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STABLE RANGE ONE FOR RINGS WITH MANY IDEMPOTENTS

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ABSTRACT. An associative ring R is said to have stable range 1 if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is a unit. The purpose of this note is to prove the following facts. Theorem 3: An exchange ring R has stable range 1 if and only if every regular element of R is unit-regular. Theorem 5: If R is a strongly π-regular ring with the property that all powers of every regular element are regular, then R has stable range 1. The latter generalizes a recent result of Goodearl and Menal [5].

Let R be an associative ring with identity. R is said to *have stable range* 1 if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is a unit. This definition is left-right symmetric by Vaserstein [9, Theorem 2]. Furthermore, by a theorem of Kaplansky, all one-sided units are two-sided in rings having stable range 1 (cf. Vaserstein [10, Theorem 2.6]). It is well known that a (von Neumann) regular ring R has stable range 1 if and only if R is unit-regular (see, for example, Goodearl [4, Proposition 4.12]).

Call a ring R *strongly π-regular* if for every element $a \in R$ there exist a number n (depending on a) and an element $x \in R$ such that $a^n = a^{n+1}x$. This is in fact a two-sided condition [3]. It is an open question whether all strongly π-regular rings have stable range 1. Goodearl and Menal [5] proved that strongly π-regular rings are unit-regular and, hence, have stable range 1 (Theorem 5.8, p. 278).

In this note we first extend the above result for von Neumann regular rings to a larger class of rings, which includes all strongly π-regular rings, π-regular rings, von Neumann regular rings, and algebraic algebras. As an application of this, we prove that a strongly π-regular ring R has stable range 1 if powers of every regular element are regular. The latter is a generalization of the above-mentioned result of Goodearl and Menal for strongly π-regular regular rings. As one can see from our proofs, rings in these classes have a large supply of idempotents.

Throughout, R stands for an associative ring with identity and $J(R)$ for the Jacobson radical of R. Modules are unitary right R-modules except otherwise specified. For other undefined terms, readers are referred to [4].

Let M_R be a right R-module. Following Crawley and Jonsson [2], M_R is said to have the *exchange property* if for every module A_R and any two

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decompositions of A_R

$$A_R = M' \oplus N = \bigoplus_{i \in I} A_i$$

where $M'_R \cong M_R$, there exist submodules $A'_i \subseteq A_i$ such that

$$A = M' \oplus \left(\bigoplus_{i \in I} A'_i \right).$$

M_R is said to have the *finite exchange property* if the above condition is satisfied whenever the index set I is finite. Many familiar classes of modules have the exchange property or the finite exchange property, see Zimmermann-Huisgen and Zimmermann [12] for a list of these classes of modules.

Warfield [8] introduced the class of exchange rings. He called a ring R an *exchange ring* if R_R has the exchange property above and proved that this definition is left-right symmetric. The class of exchange rings is quite large. Call a ring R *semiregular* (*semi-π-regular, semi-strongly π-regular*) if $R/J(R)$ is regular (π-regular , strongly π-regular) and idempotents can be lifted modulo $J(R)$. It is easy to verify that the following classes of rings (in the order of containments) are all contained in the class of exchange rings: (1) local rings; (2) semiperfect rings; (3) semiregular rings; (4) semistrongly π-regular rings; (5) semi-π-regular rings (see, for example, Stock [7, p. 440]).

The following characterizations of the finite exchange property for projective modules were given by Nicholson [6, Proposition 2.9].

Lemma 1 (Nicholson). *The following conditions are equivalent for a projective module P:*

(1) *P has the finite exchange property.*

(2) *If $P = M_1 + M_2 + \cdots + M_n$ where M_i are submodules, there is a decomposition $P = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ with $P_i \subseteq M_i$ for each i.*

(3) *If $P = M + N$ where M and N are submodules, there exists a summand P_1 of P such that $P_1 \subseteq M$ and $P = P_1 + N$.* □

The original definition of stable range 1 for an arbitrary ring R is equivalent to the condition that for any $a, x, b \in R$ satisfying $ax + b = 1$, there exists $y \in R$ such that $a + by$ is a unit in R. The next lemma says that, for exchange rings, the element b in the latter condition can be further restricted to idempotents.

