TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 7, July 1995

COMPACT COMPOSITION OPERATORS ON THE BLOCH SPACE

KEVIN MADIGAN AND ALEC MATHESON

ABSTRACT. Necessary and sufficient conditions are given for a composition operator $C_\phi f = f \circ \phi$ to be compact on the Bloch space $\mathscr{B}$ and on the little Bloch space $\mathscr{B}_0$. Weakly compact composition operators on $\mathscr{B}_0$ are shown to be compact. If $\phi \in \mathscr{B}_0$ is a conformal mapping of the unit disk $\mathbb{D}$ into itself whose image $\phi(\mathbb{D})$ approaches the unit circle $\mathbb{T}$ only in a finite number of nontangential cusps, then C_ϕ is compact on $\mathscr{B}_0$. On the other hand if there is a point of $\mathbb{T} \cap \overline{\phi(\mathbb{D})}$ at which $\phi(\mathbb{D})$ does not have a cusp, then C_ϕ is not compact.

1. INTRODUCTION

Let $\mathbb{D}$ denote the unit disk in the complex plane. A function f holomorphic in $\mathbb{D}$ is said to belong to the Bloch space $\mathscr{B}$ if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty$$

and to the little Bloch space $\mathscr{B}_0$ if

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

It is well known that $\mathscr{B}$ is a Banach space under the norm

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|$$

and that $\mathscr{B}_0$ is a closed subspace of $\mathscr{B}$. Furthermore, $\mathscr{B}$ is isometrically isomorphic to the second dual of $\mathscr{B}_0$ and the inclusion $\mathscr{B}_0 \subset \mathscr{B}$ corresponds to the canonical imbedding of $\mathscr{B}_0$ into $\mathscr{B}_0^{**}$ [ACP]. It is a simple consequence of the Schwarz-Pick lemma [A] that a holomorphic mapping ϕ of the unit disk into itself induces a bounded composition operator $C_\phi f = f\circ\phi$ on $\mathscr{B}$. Indeed, if $f \in \mathscr{B}$, then

$$\begin{aligned}(1-|z|^2)|(f\circ\phi)'(z)| &= (1-|z|^2)|f'(\phi(z))||\phi'(z)| \\ &= \frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)|(1-|\phi(z)|^2)|f'(\phi(z))|\end{aligned} \tag{1}$$

Received by the editors November 23, 1993 and, in revised form, April 27, 1994; originally communicated to the *Proceedings of the AMS* by Theodore Gamelin.

1991 *Mathematics Subject Classification*. Primary 47B38, 47B07; Secondary 30D55.

Key words and phrases. Composition operator, compact operator, Bloch space, cusp.

The authors would like to thank Dan Luecking for bringing Proposition 1 to their attention and David Stegenga for discussions about cusps.

and the Schwarz-Pick lemma guarantees that

$$\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| \leq 1. \tag{2}$$

Since the identity function $f(z) = z$ belongs to $\mathscr{B}_0$, it is clear that $\phi \in \mathscr{B}_0$ if C_ϕ maps $\mathscr{B}_0$ into itself. Conversely, if $\phi \in \mathscr{B}_0$ and $f \in \mathscr{B}_0$, it follows from (1) and (2) that $f \circ \phi \in \mathscr{B}_0$. Indeed, if $\epsilon > 0$, there exists $\delta > 0$ such that $(1-|z|^2)|f'(z)| < \epsilon$ whenever $|z|^2 > 1-\delta$. In particular, $(1-|z|^2)|(f\circ\phi)'(z)| < \epsilon$ whenever $|\phi(z)|^2 > 1-\delta$. On the other hand, if $|\phi(z)|^2 \leq 1-\delta$,

$$(1-|z|^2)|(f\circ\phi)'(z)| \leq \frac{\|f\|_{\mathscr{B}}}{\delta}(1-|z|^2)|\phi'(z)|,$$

and the right-hand side tends to 0 as $|z| \to 1$.

In Section 2 the compact composition operators on $\mathscr{B}_0$ and on $\mathscr{B}$ will be characterized in terms of the quotient $\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)|$. A bounded linear operator $T: X \to Y$ from the Banach space X to the Banach space Y is *weakly compact* if T takes bounded sets in X into relatively weakly compact sets in Y. Gantmacher's theorem [D, p. 21] asserts that T is weakly compact if and only if $T^{**}(X^{**}) \subset Y$ where T^{**} denotes the second adjoint of T. This theorem and the characterization of compact operators on $\mathscr{B}_0$ will be used to show that every weakly compact composition operator on $\mathscr{B}_0$ is compact.

