COMPACT COMPOSITION OPERATORS ON THE BLOCH SPACE

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Abstract. Necessary and sufficient conditions are given for a composition operator $C_\phi f = f \circ \phi$ to be compact on the Bloch space $B$ and on the little Bloch space $B_0$. Weakly compact composition operators on $B_0$ are shown to be compact. If $\phi \in B_0$ is a conformal mapping of the unit disk $D$ into itself whose image $\phi(D)$ approaches the unit circle $T$ only in a finite number of nontangential cusps, then $C_\phi$ is compact on $B_0$. On the other hand if there is a point of $T \cap \phi(D)$ at which $\phi(D)$ does not have a cusp, then $C_\phi$ is not compact.

1. Introduction

Let $D$ denote the unit disk in the complex plane. A function $f$ holomorphic in $D$ is said to belong to the Bloch space $B$ if

$$\sup_{z \in D}(1 - |z|^2)|f'(z)| < \infty$$

and to the little Bloch space $B_0$ if

$$\lim_{|z| \to 1} (1 - |z|^2)|f'(z)| = 0.$$ 

It is well known that $B$ is a Banach space under the norm

$$\|f\|_B = |f(0)| + \sup_{z \in D}(1 - |z|^2)|f'(z)|$$

and that $B_0$ is a closed subspace of $B$. Furthermore, $B$ is isometrically isomorphic to the second dual of $B_0$ and the inclusion $B_0 \subset B$ corresponds to the canonical imbedding of $B_0$ into $B_0^{**}$ [ACP]. It is a simple consequence of the Schwarz-Pick lemma [A] that a holomorphic mapping $\phi$ of the unit disk into itself induces a bounded composition operator $C_\phi f = f \circ \phi$ on $B$. Indeed, if $f \in B$, then

$$(1 - |z|^2)|(f \circ \phi)'(z)| = (1 - |z|^2)|f'(\phi(z))||\phi'(z)|$$

and

$$\frac{1 - |z|^2}{1 - |\phi(z)|^2}|\phi'(z)|(1 - |\phi(z)|^2)|f'(\phi(z))|$$

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and the Schwarz-Pick lemma guarantees that

\[
\frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)| \leq 1.
\]

Since the identity function \( f(z) = z \) belongs to \( \mathcal{B}_0 \), it is clear that \( \phi \in \mathcal{B}_0 \) if \( C_\phi \) maps \( \mathcal{B}_0 \) into itself. Conversely, if \( \phi \in \mathcal{B}_0 \) and \( f \in \mathcal{B}_0 \), it follows from (1) and (2) that \( f \circ \phi \in \mathcal{B}_0 \). Indeed, if \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( (1 - |z|^2)|f'(z)| < \epsilon \) whenever \( |z|^2 > 1 - \delta \). In particular, \( (1 - |z|^2)|(f \circ \phi)'(z)| < \epsilon \) whenever \( |\phi(z)|^2 > 1 - \delta \). On the other hand, if \( |\phi(z)|^2 \leq 1 - \delta \),

\[
(1 - |z|^2)|(f \circ \phi)'(z)| \leq \frac{\|f\|_\infty}{\delta} (1 - |z|^2)|\phi'(z)|,
\]

and the right-hand side tends to 0 as \( |z| \to 1 \).

In Section 2 the compact composition operators on \( \mathcal{B}_0 \) and on \( \mathcal{B} \) will be characterized in terms of the quotient \( \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)| \). A bounded linear operator \( T: X \to Y \) from the Banach space \( X \) to the Banach space \( Y \) is weakly compact if \( T \) takes bounded sets in \( X \) into relatively weakly compact sets in \( Y \). Gantmacher's theorem [D, p. 21] asserts that \( T \) is weakly compact if and only if \( T^{**}(X^{**}) \subset Y \) where \( T^{**} \) denotes the second adjoint of \( T \). This theorem and the characterization of compact operators on \( \mathcal{B}_0 \) will be used to show that every weakly compact composition operator on \( \mathcal{B}_0 \) is compact.