Lemma 2. *Let R be an exchange ring, then the following conditions are equivalent:*

(1) *R has stable range 1.*

(2) *For any $a \in R, e^2 = e \in R$, if $ax + e = 1$ for some $x \in R$, then there exists $y \in R$ such that $a + ey = u$ is a unit.*

Proof. (1) $\Rightarrow$ (2): Trivial. (2) $\Rightarrow$ (1): Assume that $aR + bR = R$. R is exchange, there exists an idempotent $e^2 = e \in bR$ such that $(1-e)R \oplus eR = R$ where $(1-e)R \subseteq aR$ and $eR \subseteq bR$, by Lemma 1. So we have $ax + e = 1$ for some $x \in R$. By assumption, there exists $y \in R$ such that $a + ey = u$ is a unit; hence, $a + bry = u$ is a unit where $e = br$. □

Recall that for a regular ring R, R has stable range 1 if and only if R is unit-regular. We now extend this to exchange rings.

Theorem 3. *An exchange rings R has stable range 1 if and only if every regular element of R is unit-regular in R.*

Proof. $\Rightarrow$: Let $axa = a$; then $ax + (1 - ax) = 1$. By the assumption on R, there exists $y \in R$ such that $a + (1 - ax)y = u$ is a unit in R. Multiplying both sides of the latter equality by ax on the left, we have that $axa = a = axu$, $au^{-1} = ax$, so $au^{-1}a = a$, and a is unit-regular.

$\Leftarrow$: By Lemma 2, we need only show that if $ax + e = 1$ with $e^2 = e$, there exists an element $y \in R$ such that $a + ey$ is a unit.

We first show that, without loss of generality, we may assume $axa = a$. In fact, if $axa \neq a$, put $f = ax$ and $r = fa - a$; then $rx = 0$. Letting $a' = a + r$, we have $a'x = ax + rx = ax + 0 = ax = f$, $a'xa' = fa' = fa + fr = fa + 0 = a + r = a'$. To see that $fr = 0$, notice that $f = ax = 1 - e$ is an idempotent. Now if $a' + ey$ is a unit for some $y \in R$ and $fr = 0$ implies $r \in (1-f)R = eR$, we have

$$a' + ey = a + r + ey = a + es + ey = a + e(s + y)$$

is a unit.

So we can assume that $ax + e = 1$, where $e^2 = e$ and $axa = a$. Notice that $axa = a$ if and only if $ea = 0$. Since we assume that every regular element is unit-regular, there exists a unit $u \in R$ such that $aua = a$. Then we have $1 - e = ax = (aua)x = (au)(ax) = au(1 - e)$.

$$\begin{aligned}(au - e)^2 &= (au - e)(au - e) = auau - aue - eau + e \\ &= au - aue - 0 + e = au(1 - e) + e = 1.\end{aligned}$$

So $au - e = v$ is a unit, therefore $a - eu^{-1} = vu^{-1}$ is a unit. □

For some other equivalent characterizations of stable range 1 for exchange rings, see Yu [11, Theorem 9].

While the question of whether all strongly π-regular rings have stable range 1 remains open, we now can reduce this to a unit-regularity problem.

Corollary 4. *A strongly π-regular ring R has stable range 1 if and only if every regular element of R is unit-regular in R.* □

Corollary 4 should be compared with an analogous result of Goodearl and Menal [5, Theorem 6.1], which says that a strongly π-regular ring R has stable range 1 if and only if every nilpotent regular element of each corner of R is unit-regular in that corner. By a *corner* of a ring R, they mean any (nonunital) subring eRe where e is an idempotent in R. While it is true that an element $x \in eRe$ is regular in eRe if and only if it is regular in R, the same is not true for unit-regularity.

One of the known cases where strongly π-regular rings have stable range 1 was Theorem 5.8 of Goodearl and Menal [5]: a strongly π-regular regular ring has stable range 1. As an application of our Theorem 3, we now extend this to the following:

Theorem 5. *Let R be a strongly π-regular ring. If all powers of every regular element are regular, then R has stable range 1.*

The proof we are going to give is a modification of Goodearl and Menal's proof in [5]. In order to make our paper self-contained, we present here a complete proof, although a portion of it is just a verbatim adoption of their argument

in von Neumann regular context. Also, we need the following lemma, which is a combination of Azumaya's Theorem 3 in [1] and Dischinger's Theorem 1 in [3].

Lemma 6 (Azumaya, Dischinger). *For every element $x \in R$ of a strongly π-regular ring R, there exist $a \in R$ and an integer $n \geq 1$ such that $xa = ax$ and $x^n = ax^{n+1} = x^{n+1}a$.* □

Proof of Theorem 5. Let $x \in R$ be a regular element with $xyx = x$. It suffices to prove, by Corollary 4, that x is unit-regular.

Set $K_i = r.ann(x^i)$ for all $i = 0, 1, 2, \ldots$.

Claim 1. There exists an integer $n \geq 1$ such that $xR + K_n = R$ and $x^n R \cap K_1 = 0$.

