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POWER REGULAR OPERATORS

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ABSTRACT. We show that for a wide class of operators T on a Banach space, including the class of decomposable operators, the sequence $\{\|T^n x\|^{1/n}\}_{n=1}^{\infty}$ converges for every x in the space to the spectral radius of the restriction of T to the subspace $\bigvee_{n=0}^{\infty}\{T^n x\}$.

1. INTRODUCTION

Throughout this paper, X will denote a complex Banach space and $\mathscr{L}(X)$ the Banach algebra of bounded linear operators on X. For an operator T in $\mathscr{L}(X)$, we denote as usual by $\sigma(T)$ its spectrum and by $r(T)$ its spectral radius. By Gelfand's formula for the spectral radius

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$$

for all T in $\mathscr{L}(X)$.

It is well known that if $\{w_n\}_{n=1}^{\infty}$ is a sequence of nonnegative numbers which is submultiplicative (that is, $w_{m+n} \leq w_m w_n$ for all m and n), then $\lim_{n\to\infty} w_n^{1/n}$ exists. Hence, the existence of the limit in the right-hand side of Gelfand's formula can be deduced from the fact that for every T in $\mathscr{L}(X)$, the sequence $\{\|T^n\|\}_{n=1}^{\infty}$ is submultiplicative. On the other hand, for T in $\mathscr{L}(X)$ and x in X, the sequence $\{\|T^n x\|\}_{n=1}^{\infty}$ is in general not submultiplicative. Nevertheless, we shall show in this paper that for a wide class of operators T in $\mathscr{L}(X)$, the sequence $\{\|T^n x\|^{1/n}\}_{n=1}^{\infty}$ is convergent for all x in X. We shall call such operators *power-regular.*

We shall prove, in particular, that all decomposable operators (see definition in §3) are power-regular. By [4] this class includes all spectral operators in Dunford's sense (hence all normal operators) and all operators with totally disconnected (hence countable) spectrum. We shall also prove that every operator T in $\mathscr{L}(X)$ for which the set $\{|\lambda|;\, \lambda \in \sigma(T)\}$ has empty interior in $[0, \infty)$ is power-regular. This class clearly contains all operators with spectrum included in a countable union of circles with centers at the origin. Moreover, we shall show that for every operator T in $\mathscr{L}(X)$ which belongs to one of these classes, the sequence $\{\|T^n x\|^{1/n}\}_{n=1}^{\infty}$ converges for all x in X to the spectral radius of the restriction of T to the subspace $\bigvee_{n=0}^{\infty}\{T^n x\}$.

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Power-regularity of compact operators and selfadjoint operators can also be deduced from the results in [11, §9]. In [5] power-regularity is also established for a more general class of operators with countable spectrum and for normal operators.

We recall that the local spectral radius of an operator T in $\mathscr{L}(X)$ at a vector x in X is defined by (cf. [5] and [13])

$$r(x, T) = \limsup_{n\to\infty} \|T^n x\|^{1/n}.$$

As shown in [5], for every T in $\mathscr{L}(X)$, the equality $r(x, T) = r(T)$ holds for quasi-all x in X (that is, for all x in the complement of a set of first category). Thus if T is power-regular,

$$r(x, T) = \lim_{n\to\infty} \|T^n x\|^{1/n}$$

for all x in X, and the limit is equal to $r(T)$ for quasi-all x in X.

We mention also that by [13], for every T in $\mathscr{L}(X)$, $\{\|T^n x\|^{1/n}\}_{n=1}^{\infty}$ converges to $r(T)$ for all x in a dense subset of X. For Hilbert spaces this follows also from [3, Theorem 2.A.1]. On the other hand, it is proved in [5] that if for T in $\mathscr{L}(X)$ and x in X the sequence $\{\|T^n x\|^{1/n}\}_{n=1}^{\infty}$ does not converge, then its set of limit points is a closed interval.

It is easy to construct weighted shifts which are not power-regular (cf. [5] and [9]). We shall also see in §6 that the backward shift on l^2 is not power-regular.