In Section 3 the results of Section 2 will be applied to certain univalent functions \( \phi \) which map \( \mathbb{D} \) into itself. It is known that such functions belong to \( \mathcal{B}_0 \) [P, p. 12]; and it will be clear from Section 2 that if \( \|\phi\|_\infty < 1 \), then \( C_\phi \) is compact on \( \mathcal{B}_0 \). On the other hand if \( \|\phi\|_\infty = 1 \) and there is a point of \( T_n \mathcal{B}(\mathbb{D}) \) at which \( \phi(\mathbb{D}) \) does not have a cusp, then \( C_\phi \) is not compact. However if \( T \cap \mathcal{B}(\mathbb{D}) \) consists of only one point at which \( \phi(\mathbb{D}) \) has a nontangential cusp, then \( C_\phi \) is compact on \( \mathcal{B}_0 \).

### 2. Compactness

Theorem 1 gives a precise description of those \( \phi \) which induce compact composition operators on \( \mathcal{B}_0 \). It will be useful first to give a criterion for compactness in \( \mathcal{B}_0 \).

**Lemma 1.** A closed set \( K \) in \( \mathcal{B}_0 \) is compact if and only if it is bounded and satisfies

\[
\lim_{|z| \to 1} \sup_{f \in K} (1 - |z|^2)|f'(z)| = 0.
\]

**Proof.** First suppose that \( K \) is compact and let \( \epsilon > 0 \). Choose an \( \epsilon/2 \)-net \( f_1, f_2, \ldots, f_n \) in \( K \). There is an \( r, 0 < r < 1 \), such that \( (1 - |z|^2)|f_i'(z)| < \epsilon/2 \) if \( |z| > r \), \( 1 \leq i \leq n \). If \( f \in K \), \( \|f - f_i\|_\infty < \epsilon/2 \) for some \( f_i \) and so

\[
(1 - |z|^2)|f'(z)| \leq \|f - f_i\|_\infty + (1 - |z|^2)|f_i'(z)| < \epsilon
\]

whenever \( |z| > r \). This establishes (3).

On the other hand if \( K \) is a closed bounded set which satisfies (3) and \( (f_n) \) is a sequence in \( K \), then by Montel's theorem there is a subsequence \( (f_{n_k}) \) which converges uniformly on compact subsets of \( \mathbb{D} \) to some holomorphic function \( f \). Then also \( (f_{n_k}') \) converges uniformly to \( f' \) on compact subsets of \( \mathbb{D} \). By (3), if
\( \epsilon > 0 \), there is an \( r \), \( 0 < r < 1 \), such that for all \( g \in K \), \( (1 - |z|^2)|g'(z)| < \epsilon/2 \) if \( |z| > r \). It follows that \( (1 - |z|^2)|f'(z)| < \epsilon/2 \) if \( |z| > r \). Since \((f_n')\) converges uniformly to \( f' \) and \((f'_n)\) converges uniformly to \( f' \) on \( |z| \leq r \), it follows that \( \limsup_{k \to \infty} \|f_{n_k} - f\|_B \leq \epsilon \). Since \( \epsilon > 0 \), \( \lim_{k \to \infty} \|f_{n_k} - f\|_B = 0 \) and so \( K \) is compact.

**Theorem 1.** If \( \phi \) is a holomorphic mapping of the unit disk \( \mathbb{D} \) into itself, then \( \phi \) induces a compact composition operator on \( \mathcal{B}_0 \) if and only if

\[
(4) \quad \lim_{|z| \to 1} \sup_{\|\phi\|_B \leq 1} \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)| = 0.
\]

**Proof.** It follows from Lemma 1 that \( C_{\phi} \) is compact on \( \mathcal{B}_0 \) if and only if \( \lim_{|z| \to 1} \sup_{\|\phi\|_B \leq 1} (1 - |z|^2)|(f \circ \phi)'(z)| = 0 \).

But

\[
(1 - |z|^2)|(f \circ \phi)'(z)| = \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)|(1 - |\phi(z)|^2)|f'(\phi(z))|,
\]

and

\[
\sup_{\|\phi\|_B \leq 1} (1 - |w|^2)|f'(w)| = 1
\]

for each \( w \in \mathbb{D} \). The theorem follows.

It should be remarked that (4) implies \( \phi \in \mathcal{B}_0 \). A similar condition characterizes compact composition operators on \( \mathcal{B} \).

