STABLE RANGE ONE FOR RINGS WITH MANY IDEMPOTENTS

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Abstract. An associative ring $R$ is said to have stable range 1 if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is a unit. The purpose of this note is to prove the following facts. Theorem 3: An exchange ring $R$ has stable range 1 if and only if every regular element of $R$ is unit-regular. Theorem 5: If $R$ is a strongly $\pi$-regular ring with the property that all powers of every regular element are regular, then $R$ has stable range 1. The latter generalizes a recent result of Goodearl and Menal [5].

Let $R$ be an associative ring with identity. $R$ is said to have stable range 1 if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is a unit. This definition is left-right symmetric by Vaserstein [9, Theorem 2]. Furthermore, by a theorem of Kaplansky, all one-sided units are two-sided in rings having stable range 1 (cf. Vaserstein [10, Theorem 2.6]). It is well known that a (von Neumann) regular ring $R$ has stable range 1 if and only if $R$ is unit-regular (see, for example, Goodearl [4, Proposition 4.12]).

Call a ring $R$ strongly $\pi$-regular if for every element $a \in R$ there exist a number $n$ (depending on $a$) and an element $x \in R$ such that $a^n = a^{n+1}x$. This is in fact a two-sided condition [3]. It is an open question whether all strongly $\pi$-regular rings have stable range 1. Goodearl and Menal [5] proved that strongly $\pi$-regular rings are unit-regular and, hence, have stable range 1 (Theorem 5.8, p. 278).

In this note we first extend the above result for von Neumann regular rings to a larger class of rings, which includes all strongly $\pi$-regular rings, $\pi$-regular rings, von Neumann regular rings, and algebraic algebras. As an application of this, we prove that a strongly $\pi$-regular ring $R$ has stable range 1 if powers of every regular element are regular. The latter is a generalization of the above-mentioned result of Goodearl and Menal for strongly $\pi$-regular regular rings. As one can see from our proofs, rings in these classes have a large supply of idempotents.

Throughout, $R$ stands for an associative ring with identity and $J(R)$ for the Jacobson radical of $R$. Modules are unitary right $R$-modules except otherwise specified. For other undefined terms, readers are referred to [4].

Let $M_R$ be a right $R$-module. Following Crawley and Jonsson [2], $M_R$ is said to have the exchange property if for every module $A_R$ and any two
decompositions of $A_R$

$$A_R = M' \oplus N = \bigoplus_{i \in I} A_i$$

where $M'_R \cong M_R$, there exist submodules $A'_i \subseteq A_i$ such that

$$A = M' \oplus \left( \bigoplus_{i \in I} A'_i \right).$$

$M_R$ is said to have the finite exchange property if the above condition is satisfied whenever the index set $I$ is finite. Many familiar classes of modules have the exchange property or the finite exchange property, see Zimmermann-Huisgen and Zimmermann [12] for a list of these classes of modules.

Warfield [8] introduced the class of exchange rings. He called a ring $R$ an exchange ring if $R_R$ has the exchange property above and proved that this definition is left-right symmetric. The class of exchange rings is quite large. Call a ring $R$ semiregular (semi-$\pi$-regular, semi-strongly $\pi$-regular) if $R/J(R)$ is regular ($\pi$-regular , strongly $\pi$-regular) and idempotents can be lifted modulo $J(R)$. It is easy to verify that the following classes of rings (in the order of containments) are all contained in the class of exchange rings: (1) local rings; (2) semiperfect rings; (3) semiregular rings; (4) semistrongly $\pi$-regular rings; (5) semi-$\pi$-regular rings (see, for example, Stock [7, p. 440]).

The following characterizations of the finite exchange property for projective modules were given by Nicholson [6, Proposition 2.9].