In Section 3 the results of Section 2 will be applied to certain univalent functions ϕ which map $\mathbb{D}$ into itself. It is known that such functions belong to $\mathscr{B}_0$ [P, p. 12]; and it will be clear from Section 2 that if $\|\phi\|_\infty < 1$, then C_ϕ is compact on $\mathscr{B}_0$. On the other hand if $\|\phi\|_\infty = 1$ and there is a point of $\mathbb{T}\cap\overline{\phi(\mathbb{D})}$ at which $\phi(\mathbb{D})$ does not have a cusp, then C_ϕ is not compact. However if $\mathbb{T}\cap\overline{\phi(\mathbb{D})}$ consists of only one point at which $\phi(\mathbb{D})$ has a nontangential cusp, then C_ϕ is compact on $\mathscr{B}_0$.

2. Compactness

Theorem 1 gives a precise description of those ϕ which induce compact composition operators on $\mathscr{B}_0$. It will be useful first to give a criterion for compactness in $\mathscr{B}_0$.

Lemma 1. *A closed set K in $\mathscr{B}_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}(1-|z|^2)|f'(z)| = 0. \tag{3}$$

Proof. First suppose that K is compact and let $\epsilon > 0$. Choose an $\epsilon/2$-net f_1, f_2, ..., f_n in K. There is an r, $0 < r < 1$, such that $(1-|z|^2)|f_i'(z)| < \epsilon/2$ if $|z| > r$, $1 \leq i \leq n$. If $f \in K$, $\|f - f_i\|_{\mathscr{B}} < \epsilon/2$ for some f_i and so

$$(1-|z|^2)|f'(z)| \leq \|f-f_i\|_{\mathscr{B}} + (1-|z|^2)|f_i'(z)| < \epsilon$$

whenever $|z| > r$. This establishes (3).

On the other hand if K is a closed bounded set which satisfies (3) and (f_n) is a sequence in K, then by Montel's theorem there is a subsequence (f_{n_k}) which converges uniformly on compact subsets of $\mathbb{D}$ to some holomorphic function f. Then also (f_{n_k}') converges uniformly to f' on compact subsets of $\mathbb{D}$. By (3), if

$\epsilon > 0$, there is an r, $0 < r < 1$, such that for all $g \in K$, $(1-|z|^2)|g'(z)| < \epsilon/2$ if $|z| > r$. It follows that $(1-|z|^2)|f'(z)| < \epsilon/2$ if $|z| > r$. Since (f_{n_k}) converges uniformly to f and (f'_{n_k}) converges uniformly to f' on $|z| \le r$, it follows that $\limsup_{k\to\infty} \|f_{n_k} - f\|_{\mathscr{B}} \le \epsilon$. Since $\epsilon > 0$, $\lim_{k\to\infty} \|f_{n_k} - f\|_{\mathscr{B}} = 0$ and so K is compact.

Theorem 1. *If ϕ is a holomorphic mapping of the unit disk $\mathbb{D}$ into itself, then ϕ induces a compact composition operator on $\mathscr{B}_0$ if and only if*

$$\lim_{|z|\to 1} \frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| = 0. \tag{4}$$

Proof. It follows from Lemma 1 that C_ϕ is compact on $\mathscr{B}_0$ if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathscr{B}}\le 1} (1-|z|^2)|(f\circ\phi)'(z)| = 0.$$

But

$$(1-|z|^2)|(f\circ\phi)'(z)| = \frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)|(1-|\phi(z)|^2)|f'(\phi(z))|,$$

and

$$\sup_{\|f\|_{\mathscr{B}}\le 1} (1-|w|^2)|f'(w)| = 1$$

for each $w \in \mathbb{D}$. The theorem follows.

It should be remarked that (4) implies $\phi \in \mathscr{B}_0$. A similar condition characterizes compact composition operators on $\mathscr{B}$.

Theorem 2. *If ϕ is a holomorphic mapping of the unit disk $\mathbb{D}$ into itself, then ϕ induces a compact composition operator on $\mathscr{B}$ if and only if for every $\epsilon > 0$, there exists r, $0 < r < 1$, such that*

$$\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| < \epsilon \tag{5}$$

whenever $|\phi(z)| > r$.

