\textbf{π}_1^1 \text{ FUNCTIONS ARE ALMOST INTERNAL}

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\textbf{Abstract.} In Analytic mappings on hyperfinite sets [Proc. Amer. Math. Soc. 2 (1993), 587–596] Henson and Ross asked for what hyperfinite sets \( S \) and \( T \) does there exists a bijection \( f \) from \( S \) onto \( T \) whose graph is a projective subset of \( S \times T \)? In particular, when is there a \( \text{Π}_1^1 \) bijection from \( S \) onto \( T \)? In this paper we prove that given an internal, bounded measure \( \mu \), any \( \text{Π}_1^1 \) function is \( L(\mu) \) a.e. equal to an internal function, where \( L(\mu) \) is the Loeb measure associated with \( \mu \). It follows that if two \( \text{Π}_1^1 \) subsets \( S \) and \( T \) of a hyperfinite set \( X \) are \( \text{Π}_1^1 \) bijective, then \( S \) and \( T \) have the same measure for every uniformly distributed counting measure \( \mu \). When \( S \) and \( T \) are internal it turns out that any \( \text{Π}_1^1 \) bijection between them must already be Borel. We also prove that if a \( \text{Π}_1^1 \) graph in the product of two hyperfinite sets \( X \) and \( Y \) is universal for all internal subsets of \( Y \), then \( |X| \gtrsim 2|Y| \), which is a partial answer to Henson and Ross’s Problem 1.5. At the end we prove some standard results about the projections and a structure of co-proper \( K \)-analytic subsets of the product of two completely regular Hausdorff topological spaces with open vertical sections. We were able to prove the above results by revealing the structure of \( \text{Π}_1^1 \) subsets of the products \( X \times Y \) of two internal sets \( X \) and \( Y \), all of whose \( Y \)-sections are \( \Sigma_1^0(\kappa) \) sets.

1. Introduction

In a recent paper [3] C.W. Henson and D. Ross proved, using Choquet’s capacitability theorem, that, in the setting of Descriptive Set Theory of Hyperfinite Sets, any function \( f \) whose graph is a \( \Sigma_1^1 \) subset of the product \( X \times Y \) of two internal sets \( X \) and \( Y \) can be well approximated by internal functions. Given any internal, *-finitely additive, bounded measure \( \mu \) there exists an internal function \( \varphi \) such that \( f = \varphi \). \( L(\mu) \)-a.e., where \( L(\mu) \) is the Loeb measure associated with \( \mu \). This in turn shows that, for hyperfinite \( X \) and \( Y \) and \( \mu \) an internal counting measure, injective \( \Sigma_1^1 \) functions preserve \( L(\mu) \) measure (the image of a \( L(\mu) \) measurable set is a \( L(\mu) \) measurable set of the same measure). In particular, if two hyperfinite sets \( S \) and \( T \) are Borel bijective, then \( |S|/|T| \approx 1 \) and, conversely, any two hyperfinite sets \( S \) and \( T \) satisfying \( |S|/|T| \approx 1 \) are Borel bijective.

It is natural to ask if the approximation of \( f \) is still possible if the function \( f \) is a member of some level of the finite projective hierarchy (Problem 1.4. in

Received by the editors August 6, 1993 and, in revised form, May 20, 1994; originally communicated to the Proceedings of the AMS by Andreas R. Blass.

1991 Mathematics Subject Classification. Primary 03H04, 03E15; Secondary 04A15, 54H05, 54J05.

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other than $Z_j$. In particular, is it true that if $f$ is $\Pi^1_1$, then $f$ is almost equal to an internal function? In this paper we consider only the latter problem and, in fact, ask and answer a slightly more general question than the one stated above. We prove that any $\Pi^1_1$ graph with internal vertical sections can be well approximated by internal graphs and any $\Pi^1_1$ graph with $\Sigma^1_1(\kappa)$ vertical sections can be well approximated by $\Sigma^0_\kappa$ graphs (for definitions see below). In fact, as we prove in Theorem 1, any such graph can be represented as a union of countably many restrictions of internal graphs to (not necessarily disjoint) $\Pi^1_1$ sets, possesses a $\Pi^1_1$ uniformization, and has a $\Pi^1_1$ domain. In particular, any $\Pi^1_1$ functions is a union of restrictions of internal functions to disjoint $\Pi^1_1$ sets. It follows that if a $\Pi^1_1$ function has internal (or more generally $\Sigma^1_1$) domain, then it must be Borel (the fact already proved in [5], Corollary 4.5 b)).

We also give a partial answer to Problem 1.5 in [3] which asked for the connection between the internal cardinalities of two hyperfinite sets $X$ and $Y$ for which there exists a $\Sigma^1_1$ graph $\Gamma \subseteq X \times Y$ universal for all $\Sigma^1_1$ subsets of $Y$. We show (Corollary 6) that if $\Gamma$ is universal for all internal subsets of $Y$, then the quotient $|X|/2^{|Y|}$ is greater than or infinitely close to 1.

At the end of the paper we apply the above results in the standard descriptive set theory of general topological spaces. We prove that a co-\(K\)-analytic graph $G$ in the product of a proper $K$-Lusin space $X$ and a proper $K$-Lusin and locally compact space $Y$, all of whose $Y$-sections are open sets in $Y$, has a co-$K$-analytic domain and is almost equal (with respect to a given Radon bounded measure $\mu$ in $X$) to a $\sigma$-compact graph in $X \times Y$ (for definitions see below). Also, the set of all points in $X$ where the vertical section of $G$ is compact and nonempty is always $\mu$ measurable.

Acknowledgments. I would like to thank Professor Ward Henson and Professor David Ross who made the early preprint of their paper [3] available to me long before the article was published. Also, I would like to thank Hermann Render for sharing his recent results with me and to the referee for a thorough reading of the paper.

2. Definitions and terminology

By a Borel set in an internal set $X$ we mean a member of the least $\sigma$-algebra generated by internal subsets of $X$. The Borel hierarchy of $\Sigma^0_\alpha$ and $\Pi^0_\alpha$ ($\alpha < \omega_1$) subsets of $X$ is introduced as usual; internal subsets of $X$ are called $\Sigma^0_\alpha$ or $\Pi^0_\alpha$ sets, and a set is $\Pi^0_\alpha$ ($\Sigma^0_\alpha$) if it is a countable intersection (union) of sets $A_n$ with each $A_n$ being a $\Sigma^0_\beta$ ($\Pi^0_\beta$) set for some $\beta < \alpha$.

