TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 7, July 1995

CHARACTERIZATION OF SUMMABILITY POINTS OF NÖRLUND METHODS

KARL-GOSWIN GROSSE-ERDMANN AND KARIN STADTMÜLLER

ABSTRACT. By a theorem of F. Leja any regular Nörlund method (N, p) sums a given power series f at most at countably many points outside its disc of convergence. This result was recently extended to a class of non-regular Nörlund methods by K. Stadtmüller. In this paper we obtain a more detailed picture showing how possible points of summability and the value of summation depend on p and f.

1. INTRODUCTION

Let $p = (p_n)_{n=0,1,\dots}$ be a sequence of complex numbers such that $P_n := \sum_{\nu=0}^{n} p_\nu \neq 0$ for all $n \in \mathbb{N}_0$. This sequence generates a Nörlund method (N, p), where the transformation matrix $A = (\alpha_{n\nu})_{n,\nu=0,1,\dots}$ is given by

$$\alpha_{n\nu} = \frac{p_{n-\nu}}{P_n} \quad \text{if} \quad 0 \le \nu \le n, \quad \alpha_{n\nu} = 0 \quad \text{if} \quad \nu > n \quad (n \in \mathbb{N}_0).$$

Thus, the (N, p)-transforms of a sequence (s_n) are given by

$$\sigma_n = \sum_{\nu=0}^{n} p_{n-\nu}\, s_\nu / P_n \qquad (n \in \mathbb{N}_0),$$

and (s_n) is (N, p)-summable to the value σ, $\sigma = (N, p)\text{-}\lim s_n$, if $\sigma_n \to \sigma$ as $n \to \infty$.

Throughout this paper let

$$(1) \qquad f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{with} \quad \overline{\lim_{k\to\infty}}\, |a_k|^{1/k} = \frac{1}{R} \quad (0 \le R \le \infty)$$

be a power series with partial sums $s_n(z) = \sum_{k=0}^{n} a_k z^k$. Its (N, p)-transforms are given by

$$\sigma_n(z) = \sum_{\nu=0}^{n} \frac{p_{n-\nu}}{P_n} s_\nu(z) = \sum_{\nu=0}^{n} \frac{P_{n-\nu}}{P_n} a_\nu z^\nu,$$

where the first equality represents the so-called sequence-sequence form and the second the series-sequence form. For Nörlund methods both transforms

Received by the editors November 3, 1993; originally communicated to the *Proceedings of the AMS* by Andrew Bruckner.

1991 *Mathematics Subject Classification*. Primary 30B10, 40G05; Secondary 40D09.

Key words and phrases. Power series, matrix transforms, Nörlund methods, summability points.

are equivalent. If $\sigma_n(z_0) \to \sigma(z_0)$ $(n \to \infty)$, we say that the power series f is (N, p)-summable at z_0 and write (N, p)-$\sum_{k=0}^{\infty} a_k z_0^k = \sigma(z_0)$; compact (= locally uniform) summability in a domain in $\mathbb{C}$ is defined accordingly.

It was shown by F. Leja [5] that a regular Nörlund method (N, p) sums any given power series (1) with $R > 0$ at most at countably many points outside the disc of convergence, and these points can only accumulate on $|z| = R$. This result was recently generalized for non-regular Nörlund methods by the second author [6]. In this paper we will deal with the problem of how these points of (N, p)-summability can be characterized and whether it is possible to prescribe summability points. Also, it was pointed out in [2] that the original proofs of Leja's and Stadtmüller's theorem contain a gap. As a by-product of our results we obtain a new and short proof of that theorem that eliminates the gap.

2. Some properties of (N, p)-methods

From the theorem of Silverman and Toeplitz (see, e.g., [4, p. 43]) we get that a Nörlund method (N, p) is regular if and only if

$$\lim_{n\to\infty} \frac{p_n}{P_n} = 0 \quad \text{and} \quad \sup_n \frac{1}{|P_n|} \sum_{\nu=0}^{n} |p_\nu| < \infty.$$

Now let (N, p) be an arbitrary Nörlund method. Then the numbers

$$\frac{p_n}{P_n} \quad (n \in \mathbb{N}_0)$$

have a strong influence on the behaviour of the method as is apparent, e.g., in [6]. We first note the following result; its simple proof is omitted.

Lemma 2.1. *Let (N, p) be a Nörlund method, and let $\alpha \in \mathbb{C}$. Then the following assertions are equivalent:*

(i) $\frac{p_n}{P_n} \to \alpha$ *as* $n \to \infty$;

(ii) $\frac{p_{n-\nu}}{P_n} \to \alpha(1-\alpha)^\nu$ *as* $n \to \infty$, *for each* $\nu \in \mathbb{N}_0$;

(iii) $\frac{P_{n-1}}{P_n} \to 1-\alpha$ *as* $n \to \infty$;

(iv) $\frac{P_{n-\nu}}{P_n} \to (1-\alpha)^\nu$ *as* $n \to \infty$, *for each* $\nu \in \mathbb{N}_0$.