Proof. First assume that (5) holds. In order to prove that C_ϕ is compact on $\mathscr{B}$ it is enough to show that if (f_n) is a bounded sequence in $\mathscr{B}$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$, then $\|f_n\circ\phi\|_{\mathscr{B}} \to 0$. Let $M = \sup_n \|f_n\|_{\mathscr{B}}$. Given $\epsilon > 0$ there exists r, $0 < r < 1$, such that $\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| < \frac{\epsilon}{2M}$ if $|\phi(z)| > r$. Since

$$\begin{aligned}(1-|z|^2)|(f_n\circ\phi)'(z)| &= \frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)|(1-|\phi(z)|^2)|f_n'(\phi(z))| \\ &\le M\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)|,\end{aligned}$$

it follows that $(1-|z|^2)|(f_n\circ\phi)'(z)| < \frac{\epsilon}{2}$ if $|\phi(z)| > r$. On the other hand, $f_n\circ\phi(0) \to 0$ and $(1-|w|^2)|f_n'(w)| \to 0$ uniformly for $|w| \le r$. Since

$$(1-|z|^2)|(f_n\circ\phi)'(z)| \le (1-|\phi(z)|^2)|f_n'(\phi(z))|,$$

it follows that for large enough n, $|f_n\circ\phi(0)| < \frac{\epsilon}{2}$ and $(1-|z|^2)|(f_n\circ\phi)'(z)| < \frac{\epsilon}{2}$ if $|\phi(z)| \le r$. Hence $\|f_n\circ\phi\|_{\mathscr{B}} < \epsilon$ for large n.

Now assume that (5) fails. Then there exists a subsequence (z_n) in $\mathbb{D}$ and an $\epsilon > 0$ such that $|z_n| \to 1$ and $\frac{1-|z_n|^2}{1-|\phi(z_n)|^2}|\phi'(z_n)| > \epsilon$ for all n. Passing to a subsequence if necessary it may be assumed that $w_n = \phi(z_n) \to w_0 \in \mathbb{T}$. Let $f_n(z) = \log \frac{1}{1-\overline{w}_n z}$. Then (f_n) converges to f_0 uniformly on compact subsets of $\mathbb{D}$. On the other hand,

$$\begin{aligned}\|C_\phi f_n - C_\phi f_0\|_{\mathscr{B}} &\geq (1-|z_n|^2)|(C_\phi f_n)'(z_n) - (C_\phi f_0)'(z_n)| \\ &= (1-|z_n|^2)|\phi'(z_n)|\left|\frac{\overline{w}_n}{1-|w_n|^2} - \frac{\overline{w}_0}{1-\overline{w}_0 w_n}\right| \\ &= \frac{(1-|z_n|^2)}{1-|w_n|^2}|\phi'(z_n)|\left|\frac{\overline{w}_n - \overline{w}_0}{1-\overline{w}_0 w_n}\right| \\ &> \epsilon\end{aligned}$$

for all n, so $C_\phi f_n$ does not converge to $C_\phi f_0$ in norm. Hence C_ϕ is not compact.

It is important to note that although (4) implies (5), since in this case C_ϕ on $\mathscr{B}$ is the second adjoint of C_ϕ on $\mathscr{B}_0$, the two conditions are not equivalent. Condition (4) implies that $\phi \in \mathscr{B}_0$, while there certainly exist functions $\phi \notin \mathscr{B}_0$ which satisfy (5). Indeed, any ϕ for which $\|\phi\|_\infty < 1$ satisfies (5) trivially.

A sequence (w_n) in $\mathbb{D}$ is said to be η-separated if $\rho(w_n, w_m) = \left|\frac{w_m - w_n}{1-\overline{w}_m w_n}\right| > \eta$ whenever $m \neq n$. Thus an η-separated sequence consists of points which are uniformly far apart in the pseudohyperbolic metric on $\mathbb{D}$, or equivalently, the hyperbolic balls $\Delta(w_n, r) = \{z \mid \rho(z, w_n) < r\}$ are pairwise disjoint for some $r > 0$. Evidently any sequence (w_n) in $\mathbb{D}$ which satisfies $|w_n| \to 1$ possesses an η-separated subsequence for any $\eta > 0$. In particular, if the sequence (w_n) in the proof of Theorem 2 is η-separated, then the calculation in the proof shows that $\|C_\phi f_m - C_\phi f_n\| > \epsilon\eta$ whenever $m \neq n$, so $(C_\phi f_n)$ has no norm convergent subsequences.