Given an infinite cardinal $\kappa$ we can define a class of $\kappa$-Borel sets using the cardinal $\kappa$ in the same manner as we defined the class of Borel sets using the cardinal $\omega$. The countable unions and the countable intersections are now going to be replaced by unions and intersections of length $\kappa$. However, only the sets $\Sigma^0_\kappa$ and $\Pi^0_\kappa$ from the first level of the $\kappa$-Borel hierarchy are going to be useful in the sequel. Thus, a set is $\Sigma^0_\kappa$ if it can be obtained as a union of $\kappa$ many internal sets and is $\Pi^0_\kappa$ if it is an intersection of $\kappa$ many internal sets.

The finite projective hierarchy of $\Sigma^1_1$ and $\Pi^1_1$ sets is defined as follows. Projections of Borel sets are called $\Sigma^1_1$ sets, and complements of $\Sigma^1_1$ sets are called $\Pi^1_1$ sets. Inductively, a $\Sigma^m_{m+1}$ set is the projection of a $\Pi^m_1$ set and complements
of $\Sigma_{m+1}$ sets are $\Pi_{m+1}$ sets. In this paper we are mostly concerned with the first level of the finite projective hierarchy. It is known that every $\Sigma_1^1$ set which is at the same time $\Pi_1^1$ must already be Borel (Henson). In particular, every element of a sequence of disjoint $\Pi_1^1$ sets whose union is $\Sigma_1^1$ must already be a Borel set. For a detailed treatment of Descriptive Set Theory of Hyperfinite Sets the reader is referred to [5].

By a graph we mean a subset of the product of two sets $X$ and $Y$. If these sets are internal, then a graph is Borel ($\Pi_1^1$ or $\Sigma_1^1$) if it is such as a subset of the product of $X$ and $Y$. Given a graph $\Gamma \subseteq X \times Y$ and a point $x \in X$ we define the Y-section $\Gamma(x)$ of $\Gamma$ at $x$ as the set $\{y \in Y : (x, y) \in \Gamma\}$. The domain $\text{dom}(\Gamma)$ of $\Gamma$ is defined as the set of all $x$ for which $\Gamma(x)$ is nonempty. Given two graphs $\Gamma$ and $\Delta$ in $X \times Y$ and a measure $\mu$ in $X$ we say that $\Gamma$ and $\Delta$ are a.e. equal with respect to $\mu$ if the set $\{x \in X : \Gamma(x) \neq \Delta(x)\}$ has $\mu$ measure zero. If, in addition, $\Delta$ is an internal (e.g., $\Sigma_1^1$, Borel) graph, then we shall call $\Delta$ an internal (e.g., $\Sigma_1^1$, Borel) lifting of $\Gamma$. (One should stress at this point that the notion of lifting defined above is different from what is usually called a lifting in Nonstandard Analysis. I have decided to keep the same name here, despite a risk of possible confusion, in order to be consistent with the definitions in [9] where this notion of lifting was originally introduced.) The restriction $\Gamma|A$ of a graph $\Gamma$ to a set $A$ is the graph $\Gamma \cap (A \times Y)$.

A uniformization of a graph $\Gamma$ is a partial function $f$ whose graph is a subset of $\Gamma$ and with the domain equal to the domain of $\Gamma$. 

To every graph $\Gamma \subseteq X \times Y$ we can associate a partial set function $f_\Gamma$, which maps $X$ into the internal power set of $Y$, by defining $f_\Gamma(x) = \Gamma(x)$, whenever $\Gamma(x)$ is internal and $\neq \emptyset$. Note that $f_\Gamma$ is defined even if not all of the $Y$-sections of $\Gamma$ are internal. If $Y$ is hyperfinite, then the upper and the lower edges of $\Gamma$ are defined as the compositions of the function $f_\Gamma$ with the functions $\min$ and $\max$ respectively.

We assume that the reader is familiar with basic notions from Nonstandard Analysis and Descriptive Set Theory of Hyperfinite Sets [4], [5]. In all the statements our nonstandard universe is at least $\omega_1$-saturated.

3. Results

Now we state and prove our main theorem which reveals the properties of $\Pi_1^1$ graphs, all of whose $Y$-sections are $\Sigma_0^1(\kappa)$ sets. The proof, in fact, uses an argument already exploited in [12, Proposition 2.4], where the structure of Borel graphs with $\Pi_0^1(\kappa)$ (or $\Sigma_0^1(\kappa)$) $Y$-sections was exhibited.

**Theorem 1.** Let $X$ and $Y$ be internal sets, and let $\Gamma \subseteq X \times Y$ be a $\Pi_1^1$ graph, all of whose $Y$-sections are $\Sigma_0^1(\kappa)$ sets. Then the following are true.

(a) $\Gamma$ can be represented as a countable union of restrictions of internal graphs to $\Pi_1^1$ sets.

(b) The domain of $\Gamma$ is a $\Pi_1^1$ set.

(c) The sets $C = \{x \in X : \Gamma(x) \neq \emptyset$ and noninternal\}$ and $D = \{x \in X : \Gamma(x) \neq \emptyset$ and internal\}$ are in the least $\sigma$-algebra generated by $\Sigma_1^1$ and $\Pi_1^1$ sets.

(d) The graph of the set function associated with $\Gamma$ is an intersection of a $\Sigma_1^1$ graph and a $\Pi_1^1$ graph. Thus, both the graph of the upper edge of $\Gamma$ and the graph of the lower edge of $\Gamma$ are intersections of a $\Sigma_1^1$ and a $\Pi_1^1$ graph.
(e) Given an internal, bounded measure $\mu$, $\Gamma$ possesses a $\Sigma_1^0$, $L(\mu)$ lifting. Also, keeping the notation from (c), the graph $\Gamma|D$ possesses an internal, $L(\mu)$ lifting. In particular, any $\Pi_1^1$ graph, all of whose Y-sections are internal sets, possesses an internal lifting and any $\Pi_1^1$ function is a.e. equal to an internal function.