Remark 2.2. If the sequence (p_n/P_n) is divergent, then the (N, p)-method sums no power series (1) with $a_1 \neq 0$ compactly in any neighbourhood of $z_0 = 0$. For if we assume that

$$\sigma_n(z) = \sum_{\nu=0}^{n} \frac{P_{n-\nu}}{P_n} a_\nu z^\nu$$

converges compactly in a neighbourhood of 0, then $(P_{n-1} a_1/P_n)$ and consequently (P_{n-1}/P_n) converges, leading to a contradiction on account of Lemma 2.1.

Thus, in this paper we will only consider Nörlund methods (N, p) with the property that (p_n/P_n) is convergent.

If (N, p) is a regular method, hence $\lim_{n\to\infty} p_n/P_n = 0$, and f is any power series (1) with $R > 0$, then f is compactly (N, p)-summable in $|z| < R$ to the limit function f. If $\lim_{n\to\infty} p_n/P_n = \alpha$ is arbitrary, we have:

Theorem A. *Let (N, p) be a Nörlund method and $\alpha \in \mathbb{C}$. Then the following two statements are equivalent:*

(i) $\lim\limits_{n\to\infty} \dfrac{p_n}{P_n} = \alpha$;

(ii) *if f is a power series* (1) *with $R > 0$, then f is compactly (N, p)-summable in $|z| < R/|1-\alpha|$ to the limit function $f((1-\alpha)z)$.*

For a proof see [6, Theorem 5]. There the limit function σ was given as

$$\sigma(z) = f(z) + \sum_{\nu=0}^{\infty} \alpha(1-\alpha)^{\nu}\{s_\nu(z) - f(z)\} = f(z) - \sum_{\nu=0}^{\infty} \alpha(1-\alpha)^{\nu} \sum_{\mu=\nu+1}^{\infty} a_\mu z^\mu$$

for small values of z. By uniform convergence we obtain

$$\sigma(z) = f(z) - \sum_{\mu=1}^{\infty} \left(\alpha \sum_{\nu=0}^{\mu-1} (1-\alpha)^{\nu} \right) a_\mu z^\mu = \sum_{\mu=0}^{\infty} a_\mu ((1-\alpha)z)^\mu = f((1-\alpha)z).$$

Since f is analytic in $|z| < R$, $f((1-\alpha)z)$ is analytic for $|z| < R/|1-\alpha|$, and by the identity theorem for holomorphic functions we get that $\sigma(z) = f((1-\alpha)z)$ in $|z| < R/|1-\alpha|$.

In the case of $\alpha = 0$ the theorem was obtained by Agnew [1, Theorem 5] for the equivalent series-sequence transform.

In the case of $\alpha = 1$ Theorem A says: If (N, p) is a Nörlund method with $\lim_{n\to\infty} p_n/P_n = 1$, then each power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with $R > 0$ is compactly (N, p)-summable in $\mathbb{C}$ to the value $f(0) = a_0$.

Thus from now on we may assume that $\lim_{n\to\infty} p_n/P_n \neq 1$.

In our further investigations we will need the following property of summability methods that is a generalization of left-translativity.

Definition 2.3. Let $\lambda \in \mathbb{C}$. A summability method A is called *λ-left-translative* if $A\text{-}\lim_{n\to\infty} s_n = \sigma$ implies that $A\text{-}\lim_{n\to\infty} s'_n = \lambda\sigma$, where $s'_0 = 0$ and $s'_n = s_{n-1}$ for $n \in \mathbb{N}$.

Theorem 2.4. *Any Nörlund method (N, p) with $\lim_{n\to\infty} p_n/P_n = \alpha$ is $(1-\alpha)$-left-translative.*

Proof. Let (s_n) be a sequence with $(N, p)\text{-}\lim_{n\to\infty} s_n = \sigma$. Then for the (N, p)-transforms of (s'_n) with $s'_0 = 0$ and $s'_n = s_{n-1}$ $(n \in \mathbb{N})$ we obtain with Lemma 2.1

$$\frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu} s'_\nu = \frac{1}{P_n}\sum_{\nu=0}^{n-1} p_{n-(\nu+1)} s_\nu = \frac{P_{n-1}}{P_n}\left(\frac{1}{P_{n-1}}\sum_{\nu=0}^{n-1} p_{n-1-\nu} s_\nu\right) \to (1-\alpha)\sigma$$

as $n \to \infty$. □

We note the following result for general λ-left-translative methods.