**Lemma 1** (Nicholson). The following conditions are equivalent for a projective module $P$:

1. $P$ has the finite exchange property.
2. If $P = M_1 + M_2 + \cdots + M_n$ where $M_i$ are submodules, there is a decomposition $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ with $P_i \subseteq M_i$ for each $i$.
3. If $P = M + N$ where $M$ and $N$ are submodules, there exists a summand $P_1$ of $P$ such that $P_1 \subseteq M$ and $P = P_1 + N$.  

The original definition of stable range 1 for an arbitrary ring $R$ is equivalent to the condition that for any $a, x, b \in R$ satisfying $ax + b = 1$, there exists $y \in R$ such that $a + by$ is a unit in $R$. The next lemma says that, for exchange rings, the element $b$ in the latter condition can be further restricted to idempotents.

**Lemma 2.** Let $R$ be an exchange ring, then the following conditions are equivalent:

1. $R$ has stable range 1.
2. For any $a \in R$, $e^2 = e \in R$, if $ax + e = 1$ for some $x \in R$, then there exists $y \in R$ such that $a + ey = u$ is a unit.

**Proof.** (1) $\Rightarrow$ (2): Trivial. (2) $\Rightarrow$ (1): Assume that $aR + bR = R$. $R$ is exchange, there exists an idempotent $e^2 = e \in bR$ such that $(1-e)R \oplus eR = R$ where $(1-e)R \subseteq aR$ and $eR \subseteq bR$, by Lemma 1. So we have $ax + e = 1$ for some $x \in R$. By assumption, there exists $y \in R$ such that $a + ey = u$ is a unit; hence, $a + bry = u$ is a unit where $e = br$.  

Recall that for a regular ring $R$, $R$ has stable range 1 if and only if $R$ is unit-regular. We now extend this to exchange rings.
Theorem 3. An exchange rings $R$ has stable range 1 if and only if every regular element of $R$ is unit-regular in $R$.

Proof. $\Rightarrow$: Let $axa = a$; then $ax + (1 - ax) = 1$. By the assumption on $R$, there exists $y \in R$ such that $a + (1 - ax)y = u$ is a unit in $R$. Multiplying both sides of the latter equality by $ax$ on the left, we have that $axa = a = axu$, so $au^{-1}a = a$, and $a$ is unit-regular.

$\Leftarrow$: By Lemma 2, we need only show that if $ax + e = 1$ with $e^2 = e$, there exists an element $y \in R$ such that $a + ey$ is a unit.

We first show that, without loss of generality, we may assume $axa = a$. In fact, if $axa \neq a$, put $f = ax$ and $r = fa - a$; then $rx = 0$. Letting $a' = a + r$, we have $a'x = ax + rx = ax + 0 = ax = f$, $a'xa' = fa' = fa + fr = fa + 0 = a + r = a'$. To see that $fr = 0$, notice that $f = ax = 1 - e$ is an idempotent. Now if $a' + ey$ is a unit for some $y \in R$ and $fr = 0$ implies $r \in (1-f)R = eR$, we have

$$a' + ey = a + r + ey = a + es + ey = a + e(s + y)$$

is a unit.

So we can assume that $ax + e = 1$, where $e^2 = e$ and $axa = a$. Notice that $axa = a$ if and only if $ea = 0$. Since we assume that every regular element is unit-regular, there exists a unit $u \in R$ such that $aua = a$. Then we have

$$1 - e = ax = (aua)x = (au)(ax) = au(1 - e).$$

$$(au - e)^2 = (au - e)(au - e) = auau - aue - eau + e = au - aue - 0 + e = au(1 - e) + e = 1.$$ 

So $au - e = v$ is a unit, therefore $a - eu^{-1} = vu^{-1}$ is a unit. $\square$

For some other equivalent characterizations of stable range 1 for exchange rings, see Yu [11, Theorem 9].

While the question of whether all strongly $\pi$-regular rings have stable range 1 remains open, we now can reduce this to a unit-regularity problem.