Another property of separated sequences is contained in the next proposition. This proposition is related to some interpolation results of Rochberg [RR1, RR2]. Since the method of proof is precisely the same as Rochberg's, a proof will only be sketched.

Proposition 1. *There is an absolute constant $R > 0$ such that if (w_n) is R-separated, then for every bounded sequence (λ_n) there is an $f \in \mathscr{B}$ such that $(1-|w_n|^2)f'(w_n) = \lambda_n$ for all n.*

The idea of the proof is to consider two operators $S: \mathscr{B} \to l^\infty$ given by

$$S(f)_n = (1-|w_n|^2)f'(w_n)$$

and $T: l^\infty \to \mathscr{B}$ given by

$$T(\lambda)(z) = \sum_{n=1}^{\infty} \lambda_n \frac{1}{3\overline{w}_n} \frac{(1-|w_n|^2)^3}{(1-\overline{w}_n z)^3}$$

where $\lambda = (\lambda_n) \in l^\infty$. The proposition will follow if it can be shown that $\|I - ST\| < 1$, for then ST will be invertible and so S will be onto. The symbol C will denote a constant whose value changes from place to place but does not depend on R. Now

$$(ST - I)(\lambda)_n = (1 - |w_n|^2) \sum_{m \neq n} \lambda_m \frac{(1 - |w_m|^2)^3}{(1 - \overline{w}_m w_n)^4}$$

and so it will be enough to estimate

$$\sup_n (1 - |w_n|) \sum_{m \neq n} \frac{(1 - |w_m|^2)^3}{|1 - \overline{w}_m w_n|^4}.$$

If $R > 1/2$, say, then there is a fixed $\delta > 0$ such that the Euclidean disk D_m of center w_m and radius $\delta(1 - |w_m|^2)$ is contained in the hyperbolic disk $\Delta_m = \Delta(w_m, R)$ and is disjoint from the hyperbolic disks Δ_n for $n \neq m$. Since $|1 - \overline{z} w_n|^{-4}$ is subharmonic and the radius of D_m is comparable to $1 - |w_m|^2$,

$$\frac{(1 - |w_m|^2)^3}{|1 - \overline{w}_m w_n|^4} \leq C \iint_{D_m} \frac{1 - |w_m|^2}{|1 - \overline{z} w_n|^4} \, dx \, dy;$$

and since $|1 - \overline{w}_n z|$ dominates $1 - |w_m|^2$ on D_m, it follows that

$$\frac{(1 - |w_m|^2)^3}{|1 - \overline{w}_m w_n|^4} \leq C \iint_{D_m} \frac{1}{|1 - \overline{w}_n z|^3} \, dx \, dy$$

and hence

$$\begin{aligned} \sup_n (1 - |w_n|) \sum_{m \neq n} \frac{(1 - |w_m|^2)^3}{|1 - \overline{w}_m w_n|^4} &\leq C \iint_{\cup_{m \neq n} D_m} \frac{1 - |w_n|^2}{|1 - \overline{w}_n z|^3} \, dx \, dy \\ &\leq C \iint_{\mathbb{D} \setminus \Delta_n} \frac{1 - |w_n|^2}{|1 - \overline{w}_n z|^3} \, dx \, dy. \end{aligned}$$

The change of variables $z = \frac{w_n + \zeta}{1 + \overline{w}_n \zeta}$ turns this into

$$\sup_n (1 - |w_n|) \sum_{m \neq n} \frac{(1 - |w_m|^2)^3}{|1 - \overline{w}_m w_n|^4} \leq C \iint_{|\zeta| > R} \frac{1}{|1 + \overline{w}_n \zeta|} \, d\xi \, d\eta,$$

and the last integral can be made arbitrarily small uniformly in n if R is chosen close enough to 1. This provides the desired estimate.

Since every sequence (w_n) with $|w_n| \to 1$ contains an R-separated subsequence (w_{n_k}), it follows that there is an $f \in \mathcal{B}$ such that $(1 - |w_{n_k}|^2) f'(w_{n_k}) = 1$ for all k. This will be used in the proof of the next theorem.