Theorem 2.5. *Let A be a λ-left-translative summability method, f a power series* (1), *and Q a polynomial. If A sums f at $z_0 \in \mathbb{C}$ to the value σ, then it sums the (formal) power series of Qf about* 0 *at z_0 to the value $Q(\lambda z_0)\sigma$.*

Proof. We prove the case $Q(z) = z$; by induction on the degree of Q and by the linearity of A the result follows for arbitrary polynomials Q.

By assumption we have $A\text{-}\sum_{k=0}^{\infty} a_k z_0^k = \sigma$ and hence $A\text{-}\sum_{k=0}^{\infty} a_k z_0^{k+1} = z_0\sigma$ by the linearity of A. Defining $s_n = \sum_{k=0}^{n} a_k z_0^{k+1}$ and $s'_0 = 0$, $s'_n = s_{n-1}$ for $n \in \mathbb{N}$, we have $s'_n = \sum_{k=1}^{n} a_{k-1} z_0^k$ $(n \in \mathbb{N}_0)$. Since A is λ-left-translative, we obtain

$$A\text{ - }\lim_{n\to\infty} s'_n = \lambda \cdot A\text{ - }\lim_{n\to\infty} s_n = \lambda z_0 \sigma.$$

But s'_n is also the n-th partial sum at z_0 of the power series of $zf(z)$ about 0. □

Vermes [7] has obtained the corresponding result for regular left-translative series-sequence methods.

3. Necessary conditions for summability and Leja's theorem

If (N, p) is a Nörlund method, then we associate to it the power series

$$p(w) = \sum_{k=0}^{\infty} p_k w^k \quad \text{and} \quad P(w) = \sum_{k=0}^{\infty} P_k w^k.$$

A short calculation shows that formally we have $p(w) = (1-w)P(w)$. If now $\lim_{n\to\infty} p_n/P_n = \alpha$, then the radius of convergence of P is $|1-\alpha|$ by Lemma 2.1, and hence p and P are holomorphic functions in $|w| < |1-\alpha|$.

Lemma 3.1. *Let (N, p) be a Nörlund method, (s_n) a sequence, and (σ_n) its (N, p)-transform. Then we have formally*

$$P(w) \cdot \sum_{k=0}^{\infty} u_k w^k = \sum_{k=0}^{\infty} P_k \sigma_k w^k,$$

where $s_n = \sum_{k=0}^{n} u_k$ for $n \in \mathbb{N}_0$.

Proof. Since, for $n \in \mathbb{N}_0$,

$$\sigma_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu} s_\nu = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu} \sum_{k=0}^{\nu} u_k = \frac{1}{P_n}\sum_{\nu=0}^{n} P_{n-\nu} u_\nu,$$

we have

$$\begin{aligned}\sum_{k=0}^{\infty} P_k \sigma_k w^k &= \sum_{k=0}^{\infty}\left(\sum_{\nu=0}^{k} P_{k-\nu} u_\nu\right) w^k \\ &= \sum_{k=0}^{\infty} P_k w^k \cdot \sum_{k=0}^{\infty} u_k w^k = P(w) \cdot \sum_{k=0}^{\infty} u_k w^k. \quad \square\end{aligned}$$

Our first main result is:

Theorem 3.2. *Let* (N, p) *be a Nörlund method with* $\lim_{n\to\infty} p_n/P_n = \alpha \neq 1$. *If* (N, p) *sums a power series* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *at a point* $z_0 \neq 0$, *then:*

(i) f *has a positive radius of convergence* R,
(ii) f *has a meromorphic continuation into* $|z| < |(1-\alpha)z_0|$, *and*
(iii) f *has a pole* ζ *with* $|\zeta| < |(1-\alpha)z_0|$ *only if* $\omega := \zeta/z_0$ *is a zero of* P *and the order of the pole* ζ *is not greater than the order of the zero* ω.

Proof. If (σ_n) is the (N, p)-transform of $\sum_{k=0}^{\infty} a_k z_0^k$, then by Lemma 3.1, setting $u_k = a_k z_0^k$, we get $P(w) \cdot \sum_{k=0}^{\infty} a_k (z_0 w)^k = \sum_{k=0}^{\infty} P_k \sigma_k w^k$, hence

$$P(w) \cdot f(z_0 w) = \sum_{k=0}^{\infty} P_k \sigma_k w^k.$$

Since, by the remark preceding Lemma 3.1, P is holomorphic in $|w| < |1-\alpha|$; and since (σ_n) is convergent, we see that $P(w) f(z_0 w)$ is holomorphic in $|w| < |1-\alpha|$. Now consider $g(z) := P(z/z_0) f(z)$. Then g is holomorphic in $|z| < |(1-\alpha)z_0|$, which implies (ii) and (iii). And (i) follows since $P(0) \neq 0$. □

Remark 3.3. In Theorem 3.2 it suffices to assume that the (N, p)-transform (σ_n) of the power series f at z_0 satisfies $\overline{\lim}_{n\to\infty} |\sigma_n|^{1/n} \leq 1$, as the proof shows.