In §2 we establish a general criterion for power-regularity from which most of our subsequent results are derived. In §3 we prove power-regularity of decomposable operators and deduce several corollaries. In §4 we introduce the class of radially decomposable operators and prove power-regularity of operators belonging to the subclass of radially super-decomposable operators. We show that this subclass contains all operators T in $\mathscr{L}(X)$ for which the set $\{|\lambda|; \lambda \in \sigma(T)\}$ has empty interior in $[0, \infty)$. In §5 we give two direct and elementary proofs of power-regularity of operators on a finite-dimensional space. Finally, in §6, we present some additional facts and examples and raise several problems.

2. A general criterion

In this section we prove a general criterion for power-regularity which is applied in the sequel to establish power-regularity of operators in some concrete classes, in particular in those mentioned in the previous section. We first need some notation.

Let T be an operator in $\mathscr{L}(X)$. We denote as usual by $\operatorname{Lat}(T)$ the collection of all closed subspaces of X which are invariant under T, and for a subspace M in $\operatorname{Lat}(T)$ we denote by T_M the restriction of T to M. We shall denote by $q(T)$ the minimum of the set $\{|\lambda|; \lambda \in \sigma(T)\}$ (for $X = \{0\}$, we define $q(T) = \infty$). For every x in X we shall denote by $r_x(T)$ the spectral radius of the restriction of T to the subspace $\bigvee_{n=0}^{\infty}\{T^n x\}$ (for $x = 0$, we define $r_x(T) = 0$). Note that by the spectral radius formula,

$$r(x, T) \le r_x(T) \le r(T)$$

for all x in X.

We can now state our general criterion for power-regularity.

Theorem 2.1. *Let T be an operator in $\mathscr{L}(X)$, and assume that for every $0 < t_1 < t_2 < \infty$ there exist a complex Banach space Y, an operator S in $\mathscr{L}(Y)$, and a bounded linear operator J from X into Y, such that setting $M = \ker J$, the following three conditions hold*:

(I) $JT = SJ$;

(II) $r(T_M) \le t_2$;

(III) $q(S) \ge t_1$.

Then for all x in X,

$$\lim_{n\to\infty} \|T^n x\|^{1/n} = r_x(T).$$

Remarks. (1) Condition (I) implies that $M \in \mathrm{Lat}(T)$.

(2) For $t_2 > r(T)$ the above conditions hold for every T in $\mathscr{L}(X)$, with $Y = X$, $S = t_2 I$, and $J = 0$.

The following fact is needed for the proof of the theorem.

Lemma 2.2. *If Y is a complex Banach space and S is an operator in $\mathscr{L}(Y)$, then for every $y \neq 0$ in Y,*

$$q(S) \le \liminf_{n\to\infty} \|T^n y\|^{1/n}.$$

Proof. The inequality is clear if $q(S) = 0$. Assume that $q(S) > 0$. Then S is invertible and $r(S^{-1}) = (q(S))^{-1}$. Hence, noticing that for every y in Y

$$\|y\| \le \|S^{-n}\| \, \|S^n y\|, \qquad n = 1, 2, \ldots,$$

by using the spectral radius formula for S^{-1} we obtain that if $y \neq 0$,

$$1 = \lim_{n\to\infty} \|y\|^{1/n} \le (q(S))^{-1} \liminf_{n\to\infty} \|S^n y\|^{1/n},$$

and the lemma is proved.

Proof of Theorem 2.1. Let $x \in X$. If $r_x(T) = 0$ the assertion is clear. Assume that $r_x(T) > 0$, and consider numbers t_1 and t_2 such that $0 < t_1 < t_2 < r_x(T)$. Let S and J be the operators which satisfy the conditions of the theorem for t_1 and t_2. We claim that $Jx \neq 0$. In fact, assuming that $x \in M = \ker J$ and remembering that $M \in \mathrm{Lat}(T)$, we obtain that $\bigvee_{n=0}^{\infty}\{T^n x\} \subset M$, and therefore $r_x(T) \le r(T_M)$, which by condition (II) contradicts the choice of t_2. So $Jx \neq 0$ (and in particular $J \neq 0$). Thus by Lemma 2.2 and condition (III),

$$t_1 \le q(S) \le \liminf_{n\to\infty} \|S^n Jx\|^{1/n}.$$

Therefore, observing that by condition (I),

$$JT^n x = S^n Jx, \qquad n = 1, 2, \ldots,$$

we obtain that

$$t_1 \le \liminf_{n\to\infty} \|S^n Jx\|^{1/n} \le \liminf_{n\to\infty} (\|J\|^{1/n} \|T^n x\|^{1/n}) = \liminf_{n\to\infty} \|T^n x\|^{1/n};$$