Corollary 4. A strongly $\pi$-regular ring $R$ has stable range 1 if and only if every regular element of $R$ is unit-regular in $R$. $\square$

Corollary 4 should be compared with an analogous result of Goodearl and Menal [5, Theorem 6.1], which says that a strongly $\pi$-regular ring $R$ has stable range 1 if and only if every nilpotent regular element of each corner of $R$ is unit-regular in that corner. By a corner of a ring $R$, they mean any (nonunital) subring $eRe$ where $e$ is an idempotent in $R$. While it is true that an element $x \in eRe$ is regular in $eRe$ if and only if it is regular in $R$, the same is not true for unit-regularity.

One of the known cases where strongly $\pi$-regular rings have stable range 1 was Theorem 5.8 of Goodearl and Menal [5]: a strongly $\pi$-regular regular ring has stable range 1. As an application of our Theorem 3, we now extend this to the following:

Theorem 5. Let $R$ be a strongly $\pi$-regular ring. If all powers of every regular element are regular, then $R$ has stable range 1.

The proof we are going to give is a modification of Goodearl and Menal's proof in [5]. In order to make our paper self-contained, we present here a complete proof, although a portion of it is just a verbatim adoption of their argument.
Lemma 6 (Azumaya, Dischinger). For every element \( x \in R \) of a strongly \( \pi \)-regular ring \( R \), there exist \( a \in R \) and an integer \( n \geq 1 \) such that \( xa = ax \) and \( x^n = ax^{n+1} = x^{n+1}a \). \( \square \)

Proof of Theorem 5. Let \( x \in R \) be a regular element with \( xyx = x \). It suffices to prove, by Corollary 4, that \( x \) is unit-regular.

Set \( K_i = r.ann(x^i) \) for all \( i = 0, 1, 2, \ldots \).

Claim 1. There exists an integer \( n \geq 1 \) such that \( xR + K_n = R \) and \( x^nR \cap K_1 = 0 \).

In fact, by Lemma 6, there exist an integer \( n > 1 \) and \( a \in R \) such that \( ax = xa \) and \( x^n = ax^{n+1} = x^{n+1}a \). Pick any \( r \in R \), we have \( x^n r = x^{n+1}ar \) and \( x^n(r - xar) = 0 \); thus \( r - xar \in K_n \) and \( r \in xR + K_n \). Therefore, \( xR + K_n = R \).

Since \( x^n = ax^{n+1} \), it is clear that \( K_n = K_{n+1} \). If \( x^n d \in x^n R \cap K_1 \), then \( xx^n d = 0 \) and \( d \in K_{n+1} = K_n \), so \( x^n d = 0 \), i.e., \( x^n R \cap K_1 = 0 \).

Claim 2. \( xR + K_i \) are direct summands of \( R_R \) for all \( i \geq 1 \).

Since \( x^i \) is regular for all \( i \geq 2 \) by our assumption on \( R \), we may assume \( x^iy_i = x^i \) for some \( y_i \in R \) for \( i \geq 2 \). Then \( K_i = (1 - y_i x^i)R \). It is easy to check that

\[
xR + (1 - y_i x^i)R = y_i x^ixR + (1 - y_i x^i)R.
\]

We check below that the element \( y_i x^ix \) is actually von Neumann regular:

\[
y_i x^ix \cdot y_i x^{i+1} \cdot y_i x^ix = y_i x^ix y_i x^{i+1} x^i x = y_i x^i x.
\]

Put \( e_i = y_i x^ix y_i x^{i+1} \) and \( f_i = 1 - y_i x^i \), then \( e_i f_i = f_i e_i = 0 \). We see that \( e_i \) and \( f_i \) are orthogonal idempotents, hence \( e_i + f_i \) is an idempotent. But \( y_i x^ixR = e_i R \), so \( xR + K_i = e_i R + f_i R = (e_i + f_i) R \) is a direct summand of \( R_R \).

Recall that we assume \( xyx = x \), so \( xR + K_1 \) is a direct summand of \( R_R \) for the same reason.

Claim 3. \( x^i R \cap K_1 \) are all direct summands of \( R_R \) for all \( i \geq 1 \).

