ON $C^*$-ALGEBRAS ASSOCIATED TO THE CONJUGATION REPRESENTATION OF A LOCALLY COMPACT GROUP

EBERHARD KANIUTH AND ANNETTE MARKFORT

Abstract. For a locally compact group $G$, let $\gamma_G$ denote the conjugation representation of $G$ in $L^2(G)$. In this paper we are concerned with nuclearity of $C^*$-algebras associated to $\gamma_G$ and the question of when these are of bounded representation type.

INTRODUCTION

Let $G$ be a locally compact group with left Haar measure and $C^*(G)$ the group $C^*$-algebra of $G$. For any unitary representation $\pi$ of $G$, there are two $C^*$-algebras associated to $\pi$. The first one is $\pi(C^*(G))$, which henceforth will be denoted $C^*(\pi)$, and the second one is $C^*(\pi(G))$, the $C^*$-algebra generated by the set of operators $\pi(x)$, $x \in G$, on the Hilbert space of $\pi$. If $G_d$ stands for the same group $G$ endowed with the discrete topology and $i_G : G_d \to G$ for the identity, then $C^*(\pi(G)) = C^*(\pi \circ i_G)$. Thus, investigating $C^*(\pi(G))$ naturally involves $G_d$.

For $\pi$ the left regular representation $\lambda_G$ of $G$, $C^*(\lambda_G)$ is called the reduced group $C^*$-algebra which is usually denoted by $C_r^*(G)$. It has been a matter of enormous interest in harmonic analysis and is one of the most important examples in the general theory of $C^*$-algebras. Very recently, Bédos [2] has drawn attention to $C^*(\lambda_G \circ i_G)$ and has shown that amenability of $G$ and of $G_d$ can both be characterized in terms of $C^*(\lambda_G \circ i_G)$.

In this paper we study $C^*$-algebras associated to the conjugation representation $\gamma_G$ of $G$ on $L^2(G)$ which is defined by

$$\gamma_G(x)f(y) = \delta(x)^{1/2}f(x^{-1}yx), \quad f \in L^2(G), \quad x, y \in G,$$

where $\delta$ denotes the modular function of $G$. We show that nuclearity of either $C^*(\gamma_G)$ or $C^*(\gamma_G \circ i_G)$ forces $G_d$ to be amenable (Theorem 1.2). Conversely, if $G_d$ is amenable then $C^*(\gamma_G)$ and $C^*(\gamma_G \circ i_G)$ are isomorphic (Theorem 1.7) and nuclear. Unfortunately, in this regard nothing substantial can be said about $C^*(\gamma_G)$ for arbitrary $G$ except that, of course, amenability of $G$ implies that $C^*(\gamma_G)$ is nuclear.

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These results will be applied in §2, where we deal with the question of when any one of the $C^*$-algebras $C^*(\gamma_G)$, $C^*(\gamma_G\cdot i_G)$ and $C^*(\gamma_G \circ i_G)$ is of bounded representation type, that is, possesses only finite-dimensional irreducible representations and there is an upper bound for the dimensions. Clearly, since $\gamma_G$ is trivial on $Z(G)$, the centre of $G$, such conditions can only be reflected by the structure of the factor group $G/Z(G)$. It turns out that, for a compactly generated Lie group $G$, any one of the above $C^*$-algebras being of bounded representation type is equivalent to the existence of an abelian subgroup of finite index in $G/Z(G)$ (Theorem 2.10).

The conjugation representation is of interest not least because of its connections to questions on inner invariant means on $L^\infty(G)$ (compare [17], [18] and [13]) and the structure of $G/Z(G)$ [14]. However, so far it is much less understood than the left regular representation. The main difficulty arising is that, even for finite groups, the support of $\gamma_G$ is generally strictly contained in the dual of $G/Z(G)$ and is intricate to determine (compare [11], [12], [13], [20], and [22]).

**Preliminaries and notation**

Let $G$ be a locally compact group. We use the same letter, for example $\pi$, for a unitary representation of $G$ and for the corresponding $*$-representation of $C^*(G)$, and $\mathcal{H}(\pi)$ always denotes the Hilbert space of $\pi$. Let $\ker\pi$ be the $C^*$-kernel of $\pi$. If $S$ and $T$ are sets of unitary representations of $G$, then $S$ is weakly contained in $T$ ($S \subset T$) if $\bigcap_{\pi \in S} \ker\sigma \supseteq \bigcap_{\tau \in T} \ker\tau$ or, equivalently, if any positive definite function associated to $S$ can be uniformly approximated on compact subsets of $G$ by sums of positive definite functions associated to $T$. $S$ and $T$ are weakly equivalent ($S \sim T$) if $S \subset T$ and $T \subset S$. The dual space $\hat{G}$ is the set of equivalence classes of irreducible representations of $G$, endowed with the Jacobson topology. As general references to dual spaces and representation theory we mention [5] and [7].

For any representation of $\pi$ of $G$, the support of $\pi$ is the closed subset $\text{supp}\pi = \{\rho \in \hat{G}; \rho \prec \pi\}$ of $\hat{G}$. In particular, the support of the left regular representation $\lambda_G$ is the reduced dual $\hat{G}_r$.

Recall that amenability of $G$ is equivalent to a number of different conditions: $C^*(\lambda_G) = C^*(G)$, $\hat{G}_r = \hat{G}$, or $1_G \prec \lambda_G$, where $1_G$ is the trivial one-dimensional representation of $G$. Concerning amenability we refer to [8], [23] and [24].

Also, we remind the reader that a $C^*$-algebra $A$ is called nuclear if there exists exactly one $C^*$-norm on the algebraic tensor product $A \otimes B$ for every $C^*$-algebra $B$. For properties equivalent to nuclearity and a short overview on this concept we refer to [23, §1.31].