Theorem 3. *Every weakly compact composition operator C_ϕ on $\mathcal{B}_0$ is compact.*

Proof. The composition operator $C_\phi : \mathcal{B}_0 \to \mathcal{B}_0$ is compact if and only if

$$\lim_{|z| \to 1} \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)| = 0$$

and, according to Gantmacher's theorem, weakly compact if and only if $C_\phi f \in \mathcal{B}_0$ for every $f \in \mathcal{B}$. If C_ϕ is not compact, there is an $\epsilon > 0$ and a sequence (z_n), $|z_n| \to 1$, such that

$$\frac{1 - |z_n|^2}{1 - |\phi(z_n)|^2} |\phi'(z_n)| \geq \epsilon$$

for all n. Since $\phi \in \mathscr{B}_0$, $|\phi(z_n)| \to 1$, and by passing to a subsequence it may be assumed that $(\phi(z_n))$ is R-separated. If $f \in \mathscr{B}$,

$$(1-|z_n|^2)|(C_\phi f)'(z_n)| = \frac{1-|z_n|^2}{1-|\phi(z_n)|^2}|\phi'(z_n)|(1-|\phi(z_n)|^2)|f'(\phi(z_n))|$$
$$\geq \epsilon(1-|\phi(z_n)|^2)|f'(\phi(z_n))|.$$

Since $(\phi(z_n))$ is R-separated, an application of Proposition 1 produces an $f \in \mathscr{B}$ such that $(1-|\phi(z_n)|^2)|(C_\phi f)'(z_n)| = 1$, for all n. Since $(1-|z_n|^2)|(C_\phi f)'(z_n)| \geq \epsilon$ and $|z_n| \to 1$, $C_\phi f \notin \mathscr{B}_0$ and so C_ϕ is not weakly compact.

A slight refinement of these arguments will show that a noncompact composition operator on $\mathscr{B}_0$ must be an isomorphism on a subspace isomorphic to the sequence space c_0. This is not surprising since $\mathscr{B}_0$ is known to be isomorphic to c_0.

3. Examples

As remarked in the introduction any holomorphic mapping ϕ of the unit disk into itself satisfying $\|\phi\|_\infty < 1$ induces a compact composition operator on $\mathscr{B}$ and also on $\mathscr{B}_0$ if $\phi \in \mathscr{B}_0$. On the other hand it is easy to see that if ϕ has a finite angular derivative at some point of $\mathbb{T}$, then C_ϕ cannot be compact. Indeed, ϕ has an angular derivative at $\zeta \in \mathbb{T}$ if the nontangential limit $\omega = f(\zeta) \in \mathbb{T}$ exists and if the quotient $\frac{f(z)-f(\zeta)}{z-\zeta}$ converges to some complex number μ as $z \to \zeta$ nontangentially. It is known that $\mu \neq 0$, and the Julia-Carathéodory lemma shows that $\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)|$ converges to $\zeta\overline{\omega}\mu \neq 0$ nontangentially. Applying Theorem 1 or 2 as appropriate shows that C_ϕ is not compact

It turns out, however, that ϕ can push the disk much more sharply into itself and still induce a noncompact composition operator. The easiest way to see this is to consider the functions $\phi_{\lambda,\alpha}(z) = 1-\lambda(1-z)^\alpha$, $0<\lambda, \alpha<1$. It is easy to see that $\phi_{\lambda,\alpha} \in \mathscr{B}_0$ and that $\phi_{\lambda,\alpha}$ maps $\mathbb{D}$ onto a region which behaves at 1 like a Stolz angle of opening $\pi\alpha$. If C_ϕ were compact on $\mathscr{B}_0$, composition with $\log\frac{1}{1-z}$ would yield a function in $\mathscr{B}_0$, but an easy calculation shows that this is not so. This leads to the consideration of cusps.