and since t_1 is an arbitrary number less than $r_x(T)$, we deduce that

$$r_x(T) \le \liminf_{n\to\infty} \|T^n x\|^{1/n}.$$

But as noticed at the beginning of this section,

$$r(x, T) = \limsup_{n\to\infty} \|T^n x\|^{1/n} \le r_x(T),$$

and we conclude that

$$\lim_{n\to\infty} \|T^n x\|^{1/n} = r_x(T).$$

This completes the proof of the theorem.

3. Decomposable operators

In this section we shall establish power-regularity of decomposable operators. We recall that according to [1] an operator T in $\mathscr{L}(X)$ is *decomposable* if for every cover of the complex plane by a pair of open sets U and V, there exist subspaces M and K in $\mathrm{Lat}(T)$ such that $M + K = X$, $\sigma(T_M) \subset U$, and $\sigma(T_K) \subset V$.

The main ingredients in the proof of power-regularity of decomposable operators are Theorem 2.1 in the preceding section and Theorem 12.15 in [7].

In the proof we shall need an additional notation. For an operator T in $\mathscr{L}(X)$ and a subspace M in $\mathrm{Lat}(T)$, we shall denote by T^M the canonical operator induced by T on the quotient space X/M.

Theorem 3.1. *If T is a decomposable operator in $\mathscr{L}(X)$, then for all x in X,*

$$\lim_{n\to\infty} \|T^n x\|^{1/n} = r_x(T).$$

Proof. Let $0 < t_1 < t_2 < \infty$, and consider the disc $G = \{\lambda \in \mathbb{C}; |\lambda| < t_2\}$. Since T is decomposable, it follows from [7, Theorem 12.15] that there exists a subspace M in $\mathrm{Lat}(T)$, such that $\sigma(T_M) \subset \overline{G}$ and $\sigma(T^M) \cap G = \varnothing$. This is equivalent to the conditions $r(T_M) \le t_2 \le q(T^M)$. From this it is readily verified that the conditions of Theorem 2.1 hold for t_1 and t_2, with $Y = X/M$, $S = T^M$, and J the canonical map of X onto X/M, and the proof is complete.

It follows from Theorem 3.1 that all operators considered in [4] are power-regular. In particular, from [4, pp. 33, 67, and 185] we obtain the following.

Corollary 3.2. *If T is an operator in $\mathscr{L}(X)$, then each of the following conditions implies that the conclusion of Theorem 3.1 holds for T.*

(1) *T is a spectral operator in Dunford's sense (hence, in particular, if T is a normal operator in Hilbert space).*

(2) *$\sigma(T)$ is totally disconnected (hence, in particular, if $\sigma(T)$ is countable).*

(3) *$\sigma(T)$ is included in the real line, and the integral*

$$\int_0^1 \ln\ln \sup_{|\operatorname{Im}\lambda|>y} \|(T-\lambda I)^{-1}\|\, dy$$

is convergent. (This condition is satisfied, in particular, if $\|(T-\lambda I)^{-1}\| \le c\exp(b/|\operatorname{Im}\lambda|^{\sigma})$, for $\operatorname{Im}\lambda \neq 0$, where α, b, c are positive constants.)

Another corollary is concerned with Banach algebras. We shall say that a Banach algebra B is power-regular, if $\lim_{n\to\infty} \|x^n y\|^{1/n}$ exists for all x and y in B.

The following is an immediate consequence of Theorem 3.1 and [4, Theorem 2.6, p. 201].

Corollary 3.3. *Every commutative semi-simple regular Banach algebra is power-regular.*

4. Radially decomposable operators

According to [7, Theorem 12.5], if T is a decomposable operator, then for every open set G in the complex plane, there exists a subspace M in $\mathrm{Lat}(T)$ such that $\sigma(T_M) \subset \overline{G}$ and $\sigma(T^M) \cap G = \varnothing$. In the proof of Theorem 3.1 we used only the fact that this holds when G is an open disc with center at the origin. This suggests that power-regularity might be true for operators T in $\mathscr{L}(X)$ which satisfy a condition that is considerably weaker than decomposability, namely, that for every $0 < t_1 < t_2 < \infty$, there exists subspaces M and K in $\mathrm{Lat}(T)$, such that $M + K = X$, $q(T_M) \geq t_1$, and $r(T_K) \leq t_2$. We shall call operators which satisfy this condition *radially decomposable*.