Let $N$ be a closed normal subgroup of $G$. Then every representation of $G/N$ can be lifted to a representation of $G$, and in this sense will also be regarded as a representation of $G$. In particular $(G/N)^{\sim} \subseteq \hat{G}$. If $H$ is a subgroup of $G$, and $\sigma$ and $\pi$ are representations of $H$ and $G$, respectively, then $\text{ind}_H^G \sigma$ denotes the representation of $G$ induced by $\sigma$ and $\pi|H$ the restriction of $\pi$ to $H$. A readable account of the theory of induced representations can be found in [7, Chapter 11]. We will use throughout the fact that inducing and restricting representations are continuous with respect to Fell’s topology [6].
Next, let
\[ \{e\} = Z_0(G) \subseteq Z_1(G) = Z_1(G) \subseteq Z_2(G) \subseteq \cdots \]
be the ascending central series and \( G_f \) the finite conjugacy class subgroup of \( G \). For any two subsets \( M, N \) of \( G \) we denote by \( C_M(N) \) the centralizer of \( N \) in \( M \). If \( M = G \) we often omit the index. Using this notation, for discrete groups \( G \), \( \gamma_G \) is weakly equivalent to the set \( \{ \text{ind}_G^G(a) \mid C(a) \setminus \{a\}; a \in G \} \) (see [13, p. 27]).

For general \( G \) the only available description of \( \text{supp} \gamma_G \) is as follows. Let \( G \) be a \( \sigma \)-compact locally compact group, and suppose that \( C^*(\lambda_G) \) is nuclear. Then by [11, Theorem]
\[
\text{supp} \gamma_G = \bigcup_{\pi \in \hat{G}} \text{supp}(\pi \otimes \pi).
\]

1. \( C^*(\gamma_G^d), C^*(\gamma_G \circ i_G), \) and Amenability

We start with a lemma which will be used in the proof of Theorem 1.2 below as well as in §2.

Lemma 1.1. For any locally compact group \( G \) and \( i_G : G_d \to G \) the identity
\[
\lambda_{G_d/(G_d)f} \prec \gamma_G \circ i_G.
\]

Proof. The proof is an adaptation of the proof of Theorem 1.8 in [13]. Let \( D = G_d \) and recall that \( \lambda_D/D_f \) is the GNS-representation defined by the characteristic function \( \chi_{D_f} \) of \( D_f \). Therefore it suffices to show that given any finite subset \( F \) of \( D \), there exists a positive definite function \( \varphi \) associated to \( \gamma_G \circ i_G \) such that \( \varphi|F = \chi_{D_f}|F \). Set \( F_1 = F \cap D_f \) and \( F_2 = F \setminus F_1 \). Then, by the proof of [13, Theorem 1.8], there exists \( a \in C(F_1) \) such that \( x^{-1}ax \neq a \) for all \( x \in F_2 \).

\( C(F_1) \) is a closed subgroup of finite index in \( G \), and hence is open. Thus we find an open neighbourhood \( V \) of \( a \) in \( G \) such that \( V \subseteq C(F_1) \) and \( x^{-1}Vx \cap V = \emptyset \) for all \( x \in F_2 \). Observe that \( \delta(x) = 1 \) for all \( x \in F_1 \) since \( x^{-1}Vx = V \). Now, let \( f = |V|^{-1/2} \chi_V \) and
\[
\varphi(x) = \langle \gamma_G(x)f, f \rangle = \delta(x)^{1/2}|V|^{-1} \int_V \chi_V(x^{-1}yx) \, dy.
\]

It follows that \( \varphi(x) = 1 \) for \( x \in F_1 \) as \( V \subseteq C(F_1) \), and \( \varphi(x) = 0 \) for \( x \in F_2 \) since \( x^{-1}Vx \cap V = \emptyset \) for \( x \in F_2 \). \( \square \)

Theorem 1.2. For a locally compact group \( G \) the following are equivalent.

(i) \( G_d \) is amenable.
(ii) \( C^*(\gamma_G \circ i_G) \) is nuclear.
(iii) \( C^*(\gamma_G^d) \) is nuclear.

Proof. (i) \( \Rightarrow \) (ii) and (i) \( \Rightarrow \) (iii) are obvious, since amenability of \( G_d \) implies that \( C^*(\gamma_G^d) \) is nuclear, and hence so are the quotients \( C^*(\gamma_G \circ i_G) \) and \( C^*(\gamma_G^d) \) of \( C^*(\lambda_{G_d/(G_d)f}) \) (compare [4, Corollary 4]).

Since \( \lambda_{G_d/(G_d)f} \prec \gamma_G \circ i_G \) (Lemma 1.1), \( C^*(\lambda_{G_d/(G_d)f}) \) is a quotient of \( C^*(\gamma_G \circ i_G) \). Thus (ii) implies nuclearity of \( C^*(\lambda_{G_d/(G_d)f}) \), and by [16, Theorem 4.2] this forces \( G_d/(G_d)f \) to be amenable. Now groups with finite conjugacy classes are well known to be amenable (see [24, Proposition 12.9 or Corollary...
As the class of amenable groups is closed under forming extensions by amenable groups, $G_d$ turns out to be amenable. (iii) $\Rightarrow$ (i) follows in the same way by appealing to Theorem 1.8 of [13] instead of Lemma 1.1.

For $\gamma_G$ replaced by the left regular representation, Theorem 1.2 has been established in [2, Theorem 3].

**Lemma 1.3.** Let $G$ and $H$ be locally compact groups, and let $j : H \to G$ be a continuous and injective homomorphism with dense range. Then $\widehat{G \circ j} \subseteq \widehat{H}$, and $\widehat{G \circ j}$ is dense in $\widehat{H}$ provided that $H$ is discrete and amenable.