(f) $\Gamma$ possesses a $\Pi_1^1$ uniformization which can be represented as a countable union of restrictions of internal functions to disjoint $\Pi_1^1$ sets. In particular, every $\Pi_1^1$ function is a countable union of restrictions of internal functions to disjoint $\Pi_1^1$ sets. Also, if the domain of $\Gamma$ is $\Sigma_1^0$, then $\Gamma$ possesses a Borel uniformization. Therefore, any $\Pi_1^1$ function with $\Sigma_1^0$ domain must be Borel.

Proof. (a) The proof of this part is essentially contained in the proof of Proposition 2.4. in [12]. It has been proved (Proposition 2.6, [12]) that, under the above assumptions, there exists a countable sequence $\Gamma_n (n \in \omega)$ of internal graphs in $X \times Y$ such that for every $x$ the $Y$-section $\Gamma_n(x)$ is a union of sets of the form $\Gamma_n(x)$. After that it is easy to obtain the desired representation of $\Gamma$.

Let $P_n = \{x \in X : \Gamma_n(x) \neq \emptyset \land \Gamma_n(x) \subseteq \Gamma(x)\}$. It is clear that $P_n$ is a $\Sigma_1^0$ set as it is defined by the conjunction of an internal property $\Gamma_n(x) \neq \emptyset$ and by a $\Pi_1^1$ formula $(\forall y)[(x, y) \in \Gamma_n \rightarrow (x, y) \in \Gamma]$. We have

$$\Gamma = \bigcup_{n \in \omega} \Gamma_n \cap P_n.$$  

(b) From the above representation we see that the domain of $\Gamma$ is a countable union of $\Pi_1^1$ sets $P_n$ and, therefore, is a $\Pi_1^1$ set.

(c) Let $\{\Gamma_n : n \in \omega\}$ be as in the proof of part (a). We can assume that $\{\Gamma_n : n \in \omega\}$ is closed with respect to taking finite unions and intersections. Therefore, $\Gamma(x)$ is going to be nonempty and noninternal if and only if it can be represented as a strictly increasing union of sets of the form $\Gamma_n(x)$. Therefore, $C = \{x \in X : \Gamma(x) \neq \emptyset$ and noninternal} is defined by the formula

$$(\exists n \in \omega)(\Gamma_n(x) \subseteq \Gamma(x) \land \Gamma_n(x) \neq \emptyset)$$

$$\land (\forall k \in \omega)(\exists p \in \omega)[p > k \land [\Gamma_k(x) \subseteq \Gamma(x)$$

$$\rightarrow ([\Gamma_k(x) \subseteq \Gamma_P(x) \land \Gamma_P(x) \subseteq \Gamma(x)])].$$

Here $\Gamma_k(x) \subseteq \Gamma_P(x)$ is an internal property expressing strict inclusion. The first formula defines a $\Pi_1^1$ set. The second formula defines a set which is obtained by taking a countable union and a countable intersection of sets defined by the formula $p > k \land [\Gamma_k(x) \subseteq \Gamma(x) \rightarrow ([\Gamma_k(x) \subseteq \Gamma_P(x) \land \Gamma_P(x) \subseteq \Gamma(x)])].$ The latter defines a union of a $\Sigma_1^0$ set and a $\Pi_1^1$ set. So $D$ is in the $\sigma$-algebra generated by $\Sigma_1^0$ and $\Pi_1^1$ sets.

The set $D = \{x \in X : \Gamma(x)$ nonempty and internal} is now the intersection of the domain of $\Gamma$, a $\Pi_1^1$ set, and the complement of $C$. So $D$ is also in the $\sigma$-algebra generated by $\Sigma_1^0$ and a $\Pi_1^1$ set.

(d) Let $f_n$ be the set function associated with the graph $\Gamma_n$. The graph of the set function $f_{\Gamma}$ associated with $\Gamma$ is defined as: $(x, A) \in f_{\Gamma}$ if and only if

$$(\exists n \in \omega)[(x, A) \in f_n \cap P_n] \land (\forall k \in \omega)(x \in P_k \rightarrow f_k(x) \subseteq A).$$

The first conjunct is $\Pi_1^1$ and the second is $\Sigma_1^0$. Taken together they define a function whose graph is an intersection of a $\Pi_1^1$ graph and a $\Sigma_1^0$ graph.
Suppose that $Y$ is hyperfinite. The upper edge of $\Gamma$ is defined as the composition of the function $\max$ and the function $f_\Gamma$. Following its definition above the graph $h$ of the upper edge of $\Gamma$ is defined as $(x, y) \in h$ if and only if

$$(\exists n \in \omega)[(x, y) \in \max (f_n | P_n)] \land (\forall k \in \omega)(x \in P_k \rightarrow \max(f_k(x)) \leq y)].$$

It is easy to see that the composition of an internal and a $\Pi^1_1$ function is again $\Pi^1_1$. Thus, the first conjunct above is $\Pi^1_1$. The second is obviously $\Sigma^1_1$, so the graph of $h$ is an intersection of a $\Sigma^1_1$ graph and a $\Pi^1_1$ graph.

We treat the lower edge similarly.

(e) Let $\mu$ be an internal, bounded measure in $X$. As $\Pi^1_1$ sets are always measurable and since $\beta$ is a bounded measure, for every $P_n$ there exists a $\Sigma^0_1$ set $S_n \subseteq P_n$ with $L(\mu)(P_n \setminus S_n) = 0$. Let $H = \bigcup_{n \in \omega} \Gamma_n | S_n$. It is easy to see that $H = \Gamma$, $L(\mu)$-a.e. Indeed, if $H(x) \neq \Gamma(x)$, then for some $n$ we must have $x \in P_n \setminus S_n$. So $\{x \in X : H(x) \neq \Gamma(x)\} \subseteq \bigcup_{n \in \omega} P_n \setminus S_n$ and the latter set is of measure zero. Therefore, $H$ is a $\Sigma^0_1$ lifting of $\Gamma$.