We conjecture that all radially decomposable operators are power-regular. This would follow from the proof of Theorem 3.1, if one could show that the conclusion of Theorem 12.15 in [7] holds for these operators for open discs with centers at the origin.

We prove in this section power-regularity of operators which belong to a somewhat more restricted class. Before describing it, we mention that, according to [12], an operator T in $\mathscr{L}(X)$ is called *super-decomposable* if for every cover of the complex plane by a pair of open sets U and V, there exists an operator A in $\mathscr{L}(X)$ which commutes with T such that $\sigma(T_{\overline{AX}}) \subset U$ and $\sigma(T_{\overline{(I-A)X}}) \subset V$.

Motivated by this terminology, we shall say that an operator T in $\mathscr{L}(X)$ is *radially super-decomposable* if for every $0 < t_1 < t_2 < \infty$, there exists an operator A in $\mathscr{L}(X)$ which commutes with T such that $q(T_{\overline{AX}}) \geq t_1$ and $r(T_{\overline{(I-A)X}}) \leq t_2$.

It is clear that a radially super-decomposable operator is radially decomposable and that a super-decomposable operator is decomposable and radially super-decomposable.

Theorem 4.1. *If T is a radially super-decomposable operator in $\mathscr{L}(X)$, then for all x in X,*

$$\lim_{n\to\infty} \|T^n x\|^{1/n} = r_x(T).$$

Proof. Let $0 < t_1 < t_2 < \infty$, and consider an operator A in $\mathscr{L}(X)$ which commutes with T such that $q(T_{\overline{AX}}) \geq t_1$ and $r(T_{\overline{(I-A)X}}) \leq t_2$. Observing that $\ker A \subset (I-A)X$, we see that $r(T_{\ker A}) \leq r(T_{\overline{(I-A)X}})$; and this implies that the conditions of Theorem 2.1 hold for t_1 and t_2, with $Y = \overline{AX}$, $S = T_Y$, and $J = A$. This completes the proof.

Remark. It follows from [12, Propositions 2.1 and 2.2] that all operators that satisfy the conditions of Corollary 3.2 are super-decomposable, and therefore the corollary also follows from Theorem 4.1. By [12, Corollary 2.4] the same is true for Corollary 3.3.

Theorem 4.2. *If T is an operator in $\mathscr{L}(X)$ such that the set $\{|\lambda| ; \lambda \in \sigma(T)\}$ has empty interior in $[0, \infty)$, then the conclusion of Theorem 4.1 holds for T.*

Proof. We shall show that T is radially super-decomposable. Let $0 < t_1 < t_2$, and consider the set $B = \{|\lambda| ; \lambda \in \sigma(T)\}$. Since B has empty interior in $[0, \infty)$, there exist $t \notin B$ such that $t_1 < t < t_2$, and therefore the set $\tau = \{\lambda \in \mathbb{C}; |\lambda| \geq t\} \cap \sigma(T)$ is a spectral set for T (that is, an open and closed subset of $\sigma(T)$). Let A denote the corresponding spectral projection $E(\tau, T)$ (see [6, p. 573]). It is well known [6, Chapter 7] that A commutes with T, $\sigma(T_{AX}) = \tau$, and $\sigma(T_{(I-A)X}) = \sigma(T)\backslash\tau$, and therefore $q(T_{AX}) \geq t > t_1$ and $r(T_{(I-A)X}) \leq t < t_2$. This shows that T is radially super-decomposable, and the assertion follows from Theorem 4.1.

The following is an immediate consequence of Theorem 4.2.