**Proof.** Let $\pi_1, \pi_2$ be representations of $G$. If $\pi_1$ and $\pi_2$ are equivalent, then $\pi_1 \circ j$ and $\pi_2 \circ j$ are equivalent representations of $H$. Conversely, if $\pi_1 \circ j$ and $\pi_2 \circ j$ are equivalent, then since $j(H)$ is dense in $G$ and representations are strongly continuous, it follows immediately that $\pi_1$ and $\pi_2$ are equivalent. Moreover, for a representation $\pi$ of $G$, $\pi$ is irreducible if and only if $\pi \circ j$ is irreducible. Thus $\pi \to \pi \circ j$ induces an injective mapping from $G$ into $H$.

It is easy to see that the Dirac function $\delta_e$ on $H$ can be pointwise approximated by positive definite functions associated to $XG \circ j$ [3, Proposition 1]. For $H$ discrete, this shows that $\lambda_H \prec \lambda_G \circ j$, and hence $\widehat{G \circ j}$ is dense in $\widehat{H}$ if, in addition, $H$ is amenable. □

**Corollary 1.4.** Suppose that $H$ is amenable and discrete, and let $G$ and $j$ be as in Lemma 1.3. Then

$$\{(\pi \circ j) \otimes (\pi \circ j) ; \pi \in \widehat{G}\} \sim \left\{\rho \otimes \check{\rho} ; \rho \in \widehat{H}\right\}.$$

**Proof.** Let $P$ and $R$ denote the set of representations on the left and on the right, respectively. It is clear from $\widehat{G \circ j} \subseteq \widehat{H}$ that $P \prec R$. On the other hand, since $\widehat{G \circ j}$ is dense in $\widehat{H}$ by Lemma 1.3, for $\rho \in \widehat{H}$ every coordinate function of the form

$$x \to ((\rho \otimes \check{\rho})(x)(\xi_1 \otimes \xi_2), \eta_1 \otimes \eta_2) = (\rho(x)\xi_1, \eta_1)(\check{\rho}(x)\xi_2, \eta_2),$$

where $\xi_1, \eta_1 \in \mathcal{H}(\rho)$ and $\xi_2, \eta_2 \in \mathcal{H}(\rho)$, can be approximated on finite subsets of $H$ by a product of functions each of which is a finite sum of positive definite functions associated to $\pi \circ j$ and $\pi \circ j$, $\pi \in \widehat{G}$, respectively. It follows that $\rho \otimes \check{\rho} \prec P$. □

We have to compare $\gamma_{Gd}$ and $\gamma_G \circ i_G$ with respect to weak equivalence. As mentioned in the proof of Lemma 1.3, for every locally compact group $G$, $\lambda_{Gd} \prec \lambda_G \circ i_G$. In general, however, $\gamma_{Gd}$ need not be weakly contained in $\gamma_G \circ i_G$. We will further comment on this in Lemma 1.8 and Remarks 1.9. But at least we have

**Corollary 1.5.** Suppose that $G$ is $\sigma$-compact and $H$ is amenable and discrete, and let $j$ be as in Lemma 1.3. Then $\gamma_H \prec \gamma_G \circ j$.

**Proof.** Since $G$ is amenable and $\sigma$-compact, $\gamma_G \sim \{\pi \otimes \check{\pi} ; \pi \in \widehat{G}\}$ by the theorem of [11]. Corollary 1.4 yields

$$\gamma_G \circ j \sim \{(\pi \circ j) \otimes (\pi \circ j) ; \pi \in \widehat{G}\} \sim \{\rho \otimes \check{\rho} ; \rho \in \widehat{H}\},$$

and this latter set weakly contains $\gamma_H$ [11, Corollary 1]. □
Lemma 1.6. Let \( G \) be a second countable group such that \( G_d \) is amenable. Then \( \gamma_G \circ i_G \prec \gamma_{G_d} \).

Proof. There exists a countable dense subset in \( G \) as \( G \) is second countable. Thus every finite subset of \( G \) is contained in some countable dense subgroup \( H \) of \( G \). For any such \( H \), \( \{ \rho \otimes \bar{\rho} : \rho \in \hat{H}_d \} \sim \gamma_{H_d} \), and hence by Corollary 1.4,

\[
\gamma_G \circ j_H \sim \{ (\pi \circ j_H) \otimes (\pi \circ j_H) : \pi \in \hat{G} \} \sim \{ \rho \otimes \bar{\rho} : \rho \in \hat{H}_d \} \sim \gamma_{H_d},
\]

where \( j_H \) denotes the inclusion \( H_d \hookrightarrow G \). On the other hand, \( \gamma_{H_d} \) is a subrepresentation of \( \gamma_{G_d} | H_d \) and

\[
\langle \gamma_G \circ i_G(x)f, f \rangle = \langle \gamma_{H_d}(x)f, f \rangle
\]

for all \( x \in H \) and \( f \in L^2(G) \). This proves \( \gamma_G \circ i_G \prec \gamma_{G_d} \). \( \Box \)

Theorem 1.7. Let \( G \) be a locally compact group. If \( G_d \) is amenable, then \( \gamma_G \circ i_G \sim \gamma_{G_d} \), and \( C^*(\gamma_G \circ i_G) \) and \( C^*(\gamma_{G_d}) \) are isomorphic.