**Theorem 2.** If \( \phi \) is a holomorphic mapping of the unit disk \( \mathbb{D} \) into itself, then \( \phi \) induces a compact composition operator on \( \mathcal{B} \) if and only if for every \( \epsilon > 0 \), there exists \( r \), \( 0 < r < 1 \), such that

\[
(5) \quad \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)| < \epsilon
\]

whenever \( |\phi(z)| > r \).

**Proof.** First assume that (5) holds. In order to prove that \( C_{\phi} \) is compact on \( \mathcal{B} \) it is enough to show that if \( (f_n) \) is a bounded sequence in \( \mathcal{B} \) which converges to 0 uniformly on compact subsets of \( \mathbb{D} \), then \( \|f_n \circ \phi\|_B \to 0 \). Let \( M = \sup_n \|f_n\|_B \). Given \( \epsilon > 0 \) there exists \( r \), \( 0 < r < 1 \), such that \( \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)| < \frac{\epsilon}{2M} \) if \( |\phi(z)| > r \). Since

\[
(1 - |z|^2)|(f_n \circ \phi)'(z)| = \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)|(1 - |\phi(z)|^2)|f'_n(\phi(z))| \leq M \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)|,
\]

it follows that \( (1 - |z|^2)|(f_n \circ \phi)'(z)| < \frac{\epsilon}{2} \) if \( |\phi(z)| > r \). On the other hand, \( f_n \circ \phi(0) \to 0 \) and \( (1 - |w|^2)|f'_n(w)| \to 0 \) uniformly for \( |w| \leq r \). Since

\[
(1 - |z|^2)|(f_n \circ \phi)'(z)| \leq (1 - |\phi(z)|^2)|f'_n(\phi(z))|,
\]

it follows that for large enough \( n \), \( |f_n \circ \phi(0)| < \frac{\epsilon}{2} \) and \( (1 - |z|^2)|(f_n \circ \phi)'(z)| < \frac{\epsilon}{2} \) if \( |\phi(z)| \leq r \). Hence \( \|f_n \circ \phi\|_B < \epsilon \) for large \( n \).
Now assume that (5) fails. Then there exists a subsequence \( (z_n) \) in \( \mathbb{D} \) and an \( \epsilon > 0 \) such that \( |z_n| \to 1 \) and \( \frac{1-|z_n|^2}{1-|\phi(z_n)|^2} |\phi'(z_n)| > \epsilon \) for all \( n \). Passing to a subsequence if necessary it may be assumed that \( w_n = \phi(z_n) \to w_0 \in \mathbb{T} \). Let \( f_n(z) = \log \frac{1-\overline{w}_n z}{1-w_n z} \). Then \( (f_n) \) converges to \( f_0 \) uniformly on compact subsets of \( \mathbb{D} \). On the other hand,

\[
\|C_\phi f_n - C_\phi f_0\|_B \geq (1-|z_n|^2)|(C_\phi f_n)'(z_n) - (C_\phi f_0)'(z_n)|
\]

\[
= \frac{1-|z_n|^2}{1-|w_n|^2} |\phi'(z_n)| \left| \frac{\overline{w}_n}{1-|w_n|^2} - \frac{\overline{w}_0}{1-w_0 w_n} \right|
\]

\[
= \frac{(1-|z_n|^2)}{1-|w_n|^2} |\phi'(z_n)| \left| \frac{\overline{w}_n - \overline{w}_0}{1-w_0 w_n} \right|
\]

\[
> \epsilon
\]

for all \( n \), so \( C_\phi f_n \) does not converge to \( C_\phi f_0 \) in norm. Hence \( C_\phi \) is not compact.

It is important to note that although (4) implies (5), since in this case \( C_\phi \) on \( \mathcal{B} \) is the second adjoint of \( C_\phi \) on \( \mathcal{B}_0 \), the two conditions are not equivalent. Condition (4) implies that \( \phi \in \mathcal{B}_0 \), while there certainly exist functions \( \phi \notin \mathcal{B}_0 \) which satisfy (5). Indeed, any \( \phi \) for which \( \|\phi\|_\infty < 1 \) satisfies (5) trivially.