We define for convenience:

Definition 3.4. Let f be a power series (1). Then the number

$$R_m := \sup\{r > 0 : f \text{ is holomorphic at } 0 \text{ and meromorphic in } |z| < r\}$$

(with $\sup \emptyset = 0$) is called the *radius of meromorphy* of f.

From Theorem 3.2 we get immediately:

Corollary 3.5. *Let* (N, p) *be a Nörlund method with* $\lim_{n\to\infty} p_n/P_n = \alpha \neq 1$ *and* f *a power series* (1). *Then:*

(i) *The method* (N, p) *does not sum* f *at any point* z *with* $|z| > R_m/|1-\alpha|$.
(ii) *If* P *has no zeros in* $|w| < |1-\alpha|$, *then* (N, p) *does not sum* f *at any point* z *with* $|z| > R/|1-\alpha|$.

In the particular case of regular Nörlund methods, when $\alpha = 0$, assertion (ii) was already noted by Leja [5]. It applies in particular to the Cesàro methods C_α $(\alpha \geq 0)$. See also Bouligand [3].

Theorem 3.2 also leads to a new and short proof of Leja's theorem and its generalization due to the second author.

Theorem 3.6 [6, Theorem 8]. *Let* (N, p) *be a Nörlund method with*

$$\lim_{n\to\infty} p_n/P_n = \alpha \neq 1$$

and f *a power series* (1). *Then for each* $\varepsilon > 0$ *the method* (N, p) *sums* f *at most at finitely many points* z *with* $|z| > R/|1-\alpha| + \varepsilon$.

Proof. By Corollary 3.5(i) we may assume that $R_m > R > 0$. Hence there exists a pole ζ_0 of f with $|\zeta_0| = R$. Now let $\varepsilon > 0$. If z is a summability point with

$|z| > R/|1-\alpha| + \varepsilon$, then we have $|\zeta_0| = R < |(1-\alpha)z|$, so that by Theorem 3.2 there is a zero ω of P with $\omega = \zeta_0/z$. Hence

$$(2) \qquad |\omega| < \frac{R}{R/|1-\alpha|+\varepsilon} = \frac{|1-\alpha|}{1+\varepsilon|1-\alpha|/R}.$$

Since P is holomorphic in $|w| < |1-\alpha|$, it has only finitely many zeros ω satisfying (2). Hence there can be only finitely many summability points z with $|z| > R/|1-\alpha| + \varepsilon$. □

Remark 3.7. In fact, by Remark 3.3 we have the following stronger result: For every $\varepsilon > 0$ there can be at most finitely many points z with $|z| > R/|1-\alpha| + \varepsilon$ for which the (N, p)-transform (σ_n) of the power series f at z satisfies $\overline{\lim}_{n\to\infty}|\sigma_n|^{1/n} \leq 1$. This corresponds to a recent result of Borwein and Jakimovski [2] for general summability methods.

By Theorem A we know that the (N, p)-transforms of a power series (1) are compactly convergent in $|z| < R/|1-\alpha|$ to the limit function $f((1-\alpha)z)$. The next theorem tells us that if, more generally, z is a summability point with $|z| < R_m/|1-\alpha|$, then the (N, p)-sum is also $f((1-\alpha)z)$.

Theorem 3.8. *Let (N, p) be a Nörlund method with $\lim_{n\to\infty} p_n/P_n = \alpha \neq 1$ and f a power series* (1) *with $R > 0$. If (N, p) sums f at a point z_0 with $|z_0| < R_m/|1-\alpha|$, then $(1-\alpha)z_0$ is no pole of f and (N, p)-$\sum_{k=0}^{\infty} a_k z_0^k = f((1-\alpha)z_0)$.*

Proof. Since $|(1-\alpha)z_0| < R_m$, there is a polynomial Q such that $g = Qf$ is holomorphic in $|z| \leq |(1-\alpha)z_0|$.

(a) We assume that $(1-\alpha)z_0$ is a pole of f. Then we can choose Q so that $g((1-\alpha)z_0) \neq 0$. Now, if $g(z) = \sum_{k=0}^{\infty} b_k z^k$, then Theorem A implies that

$$(N, p)\text{-}\sum_{k=0}^{\infty} b_k z_0^k = g((1-\alpha)z_0).$$

On the other hand, from the $(1-\alpha)$-left-translativity of (N, p) (see Theorem 2.4) we get by Theorem 2.5

$$(N, p)\text{-}\sum_{k=0}^{\infty} b_k z_0^k = Q((1-\alpha)z_0)\sigma,$$

where $\sigma = (N, p)$-$\sum_{k=0}^{\infty} a_k z_0^k$. Hence $Q((1-\alpha)z_0)\sigma = g((1-\alpha)z_0) \neq 0$. This implies that $Q((1-\alpha)z_0) \neq 0$, which contradicts the assumption that f has a pole at $(1-\alpha)z_0$.