Corollary 4.3. *If T is an operator in $\mathscr{L}(X)$ such that $\sigma(T)$ is included in a countable union of circles with centers at the origin, then T is power-regular.*

Corollary 4.3 implies that all operators considered in [2] which are annihilated by a nonzero analytic function are power-regular, since by [2, Theorem 3(a)], if T is such an operator, then the set $\{|\lambda|; \lambda \in \sigma(T)\}$ is countable. This includes, in particular, operators of class C_0, that is, completely nonunitary contractions in Hilbert space, which are annihilated by a nonzero bounded analytic function in the unit disc. For this class power-regularity follows also from Theorem 3.1, since by a result of Foiaş [8], operators of class C_0 are decomposable.

5. Finite-dimensional spaces

If X is finite dimensional, then every operator in $\mathscr{L}(X)$ has finite spectrum and hence is power-regular by Corollaries 3.2 or 4.3. This also follows from [11, p. 116] where the more general case of compact operators is considered. In that proof the Jordan canonical form is used. We give here two direct and elementary proofs.

Theorem 5.1. *If X is finite dimensional and T is in $\mathscr{L}(X)$, then for all x in X,*

$$\lim_{n\to\infty} \|T^n x\|^{1/n} = r_x(T).$$

First proof. Let $x \in X$. By considering the restriction of T to the subspace spanned by the vectors $x, Tx, T^2x, \ldots$, we may assume that x is a cyclic vector for T. Let λ be an eigenvalue of T such that $|\lambda| = r(T)$, and let v be a corresponding unit eigenvector. Since x is a cyclic vector for T, there exists a polynomial p such that $p(T)x = v$, and therefore for every positive integer n,

$$(r(T))^n = |\lambda|^n = \|p(T)T^n x\| \leq \|p(T)\|\,\|T^n x\|.$$

Hence noticing that $p(T) \neq 0$ (since $v \neq 0$), we obtain that

$$r(T) \leq \liminf_{n\to\infty}(\|p(T)\|^{1/n}\|T^n x\|^{1/n}) = \liminf_{n\to\infty} \|T^n x\|^{1/n}.$$

Combining this with the fact that

$$\limsup_{n\to\infty} \|T^n x\|^{1/n} \leq \lim_{n\to\infty} \|T^n\|^{1/n} = r(T),$$

the assertion follows.

Second proof. Assume again that x is a cyclic vector for T. Let $\mathscr{A}$ be the algebra generated in $\mathscr{L}(X)$ by T and the identity operator, that is, $\mathscr{A} = \operatorname{span}\{T^n; n = 0, 1, \ldots\}$. Consider the linear mapping $L : \mathscr{A} \to X$ defined by

$L(S) = Sx$, $S \in \mathscr{A}$. Since x is a cyclic vector for T, L is an isomorphism between the finite-dimensional Banach spaces $\mathscr{A}$ and X. Therefore, there exists a constant $c > 0$ such that

$$c^{-1}\|S\| \leq \|Sx\| \leq c\|S\|,$$

for all S in $\mathscr{A}$; hence in particular,

$$c^{-1}\|T^n\| \leq \|T^n x\| \leq c\|T^n\|, \quad n = 1, 2, \ldots.$$

This implies that

$$\lim_{n\to\infty} \|T^n x\|^{1/n} = \lim_{n\to\infty} \|T^n\|^{1/n} = r(T),$$

and the proof is complete.

Remarks. (1) The second proof is valid also for finite-dimensional spaces over the real field (in this case the spectral radius of an operator is defined by Gelfand's formula).

(2) For general X and all T in $\mathscr{L}(X)$, it is easily verified that for every $t > 0$, the set $\{x \in X; r(x, T) < t\}$ is a linear space (which is not generally closed in X). Hence if X is finite dimensional and T is in $\mathscr{L}(X)$, then $\lim_{n\to\infty} \|T^n x\|^{1/n} = r(T)$ for all x in X, except (if $r(T) > 0$) for x in the subspace $\{x \in X; r(x, T) < r(T)\}$. Thus, in particular, this equality holds with probability one (with respect to normalized area measure) for x in the Euclidean unit sphere of X.

6. Additional facts and problems

It is easily verified that power-regularity is preserved by similarity and (finite) direct sums but is not in general preserved by sums and products. In fact, every Hilbert space operator A is the sum of two power-regular operators $\frac{1}{2}(A + A^*)$ and $\frac{1}{2}(A - A^*)$ (recall that selfadjoint operators are power-regular), and every weighted shift is the product of a diagonal operator and an isometry (which are clearly power-regular), but as already mentioned in §1, there exist weighted shifts which are not power-regular. For commuting operators the situation is less clear.