A sequence \( (w_n) \) in \( \mathbb{D} \) is said to be \( \eta \)-separated if \( \rho(w_n, w_m) = \frac{|w_m - w_n|}{1-|w_m w_n|} > \eta \) whenever \( m \neq n \). Thus an \( \eta \)-separated sequence consists of points which are uniformly far apart in the pseudohyperbolic metric on \( \mathbb{D} \), or equivalently, the hyperbolic balls \( \Delta(w_n, r) = \{z \mid \rho(z, w_n) < r\} \) are pairwise disjoint for some \( r > 0 \). Evidently any sequence \( (w_n) \) in \( \mathbb{D} \) which satisfies \( |w_n| \to 1 \) possesses an \( \eta \)-separated subsequence for any \( \eta > 0 \). In particular, if the sequence \( (w_n) \) in the proof of Theorem 2 is \( \eta \)-separated, then the calculation in the proof shows that \( \|C_\phi f_m - C_\phi f_n\| > \epsilon \eta \) whenever \( m \neq n \), so \( (C_\phi f_n) \) has no norm convergent subsequences.

Another property of separated sequences is contained in the next proposition. This proposition is related to some interpolation results of Rochberg [RR1, RR2]. Since the method of proof is precisely the same as Rochberg’s, a proof will only be sketched.

**Proposition 1.** There is an absolute constant \( R > 0 \) such that if \( (w_n) \) is \( R \)-separated, then for every bounded sequence \( (\lambda_n) \) there is an \( f \in \mathcal{B} \) such that \( (1-|w_n|^2)f'(w_n) = \lambda_n \) for all \( n \).

The idea of the proof is to consider two operators \( S: \mathcal{B} \to l^\infty \) given by

\[
S(f)_n = (1-|w_n|^2)f'(w_n)
\]

and \( T: l^\infty \to \mathcal{B} \) given by

\[
T(\lambda)(z) = \sum_{n=1}^\infty \lambda_n \frac{1}{3w_n^2} \frac{(1-|w_n|^2)^3}{(1-\overline{w}_n z)^3}
\]

where \( \lambda = (\lambda_n) \in l^\infty \). The proposition will follow if it can be shown that \( \|I - ST\| < 1 \), for then \( ST \) will be invertible and so \( S \) will be onto. The symbol \( C \) will denote a constant whose value changes from place to place but
does not depend on $R$. Now

$$(ST - I)(\lambda)_n = (1 - |w_n|^2) \sum_{m \neq n} \lambda_m \frac{(1 - |w_m|^2)^3}{(1 - \overline{w}_m w_n)^4}$$

and so it will be enough to estimate

$$\sup_n (1 - |w_n|) \sum_{m \neq n} \frac{(1 - |w_m|^2)^3}{|1 - \overline{w}_m w_n|^4}.$$

If $R > i/2$, say, then there is a fixed $\delta > 0$ such that the Euclidean disk $D_m$ of center $w_m$ and radius $\delta(1 - |w_m|^2)$ is contained in the hyperbolic disk $\Delta_m = \Delta(w_m, R)$ and is disjoint from the hyperbolic disks $\Delta_n$ for $n \neq m$. Since $|1 - \overline{w}_m z|^4$ is subharmonic and the radius of $D_m$ is comparable to $1 - |w_m|^2$,

$$\frac{(1 - |w_m|^2)^3}{|1 - \overline{w}_m w_n|^4} \leq C \int_{D_m} \frac{1 - |w_m|^2}{|1 - \overline{w}_m z|^3} \, dx \, dy;$$

and since $|1 - \overline{w}_n z|$ dominates $1 - |w_m|^2$ on $D_m$, it follows that

$$\frac{(1 - |w_m|^2)^3}{|1 - \overline{w}_m w_n|^4} \leq C \int_{D_m \setminus \Delta_m} \frac{1 - |w_n|^2}{|1 - \overline{w}_n z|^3} \, dx \, dy$$

and hence

$$\sup_n (1 - |w_n|) \sum_{m \neq n} \frac{(1 - |w_m|^2)^3}{|1 - \overline{w}_m w_n|^4} \leq C \int_{\cup_{m \neq n} D_m} \frac{1 - |w_n|^2}{|1 - \overline{w}_n z|^3} \, dx \, dy.$$ 

The change of variables $z = \frac{w_n + \zeta}{1 + w_n \zeta}$ turns this into

$$\sup_n (1 - |w_n|) \sum_{m \neq n} \frac{(1 - |w_m|^2)^3}{|1 - \overline{w}_m w_n|^4} \leq C \int_{|\zeta| > R} \frac{1}{|1 + \overline{w}_n \zeta|} \, d\zeta \, d\eta,$$

and the last integral can be made arbitrarily small uniformly in $n$ if $R$ is chosen close enough to 1. This provides the desired estimate.