Proof. We first reduce to the \( \sigma \)-compact case. To that end, let \( \mathfrak{H} \) denote the set of all \( \sigma \)-compact open subgroups \( H \) of \( G \), and suppose that we already know \( \gamma_H \circ i_H \sim \gamma_{H_d} \) for every \( H \in \mathfrak{H} \). To show that \( \gamma_G \circ i_G \prec \gamma_{G_d} \), let a finite subset \( F \) of \( G \) and \( f \in L^2(G) \) be given and consider the function \( \varphi(x) = \langle \gamma_G(x)f, f \rangle \). Choose \( H \in \mathfrak{H} \) such that \( F \subseteq H \) and \( f | G \setminus H = 0 \). Then \( \varphi(x) = \langle \gamma_H(x)f | H, f | H \rangle \) for all \( x \in H \). Since \( \gamma_H \circ i_H \prec \gamma_{H_d} \), \( \varphi \) can be approximated on \( F \) by sums of positive definite functions associated to \( \gamma_{H_d} \).

It follows that

\[
\gamma_G \circ i_H \prec \gamma_{H_d} \sim \gamma_{G_d} | H_d ,
\]

and hence \( \gamma_G \circ i_G \prec \gamma_{G_d} \). That, conversely, \( \gamma_{G_d} \prec \gamma_G \circ i_G \) is seen in the same way.

Recall next that, by Corollary 1.5, \( \gamma_H \circ i_H \prec \gamma_{H_d} \) for each \( H \in \mathfrak{H} \). From Lemma 1.6 we know that conversely \( \gamma_H \circ i_H \prec \gamma_{H_d} \) provided that \( H \) is second countable. Thus it remains to extend this to the case of a \( \sigma \)-compact group \( H \).

Being \( \sigma \)-compact, \( H \) is a projective limit of second countable groups \( H_\alpha = H/K_\alpha \), \( \alpha \in A \), where the \( K_\alpha \) are compact. Now, the set

\[
\{ f \in C_c(H) : \text{ for some } \alpha \in A, f(xk) = f(x) \text{ for all } x \in H \text{ and } k \in K_\alpha \}
\]

is dense in \( C_c(H) \) in the inductive limit topology. Therefore it suffices to approximate a function \( x \rightarrow \langle \gamma_H(x)f, f \rangle \), where \( f \in C_c(H) \) is constant on cosets of some \( K = K_\alpha \), on finite subsets of \( H \) by sums of positive definite functions associated to \( \gamma_{H_d} \). Define \( g \) on \( H/K \) by \( g(xK) = f(x) \) for \( x \in H \). Then

\[
\langle \gamma_H(x)f, f \rangle = \langle \gamma_{H/K}(xK)g, g \rangle ,
\]

and by Lemma 1.6 the function on the right can be approximated on finite subsets of \( H/K \) by sums of positive definite functions associated to \( \gamma_{(H/K)_d} \).

Now, \((H/K)_d = H_d/K_d\), and by [20, Lemma 1.1], \( \gamma_{H_d/K_d} \prec \gamma_{H_d} \) since \( H_d \) is amenable. This shows that \( \gamma_H \circ i_H \prec \gamma_{H_d} \) and finishes the proof. \( \Box \)

Obviously, if \( G_d \) is amenable, then so is \( G \). As to the regular representation, it has been observed in [2, Theorem 3] that if \( G \) is amenable and \( \lambda_{G_d} \sim \lambda_G \circ i_G \), then \( G_d \) is amenable. In fact, under these assumptions,

\[
1_{G_d} = 1_G \circ i_G \prec \lambda_{G_d} \circ i_G \sim \lambda_{G_d},
\]
Although it is conceivable, we do not know whether, as a converse to Theorem 1.7, amenability of $G$ and $\gamma_{G_d} \sim \gamma_G \circ i_G$ imply that $G_d$ is amenable.

We conclude this section by returning to the question of when $\gamma_{G_d} \prec \gamma_G \circ i_G$. Recall that a locally compact group is said to be an [SIN]-group if $G$ has a system of neighbourhoods $V$ of the identity such that $x^{-1}Vx = V$ for all $x \in G$.

**Lemma 1.8.** If $G$ is an [SIN]-group, then $\gamma_{G_d} \prec \gamma_G \circ i_G$.

**Proof.** It suffices to approximate the function $x \mapsto \chi_{C(a)}(x) = \langle \gamma_{G_d}(x) \delta_a, \delta_a \rangle$, $a \in G$, on finite subsets $F$ of $G$ by positive definite functions associated to $\gamma_G \circ i_G$. Now, given such an $F$, there exists an invariant symmetric neighbourhood $V$ of $e$ in $G$ such that $x^{-1}ax \not\in V^2a$ for all $x \in F \setminus C(a)$. Let $\phi = |V|^{-1/2} \chi_V$; then it is easily verified that

$$\langle \gamma_G(x) \phi , \phi \rangle = |V|^{-1} \int_V \chi_V(x^{-1}vax) dv$$

is equal to 1 for all $x \in C(a)$ and equal to 0 for all $x \in F \setminus C(a)$.

**Remarks 1.9.** (i) Suppose that $C^*(\lambda_G)$ is nuclear and that $\gamma_{G_d} \prec \gamma_G \circ i_G$. Then $G$ is amenable. This can be seen as follows. Since $\lambda^*_G \prec \gamma_{G_d} \prec \gamma_G \circ i_G \prec \lambda_G \circ i_G$

[11, Proposition 1], there is a homomorphism of $C^*(\lambda_G(G)) = C^*(\lambda_G \circ i_G)$ onto $\mathcal{C}$. By [2, Theorem 1] this implies that $G$ is amenable. In particular, for any noncompact connected semisimple Lie group $G$, $\gamma_{G_d}$ is not weakly contained in $\gamma_G \circ i_G$.

(ii) By Lemma 1.8 for $G$ compact, $\gamma_{G_d} \prec \gamma_G \circ i_G$. Moskowitz [22] has shown that, for $G$ a compact connected Lie group, $\text{supp} \gamma_G = (G/Z(G))^\sim$. This can be used to compare the sets $\text{supp}(\gamma_G \circ i_G)$, $(\text{supp} \gamma_G) \circ i_G$, and $\text{supp} \gamma_{G_d}$. As an illustrating example let us look at $G = SO(3)$. Then $(\text{supp} \gamma_G) \circ i_G = \widehat{G} \circ i_G$, and $\widehat{G} \circ i_G$ fails to be dense in $\widehat{G}$ (see [3, Corollary 1]).