Throughout the remainder of this section ϕ will denote a univalent mapping of the unit disk $\mathbb{D}$ into itself with image $G = \phi(\mathbb{D})$. For simplicity it will be assumed that $\overline{G}\cap\mathbb{T} = \{1\}$. The region G is said to have a cusp at 1 [P, p. 256] if

$$\operatorname{dist}(w, \partial G) = o(|1-w|) \tag{6}$$

as $w \to 1$ in G. Otherwise G does not have a cusp at 1. The cusp is said to be nontangential if G lies inside a Stolz angle near 1, i.e., there exist $r, M > 0$ such that

$$|1-w| \leq M(1-|w|^2) \tag{7}$$

if $|1-w| < r$, $w \in G$. Finally the following geometric property of the conformal mapping ϕ will be needed. If ϕ is a conformal mapping with domain $\mathbb{D}$, then

$$\frac{1}{4}(1-|z|^2)|\phi'(z)| \le \operatorname{dist}(\phi(z), \partial G) \le (1-|z|^2)|\phi'(z)|. \tag{8}$$

This inequality, known as the Koebe distortion theorem, is an elementary consequence of the Schwarz lemma and Koebe's one-quarter theorem [G, p. 13]. It can be used to prove that bounded univalent functions lie in $\mathscr{B}_0$. Indeed, if $\phi \notin \mathscr{B}_0$, there is a $\delta > 0$ and a sequence (z_n) in $\mathbb{D}$ with $|z_n| \to 1$ and $(1-|z_n|)|\phi'(z_n)| > \delta$ for all n. Hence $\operatorname{dist}(\phi(z_n), \partial G) > \delta/4$, so $\phi(z_n)$ has a cluster point in G, contradicting the fact that ϕ is a proper map. Theorem 4 provides a negative result.

Theorem 4. *If ϕ is univalent and $G = \phi(\mathbb{D})$ satisfies $\overline{G} \cap \mathbb{T} = \{1\}$ but does not have a cusp at* 1, *then C_ϕ is not compact on $\mathscr{B}_0$.*

Proof. Since G does not have a cusp at 1, (6) fails. Hence there is a $\delta > 0$ and a sequence (z_n) in $\mathbb{D}$ such that $|z_n| \to 1$, but

$$\operatorname{dist}(\phi(z_n), \partial G) \ge \delta|1-\phi(z_n)|.$$

Hence

$$\delta(1-|\phi(z_n)|^2) \le 2\delta(1-|\phi(z_n)|) \le 2\operatorname{dist}(\phi(z_n), \partial G) \le 2(1-|z_n|^2)|\phi'(z_n)|,$$

so $\frac{1-|z_n|^2}{1-|\phi(z_n)|^2}|\phi'(z_n)| \ge \frac{\delta}{2}$. Since $|z_n| \to 1$, Theorem 1 shows that C_ϕ is not compact.

The next theorem shows how to produce compact composition operators on $\mathscr{B}_0$ from univalent mappings ϕ with $\|\phi\|_\infty = 1$.

Theorem 5. *If ϕ is univalent and if G has a nontangential cusp at* 1 *and touches the unit circle at no other point, then C_ϕ is a compact operator on $\mathscr{B}_0$.*

Proof. As $\phi \in \mathscr{B}_0$, it will be enough to show that

$$\lim_{|z|\to 1} \frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| = 0,$$

since the theorem will then follow from Theorem 1. Since G has a nontangential cusp at 1, there exist $r, M > 0$ such that

$$|1-w| \le M(1-|w|^2)$$

if $|1-w| < r$, $w \in G$. Let $\epsilon > 0$. Since G has a cusp at 1, there is a $\delta > 0$ such that

$$\operatorname{dist}(w, \partial G) \le \frac{\epsilon}{4M}|1-w|$$

if $|1-w| < \delta$, $w \in G$. Let $\eta = \min(\delta, r)$. If $|1-\phi(z)| < \eta$, it follows that

$$\begin{aligned} \frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| &\le \frac{4\operatorname{dist}(\phi(z), \partial G)}{1-|\phi(z)|^2} \\ &\le \frac{\epsilon}{M}\frac{|1-\phi(z)|}{1-|\phi(z)|^2} \\ &< \epsilon. \end{aligned}$$

On the other hand if $|1-\phi(z)| \geq \eta$, there is a constant $N > 0$ such that $|\phi'(z)| \leq N$ by the smoothness assumption and a $\rho > 0$ such that $1-|\phi(z)|^2 \geq \rho$. In this case

$$\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| \leq \frac{N}{\rho}(1-|z|^2),$$

and this is less than ϵ if $|z|^2 > 1-\frac{\rho\epsilon}{N}$. That completes the proof.