Since every sequence $(w_n)$ with $|w_n| \to 1$ contains an $R$-separated subsequence $(w_{n_k})$, it follows that there is an $f \in \mathcal{B}$ such that $(1 - |w_{n_k}|^2)f'(w_{n_k}) = 1$ for all $k$. This will be used in the proof of the next theorem.

**Theorem 3.** Every weakly compact composition operator $C_{\phi}$ on $\mathcal{B}_0$ is compact.

**Proof.** The composition operator $C_{\phi} : \mathcal{B}_0 \to \mathcal{B}_0$ is compact if and only if

$$\lim_{|z| \to 1} \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)| = 0$$

and, according to Gantmacher’s theorem, weakly compact if and only if $C_{\phi}f \in \mathcal{B}_0$ for every $f \in \mathcal{B}$. If $C_{\phi}$ is not compact, there is an $\epsilon > 0$ and a sequence $(z_n)$, $|z_n| \to 1$, such that

$$\frac{1 - |z_n|^2}{1 - |\phi(z_n)|^2} |\phi'(z_n)| \geq \epsilon$$
for all \( n \). Since \( \phi \in \mathcal{B}_0 \), \( |\phi(z_n)| \to 1 \), and by passing to a subsequence it may be assumed that \((\phi(z_n))\) is \( R \)-separated. If \( f \in \mathcal{B} \),

\[
(1 - |z_n|^2)|(C_\phi f)'(z_n)| = \frac{1 - |z_n|^2}{1 - |\phi(z_n)|^2} |\phi'(z_n)|(1 - |\phi(z_n)|^2)|f'(\phi(z_n))| \
\geq \epsilon (1 - |\phi(z_n)|^2)|f'(\phi(z_n))|.
\]

Since \((\phi(z_n))\) is \( R \)-separated, an application of Proposition 1 produces an \( f \in \mathcal{B} \) such that \((1 - |\phi(z_n)|^2)|(C_\phi f)'(z_n)| = 1\), for all \( n \). Since \((1 - |z_n|^2)|(C_\phi f)'(z_n)| \geq \epsilon \) and \( |z_n| \to 1 \), \( C_\phi f \notin \mathcal{B}_0 \) and so \( C_\phi \) is not weakly compact.

A slight refinement of these arguments will show that a noncompact composition operator on \( \mathcal{B}_0 \) must be an isomorphism on a subspace isomorphic to the sequence space \( c_0 \). This is not surprising since \( \mathcal{B}_0 \) is known to be isomorphic to \( c_0 \).

3. Examples

As remarked in the introduction any holomorphic mapping \( \phi \) of the unit disk into itself satisfying \( \|\phi\|_\infty < 1 \) induces a compact composition operator on \( \mathcal{B} \) and also on \( \mathcal{B}_0 \) if \( \phi \in \mathcal{B}_0 \). On the other hand it is easy to see that if \( \phi \) has a finite angular derivative at some point of \( T \), then \( C_\phi \) cannot be compact. Indeed, \( \phi \) has an angular derivative at \( \zeta \in T \) if the nontangential limit \( \omega = f(\zeta) \in T \) exists and if the quotient \( \frac{f(z) - f(\zeta)}{z - \zeta} \) converges to some complex number \( \mu \) as \( z \to \zeta \) nontangentially. It is known that \( \mu \neq 0 \), and the Julia-Carathéodory lemma shows that \( \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)| \) converges to \( \zeta \omega \mu \neq 0 \) nontangentially. Applying Theorem 1 or 2 as appropriate shows that \( C_\phi \) is not compact.