First, we show \( x^i R \cap K_1 = x^i K_{i+1} \). Since \( x^i K_{i+1} \subset x^i R \) and \( x^i K_{i+1} \subset K_1 \), \( x^i K_{i+1} \subset x^i R \cap K_1 \); on the other hand, pick any \( x^ir \in x^i R \cap K_1 \), \( xx^ir = x^{i+1}r = 0 \), \( r \in K_{i+1} \), so \( x^ir \in x^i K_{i+1} \), \( x^i R \cap K_1 \subset x^i K_{i+1} \).

Second, recall that we assume \( x^iy_i x^i = x^i \), so that \( K_{i+1} = (1 - y_{i+1} x^{i+1}) R \), and we see that \( x^i R \cap K_1 = x^i K_{i+1} = x^i (1 - y_{i+1} x^{i+1}) R \). We check below that \( x^i (1 - y_{i+1} x^{i+1}) \) is von Neumann regular:

\[
x^i (1 - y_{i+1} x^{i+1}) y_i x^i (1 - y_{i+1} x^{i+1}) = (x^i - x^iy_{i+1} x^{i+1}) y_i x^i (1 - y_{i+1} x^{i+1})
\]

\[
= (1 - x^i y_{i+1} x) x^iy_i x^i (1 - y_{i+1} x^{i+1}) = (1 - x^i y_{i+1} x) x^i (1 - y_{i+1} x^{i+1})
\]

\[
= (x^i - x^i y_{i+1} x^{i+1}) (1 - y_{i+1} x^{i+1}) = x^i (1 - y_{i+1} x^{i+1}) (1 - y_{i+1} x^{i+1})
\]

\[
= x^i (1 - y_{i+1} x^{i+1}).
\]

Therefore \( x^i R \cap K_1 = x^i K_{i+1} \) is a direct summand of \( R_R \).

Inasmuch as \( xyx = x \), \( xR \cap K_1 = xK_2 \) is a direct summand of \( R_R \).
Claim 4. \((xR + K_m)/xR \cong K_1/x^mR \cap K_1\) for all \(m\).

Every right ideal involved here is a direct summand of \(R_R\) by Claims 2 and 3. We have the ascending and descending chains of direct summands

\[
xR \subset xR + K_1 \subset xR + K_2 \subset \ldots \subset xR + K_m,
\]

\[
K_1^{\oplus} \supset xR \cap K_1^{\oplus} \supset x^2R \cap K_1^{\oplus} \supset \ldots \supset x^mR \cap K_1
\]

which give us the decompositions

\[
(xR + K_m)/xR \cong \bigoplus_{i=0}^{m-1} (xR + K_{i+1})/(xR + K_i),
\]

\[
K_1/(x^mR \cap K_1) \cong \bigoplus_{i=0}^{m-1} (x^iR \cap K_1)/(x^{i+1}R \cap K_1).
\]

So if we can show that

\[
(xR + K_{i+1})/(xR + K_i) \cong (x^iR \cap K_1)/(x^{i+1}R \cap K_1)
\]

for all \(i\), we are done.

First we note that

\[
(xR + K_{i+1})/(xR + K_i) = (xR + K_i + K_{i+1})/(xR + K_i)
\]

\[
\cong K_{i+1}/[(xR + K_i) \cap K_{i+1}] = K_{i+1}/[(xR \cap K_{i+1}) + K_i].
\]

As \(x^iK_{i+1} \subseteq x^iR \cap K_1\) and \(x^i[(xR \cap K_{i+1}) + K_i] \subseteq x^{i+1}R \cap K_1\), left multiplication by \(x^i\) gives a module homomorphism

\[
f : K_{i+1}/[(xR \cap K_{i+1}) + K_i] \rightarrow (x^iR \cap K_1)/(x^{i+1}R \cap K_1).
\]

The map \(f\) is epic: Pick any \(r \in x^iR \cap K_1\), \(r = x^ia\) for some \(a \in R\). But \(x^{i+1}a = xr = 0\); then \(a \in K_{i+1}\). So \(f(a) = r\).