Considering $G_d$, it follows from [13, Corollary 1.9] that $\text{supp} \gamma_{G_d} = (G_d)^\sim \cup \{1_{G_d}\}$ since $(G_d)^r$ is trivial and the centralizer of each matrix in $SO(3) \setminus \{E\}$ has a subgroup of index 2, which is conjugate to $SO(2)$. Thus $\text{supp} \gamma_{G_d} \cap (\text{supp} \gamma_G) \circ i_G = \{1_{G_d}\}$ and $\text{supp} \gamma_{G_d}$ is strictly contained in $\text{supp}(\gamma_G \circ i_G)$, since $1_{G_d}$ is the only finite-dimensional representation in $\text{supp} \gamma_{G_d}$.

2. When is $C^*(\gamma_G)$ of bounded representation type?

Let $A$ be a $C^*$-algebra and $\widehat{A}$ its dual space. $A$ is said to be of bounded representation type (b.r.t.) if every $\pi \in \widehat{A}$ is finite dimensional and if there is an upper bound for these dimensions. The analogous notion applies to representations. Thus, a representation $\rho$ of $A$ is of b.r.t. if $\rho(A)$ is of b.r.t. Moreover, a locally compact group $G$ is of bounded representation type if $C^*(G)$ has this property. The first paper dealing with such groups that we are aware of is [15]. Groups of b.r.t. have finally been identified by Moore [21] as precisely those which have an abelian subgroup of finite index.

In this section we are interested in the question of when the $C^*$-algebras $C^*(\gamma_G)$, $C^*(\gamma_{G_d})$ and $C^*(\gamma_G \circ i_G)$ are of bounded representation type. For
such a particular representation, this appears to be a rather intricate problem. We succeeded in resolving it for compactly generated Lie groups, where by Lie group we mean a locally compact group $G$ whose connected component $G_0$ of the identity is open and is an analytic group. However, we were unable to characterize non-finitely-generated discrete groups $G$ or totally disconnected compact groups $G$ with $C^*(\gamma_G)$ of b.r.t.

It is worth commenting here on the same question for the left regular representation. Now, for any locally compact group $H$, $C^*(\lambda_H)$ being of b.r.t. implies that $H$ has an abelian subgroup of finite index. Indeed, this follows from [26, Satz 2] and can also be deduced from Moore’s results [21]. As to $C^*(\lambda_H \circ i_H)$, notice that by [2, Lemma 2] $\lambda_H$ is weakly contained in $\lambda_H \circ i_H$, so that $C^*(\lambda_H)$ is of b.r.t. provided that $C^*(\lambda_H \circ i_H)$ is.

**Remarks 2.1.** (i) If $\gamma_G$ is of bounded representation type (b.r.t.), then $\gamma_G | H$ is of b.r.t. for every closed subgroup $H$ of $G$. Indeed, let

$$T = \bigcup_{\pi \in \text{supp} \gamma_G} \text{supp}(\pi | H) \subseteq \hat{H},$$

and suppose that $\dim \pi \leq d$ for all $\pi \in \text{supp} \gamma_G$. Then $\dim \tau \leq d$ for all $\tau \in T$, and hence for all $\tau \in \hat{T}$. On the other hand, $\hat{T} = \text{supp}(\gamma_G | H)$ since $T$ is weakly equivalent to $\gamma_G | H$.

(ii) Let $H$ be an open subgroup of $G$. If $\gamma_G$ is of b.r.t., then so is $\gamma_H$. In fact, by (i) $\gamma_G | H$ is of b.r.t., and $\gamma_H$ is a subrepresentation of $\gamma_G | H$ as $L^2(H)$ is a subspace of $L^2(G)$. Notice, however, that in general for a closed subgroup $H$ of $G$, $\gamma_H$ need not even be weakly contained in $\gamma_G | H$ (see [14]).

(iii) If $\gamma_G$ is of b.r.t. and $C^*(\lambda_G)$ is nuclear, then $G$ is amenable. The nuclearity assumption guarantees that $\gamma_G \prec \lambda_G$ [11, Proposition 1]. Now, it is well known that $G$ is amenable provided that $\lambda_G$ weakly contains a finite-dimensional representation. Recall that $C^*(\lambda_G) = C^*(G)$, as a matter of fact) is nuclear if $G/\Gamma_0$ is amenable.

If $N$ is a closed normal subgroup of $G$, then $G$ acts on $\hat{N}$ by $(x, \lambda) \rightarrow \lambda^x$, where $\lambda^x(n) = \lambda(\lambda^{-1}nx)$ for $x \in G$ and $n \in N$, and $G^x$ denotes the stability subgroup of $\lambda$ in $G$ under this action.

**Lemma 2.2.** Let $G$ be a locally compact group, and suppose that $\text{supp} \gamma_G$ contains a dense subset of finite-dimensional representations. Let $N$ be a closed normal subgroup of $G$ such that $N/N \cap Z(G)$ is a vector group. Then there exists a closed subgroup $H$ of finite index in $G$ such that $N \subseteq Z_2(H)$.