It turns out, however, that \( \phi \) can push the disk much more sharply into itself and still induce a noncompact composition operator. The easiest way to see this is to consider the functions \( \phi_{\lambda, \alpha}(z) = 1 - \lambda(1 - z)^\alpha \), \( 0 < \lambda, \alpha < 1 \). It is easy to see that \( \phi_{\lambda, \alpha} \in \mathcal{B}_0 \) and that \( \phi_{\lambda, \alpha} \) maps \( \mathbb{D} \) onto a region which behaves at 1 like a Stolz angle of opening \( \pi \alpha \). If \( C_\phi \) were compact on \( \mathcal{B}_0 \), composition with \( \log \frac{1}{1 - z} \) would yield a function in \( \mathcal{B}_0 \), but an easy calculation shows that this is not so. This leads to the consideration of cusps.

Throughout the remainder of this section \( \phi \) will denote a univalent mapping of the unit disk \( \mathbb{D} \) into itself with image \( G = \phi(\mathbb{D}) \). For simplicity it will be assumed that \( \overline{G} \cap T = \{1\} \). The region \( G \) is said to have a cusp at 1 [P, p. 256] if

\[
(6) \quad \text{dist}(w, \partial G) = o(|1 - w|)
\]
as \( w \to 1 \) in \( G \). Otherwise \( G \) does not have a cusp at 1. The cusp is said to be nontangential if \( G \) lies inside a Stolz angle near 1, i.e., there exist \( r, M > 0 \) such that

\[
(7) \quad |1 - w| \leq M(1 - |w|^2)
\]
if \( |1 - w| < r \), \( w \in G \). Finally the following geometric property of the conformal mapping \( \phi \) will be needed. If \( \phi \) is a conformal mapping with domain \( \mathbb{D} \),
This inequality, known as the Koebe distortion theorem, is an elementary consequence of the Schwarz lemma and Koebe's one-quarter theorem [G, p. 13]. It can be used to prove that bounded univalent functions lie in $\mathcal{B}_0$. Indeed, if $\phi \notin \mathcal{B}_0$, there is a $\delta > 0$ and a sequence $(z_n)$ in $\mathbb{D}$ with $|z_n| \to 1$ and $(1 - |z_n|)|\phi'(z_n)| > \delta$ for all $n$. Hence $\text{dist}(\phi(z_n), \partial G) > \delta/4$, so $\phi(z_n)$ has a cluster point in $G$, contradicting the fact that $\phi$ is a proper map. Theorem 4 provides a negative result.

**Theorem 4.** If $\phi$ is univalent and $G = \phi(\mathbb{D})$ satisfies $G \cap \mathbb{T} = \{1\}$ but does not have a cusp at 1, then $C_\phi$ is not compact on $\mathcal{B}_0$.

**Proof.** Since $G$ does not have a cusp at 1, (6) fails. Hence there is a $\delta > 0$ and a sequence $(z_n)$ in $\mathbb{D}$ such that $|z_n| \to 1$, but

$$\text{dist}(\phi(z_n), \partial G) \geq \delta|1 - \phi(z_n)|.$$

Hence

$$\delta(1 - |\phi(z_n)|^2) \leq 2\delta(1 - |\phi(z)|) \leq 2\text{dist}(\phi(z_n), \partial G) \leq 2(1 - |z_n|^2)|\phi'(z_n)|,$$

so

$$\frac{1 - |z_n|^2}{1 - |\phi(z_n)|^2} |\phi'(z_n)| \geq \frac{\delta}{2}.$$

Since $|z_n| \to 1$, Theorem 1 shows that $C_\phi$ is not compact.

The next theorem shows how to produce compact composition operators on $\mathcal{B}_0$ from univalent mappings $\phi$ with $\|\phi\|_\infty = 1$.

**Theorem 5.** If $\phi$ is univalent and if $G$ has a nontangential cusp at 1 and touches the unit circle at no other point, then $C_\phi$ is a compact operator on $\mathcal{B}_0$.