(b) Now, since $(1-\alpha)z_0$ cannot be a pole of f, we can choose Q so that $Q((1-\alpha)z_0) \neq 0$. Then similar conclusions as in (a) lead to

$$Q((1-\alpha)z_0)\sigma = g((1-\alpha)z_0) = Q((1-\alpha)z_0)f((1-\alpha)z_0),$$

hence

$$(N, p)\text{-}\sum_{k=0}^{\infty} a_k z_0^k = \sigma = f((1-\alpha)z_0). \quad \square$$

4. Characterization of summability points

We have seen that if (N, p) is a Nörlund method with $\lim_{n\to\infty} p_n/P_n = \alpha \neq 1$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is a power series with $0 < R < R_m$, then one has compact summability for $|z| < R/|1-\alpha|$ and no summability for $|z| > R_m/|1-\alpha|$. In this section we want to characterize the points z with $R/|1-\alpha| \leq |z| < R_m/|1-\alpha|$ at which summability takes place.

Lemma 4.1. *Let (P_n) be any sequence of complex numbers such that $P_n \neq 0$ for almost all n and $\lim_{n\to\infty} P_{n-1}/P_n = \beta \neq 0$. Let w_0 be a point with $0 < |w_0| < |\beta|$ and $\sum_{k=0}^{\infty} P_k w_0^k = 0$. Define a sequence (Q_n) by*

$$\sum_{k=0}^{\infty} Q_k w^k = \frac{1}{1 - w/w_0} \sum_{k=0}^{\infty} P_k w^k.$$

Then $Q_n \neq 0$ for sufficiently large n, $\lim_{n\to\infty} Q_{n-1}/Q_n = \beta$ and $\lim_{n\to\infty} Q_n/P_n$ exists.

Proof. Since, in a neighbourhood of 0,

$$\sum_{k=0}^{\infty} Q_k w^k = \sum_{k=0}^{\infty} \left(\frac{w}{w_0}\right)^k \sum_{k=0}^{\infty} P_k w^k = \sum_{k=0}^{\infty} \left(\sum_{\nu=0}^{k} \frac{P_\nu}{w_0^{k-\nu}}\right) w^k,$$

we have for $n \in \mathbb{N}_0$

$$\begin{aligned} Q_n &= \frac{1}{w_0^n} \sum_{k=0}^{n} P_k w_0^k = \frac{1}{w_0^n} \sum_{k=0}^{\infty} P_k w_0^k - \frac{1}{w_0^n} \sum_{k=n+1}^{\infty} P_k w_0^k \\ &= -\frac{1}{w_0^n} \sum_{k=n+1}^{\infty} P_k w_0^k = -\sum_{k=0}^{\infty} P_{n+k+1} w_0^{k+1}. \end{aligned}$$

Hence

$$\frac{Q_n}{P_n} = -\sum_{k=0}^{\infty} \frac{P_{n+k+1}}{P_n} w_0^{k+1}$$

holds for sufficiently large n. We put $\varphi_k(n) = P_{n+k+1} w_0^{k+1}/P_n$ for these n. Then we have $\lim_{n\to\infty} \varphi_k(n) = w_0^{k+1}/\beta^{k+1}$ for $k \in \mathbb{N}_0$ by Lemma 2.1, and one verifies that for a fixed r, $|w_0|/|\beta| < r < 1$, there is some $M > 0$ such that $|\varphi_k(n)| \leq M r^k$ for all k, n. Hence the Weierstrass M-test implies that

$$\frac{Q_n}{P_n} \to -\sum_{k=0}^{\infty} \frac{w_0^{k+1}}{\beta^{k+1}} \neq 0$$

as $n \to \infty$. Thus we have $Q_n \neq 0$ for sufficiently large n and

$$\frac{Q_{n-1}}{Q_n} = \frac{Q_{n-1}}{P_{n-1}} \frac{P_{n-1}}{P_n} \frac{P_n}{Q_n} \to \beta$$

as $n \to \infty$. □

Lemma 4.2. *Let (N, p) be a Nörlund method with $\lim_{n\to\infty} p_n/P_n = \alpha \neq 1$. Let $N \in \mathbb{N}$ and $0 < |w_0| < |1-\alpha|$. If P has a zero of order $M \geq N$ at ω_0, then*

(N, p) sums the power series

$$f(z)=\sum_{k=0}^{\infty} b_k z^k=\frac{1}{(1-z)^N}$$

at $z_0 = 1/\omega_0$.