**Proof.** Let $\Pi = \{ \pi \in \text{supp} \gamma_G : \dim \pi < \infty \}$ and $\Lambda = \bigcup_{\pi \in \Pi} \text{supp}(\pi | N)$. By hypothesis, $\gamma_G | N \sim \Pi | N \sim \Lambda$, so that $\Lambda$ separates the points of $V = N/N \cap Z(G)$. $V$ and hence $\hat{V}$ being a vector group, $\Lambda$ contains a basis $\{ \lambda_1, \ldots, \lambda_m \}$ of $\hat{V}$. Now, $H = \bigcap_{j=1}^m G_{\lambda_j}$ has finite index in $G$ and $\lambda^h = \lambda_j$ for all $h \in H$ and $1 \leq j \leq m$. Since continuous automorphisms of vector groups are linear, it follows that $\lambda^h = \lambda$ for all $\lambda \in \hat{V}$ and $h \in H$. This implies that $N/N \cap Z(G) \subseteq Z(H/N \cap Z(G))$ and hence $N \subseteq Z_2(H)$. $\Box$

**Lemma 2.3.** Let $G$ and $\gamma_G$ be as in Lemma 2.2. Let $N$ be a closed normal subgroup of $G$ such that $N/N \cap Z(G) = \mathbb{T}^m$ for some $m \in \mathbb{N}$. Then $N \subseteq Z_2(H)$ for some subgroup $H$ of finite index in $G$. 

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Proof. Let \( \Pi \) and \( \Lambda \) be as in the proof of the previous lemma. Then \( \Lambda \) generates \((N/N \cap Z(G))^\sim = Z^m\), so that \( G_1 \) has finite index in \( G \) for each \( \lambda \in Z^m \). As \( Z^m \) is finitely generated, we find a subgroup \( H \) of finite index in \( G \) such that \( \lambda^h = \lambda \) for all \( \lambda \in Z^m \) and all \( h \in H \). This proves that \( N/N \cap Z(G) \subseteq Z(H/N \cap Z(G)) \) and hence \( N \subseteq Z_2(H) \). \( \square \)

Lemma 2.4. Let \( K \) be a compact connected normal subgroup of the Lie group \( G \). If \( \gamma_G \) is of b.r.t., then the commutator subgroup \( K' \) of \( K \) is contained in the centre of \( G \).

Proof. It suffices to show that \( K' \subseteq Z(H) \) for every \( \sigma \)-compact open subgroup \( H \) of \( G \). Since \( \gamma_H \) is of b.r.t. for every such \( H \), we can assume that \( G \) is \( \sigma \)-compact and hence second countable as it is a Lie group. Recall that by [19, Lemma 3.1], for any second countable group \( G \), \( \gamma_G \) is unitarily equivalent to the restriction of \( ind_{\Delta_G}^{G \times G} 1_{\Delta_G} \) to \( \Delta_G \) where \( \Delta_G \) denotes the diagonal subgroup of \( G \times G \). Since \( K \) is compact and \( G \) is second countable, \( \Delta_K \) and \( \Delta_G \) are regularly related in \( G \times G \) in the sense of Mackey. Therefore, by [6, Theorem 5.3], with \( \Delta_G^u = u\Delta_G u^{-1} \) for \( u \in G \times G \),

\[
\gamma_G | K = \text{ind}_{\Delta_G}^{G \times G} 1_{\Delta_G} | \Delta_K \sim \{ \text{ind}_{u^{-1}\Delta_G \cap \Delta_K}^{\Delta_G} 1_{u^{-1}\Delta_G \cap \Delta_K} ; u \in G \times G \}
\]

\[
= \{ \text{ind}_{C(a) \cap K}^{K} 1_{C(a) \cap K} ; a \in G \} = \{ \text{ind}_{C(a)}^{K(a)} ; a \in G \}.
\]

Fix \( a \in G \), and let \( N(a) \) denote the greatest normal subgroup of \( K \) contained in \( C_K(a) \). There exist finitely many \( x_1, \ldots, x_m \in K \) such that

\[
N(a) = \bigcap_{j=1}^m x_j^{-1} C_K(a) x_j
\]

(compare [1, Proposition 2.1]). By [6, Theorem 5.5] the \( m \)-fold tensor product \( (\gamma_G | K)^{\otimes m} \) weakly contains

\[
\text{ind}_{x_1^{-1} C_K(a) x_1 \cap \cdots \cap x_m^{-1} C_K(a) x_m}^{C(a) \cap K} 1_{x_1^{-1} C_K(a) x_1 \cap \cdots \cap x_m^{-1} C_K(a) x_m} = \text{ind}_{N(a)}^{K(a)} 1_{N(a)}.
\]

Now tensor products of representations of b.r.t. are again of b.r.t. [25, Lemma 5]. Thus \( \text{ind}_{N(a)}^{K(a)} 1_{N(a)} \) is of b.r.t., and since \( K \) is connected this yields that \( K/\langle a \rangle \) is abelian. It follows that

\[
K/ \left( \bigcap_{a \in G} (C(a) \cap K) \right) = K/ \bigcap_{a \in G} N(a)
\]

is abelian. This proves \( K' \subseteq \bigcap_{a \in G} C(a) = Z(G) \). \( \square \)

Proposition 2.5. Let \( G \) be a Lie group and \( N \) a connected closed normal subgroup of \( G \). If \( C^*(\gamma_G) \) is of b.r.t., then there exists a subgroup \( H \) of finite index in \( G \) such that \( N \subseteq Z_6(H) \).

Proof. Let \( M = N \cap Z(G) \). Since \( \gamma_G | N \) separates the points of \( N/M \), \( N/M \) is a maximally almost periodic connected Lie group. By the Freudenthal-Weil theorem [5, Théorème 16.4.6] \( N/M \) is a direct product of a vector group \( W \) and a compact connected Lie group \( K \).