To get an internal lifting of $\Gamma \upharpoonright D$ notice that $D$ is $L(\mu)$ measurable. We can find a $\Sigma^0_1$ subset $S$ of $D$ with $L(\mu)(D \setminus S) = 0$ and with the graph $H \upharpoonright S$ having all of its $Y$-sections internal sets. Using $\omega_1$-saturation we can then easily find an internal lifting of $H \upharpoonright S$ and thus of $\Gamma \upharpoonright D$.

(f) Using Reduction Theorem for $\Pi^1_1$ sets (see [5] for example), there exists a sequence $R_n \subseteq P_n$ of disjoint $\Pi^1_1$ sets with $\bigcup_{n \in \omega} R_n = \bigcup_{n \in \omega} P_n$. Now, we can take any internal uniformization $\gamma_n$ of $\Gamma_n$ and observe that $\bigcup_{n \in \omega} \gamma_n | R_n$ is a $\Pi^1_1$ uniformization of $\Gamma$ of desired form. In addition, if the domain of $\Gamma$ is $\Sigma^1_1$, then every $R_n$ must be Borel. □

Remark 2. Suppose that $\Gamma$ is a $\Pi^1_1$ graph in $X \times Y$. The proof of (a) of Theorem 1 then shows that if there exists a sequence of Borel graphs $\Gamma_n$ ($n \in \omega$) with internal $Y$-sections such that for every $x$ $\Gamma_n(x)$ contains $\Gamma_n(x)$ for some $n$, with $\Gamma_n(x) \neq \emptyset$, then the domain of $\Gamma$ must be $\Pi^1_1$. Indeed, if we define the sets $P_n$ as in the proof of Theorem 1 (a), then $P_n$ are still going to be $\Pi^1_1$. It is known (see [9] or [12]) that a Borel graph with internal $Y$-sections must have a Borel domain. Therefore, the property $\Gamma_n(x) \neq \emptyset$ in the definition of $P_n$ is Borel and, thus, $P_n$ is $\Pi^1_1$. By our assumption, $\text{dom}(\Gamma) = \bigcup_{n \in \omega} P_n$, so the domain of $\Gamma$ is $\Pi^1_1$.

So, by Theorem 1(f), two hyperfinite sets cannot be $\Pi^1_1$ bijective unless that bijection is already Borel. It would be interesting to give a sufficient and necessary condition for two true (i.e., non-Borel) $\Pi^1_1$ subsets $P$ and $Q$ of a hyperfinite set $X$ that ensure the existence of a $\Pi^1_1$ bijection between them. For Borel $P$ and $Q$ it was shown in [8] that $P$ and $Q$ are Borel bijective if and only if $P$ and $Q$ have the same nonvanishing $L(\mu)$ measure for some counting, uniformly distributed internal measure $\mu$. (Here, the word nonvanishing means that the measure of $P$ and $Q$ are not 0 or $\infty$.)

It is not true in general that a $\Pi^1_1$ graph with internal $Y$-sections can be represented as a countable union of restrictions of internal graph to disjoint $\Pi^1_1$ sets, i.e. our representation (1) in Theorem 1. is best possible. The key fact is that any graph that allows such a representation and has a $\Sigma^1_1$ domain must already be Borel because each $P_n$ is, in this case, Borel.
Example 3. Let $X$ be an internal set, $Y = \{1, 2\}$, and $P \subseteq X$ be a true $\Pi_1^0$ set (i.e., a $\Pi_1^0$ set which is not Borel). Let $f$ be a function defined to be $1$ on $P$ and undefined otherwise. Let $g$ be defined identically to be $2$ on all of $X$. Define $\Gamma$ to be the union of the graphs of $f$ and $g$. We show that $\Gamma$ is a $\Pi_1^0$ graph with internal $Y$-sections which cannot be represented as a countable union of restrictions of internal graphs to disjoint $\Pi_1^0$ sets. In addition, the set function associated with $\Gamma$ as well as the lower edge of $\Gamma$ are not $\Pi_1^0$.

It is clear that $\Gamma$ is a $\Pi_1^0$ graph with all of its $Y$-sections internal sets. First note that the domain of $\Gamma$ is Borel (in fact, internal). Now if $\Gamma$ is a countable union of restrictions of internal graphs to disjoint $\Pi_1^0$ sets $P_n$, then each $P_n$ would be Borel. Thus, $\Gamma$ would be a Borel graph. This in turn means that the graph of $f$ would also be Borel and therefore $P$ is Borel. This is a contradiction.

If the set function $f_\Gamma$ is $\Pi_1^0$, then it must be Borel because it has a Borel domain. It follows that $\Gamma$ is Borel because $(x, y) \in \Gamma$ if and only if $(\exists A)((x, A) \in f_\Gamma \wedge y \in A))$ (a $\Sigma_1^1$ formula) and $(x, y) \notin \Gamma$ if and only if $(\exists A)((x, A) \notin f_\Gamma \wedge y \in A))$ (again a $\Sigma_1^1$ formula).

Similarly, if the lower edge of $\Gamma$ is $\Pi_1^0$, then it must be Borel and again $P$ would be Borel.

An easy variation of the above example also shows that, in addition, the answer to the open question 1 in [9] is negative. The question, motivated by a similar result for $\Sigma_1^1$ functions, asked if every $\Sigma_1^1$ graph in $X \times Y$, all of whose $Y$-sections are internal sets, possesses a complete Borel extension. That is, does there exist a Borel graph $\Delta \supseteq \Gamma$ with $\text{dom}(\Delta) = X$ such that all of the $Y$-sections of $\Delta$ are internal and for every $x \in \text{dom}(\Gamma)$, $\Delta(x) = \Gamma(x)$? If this were true for every $\Gamma$, then the set function associated with $\Gamma$ would be $\Sigma_1^1$ because it is equal to the restriction of the set function $f_\Delta$ (which was proved to be Borel in [9]) to the domain of $\Gamma$. However, if in the above example we take $P$ to be a true $\Sigma_1^0$ set instead of a true $\Pi_1^1$, then $f_\Gamma$ would be Borel as its domain is Borel (a $\Sigma_1^1$ function with a Borel domain must be Borel). We conclude that $\Gamma$ is Borel and obtain a contradiction as above.