Proof. Let (σ_n) be the (N, p)-transform of the given power series at $1/\omega_0$. Then Lemma 3.1 implies that

$$\sum_{k=0}^{\infty} P_k\sigma_k w^k=\frac{1}{(1-w/\omega_0)^N}\sum_{k=0}^{\infty} P_k w^k$$

for small values of w. Since $\lim_{n\to\infty} P_{n-1}/P_n = 1-\alpha \neq 0$ (Lemma 2.1), an N-fold application of Lemma 4.1 implies the existence of

$$\lim_{n\to\infty}\frac{P_n\sigma_n}{P_n}=\lim_{n\to\infty}\sigma_n. \quad \square$$

We can now obtain the desired characterization of summability points.

Theorem 4.3. *Let (N, p) be a Nörlund method with $\lim_{n\to\infty} p_n/P_n = \alpha \neq 1$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ a power series with $0 < R < R_m$, and z_0 a point with $R/|1-\alpha| \le |z_0| < R_m/|1-\alpha|$. Then (N, p) sums f at z_0 if and only if the following assertions hold:*

(i) *if ζ is a pole of f with $|\zeta| < |(1-\alpha)z_0|$ of order $N \in \mathbb{N}$, then P has a zero at $\omega := \zeta/z_0$ of order $M \ge N$;*

(ii) *if $\zeta_1, \dots, \zeta_l$ are the poles of f on $|z| = |(1-\alpha)z_0|$ (if there are any) with orders $N_1, \dots, N_l$, and if*

$$g(z)=\sum_{k=0}^{\infty} b_k z^k=\sum_{j=1}^{l}\sum_{\nu=1}^{N_j}\frac{c_\nu^{(j)}}{(z-\zeta_j)^\nu}$$

is the sum of the principal parts of f at these poles, then (N, p) sums the power series g at z_0.

Proof. Necessity: Assume that (N, p) sums f at z_0. Then condition (i) follows from Theorem 3.2. To derive (ii) we write

$$\begin{aligned} f(z) &= g(z)+\tilde{g}(z)+h(z)\\ &=\sum_{j=1}^{l}\sum_{\nu=1}^{N_j}\frac{c_\nu^{(j)}}{(z-\zeta_j)^\nu}+\sum_{j=l+1}^{m}\sum_{\nu=1}^{N_j}\frac{c_\nu^{(j)}}{(z-\zeta_j)^\nu}+h(z),\end{aligned}$$

where $\zeta_{l+1}, \dots, \zeta_m$ are the poles of f in $|z| < |(1-\alpha)z_0|$ (if there are any) with orders $N_{l+1}, \dots, N_m$, and $\tilde{g}$ is the sum of the principal parts of f at these poles. Since

$$\frac{c_\nu^{(j)}}{(z-\zeta_j)^\nu}=\frac{(-1)^\nu c_\nu^{(j)}/\zeta_j^\nu}{(1-z/\zeta_j)^\nu},$$

(i) and Lemma 4.2 imply that (N, p) sums the power series of $\tilde{g}$ about 0 at z_0. Since h is holomorphic in $|z| \le |(1-\alpha)z_0|$, Theorem A implies that

(N, p) also sums its power series about 0 at z_0. Hence (N, p) sums the power series of $g = f - \tilde{g} - h$ about 0 at z_0.

Sufficiency: Now assume that (i) and (ii) hold. As above we write $f = g + \tilde{g} + h$ and note that (ii), (i) with Lemma 4.2, and Theorem A imply that (N, p) sums the power series of g, $\tilde{g}$, and h, respectively, about 0 at the point z_0. Hence the result follows. □

Remark 4.4. The theorem solves the problem of characterization of summability points completely if f has no poles on $|z| = |(1-\alpha)z_0|$. In that case the zeros of P govern the summability behaviour. In the general case the problem is reduced to the question when (N, p) sums a linear combination of functions $1/(z-\zeta)^\nu$ with $|\zeta| = |(1-\alpha)z_0|$ and $\nu \in \mathbb{N}$ at the point z_0. Also, the theorem leaves open the problem of characterizing summability points z_0 with $|z_0| = R_m/|1-\alpha|$.

5. Prescribing summability points

A Nörlund method (N, p) with $\lim_{n\to\infty} p_n/P_n = \alpha \neq 1$ sums any given power series (1) at most at finitely many points in $|z| > R/|1-\alpha| + \varepsilon$, hence at most at countably many points in $|z| > R/|1-\alpha|$. Now we ask if one can prescribe summability points z_0 there. If we assume that $|z_0| \neq R_m/|1-\alpha|$, Corollary 3.5 and Theorem 3.8 tell us that we must have $|z_0| < R_m/|1-\alpha|$ and that $(1-\alpha)z_0$ is no pole of f. Under the given assumption these turn out to be the only restrictions. We first need:

Lemma 5.1. *Let* $T(w) = \sum_{k=0}^{\infty} T_k w^k$ *be a polynomial with* $T(0) \neq 0$ *and* $\lambda \in \mathbb{C}, \lambda \neq 0$, *with* $T(\lambda) \neq 0$. *Then there exists a polynomial* U *such that the polynomial* $\widetilde{T}(w) := \sum_{k=0}^{\infty} \widetilde{T}_k w^k := U(w)T(w)$ *satisfies* $\widetilde{T}(0) \neq 0$, $\widetilde{T}(\lambda) \neq 0$, *and* $\sum_{k=0}^{n} \widetilde{T}_k \lambda^k \neq 0$ *for all* $n \in \mathbb{N}_0$.