Let \( q : G \to G/M \) be the quotient homomorphism. As \( K \) is normal in \( G/M \), it follows from Lemma 2.4 that \( K' \subseteq Z(G/M) \) and hence \( q^{-1}(K') \subseteq \)
$Z_2(G)$. Applying Lemma 1.1 in [14] twice gives $\gamma_{G/q^{-1}(K')} \prec \gamma_G$, so that $\gamma_{G/q^{-1}(K')}$ is of b.r.t. Now, $K/K'$ is a normal torus in $G/q^{-1}(K')$. It follows from Lemma 2.3 that $K/K' \subseteq Z_2(H_1/q^{-1}(K'))$ for some subgroup $H_1$ of finite index in $G$. Thus $q^{-1}(K) \subseteq Z_4(H_1)$.

Now, moving to $G/q^{-1}(K)$, similar arguments apply to the normal vector subgroup $W$ of $G/q^{-1}(K)$. Again, since continuous automorphisms of vector groups are linear, $W \cap Z(G/q^{-1}(K))$ is a vector group and hence so is $W/W \cap Z(G/q^{-1}(K))$. Lemma 2.2 yields that $W \subseteq Z_2(H_2/q^{-1}(K))$ for some subgroup $H_2$ of finite index in $G$ containing $q^{-1}(K)$. With $H = H_1 \cap H_2$, we obtain that $N \subseteq Z_6(H)$.

Remark 2.6. Let $D$ be a discrete group with $\gamma_D$ of b.r.t. Then, since $\lambda_{D/D_f} \prec \gamma_D$ [13, Theorem 1.8], $\lambda_{D/D_f}$ is of b.r.t. and therefore $D/D_f$ has an abelian subgroup of finite index (compare [26, Satz 1]). In particular, $D$ is amenable. It is worthwhile to remind the reader that in order to conclude that a discrete group $G$ is almost abelian it is only required that $\lambda_G$ is of type I [10].

Corollary 2.7. If $G$ is a Lie group with $C^*(\gamma_G)$ of b.r.t., then $G_d$ is amenable and $C^*(\gamma_{G_d})$ and $C^*(\gamma_G \circ i_G)$ are both of b.r.t.

Proof. By Proposition 2.5, $G_0 \subseteq Z_m(H)$ for some $m \in \mathbb{N}$ and some subgroup $H$ in $G$ of finite index. In particular, $G_0$ is nilpotent. Let $D = G/G_0$; then repeated application of [14, Lemma 1.1] gives $\gamma_D \prec \gamma_G$. Thus $\gamma_D$ is of b.r.t., and hence $D$ is amenable (Remark 2.6). Since $(G_0)_d$ and $G_0/G_0$ are amenable, $G_d$ is amenable.

By what we have seen in Theorem 1.7, $\gamma_{G_d} \sim \gamma_G \circ i_G$, and $G^*(\gamma_{G_d})$ and $C^*(\gamma_G \circ i_G)$ are isomorphic. Thus it remains to recognize that $\gamma_{G_d}$ is of b.r.t. But this follows because $\gamma_G$ is of b.r.t. and $\text{supp} \gamma_{G_d}$ is contained in the closure of $(\text{supp} \gamma_G) \circ i_G$ in $\widehat{G}_d$. □

Corollary 2.8. For a connected group $G$, $C^*(\gamma_G)$ is of bounded representation type if and only if $G$ is 2-step nilpotent.

Proof. Clearly, if $G/Z(G)$ is abelian, then every $\pi \in \text{supp} \gamma_G$ is one-dimensional. Conversely, suppose that $G$ is connected and $\gamma_G$ is of b.r.t. Then $G$ is a projective limit of Lie groups $G_i = G/K_i$, $i \in I$, where the $K_i$ are compact, and every $\gamma_{G_i}$ is of b.r.t. Let $q_i : G \to G_i$ denote the quotient homomorphism. Since $Z(G) = \bigcap_{i \in I} q_i^{-1}(Z(G_i))$, $G$ is 2-step nilpotent if all $G_i$ are. Therefore we can assume that $G$ is a Lie group.

By Corollary 2.7, $\gamma_{G_f}$ is of b.r.t., and hence $G/G_f$ has an abelian subgroup of finite index. For any $x \in G_f$, $C(x)$ is a closed subgroup of finite index in $G$, so that $x \in Z(G)$. It follows that $\overline{G}_f \subseteq Z(G)$, and $G/\overline{G}_f$ has a closed abelian subgroup of finite index. $G$ being connected, we obtain that $G/Z(G)$ is abelian. □

Lemma 2.9. Let $D$ be a discrete group such that $\gamma_D$ is of b.r.t. For $x \in D$ let $N(x)$ denote the greatest normal subgroup of $D$ contained in $C(x)$. Suppose that for some finite subset $F$ of $D$, $\bigcap_{x \in F} N(x) = Z(D)$. Then $D/Z(D)$ has an abelian subgroup of finite index.
Proof. Since \( \text{ind}_{C(x)}^D 1_{C(x)} \prec \gamma_D \) for each \( x \in D \), all these quasi-regular representations are of b.r.t. The kernel of \( \text{ind}_{C(x)}^D 1_{C(x)} \) is \( N(x) \) as is easily verified. Now, \( \text{ind}_{C(x)}^D 1_{C(x)} \) being of b.r.t. is equivalent to the algebra generated by the operators \( \text{ind}_{C(x)}^D 1_{C(x)}(y), y \in D \), on \( l^2(D/C(x)) \) satisfying a standard polynomial identity (see \([15]\) and \([21]\)).

Therefore, by Satz 1 of \([26]\), the factor group \( D/N(x) \), which is isomorphic to \( \text{ind}_{C(x)}^D 1_{C(x)}(D) \), has an abelian subgroup \( A(x)/N(x) \) of finite index. With \( A = \bigcap_{x \in F} A(x) \)

it follows that \( A \) has finite index in \( D \) and

\[
A' \subseteq \bigcap_{x \in F} A(x)' \subseteq \bigcap_{x \in F} N(x) = Z(D). \quad \Box
\]

Theorem 2.10. For a compactly generated Lie group \( G \) the following conditions are equivalent:

(i) \( C^*(\gamma_G) \) is of bounded representation type.