Remark 4. Any $\Pi_1^1$ subset $P$ of an internal set $X$ can be generically expressed as the union of an increasing sequence $B_\alpha$ ($\alpha < \omega_1$) of Borel sets such that every $\Sigma_1^1$ subset $S \subseteq P$ is contained in some $B_\beta$, $\beta < \omega_1$ (see [11]). The sets $B_\alpha$ are called constituents of $P$. Now when we know that the domain of any $\Pi_1^1$ function $f$ is $\Pi_1^1$ we can ask for the connection between the constituents of $f$ and the constituents of $\text{dom}(f)$. In fact, it is easy to see that if $f_\alpha$ ($\alpha < \omega_1$) are constituents of $f$ then $\text{dom}(f_\alpha)$, ($\alpha < \omega_1$) are constituents of $\text{dom}(f)$. Indeed, $\text{dom}(f_\alpha)$ are Borel because the domains of Borel functions are always Borel. Given a $\Sigma_1^1$ subset $S$ of $\text{dom}(f)$ the function $f|S$ must Borel (Theorem 1(f)) and, thus, is a subset of some $f_\beta$. It follows that $S \subseteq \text{dom}(f_\beta)$. Conversely, suppose that $B_\alpha$ ($\alpha < \omega_1$) are constituents of $\text{dom}(f)$. Then, again $f_\alpha = f|B_\alpha$ are Borel functions and, as one can easily see, they are constituents of $f$.

The following corollary is obtained by passing to complements.

Corollary 5. Let $\Gamma$ be a $\Sigma_1^1$ graph in $X \times Y$. Suppose that all of the $Y$-sections of $\Gamma$ are $\Pi_1^0(\kappa)$ sets. Then the following hold.
(a) $\Gamma$ can be represented as a monotone, countable intersection of the form

$$\Gamma = \bigcap_{n \in \omega} (\Gamma_n^\cap | S_n^\cap \cup \ldots \cup \Gamma_n^{S_n^\cap})$$

where $\Gamma_n^\cap, \ldots, \Gamma_k^\cap$ are internal graphs and $S_n^\cap, \ldots, S_k^\cap$ are sets.

(b) The sets $C = \{x \in X : \Gamma(x) \neq \varnothing \text{ and noninternal}\}$ and $D = \{x \in X : \Gamma(x) \neq \varnothing \text{ and internal}\}$ are in the least $\sigma$-algebra generated by $\Sigma_1^\cap$ and $\Pi_1^\cap$ sets.

(c) For every bounded internal measure $\mu$, $\Gamma$ has $\Pi_1^0, \Lambda(\mu)$ lifting and $\Gamma|D$ has an internal lifting.

(d) The graph of the set function associated with $\Gamma$ is an intersection of a $\Sigma_1^\cap$ graph and a $\Pi_1^\cap$ graph.

Proof. We can assume that $\Gamma(x) \neq Y$, by enlarging $Y$ if necessary.

(a) The complement $\Gamma^c$ of $\Gamma$ can be represented as (1). In other words

$$\Gamma^c = \bigcup_{n \in \omega} \Gamma_n^c | P_n$$

for some internal graphs $\Gamma_n$ and some $\Pi_1^\cap$ sets $P_n$. Therefore, $\Gamma$ can be expressed as

$$\Gamma = \bigcap_{n \in \omega} (\Gamma_n \cap (P_n \times Y))^c = \bigcap_{n \in \omega} (\Gamma_n^c \cup (P_n^c \times Y)).$$

Now, the finite intersection of sets of the form $\Gamma_n^c \cup (P_n^c \times Y)$ is easily seen to be a finite union of restrictions of internal graphs to $\Sigma_1^\cap$ sets and the representation (2) follows.

Part (b) follows again by looking at the complement of $\Gamma$. Part (c) follows from the above fact that $\Gamma$ can be represented as a countable intersection of finite unions of graphs, each of which is a restriction of an internal graph to a $\Sigma_1^\cap$ set. The latter graphs obviously have internal liftings so that, at the end, $\Gamma$ has a $\Pi_1^0$ lifting. The same lifting can be used to obtain an internal lifting of $\Gamma|D$ in an obvious manner.

(d) Let $P_n$ and $f_n$ be as in the proof of parts (a) and (d) of Theorem 1 applied to the complement of $\Gamma$. Then the graph of $f_\Gamma$ can be defined as $(x, A) \in f_\Gamma$ if and only if

$$x \in \text{dom}(\Gamma) \land (\forall y)((x, y) \in \Gamma \rightarrow y \in A) \land (\forall n \in \omega)[x \in P_n \rightarrow A \cap f_n(x) = \varnothing].$$

The property $x \in \text{dom}(\Gamma)$ is $\Sigma_1^\cap$ and ensures that $\Gamma(x)$ is not empty. The formula $(\forall y)((x, y) \in \Gamma \rightarrow y \in A)$ asserts that $\Gamma(x) \subseteq A$ and is $\Pi_1^\cap$. Finally, $(\forall n \in \omega)[x \in P_n \rightarrow A \cap \varphi_n(x) = \varnothing]$ is a $\Sigma_1^\cap$ formula expressing $A \subseteq \Gamma(x)$.

It is natural to ask if the structural results of Corollary 5 can be strengthened in the case when all of the $Y$-sections of $\Gamma$ are internal rather than $\Pi_1^0(\kappa)$ sets. For example, is it true that a $\Sigma_1^\cap$ graph $\Gamma$, all of whose $Y$-sections are internal sets, has a representation (1) with $P_n$ being $\Sigma_1^\cap$ sets? It is known that any $\Sigma_1^\cap$ function $f$ can be extended to a Borel function [10] and that any Borel function is a countable union of restrictions of internal functions to Borel sets [9]. It then follows, by intersecting with the domain of $f$, that any $\Sigma_1^\cap$ function is a countable union of restrictions of internal functions to disjoint $\Sigma_1^\cap$ sets. The same is true for $\Sigma_1^\cap$ graphs, all of whose $Y$-sections are of cardinality $\leq n$ for a
fixed, finite integer \( n \) or, more generally, when the set function associated with \( \Gamma \) is itself \( \Sigma_1^1 \).