In fact, by Lemma 6, there exist an integer $n \geq 1$ and $a \in R$ such that $ax = xa$ and $x^n = ax^{n+1} = x^{n+1}a$. Pick any $r \in R$, we have $x^n r = x^{n+1}ar$ and $x^n(r - xar) = 0$; thus $r - xar \in K_n$ and $r \in xR + K_n$. Therefore, $xR + K_n = R$.

Since $x^n = ax^{n+1}$, it is clear that $K_n = K_{n+1}$. If $x^n d \in x^n R \cap K_1$, then $xx^n d = 0$ and $d \in K_{n+1} = K_n$, so $x^n d = 0$, i.e., $x^n R \cap K_1 = 0$.

Claim 2. $xR + K_i$ are direct summands of R_R for all $i \geq 1$.

Since x^i is regular for all $i \geq 2$ by our assumption on R, we may assume $x^i y_i x^i = x^i$ for some $y_i \in R$ for $i \geq 2$. Then $K_i = (1 - y_i x^i)R$. It is easy to check that

$$xR + (1 - y_i x^i)R = y_i x^i xR + (1 - y_i x^i)R.$$

We check below that the element $y_i x^i x$ is actually von Neumann regular:

$$y_i x^i x \cdot y_{i+1} x^i \cdot y_i x^i x = y_i x^i x y_{i+1} x^i x = y_i x^i x.$$

Put $e_i = y_i x^i x y_{i+1} x^i$ and $f_i = 1 - y_i x^i$, then $e_i f_i = f_i e_i = 0$. We see that e_i and f_i are orthogonal idempotents, hence $e_i + f_i$ is an idempotent. But $y_i x^i xR = e_i R$, so $xR + K_i = e_i R + f_i R = (e_i + f_i)R$ is a direct summand of R_R.

Recall that we assume $xyx = x$, so $xR + K_1$ is a direct summand of R_R for the same reason.

Claim 3. $x^i R \cap K_1$ are all direct summands of R_R for all $i \geq 1$.

First, we show $x^i R \cap K_1 = x^i K_{i+1}$. Since $x^i K_{i+1} \subset x^i R$ and $x^i K_{i+1} \subset K_1$, $x^i K_{i+1} \subset x^i R \cap K_1$; on the other hand, pick any $x^i r \in x^i R \cap K_1$, $xx^i r = x^{i+1} r = 0$, $r \in K_{i+1}$, so $x^i r \in x^i K_{i+1}$, $x^i R \cap K_1 \subset x^i K_{i+1}$.

Second, recall that we assume $x^i y_i x^i = x^i$, so that $K_{i+1} = (1 - y_{i+1} x^{i+1})R$, and we see that $x^i R \cap K_1 = x^i K_{i+1} = x^i(1 - y_{i+1} x^{i+1})R$. We check below that $x^i(1 - y_{i+1} x^{i+1})$ is von Neumann regular:

$$\begin{aligned}
x^i(1 - y_{i+1}x^{i+1}) \cdot y_i \cdot x^i(1 - y_{i+1}x^{i+1}) &= (x^i - x^i y_{i+1} x^{i+1}) y_i x^i (1 - y_{i+1} x^{i+1}) \\
&= (1 - x^i y_{i+1} x) x^i y_i x^i (1 - y_{i+1} x^{i+1}) = (1 - x^i y_{i+1} x) x^i (1 - y_{i+1} x^{i+1}) \\
&= (x^i - x^i y_{i+1} x^{i+1})(1 - y_{i+1} x^{i+1}) = x^i (1 - y_{i+1} x^{i+1})(1 - y_{i+1} x^{i+1}) \\
&= x^i (1 - y_{i+1} x^{i+1}).
\end{aligned}$$

Therefore $x^i R \cap K_1 = x^i K_{i+1}$ is a direct summand of R_R.

Inasmuch as $xyx = x$, $xR \cap K_1 = xK_2$ is a direct summand of R_R.

Claim 4. $(xR+K_m)/xR \cong K_1/x^mR \cap K_1$ for all m.

Every right ideal involved here is a direct summand of R_R by Claims 2 and 3. We have the ascending and descending chains of direct summands

$$xR \subset^{\oplus} xR+K_1 \subset^{\oplus} xR+K_2 \subset^{\oplus} \cdots \subset^{\oplus} xR+K_m,$$
$$K_1^{\oplus} \supset xR \cap K_1^{\oplus} \supset x^2R \cap K_1^{\oplus} \supset \cdots^{\oplus} \supset x^mR \cap K_1$$

which give us the decompositions

$$(xR+K_m)/xR \cong \bigoplus_{i=0}^{m-1} (xR+K_{i+1})/(xR+K_i),$$
$$K_1/(x^mR \cap K_1) \cong \bigoplus_{i=0}^{m-1} (x^iR \cap K_1)/(x^{i+1}R \cap K_1).$$

So if we can show that

$$(xR+K_{i+1})/(xR+K_i) \cong (x^iR \cap K_1)/(x^{i+1}R \cap K_1)$$

for all i, we are done.