**Proof.** As $\phi \in \mathcal{B}_0$, it will be enough to show that

$$\lim_{|z| \to 1} \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)| = 0,$$

since the theorem will then follow from Theorem 1. Since $G$ has a nontangential cusp at 1, there exist $r, M > 0$ such that

$$|1 - w| \leq M(1 - |w|^2)$$

if $|1 - w| < r$, $w \in G$. Let $\epsilon > 0$. Since $G$ has a cusp at 1, there is a $\delta > 0$ such that

$$\text{dist}(w, \partial G) \leq \frac{\epsilon}{4M} |1 - w|$$

if $|1 - w| < \delta$, $w \in G$. Let $\eta = \min(\delta, r)$. If $|1 - \phi(z)| < \eta$, it follows that

$$\frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)| \leq \frac{4\text{dist}(\phi(z), \partial G)}{1 - |\phi(z)|^2} \leq \frac{\epsilon}{M} \frac{|1 - \phi(z)|}{1 - |\phi(z)|^2} < \epsilon.$$
On the other hand if \(|1 - \phi(z)| \geq \eta\), there is a constant \(N > 0\) such that \(|\phi'(z)| \leq N\) by the smoothness assumption and a \(\rho > 0\) such that \(1 - |\phi(z)|^2 \geq \rho\). In this case

\[
\frac{1 - |z|^2}{1 - |\phi(z)|^2}|\phi'(z)| \leq \frac{N}{\rho}(1 - |z|^2),
\]
and this is less than \(\epsilon\) if \(|z|^2 > 1 - \frac{\rho \epsilon}{N}\). That completes the proof.

It is possible to describe regions \(G\) with tangential cusp such that the Riemann mapping \(\phi: \mathbb{D} \to G\) admits either possibility. Indeed, suppose that \(h(\theta)\) and \(k(\theta)\) are positive continuous functions on \([0, \theta_0]\) with \(h(\theta) = o(\theta)\) and \(k(\theta) = o(\theta)\). Let

\[
G = \{ re^{i\theta} \mid 0 < \theta < \theta_0, \ h(\theta) < 1 - r < h(\theta) + k(\theta) \}.
\]

Then clearly \(G\) has a tangential cusp at 1. If \(k(\theta) = o(h(\theta))\), then, for \(w = re^{i\theta} = \phi(z)\),

\[
(1 - |z|^2)|\phi'(z)| \leq \text{dist}(w, \partial G) \leq k(\theta)
\]

and

\[
1 - |w|^2 \geq 1 - |w| > h(\theta),
\]

so \(\frac{1 - |z|^2}{1 - |\phi(z)|^2}|\phi'(z)| \to 0\) as \(|\phi(z)| \to 1\). Since \(\phi\) is univalent, the argument of Theorem 5 shows that \(C_\phi\) is compact. On the other hand if \(k(\theta) = 2h(\theta)\) and \(w(\theta) = (1 - 2h(\theta))e^{i\theta} = \phi(z(\theta))\), then evidently \(\text{dist}(w(\theta), \partial G) > ch(\theta)\) for some constant \(c\), and since \((1 - |z|^2)|\phi'(z)| \geq \text{dist}(\phi(z), \partial G)\), it follows that \(\frac{1 - |z(\theta)|^2}{1 - |w(\theta)|^2}|\phi'(z(\theta))| \geq \frac{c}{\epsilon}\), and so \(C_\phi\) is not compact.

4. Conclusion

Although the conditions of Theorems 1 and 2 provide succinct analytic conditions on a function \(\phi\) in order that it induce compact composition operators, it is desirable to have more geometric conditions. For example, it is clear from Section 3 that if \(\phi\) is a conformal mapping which has only a finite number of nontangential cusps on the unit circle \(T\) and no other points of contact, then \(C_\phi\) will be compact on \(\mathcal{B}_0\). This raises the question of whether or not there is a \(\phi \in \mathcal{B}_0\) such that \(\phi(\mathbb{D}) \cap T\) is infinite and \(C_\phi\) is compact on \(\mathcal{B}_0\). In this regard, it is known that if \(\phi\) has nontangential limit of modulus one on a set of positive measure, then \(\phi\) has an angular derivative at some point and so \(C_\phi\) is not compact [Sh, p. 71]. Further information about compact operators considered from a geometric point of view, especially on \(H^2\), can be found in [Sh] and [SSS].

Finally, if \(\phi \in \mathcal{B}_0\) and \(C_\phi\) is compact, then \(\log \frac{1}{1 - \phi(z)} \in \mathcal{B}_0\) for all \(w \in T\). Is the converse of this true?

References


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