Problem 1. *Are the sum and product of two commuting power-regular operators also power-regular?*

We show next that power-regularity is not preserved in general by adjoint operators. Consider the unilateral shift S on l^2 (that is, $Se_n = e_{n+1}$, $n = 0, 1, \ldots$, where $\{e_n\}_{n=0}^{\infty}$ is the standard orthonormal basis in l^2). Since S is an isometry, it is power-regular. We claim that its adjoint S^* (the backward shift) is not power-regular. To see this, consider the sequence $\{a_n\}_{n=0}^{\infty}$ in l^2 defined by $a_0 = 1$ and $a_n = \exp(-2^k)$, $2^{k-1} \leq n < 2^k$, $k = 1, 2, \ldots$. Then $x = \{(a_n^2 - a_{n-1}^2)^{1/2}\}_{n=0}^{\infty}$ is in l^2, and $\|S^{*n}x\| = a_n$, $n = 1, 2, \ldots$. Since the sequence $\{a_n^{1/n}\}_{n=1}^{\infty}$ is not convergent, this shows that S^* is not power-regular.

It is clear that power-regularity is preserved by restrictions to invariant subspaces. Hence, for example, power-regularity of subnormal operators follows from that of normal operators. On the other hand, the preceding example implies that power-regularity is not preserved in general by passing to quotient spaces. In fact, if T is the unitary operator on $l^2(\mathbb{Z})$ defined by $T\{a_n\}_{n=-\infty}^{\infty} =$

$\{a_{n+1}\}_{n=-\infty}^{\infty}$, then the subspace M, which consists of all sequences $\{c_n\}_{n=-\infty}^{\infty}$ in $l^2(\mathbb{Z})$ such that $c_n = 0$ for $n \geq 0$, is in $\mathrm{Lat}(T)$, and T^M can be identified in an obvious way with the backward shift S^*.

For all operators T in $\mathscr{L}(X)$ for which power-regularity was proved in the preceding sections, the equality

$$\lim_{n\to\infty} \|T^n x\|^{1/n} = r_x(T)$$

was also established for all x in X. We do not know whether this is always the case.

Problem 2. *Assume that* T *is a power-regular operator in* $\mathscr{L}(X)$. *Is the above equality satisfied for all* x *in* X?

For general operators the equality is not always true, even if the limit on the left-hand side exists. To see this, consider again the backward shift S^* on l^2, and let x denote the sequence $\{1/(n+1)!\}_{n=0}^{\infty}$ in l^2. It follows from [10, p. 282] that x is a cyclic vector for S^*, and therefore $r_x(S^*) = r(S^*) = 1$. On the other hand, a simple estimate shows that

$$\|S^{*n}x\| \leq \frac{1}{n!}, \qquad n = 1, 2, \dots,$$

and therefore $\lim_{n-\infty} \|S^{*n}x\|^{1/n} = 0$. Hence the equality is not satisfied in this case.

It follows from Corollary 4.3 that every operator whose spectrum is included in a circle with center at the origin is power regular. We do not know whether the same is true for operators with spectrum included in a smooth curve, even for curves of very special type.

Problem 3. *Is every operator in* $\mathscr{L}(X)$ *with spectrum included in a circle or in the real line power-regular?*

We conclude with some comments about power-regular Banach algebras. It is clear that a closed subalgebra of a power-regular Banach algebra is also power-regular. Thus, by Corollary 3.3, every closed subalgebra of a commutative semisimple regular Banach algebra is power-regular. Also, for a general commutative Banach algebra B with identity, one can show that for every y in B whose Gelfand transform does not vanish on any nonempty open subset of the maximal ideal space of B, and for all x in B,

$$\lim_{n\to\infty} \|x^n y\|^{1/n} = r(x)$$

(where $r(x)$ denotes the spectral radius of x). Thus, in particular, if every nonzero element y of B has this property, then B is power-regular. Therefore, every Banach algebra with identity of analytic functions on some domain in the complex plane, which is dense in its maximal ideal space, is power-regular.

In a lecture at the Banach center we raised the question whether every commutative semisimple Banach algebra with identity is power-regular. Vladimir Müller constructed an example which shows that this is not the case.

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