First we note that

$$\begin{aligned}(xR+K_{i+1})/(xR+K_i) &= (xR+K_i+K_{i+1})/(xR+K_i)\\ &\cong K_{i+1}/[(xR+K_i)\cap K_{i+1}] = K_{i+1}/[(xR\cap K_{i+1})+K_i].\end{aligned}$$

As $x^iK_{i+1} \subseteq x^iR \cap K_1$ and $x^i[(xR \cap K_{i+1})+K_i] \subseteq x^{i+1}R \cap K_1$, left multiplication by x^i gives a module homomorphism

$$f\colon K_{i+1}/[(xR\cap K_{i+1})+K_i] \to (x^iR\cap K_1)/(x^{i+1}R\cap K_1).$$

The map f is epic: Pick any $r \in x^iR \cap K_1$, $r = x^ia$ for some $a \in R$. But $x^{i+1}a = xr = 0$; then $a \in K_{i+1}$. So $f(a) = r$.

The map f is monic: Suppose $z \in K_{i+1}$ and $x^iz \in x^{i+1}R \cap K_1$; then we have $x^iz = x^{i+1}b$ for some $b \in R$ and $x^{i+1}b = x(x^iz) = 0$, whence $xb \in K_{i+1} \cap xR$. Since $x^i(z-xb) = 0$, $z-xb \in K_i$; thus $z \in (xR\cap K_{i+1})+K_i$, i.e. f is monic.

We have proved that f is an isomorphism.

Claim 5. x is unit-regular, i.e., there exists a unit $u \in R$ such that $xux = x$.

It follows from Claims 1 and 4 that

$$(xR+K_n)/xR = R/xR \cong K_1/(x^nR\cap K_1) = K_1/0 = K_1.$$

It is assumed that $xyx = x$; hence

$$R = yxR \oplus K_1 = xR \oplus (1-xy)R.$$

So $K_1 \cong (1-xy)R$. Denote this isomorphism by α. Also, the restriction of the left multiplication by x gives an isomorphism β from yxR to xR. Define $u \in \mathrm{end}(R_R) = R$ to be the direct sum of α and β^{-1}; it is easy to check that u is a unit in R and $xux = x$. □

The above proof actually proves the following more general statement:

Theorem 7. *For an exchange ring R, if powers of every von Neumann regular element are von Neumann regular and for every von Neumann regular element $x \in R$ there exists an integer $n \geq 1$ such that $x^n R = x^{n+1} R$ and $Rx^n = Rx^{n+1}$, then R has stable range one.*

Proof. $x^n R = x^{n+1} R$ implies $xR + K_n = R$ and $Rx^n = Rx^{n+1}$ implies $K_n = K_{n+1}$; hence $x^n R \cap K_1 = 0$. So Claim 1 is valid. Claims 2 and 3 use only the property that powers of every regular element are regular and so are still valid in this case. Claims 4 and 5 have nothing to do with the strongly π-regularity of R and therefore are also valid here. Finally, the conclusion follows from Theorem 3. □

We conclude this note by giving two examples. One shows that our generalization of Goodearl and Menal's result on strongly π-regular regular rings to Theorem 5 is nontrivial, the other shows that the converse of Theorem 5 is false.

Example 8. Let F be any field, $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$.

R is obviously Artinian and hence strongly π-regular. One checks that an element $x \in R$ is von Neumann regular in R if and only if x is not nilpotent. So powers of every regular element in R are regular. But $J(R) = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right) \neq 0$, therefore R is not regular. This shows that our generalization of Goodearl and Menal's result on regular strongly π-regular rings to Theorem 5 is nontrivial.

Example 9. Let R be the 2×2 matrix ring over $F[x]/(x^2)$, where F is a field.

Clearly, R is a finite-dimensional algebra and hence strongly π-regular. Of course, R has stable range 1. But not all the powers of every regular element are regular in R. Take $a = \left(\begin{smallmatrix} 0 & 1 \\ 0 & x \end{smallmatrix}\right)$ and $u = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; it is easy to see that $aua = a$. But $a^2 = \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right)$ is not regular. So the condition that powers of every regular element are regular is sufficient but not necessary for strongly π-regular rings to have stable range 1.

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