(ii) \( C^*(\gamma_{G_d}) \) is of bounded representation type.

(iii) \( C^*(\gamma_G \circ i_G) \) is of bounded representation type.

(iv) \( G/Z(G) \) possesses an abelian subgroup of finite index.

Proof. (iv) \( \Rightarrow \) (i), (ii), (iii) are clear since all three representations \( \gamma_G \), \( \gamma_{G_d} \), and \( \gamma_G \circ i_G \) are trivial on \( Z(G) = Z(G_d) \). (i) \( \Rightarrow \) (ii) and (i) \( \Rightarrow \) (iii) are consequences of Corollary 2.7.

Notice next that (iii) \( \Rightarrow \) (ii). In fact, if \( C^*(\gamma_G \circ i_G) \) is of b.r.t., then so is \( C^*(\lambda_{G_d/(G_d)_f}) \) by Lemma 1.1. This implies that \( G_d/(G_d)_f \) is almost abelian and hence \( G_d \) is amenable. Theorem 1.8 now shows that \( C^*(\gamma_{G_d}) \) is of b.r.t.

It remains to show (ii) \( \Rightarrow \) (iv). For that we want to apply Lemma 2.9. Thus we have to produce a finite subset \( F \) of \( G \) such that \( \bigcap_{x \in F} N(x) = Z(G) \).

To construct \( F \) let \( Z_0 = Z(G) \cap G_0 \) and notice that \( \gamma_G | G_0 \) separates the points of \( G_0/Z_0 \) and is of b.r.t. by Remarks 2.1 (i). Therefore \( G_0/Z_0 \) is a maximally almost periodic connected Lie group. It follows from the Freudenthal-Weil theorem (see \([5, \text{Théorème 16.4.6}]\)) that \( G_0/Z_0 \) is a direct product of a compact Lie group \( K \) and some \( \mathbb{R}^m \). Now, \( \gamma_{G_0/Z_0} \) is of b.r.t. and \( K \) is normal in \( G/Z_0 \). An application of Lemma 2.4 yields that \( K \) is 2-step nilpotent. As is well known this implies that \( K \), being a compact connected Lie group, is a torus \( \mathbb{T}^n \).

Let \( q : G \to G/G_0 \) and \( h : G \to G/Z_0 \) denote the quotient homomorphisms. Choose a finite subset \( F_1 \) of \( G \) such that \( q(F_1) \) generates \( G/G_0 \) as a group. Both \( \mathbb{R}^m \) and \( \mathbb{T}^n \) contain finitely generated dense subgroups. Thus there exist finite subsets \( F_2 \) and \( F_3 \) of \( G_0 \) such that \( h(F_2) \) and \( h(F_3) \) generate a dense subgroup of \( \mathbb{R}^m \) and \( \mathbb{T}^n \), respectively. Finally, let \( F = F_1 \cup F_2 \cup F_3 \). It is now obvious that \( F \cup Z_0 \) generates a dense subgroup of \( G \), whence \( C(F) = Z(G) \). This completes the proof. \( \Box \)

One might well expect that Theorem 2.10 remains true when the assumption that \( G \) be compactly generated is dropped. However, as mentioned earlier, we did not succeed in proving that if \( G \) is a (not necessarily finitely generated)
discrete group with $C^*(\gamma_G)$ of b.r.t., then $G/Z(G)$ must be almost abelian. This is surprising since we already know that $G/G$ is almost abelian (Remark 2.6). The point is that it seems to be difficult to handle discrete groups with finite conjugacy classes (so-called $[FC]$-groups). We finish the paper by looking at a special class of $[FC]$-groups.

**Example 2.11.** Let $G$ be the restricted direct product of finite groups $G_i$, $i \in I$. We claim that the following conditions are equivalent:

(i) $C^*(\gamma_G)$ is of bounded representation type.
(ii) $\dim \pi < \infty$ for every $\pi \in \text{supp} \gamma_G$.
(iii) $G_i$ is 2-step nilpotent for almost all $i \in I$.

Condition (iii) implies that $G/Z(G)$ is finite, whence (i) follows. To verify (ii) $\Rightarrow$ (iii), first consider a finite group $F$. Suppose that $\text{supp}(\sigma \otimes \sigma) \subseteq (F/F')^\sim$ for all $\sigma \in \hat{F}$. Then $\sigma \mid F'$ has to be a multiple of a $G$-invariant character for all $\sigma \in \hat{F}$, and this yields $F' \subseteq Z(F)$. Thus, if $F$ fails to be 2-step nilpotent, then for at least one $\sigma \in \hat{F}$, $\sigma \otimes \sigma$ has an irreducible subrepresentation of dimension $\geq 2$.

Now, suppose that $G_i$ is not 2-step nilpotent for all $i$ in some infinite subset $J$ of $I$. For each $i \in J$, choose $\sigma_i \in \hat{G}_i$ and $\tau_i \in \text{supp}(\sigma_i \otimes \sigma_i)$ with $\dim \tau_i \geq 2$. For $i \in I \setminus J$, let $\sigma_i = \tau_i = 1_{G_i}$. The infinite tensor products $\pi = \bigotimes_{i \in I} \sigma_i$ and $\rho = \bigotimes_{i \in I} \tau_i$ are irreducible $[9, \S 11]$. $\rho$ is infinite dimensional, and $\rho \in \text{supp}(\pi \otimes \pi)$. This contradicts (ii).

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**References**


Fachbereich Mathematik/Informatik, Universität Paderborn, D-33095 Paderborn, Germany
E-mail address: kaniuth@uni-paderborn.de