It is possible to describe regions G with tangential cusp such that the Riemann mapping $\phi: \mathbb{D} \to G$ admits either possibility. Indeed, suppose that $h(\theta)$ and $k(\theta)$ are positive continuous functions on $[0, \theta_0]$ with $h(\theta) = o(\theta)$ and $k(\theta) = o(\theta)$. Let

$$G = \{ re^{i\theta} \mid 0 < \theta < \theta_0,\ h(\theta) < 1-r < h(\theta)+k(\theta) \}.$$

Then clearly G has a tangential cusp at 1. If $k(\theta) = o(h(\theta))$, then, for $w = re^{i\theta} = \phi(z)$,

$$(1-|z|^2)|\phi'(z)| \leq \operatorname{dist}(w, \partial G) \leq k(\theta)$$

and

$$1-|w|^2 \geq 1-|w| > h(\theta),$$

so $\frac{1-|z|^2}{1-|\phi(z)|^2}|\phi'(z)| \to 0$ as $|\phi(z)| \to 1$. Since ϕ is univalent, the argument of Theorem 5 shows that C_ϕ is compact. On the other hand if $k(\theta) = 2h(\theta)$ and $w(\theta) = (1-2h(\theta))e^{i\theta} = \phi(z(\theta))$, then evidently $\operatorname{dist}(w(\theta), \partial G) > ch(\theta)$ for some constant c, and since $(1-|z|^2)|\phi'(z)| \geq \operatorname{dist}(\phi(z), \partial G)$, it follows that $\frac{1-|z(\theta)|^2}{1-|w(\theta)|^2}|\phi'(z(\theta))| \geq \frac{c}{4}$, and so C_ϕ is not compact.

4. Conclusion

Although the conditions of Theorems 1 and 2 provide succinct analytic conditions on a function ϕ in order that it induce compact composition operators, it is desirable to have more geometric conditions. For example, it is clear from Section 3 that if ϕ is a conformal mapping which has only a finite number of nontangential cusps on the unit circle $\mathbb{T}$ and no other points of contact, then C_ϕ will be compact on $\mathscr{B}_0$. This raises the question of whether or not there is a $\phi \in \mathscr{B}_0$ such that $\overline{\phi(\mathbb{D})} \cap \mathbb{T}$ is infinite and C_ϕ is compact on $\mathscr{B}_0$. In this regard, it is known that if ϕ has nontangential limit of modulus one on a set of positive measure, then ϕ has an angular derivative at some point and so C_ϕ is not compact [Sh, p. 71]. Further information about compact operators considered from a geometric point of view, especially on H^2, can be found in [Sh] and [SSS].

Finally, if $\phi \in \mathscr{B}_0$ and C_ϕ is compact, then $\log \frac{1}{1-\overline{w}\phi(z)} \in \mathscr{B}_0$ for all $w \in \mathbb{T}$. Is the converse of this true?

References

[A] Lars V. Ahlfors, *Conformal invariants*, McGraw-Hill, New York, 1973.

[ACP] J. M. Anderson, J. Clunie and Ch. Pommerenke, *On Bloch functions and normal functions*, J. Reine Angew. Math. **270** (1974), 12–37.

[D] Joseph Diestel, *Sequences and series in Banach spaces*, Springer-Verlag, New York, 1984.

[G] John Garnett, *Applications of harmonic measure*, Univ. Arkansas Lecture Notes in the Math. Sci., vol. 8, John Wiley & Sons, New York, 1986.

[P] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Springer-Verlag, Berlin, 1992.

[RR1] R. Rochberg, *Decomposition theorems for Bergman spaces and their applications*, Operators and Function Theory (S. C. Power, ed.), Reidel, Dordrecht, 1985, pp. 225–277.

[RR2] R. Rochberg, *Interpolation by functions in the Bergman spaces*, Michigan Math. J. **29** (1982), 229–236.

[Sh] J. Shapiro, *Composition operators and classical function theory*, Springer-Verlag, New York, 1993.

[SSS] J. H. Shapiro, W. Smith, and D. A. Stegenga, *Geometric models and compactness of composition operators*, J. Funct. Anal. (to appear).

Department of Mathematics, Northwestern University, Evanston, Illinois 60208
E-mail address: madigan@math.nwu.edu

Department of Mathematics, Lamar University, Beaumont, Texas 77710
E-mail address: matheson@math.lamar.edu