Proof. Consider the numbers $\tau_n := \sum_{k=0}^{n} T_k \lambda^k \ (n \in \mathbb{N}_0)$. If $\tau_n \neq 0$ for all n, then we may take $U(w) \equiv 1$. Else there are $n_0, n_1 \in \mathbb{N}_0$, $n_0 < n_1$, such that $\tau_n \neq 0$ if $n \leq n_0$ and $n \geq n_1$. For $c \in \mathbb{C}$ put $\widetilde{T}(w) := \sum_{k=0}^{\infty} \widetilde{T}_k w^k := (w - c)T(w)$. Then we have $\tilde{\tau}_n := \sum_{k=0}^{n} \widetilde{T}_k \lambda^k = \lambda\tau_{n-1} - c\tau_n$ for $n \in \mathbb{N}_0$ (with $\tau_{-1} := 0$). Hence we can choose c in such a way that $\tilde{\tau}_n \neq 0$ for $n \leq n_0 + 1$ and $n \geq n_1$. Repeating this process if necessary we arrive at the desired polynomial. □

Theorem 5.2. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be a power series with* $0 < R < R_m$, *and let* $\alpha \neq 1$. *Let* S *be a finite set of points in* $R/|1-\alpha| \leq |z| < R_m/|1-\alpha|$ *such that* $(1-\alpha)z$ *is no pole of* f *for all* $z \in S$. *Then there exists a Nörlund method* (N, p) *with* $\lim_{n\to\infty} p_n/P_n = \alpha$ *that sums* f *at every point of* S.

Proof. Since for $z \in S$ we have $|(1-\alpha)z| < R_m$, there are only finitely many poles ζ of f with $|\zeta| \leq |(1-\alpha)z|$. Hence there exists a poynomial $T(w) = \sum_{k=0}^{\infty} T_k w^k$ with $T(0) \neq 0$ and $T(1-\alpha) \neq 0$ that has a zero at every point ω of the form $\omega = \zeta/z$ where $z \in S$ and ζ is a pole of f with $|\zeta| \leq |(1-\alpha)z|$ and the order of the zero ω is not smaller than the order of the pole ζ. By Lemma 5.1 we can assume that in addition $\sum_{k=0}^{n} T_k(1-\alpha)^k \neq 0$ for all $n \in \mathbb{N}_0$.

We put

$$P(w) = \frac{1}{1 - w/(1-\alpha)} T(w)$$

and claim that (N, p) is the desired Nörlund method.

First note that $P_n = \sum_{k=0}^{n} T_k(1-\alpha)^k/(1-\alpha)^n \neq 0$ for $n \in \mathbb{N}_0$ and that $\lim_{n\to\infty} P_{n-1}/P_n = 1-\alpha$, hence $\lim_{n\to\infty} p_n/P_n = \alpha$ by Lemma 2.1.

Now let $z_0 \in S$. We have to show that (N, p) sums f at z_0. Applying Theorem 4.3 we need to show that conditions (i) and (ii) hold there. Condition (i) follows from the construction of T and P. As for (ii), let ζ be any pole of f with $|\zeta| = |(1-\alpha)z_0|$ with order N. It now suffices to show that (N, p) sums the power series

$$g(z) = \sum_{k=0}^{\infty} b_k z^k = \frac{1}{(z-\zeta)^\nu}$$

at z_0 for any $\nu = 1, 2, \dots, N$. If (σ_n) is the (N, p)-transform of g at z_0, then by Lemma 3.1 we have for small values of w

$$P(w) \cdot \frac{1}{(z_0 w - \zeta)^\nu} = \sum_{k=0}^{\infty} P_k \sigma_k w^k.$$

We put $Q_n := P_n \sigma_n$ $(n \in \mathbb{N}_0)$ and

$$\widetilde{Q}(w) := \sum_{k=0}^{\infty} \widetilde{Q}_k w^k := P(w) \cdot \frac{1 - w/(1-\alpha)}{(z_0 w - \zeta)^\nu} = \frac{T(w)}{(z_0 w - \zeta)^\nu}.$$