In the same way, does a \( \Sigma_1^1 \) graph \( \Gamma \subseteq X \times Y \), all of whose \( Y \) sections are internal \( (\Pi_0^0(\kappa)) \) sets, possess a complete \( \Sigma_1^1 \) extension? In other words, does there exist a \( \Sigma_1^1 \) graph \( \Delta \supseteq \Gamma \), all of whose \( Y \) sections are internal \( (\Pi_0^0(\kappa)) \) sets, such that \( \text{dom}(\Delta) = X \) and \( \Delta(x) = \Gamma(x) \) for every \( x \in \text{dom}(\Gamma) \)? We know that the answer is positive, in one special case, when \( \Gamma \) is a finite union of restrictions of internal \( (\Pi_0^0(\kappa)) \) graphs to \( \Sigma_1^1 \) sets.

Next, we give a partial answer to Problem 1.5 in [3]. A graph \( \Gamma \subseteq X \times Y \) is said to be universal for internal subsets of \( Y \) if every internal subset of \( Y \) can be obtained as a \( Y \) section of \( \Gamma \). The internal power set of \( Y \) is denoted as \( 2^Y \).

**Corollary 6.** Let \( X \) and \( Y \) be hyperfinite sets, and let \( \Gamma \) be a \( \Pi_1^1 \) \( (\Sigma_1^1) \) graph in \( X \times Y \) universal for all internal subsets of \( Y \). Then \( |X|/2^{|Y|} \geq \delta \) or \( \approx 1 \).

**Proof.** We can only concentrate on \( \Pi_1^1 \) graphs because if \( \Gamma \) is universal for internal subsets of \( Y \), then its complement is also universal for internal subsets of \( Y \).

We first prove that there exists a \( \Pi_1^1 \) graph \( H \subseteq \Gamma \) universal for internal subsets of \( Y \) and with all of its \( Y \) sections \( \Sigma_1^0 \) sets. As in the proof of Theorem 1, there exists a sequence \( \Gamma_n \) \( (n \in \omega) \) of internal graphs in \( X \times Y \) such that for every \( x \) if \( \Gamma(x) \) is internal, then there exists \( n \) such that \( \Gamma(x) = \Gamma_n(x) \).

Let \( P_n = \{ x \in X : \Gamma_n(x) \subseteq \Gamma(x) \} \). Note that \( P_n \) are all \( \Pi_1^1 \) sets. Now, define \( H = \bigcup_{n\in\omega} \Gamma_n \). It is clear that \( H \) is a \( \Pi_1^1 \) graph with \( \Sigma_1^0 \) \( Y \)-sections, universal for internal subsets of \( Y \).

Define an internal measure \( \mu \) by setting \( \mu(A) = |A|/2^{|Y|} \) for every hyperfinite \( A \). If \( X \) has unbounded \( \mu \) measure, then we are done. If the measure of \( X \) is finite, then we proceed as follows. Let \( D = \{ x \in X : H(x) \text{ internal and } \neq \emptyset \} \). Then \( D \) is \( L(\mu) \) measurable and \( H(D) \) has an internal lifting \( G \). Let \( f_n \) be the set functions associated with \( \Gamma_n|P_n \), and let \( f_G \) be the set function associated with \( G \). We prove that the range of \( f_G \) is equal almost to all of the set \( 2^Y \). Indeed, if a set \( B \) is not in the range of \( f_G \), then it must be of the form \( f_n(x) \) for some \( n \) and some \( x \in D \Delta \text{dom}(G) \). But \( f_n \) maps measure-zero sets onto measure zero sets, being a restriction of an internal function to a measurable set. Therefore, the complement of the range of \( f_G \) in \( 2^Y \) is of \( L(\mu) \) measure zero. Now we are done because \( L(\mu)(2^Y) = 1 = L(\mu)(\text{range}(f_G)) \leq L(\mu)(X) \approx |X|/2^{|Y|} \).

At the end we give some application of Theorem 1 and its corollaries to the standard Descriptive Set Theory. We call a set in arbitrary topological space \( X \) a \( F_\sigma \) set if it is a countable union of closed sets in \( X \). The standard part map associated with the topology in \( X \) is denoted as \( \text{st}_X \), whereas the standard part map in the product \( X \times Y \) of two spaces \( X \) and \( Y \) is denoted simply as \( \text{st} \), if there is no danger of confusion. The set of near standard elements is denoted as \( \text{n.s.}(X) \). Given a Borel measure \( \mu \) in \( X \) we say that the standard part map \( \text{st}_X : \text{n.s.}(X) \rightarrow X \) is measure preserving if for every set \( M \subseteq X \), we have \( \mu(\text{st}_X^{-1}(M)) = L(\mu)(\text{range}(f_G)) \leq L(\mu)(X) \approx |X|/2^{|Y|} \).
Theorem 7. Let $X$ and $Y$ be Hausdorff topological spaces, and let $G \subseteq X \times Y$ be a graph such that

(i) $\text{st}^{-1}(G)$ is a $\Pi_1^1$ subset of $\mathcal{X} \times \mathcal{Y}$

in some $\kappa^+$-saturated nonstandard universe, where $\kappa$ is greater than the maximum of the cardinalities of the basis in $X$ and $Y$. Suppose that

(ii) $\text{st}^{-1}_y(G(x))$ is $\Sigma_1^0(\kappa)$ for every $x$.

Suppose that $\mu$ is a complete, Borel probability measure in $X$ such that

(iii) $\text{st}_x : \text{n.s.}(\mathcal{X}) \to X$ is measure preserving.

Then, the following hold.

(a) $\text{st}^{-1}_x(\text{dom}(G))$ is a $\Pi_1^1$ set.
(b) There exists an $F_\sigma$ graph $H$ in $X \times Y$ such that $H = G$, $\mu$-a.e. If $X$ and $Y$ are regular, then $H$ can be chosen to be $\sigma$-compact.
(c) If $Y$ is regular, then the set $C = \{x \in X : G(x)$ compact and nonempty$\}$ is $\mu$ measurable and $G|C$ is a.e. equal to a compact graph.