The map \(f\) is monic: Suppose \(z \in K_{i+1}\) and \(x^i z \in x^{i+1}R \cap K_1\); then we have \(x^i z = x^{i+1}b\) for some \(b \in R\) and \(x^{i+1}b = x(x^i z) = 0\), whence \(xb \in K_{i+1} \cap xR\).

Since \(x^i(z - xb) = 0\), \(z - xb \in K_i\); thus \(z \in (xR \cap K_{i+1}) + K_i\), i.e., \(f\) is monic.

We have proved that \(f\) is an isomorphism.

Claim 5. \(x\) is unit-regular, i.e., there exists a unit \(u \in R\) such that \(xux = x\).

It follows from Claims 1 and 4 that

\[(xR + K_n)/xR = R/xR \cong K_1/x^nR \cap K_1 = K_1/0 = K_1.
\]

It is assumed that \(x\) is a unit in \(R\); hence

\[R = yxR \oplus K_1 = xR \oplus (1 - xy)R.
\]

So \(K_1 \cong (1 - xy)R\). Denote this isomorphism by \(\alpha\). Also, the restriction of the left multiplication by \(x\) gives an isomorphism \(\beta\) from \(yxR\) to \(xR\). Define \(u \in \text{end}(R_R) = R\) to be the direct sum of \(\alpha\) and \(\beta^{-1}\); it is easy to check that \(u\) is a unit in \(R\) and \(xux = x\). \(\square\)

The above proof actually proves the following more general statement:
Theorem 7. For an exchange ring $R$, if powers of every von Neumann regular element are von Neumann regular and for every von Neumann regular element $x \in R$ there exists an integer $n \geq 1$ such that $x^n R = x^{n+1} R$ and $R x^n = R x^{n+1}$, then $R$ has stable range one.

Proof. $x^n R = x^{n+1} R$ implies $x R + K_n = R$ and $R x^n = R x^{n+1}$ implies $K_n = K_{n+1}$; hence $x^n R \cap K_1 = 0$. So Claim 1 is valid. Claims 2 and 3 use only the property that powers of every regular element are regular and so are still valid in this case. Claims 4 and 5 have nothing to do with the strongly $\pi$-regularity of $R$ and therefore are also valid here. Finally, the conclusion follows from Theorem 3. \[ \square \]

We conclude this note by giving two examples. One shows that our generalization of Goodearl and Menal's result on strongly $\pi$-regular regular rings to Theorem 5 is nontrivial, the other shows that the converse of Theorem 5 is false.

Example 8. Let $F$ be any field, $R = \left( \begin{array}{cc} F & F \\ 0 & F \end{array} \right)$. 

$R$ is obviously Artinian and hence strongly $\pi$-regular. One checks that an element $x \in R$ is von Neumann regular in $R$ if and only if $x$ is not nilpotent. So powers of every regular element in $R$ are regular. But $J(R) = \left( \begin{array}{c} 0 \\ F \end{array} \right) \neq 0$, therefore $R$ is not regular. This shows that our generalization of Goodearl and Menal's result on regular strongly $\pi$-regular rings to Theorem 5 is nontrivial.

Example 9. Let $R$ be the $2 \times 2$ matrix ring over $F[x]/(x^2)$, where $F$ is a field.

Clearly, $R$ is a finite-dimensional algebra and hence strongly $\pi$-regular. Of course, $R$ has stable range 1. But not all the powers of every regular element are regular in $R$. Take $a = \left( \begin{array}{cc} 1 & 0 \\ 0 & x \end{array} \right)$ and $u = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$; it is easy to see that $a u a = a$. But $a^2 = \left( \begin{array}{cc} 0 & 1 \\ 1 & x \end{array} \right)$ is not regular. So the condition that powers of every regular element are regular is sufficient but not necessary for strongly $\pi$-regular rings to have stable range 1.

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REFERENCES


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