By construction of T we see that $\widetilde{Q}$ is a polynomial with $\widetilde{Q}(1-\alpha) \neq 0$. On the one hand we now have

$$\sum_{k=0}^{\infty} Q_k w^k = \sum_{k=0}^{\infty} P_k \sigma_k w^k = \frac{1}{1 - w/(1-\alpha)} \widetilde{Q}(w),$$

so that

$$(1-\alpha)^n Q_n = (1-\alpha)^n \sum_{k=0}^{n} \frac{1}{(1-\alpha)^{n-k}} \widetilde{Q}_k = \sum_{k=0}^{n} \widetilde{Q}_k (1-\alpha)^k \to \widetilde{Q}(1-\alpha)$$

as $n \to \infty$. This shows that $Q_n \neq 0$ for large n and that $\lim_{n\to\infty} Q_{n-k}/Q_n = (1-\alpha)^k$ for every $k \in \mathbb{N}_0$. On the other hand we have

$$\sum_{k=0}^{\infty} P_k w^k = \sum_{k=0}^{\infty} Q_k w^k \cdot (z_0 w - \zeta)^\nu,$$

hence

$$\begin{aligned}\frac{P_n}{Q_n} &= \sum_{k=0}^{\nu} \frac{Q_{n-k}}{Q_n} \binom{\nu}{k} z_0^k (-\zeta)^{\nu-k} \\ &\to \sum_{k=0}^{\nu} \binom{\nu}{k} \big((1-\alpha)z_0\big)^k (-\zeta)^{\nu-k} = \big((1-\alpha)z_0 - \zeta\big)^\nu \neq 0;\end{aligned}$$

note that $(1-\alpha)z_0 \neq \zeta$ by assumption. This implies that

$$\sigma_n = \frac{Q_n}{P_n}$$

converges as $n \to \infty$, which had to be shown. □

Remark 5.3. If $\alpha = 0$, the Nörlund method (N, p) constructed in the above proof is even regular. For in that case we have $p(w) = (1-w)P(w) = T(w)$, which is a polynomial, so that $\sup_n \sum_{\nu=0}^{n} |p_\nu|/|P_n| < \infty$ (see Section 2). For the function $f(z) = 1/(R-z)$ and $\alpha = 1 - R$ Theorem 5.2 was obtained in [2, Section 5].

6. Regular Nörlund methods

We briefly summarize here our main results for regular Nörlund methods (N, p). In that case we have $\lim_{n\to\infty} p_n/P_n = 0$ (see Section 2).

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be a power series with* $0 \leq R \leq R_m \leq \infty$, *and let* (N, p) *be a regular Nörlund method. Then*:

- *If* $R = 0$, *then* (N, p) *sums* f *at no point of* $|z| > 0$ (*Theorem* 3.2).

Now assume that $R > 0$. *Then*:

- (N, p) *sums* f *compactly in* $|z| < R$ *to the limit function* f (*Theorem* A).
- (N, p) *sums* f *at most at finitely many points in* $R + \varepsilon < |z| \leq R_m$ $(\varepsilon > 0)$, *hence at most at countably many points in* $R < |z| \leq R_m$ (*Theorem* 3.6). *Moreover, if* $|z| < R_m$, *then a summation point is not a pole of* f *and the value of summation is* $f(z)$ (*Theorem* 3.8).
- (N, p) *cannot sum* f *at any point of* $|z| > R_m$; *if* P *has no zeros in* $|w| < 1$, *then* (N, p) *does not sum* f *at any point of* $|z| > R$ (*Corollary* 3.5).

Conversely:

- *If* S *is a finite set of points* z *with* $R \leq |z| < R_m$ *that does not contain any pole of* f, *then there exists a regular Nörlund method* (N, p) *that sums* f *at every point of* S (*Theorem* 5.2 *and Remark* 5.3).

References

1. R. P. Agnew, *Summability of power series*, Amer. Math. Monthly **53** (1946), 251–259.
2. D. Borwein and A. Jakimovski, *Matrix transformations of power series*, Proc. Amer. Math. Soc. **122** (1994), 511–523.
3. G. Bouligand, *Sur la comparaison de certains procédés de sommation des séries divergentes*, Enseign. Math. **26** (1927), 15–27.
4. G. H. Hardy, *Divergent series*, Clarendon Press, Oxford, 1949.
5. F. Leja, *Sur la sommation des séries entières par la méthode des moyennes*, Bull. Sci. Math. **54** (1930), 239–245.

6. K. Stadtmüller, *Summability of power series by non-regular Nörlund-methods*, J. Approx. Theory **68** (1992), 33–44.

7. P. Vermes, *The application of γ-matrices to Taylor series*, Proc. Edinburgh Math. Soc. (2) **8** (1948), 43–49.

Fachbereich Mathematik, FernUniversität Hagen, 58084 Hagen, Germany
E-mail address: KG.Grosse-Erdmann@FernUni-Hagen.de

Abteilung Mathematik III, Universität Ulm, 89069 Ulm, Germany