Proof. Define $\Gamma = \text{st}^{-1}(G)$. It is easy to see that all of the $Y$-sections of $\Gamma$ are $\Sigma_1^0(\kappa)$. Indeed, let $\alpha \in \mathcal{X}$ be in the domain of $\Gamma$. Suppose that $\text{st}^{-1}_y(\alpha) = x$. We have $\Gamma(\alpha) = [m(x) \times \text{st}^{-1}_y(G(x))] \cap (\alpha \times \mathcal{Y}) = \text{st}^{-1}_y(G(x))$. This is a $\Sigma_1^0(\kappa)$ set.

(a) By Theorem 1 (b), the domain of $\Gamma$ is $\Pi_1^1$. Now, we always have $\text{st}^{-1}_x(\text{dom}(G))$ equal to $\text{dom}(\text{st}^{-1}_x(G))$ which is, in turn, equal to the domain of $\Gamma$ and we are done.

(b) By Theorem 1 (c) there exists a $\Sigma_1^0$, $L(\mu)$ lifting $F \subseteq \Gamma$ of $\Gamma$. As the standard part image of an internal set in any topological space is closed, $\text{st}(F)$ is a countable union of closed sets in $X \times Y$ $\mu$-a.e. equal to $G$. Also, if $X$ and $Y$ are regular, then its product is regular and in any regular topological space the standard part image of any internal set all of whose points are near standard is compact.

(c) We prove that

$$\{\alpha \in \mathcal{X} : \Gamma(\alpha)$ internal and $\neq \emptyset\} = \text{st}^{-1}(\{x \in X : G(x) compact$ and $\neq \emptyset\})$$

Indeed, let $\Gamma(\alpha)$ be internal and nonempty. As every point of $\Gamma(\alpha)$ is near standard and $Y$ regular, $\text{st}_y(\Gamma(\alpha))$ is compact. Also, $\text{st}_y(\Gamma(\alpha)) = G(x)$, where $x = \text{st}_x(\alpha)$. We showed that $\alpha \in \text{st}^{-1}(\{x \in X : G(x) compact$ and $\neq \emptyset\})$.

Conversely, suppose that $\alpha \in \text{st}^{-1}(\{x \in X : G(x) compact$ and $\neq \emptyset\})$. Let $x$ be such that $\text{st}_x(\alpha) = x$ and $G(x)$ compact. Now, $\Gamma(\alpha) = \text{st}^{-1}_y(G(x))$ is a $\Pi_1^0(\kappa)$ set because $G(x)$ is compact. On the other hand, by our assumption, $\text{st}^{-1}_y(G(x))$ is $\Sigma_1^0(\kappa)$. By $\kappa^+$ saturation $\text{st}^{-1}_y(G(x))$ must be internal. This establishes the above equality.

Now, the set $\{\alpha \in \mathcal{X} : \Gamma(\alpha)$ internal and $\neq \emptyset\}$ is a member of the least $\sigma$-algebra containing $\Pi_1^1$ and $\Sigma_1^1$ sets and therefore is $L(\mu)$-measurable. As $\text{st}$ is $\mu$-measure-preserving the set $\text{st}^{-1}(\{x \in X : G(x) compact$ and $\neq \emptyset\})$ is $\mu$ measurable.

To prove the second part of (c) notice that by Theorem 1 (c) $\Gamma(\alpha)$ internal and $\neq \emptyset$ has an internal lifting. The standard part image of that lifting is a compact lifting of $G|C$. \hspace{1cm} \Box

In the same vein we prove the following extension of Proposition 3.9 from [12].
Theorem 8. Let $X$ and $Y$ be Hausdorff topological spaces, and let $G \subseteq X \times Y$ be a graph such that

(i) $\text{st}^{-1}(G)$ is a $\Sigma^1_1$ subset of $^*X \times ^*Y$

in some $\kappa^+$-saturated nonstandard universe, where $\kappa$ is greater than the maximum of the cardinalities of the basis in $X$ and $Y$. Suppose that

(ii) $\text{st}^{-1}(G(x))$ is $\Pi^0_0(\kappa)$ for every $x$.

Suppose that $\mu$ is a complete, Borel probability measure in $X$ such that

(iii) $\text{st}^{-1}_X : \text{n.s.}(^*X) \to X$ is measure-preserving.

Then, for every $\epsilon > 0$ there exists a closed graph $H_\epsilon$ in $X \times Y$ such that $H_\epsilon \subseteq G$; $\text{dom}(H_\epsilon)$ is $\mu$ measurable; $\mu(\text{dom}(H_\epsilon)) \geq \text{dom}(G) - \epsilon$, and for all $x \in \text{dom}(H_\epsilon)$, $H_\epsilon(x) = G(x)$.

If $X$ and $Y$ are regular, then $H_\epsilon$ can be chosen to be compact.

Proof. The proof follows the same lines as the proof of Proposition 3.9. in [12] once we realize that, following Corollary 5 (c) $\text{st}^{-1}(G)$ possesses a $\Pi^0_1$ lifting. \qed

We shall now specify those topological spaces $X$ and $Y$ and those graphs $G$ for which conditions (i), (ii) and (iii) of Theorems 7 and 8 are fulfilled. For that purpose we restrict ourselves only to completely regular, Hausdorff spaces. A completely regular, Hausdorff space $X$ is called proper $K$-Lusin if it is a Baire set in each of its compactifications. A subset $S$ of $X$ is called proper $K$-analytic if it can be obtained by applying the Souslin operation to Baire sets in any compactification $Y$ of $X$. In this paper the complements of proper $K$-analytic sets are called co-proper $K$-analytic (for more details see [7]).

Ward Henson proved (see [2]) that $S \subseteq X$ is proper $K$-Lusin if and only if $\text{st}^{-1}_X(S)$ is Borel in $^*X$ and in particular, $X$ is proper $K$-Lusin if and only if the set n.s.($^*X$) of near-standard points in $^*X$ is Borel in $^*X$. Among metrizable spaces, proper $K$-Lusin spaces include separable absolute Borel spaces (a space is absolute Borel if it is Borel in each of its metrizable extensions) and in particular every Polish space (complete, separable metric space).

