(x) = x^p$, $z_2(x) = (q + 1)^{-1}x^{q+1}$; by the assumption on p and q we obtain $z_1(x), z_2(x) \in W^{2,1}(0, 1)\backslash W^{2,\infty}[0, 1]$.

When $\delta > \frac{1+2(p+q)}{1-q}$, we have $\gamma > p$, $\gamma > q$ and then

$$m_0(s) = n_0(s) = M_0(s) = N_0(s) = 1;$$

hence for every fixed $u \in \mathscr{A}^1_\infty$ we obtain

$$L_1(u) = k^{\delta-1}\liminf_{x\to 0^+}\left|\frac{u'(x)}{x^{\gamma-1}}\right|^\delta;$$

this functional is not identically equal to 0: for instance, $L_1(x^p) = +\infty$ and $L_1((q + 1)^{-1}x^{q+1}) = +\infty$.

When $\delta = \frac{1+2(p+q)}{1-q}$, by a computation similar to the previous case, we obtain

$$m_0(s) = M_0(s) = 1, \quad n_0(s) = N_0(s) = (s-1)^2.$$

Then, for every fixed $u \in \mathscr{A}^1_\infty$ we have

$$L_1(u) = \delta k^{\delta-1} \left| \liminf_{x\to 0^+} \int_0^{u'(x)x^{1-\gamma}} (\xi - 1)^2 |\xi|^{\delta-1}\, d\xi \right| ;$$

also in this case this functional is not identically equal to 0: for instance $L_1(x^p) = +\infty$, while $L_1((q+1)^{-1}x^{q+1}) = 2k^{\delta-1}/(\delta+1)(\delta+2)$.

Finally, when $\delta < \frac{1+2(p+q)}{1-q}$, Theorem 3.1 does not apply because the functions n_y, N_y are not locally bounded. However it is possible to show that in this case the gap phenomenon does not occur (see [Be]): for every $u \in \mathscr{A}^1_{W^{2,1}}$ we construct u_ε in $W^{2,\infty}(0,1)$ by (3.11) and we prove that, if $F(u) < +\infty$, then $F(u_\varepsilon) \to F(u)$ as $\varepsilon \to 0$, i.e.

$$\int_0^1 f(x, u, u', u'')\, dx < +\infty \Rightarrow L(u) = 0.$$

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