Also, the standard part image of a $\Sigma^1_1$ set in $^*X$ is always proper $K$-analytic [2]. It follows that if $\text{st}^{-1}_X(S)$ is $\Pi^0_1$ in $^*X$, then the complement $X \setminus S$ of $S$ in $X$ is the st$^{-1}_X$ image of the complement of a $\Pi^0_1$ set $\text{st}^{-1}_X(S)$ in $^*X$. Thus, $S$ is co-proper $K$-analytic. However, the converse seems not to be true for arbitrary completely regular $X$. Recently, Herman Render [6] proved, answering an old conjecture of Henson, that for a subset $S$ of $X$, $\text{st}^{-1}_X(S)$ is $\Sigma^1_1$ in $^*X$ if and only if $S$ is proper $K$-analytic. Thus, if $X$ is proper $K$-Lusin space, then $\text{st}^{-1}_X(S)$ is $\Pi^0_1$ in $^*X$ if and only if $S$ is co-proper $K$-analytic.

It is shown in [12] that for $S \subseteq X$, $\text{st}^{-1}_X(S)$ is $\Pi^0_0(\kappa)$, where $\kappa$ is greater than the cardinality of the base in $X$ if and only if $S$ is compact. Also, it is easy to see that if $X$ is a locally compact topological space, then $S$ is open in $X$ if and only if $\text{st}^{-1}_X(S)$ is $\Sigma^0_0(\kappa)$ in $^*X$.

Condition (iii) is satisfied if $\mu$ is taken to be a Radon measure in $X$, as was shown in [1]. (A Borel measure $\mu$ is called Radon if

$$\mu(B) = \sup\{\mu(C) : C \subseteq B, C \text{ compact}\} = \inf\{\mu(O) : O \supseteq B, O \text{ open}\}.$$"

Putting all the above together we obtain
Corollary 9. Let $X$ and $Y$ be proper $K$-Lusin spaces with $Y$ being, in addition, locally compact. Let $G \subseteq X \times Y$ be a co-proper $K$-analytic graph, all of whose $Y$-sections are open sets. Suppose that $\mu$ is a Radon probability measure in $X$. Then

(a) the domain of $G$ is a co-proper $K$-analytic set;
(b) there exists an $\sigma$-compact graph $H$ in $X \times Y$ such that $H = G$, $\mu$-a.e.;
(c) the set $\{x \in X : G(x) \text{ compact and } \neq \emptyset\}$ is $\mu$-measurable.

Proof. Condition (i) of Theorem 7 is fulfilled because the product of two proper $K$-Lusin spaces is again proper $K$-Lusin (as one easily can check by using the above described nonstandard characterization of being proper $K$-Lusin for completely regular spaces). Condition (ii) is satisfied because $Y$ is locally compact, while condition (iii) is always satisfied for Radon measures. It follows that $st_x^{-1}(\text{dom}(G))$ is $\Pi^1_1$ in $\ast X$, so $\text{dom}(G)$ is co-proper $K$-analytic.

Parts (b) and (c) follow from the corresponding parts of Theorem 7. □

In the same manner, but this time using Theorem 8, we obtain

Corollary 10. Let $X$ and $Y$ be completely regular topological space. Let $G \subseteq X \times Y$ be a proper $K$-analytic graph, all of whose $Y$-sections are compact sets. Suppose that $\mu$ is a Radon probability measure in $X$. Then for every $\epsilon > 0$ there exists a compact graph $H_\epsilon$ in $X \times Y$ such that $H_\epsilon \subseteq G$; $\mu(\text{dom}(H_\epsilon)) \geq \text{dom}(G) - \epsilon$, and for all $x \in \text{dom}(H_\epsilon)$, $H_\epsilon(x) = G(x)$. □

Example 11. Let $E$ and $F$ be Polish spaces, and let $f$ be a functions whose graph is a true $\Pi^1_1$ subset of $E \times F$. We identify functions with their graphs and set $X = \ast E$, $Y = \ast F$. Suppose that the domain of $f$ is Borel. Then $\Gamma = st_{E \times F}(f)$ is an example of a $\Pi^1_1$ graph with $\Pi^0_1$ $Y$-sections which cannot be expressed as a countable intersection of $\Pi^1_1$ graphs with internal $Y$-sections.

Indeed, let $\Gamma = \bigcap_{n \in \omega} \Gamma_n$ with $\Gamma_n$ being $\Pi^1_1$ graphs with internal $Y$-sections. We may suppose that $\text{dom}(\Gamma_n) = \text{dom}(\Gamma)$. By Theorem 1(f), $\Gamma_n$ possesses a Borel uniformization $g_n$. Let $G_n = st_{E \times F}(g_n)$. Then $G_n$ is a $\Sigma^1_1$ subset of $E \times F$ with $\text{dom}(G_n) = \text{dom}(f)$. Graphs $G_n$ have the property that for every $x \in \text{dom}(f)$ and every $\epsilon > 0$ there exists $n$ such that $G_n(x)$ is at distance less than $\epsilon$ from $f(x)$. Therefore, $f$ is defined as $(x, y) \in f$ if and only if $(\forall n \in \omega)(\exists y_n)[(x, y_n) \in G_n \land d(y, y_n) < 1/n]$ where $d$ is the distance function in $F$. This formula defines a $\Sigma^1_1$ function, so $f$ is Borel—a contradiction.

There exists a standard example of a closed subset $C$ of $\omega^\omega \times \omega^\omega$ with the domain equal to $\omega^\omega$ which does not possess a Borel uniformization. By the Kondo-Addison uniformization theorem, there exists a $\Pi^1_1$ uniformization of $C$ whose graph, therefore, is not Borel.

Another example of a true $\Pi^1_1$ function with a Borel domain arising in "real" mathematics is communicated to me by Professor Kechris. Let $G$ be the set of all permutations of integers understood as a Polish group, and let $X$ be the space of all closed subgroups of $G$. We consider $X$ to be endowed with Effros Borel structure. Then the function $H \mapsto \{p \in G : pHp^{-1} = H\}$ which sends a closed subgroup to its normalizer is a true $\Pi^1_1$ function with the domain equal to $X$. 
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