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ON THE COHOMOLOGY OF Γ_p

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ABSTRACT. Let Γ_g denote the mapping class group of genus g. In this paper, we calculate p-torsion of Farrell cohomology $\widehat{H}^*(\Gamma_p)$ for any odd prime p.

INTRODUCTION

The mapping class group Γ_g^s of a connected oriented surface F_g^s of genus g with s punctures is defined as the group of connected components of the group of orientation-preserving diffeomorphisms of F_g^s which possibly permute s punctures. We will also denote Γ_g^0 simply by Γ_g. The cohomology $H^*(\Gamma_g)$ is one of the central topics in contemporary mathematics since it is closely related to algebraic topology, algebraic geometry, the theory of Riemann surfaces, the theory of three-dimensional manifolds, the theory of combinatorial groups and physics. It is well known that Γ_1 is the special linear group $SL_2(\mathbb{Z})$ and the cohomology $H^*(\Gamma_1; \mathbb{Z}) = H^*(SL(2, \mathbb{Z}); \mathbb{Z}) = \mathbb{Z}[u]/\langle 12u\rangle$, where u is a generator of degree 2. The cohomology $H^*(\Gamma_2; \mathbb{Z})$ was completely calculated by Benson and Cohen in [BC]. Recently, Looijenga obtained $H^*(\Gamma_3; \mathbb{Q})$ with rational coefficient [L]. Recall that Farrell and ordinary cohomologies of Γ_3 coincide above the $\mathrm{vcd}(\Gamma_3) = 7$ (see [Br]). It is easy to see that the Farrell cohomology $\widehat{H}^*(\Gamma_3; \mathbb{Z})$ contains only 2, 3 and 7 torsion since Γ_3 does. The 7-component $\widehat{H}^*(\Gamma_3; \mathbb{Z})_{(7)}$ is included in a general result of $\widehat{H}^*(\Gamma_{(p-1)/2}; \mathbb{Z})_{(p)}$ by the author in [X1]. The 2-component $\widehat{H}^*(\Gamma_3; \mathbb{Z})_{(2)}$ is more difficult to calculate and remains open. In this note, we give the 3-component $\widehat{H}^*(\Gamma_3; \mathbb{Z})_{(3)}$.

Let π_1 and π_2 denote representatives of the two different conjugacy classes of order 3 subgroups of Γ_3. We describe explicitly the quotients $N(\pi_1)/\pi_1$ and $N(\pi_2)/\pi_2$ as finite index subgroups of Γ_1^2 and Γ_0^5, where $N(-)$ stands for the normalizer. The cohomology $H^*(\Gamma_1^2)$ is completely calculated. The Shapiro lemma and a result of Cohen about $H^*(\Gamma_0^5; \mathbb{Z})$ as Σ_5-module are employed for computing $H^*(N(\pi_1))_{(3)}$ and $H^*(N(\pi_2))_{(3)}$ respectively. Then, the Farrell cohomology $\widehat{H}^*(\Gamma_3; \mathbb{Z})_{(3)}$ follows immediately because Γ_3 is 3-periodic. It is generally believed that $\widehat{H}^*(\Gamma_g)$ (and $H^*(\Gamma_g)$) might be calculated inductively via $H^*(\Gamma_h^n)$'s $(h < g)$, the mapping class groups of lower genus with punctures. For a fixed prime $p > 2$, the first two genera g's such that Γ_g contains a cyclic subgroup of order p are $(p-1)/2$ and $p-1$. We have completed the

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calculations of the p-component of $\widehat{H}^*(\Gamma_{(p-1)/2};\mathbb{Z})$ and $\widehat{H}^*(\Gamma_{p-1};\mathbb{Z})$ in our previous papers [X1] and [X2] respectively. Next, the third genus g such that Γ_g contains a cyclic subgroup of order p is p. As one more successful example along these basic lines, we finish by calculating the p-component of $\widehat{H}^*(\Gamma_p;\mathbb{Z})$ for any prime $p \geq 3$ (not only $p=3$) in this note. Note that the 2-component of $H^*(\Gamma_2;\mathbb{Z})$ is given in [BC].

The main results of this note are as follows.

Theorem 5.4.
$$\widehat{H}^n(\Gamma_3;\mathbb{Z})_{(3)} = \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9$$
for $n \equiv 0 \pmod 4$;
$$\widehat{H}^n(\Gamma_3;\mathbb{Z})_{(3)} = \mathbb{Z}/3$$
for n *odd*;
$$\widehat{H}^n(\Gamma_3;\mathbb{Z})_{(3)} = \mathbb{Z}/3 \oplus \mathbb{Z}/9$$
for $n \equiv 2 \pmod 4$.

It is easy to see a dihedral subgroup D_{2p} of order $2p$ sitting in Γ_p for any prime $p > 2$.

Theorem 6.5. *For any prime* $p > 3$, *the restriction map*
$$R : \widehat{H}^n(\Gamma_p;\mathbb{Z})_{(p)} \to \widehat{H}^n(D_{2p};\mathbb{Z})_{(p)}$$
is an isomorphism for any n. *Namely,*
$$\widehat{H}^n(\Gamma_p;\mathbb{Z})_{(p)} = \mathbb{Z}/p$$
for $n \equiv 0 \pmod 4$;
$$\widehat{H}^n(\Gamma_p;\mathbb{Z})_{(p)} = 0$$
for other n*'s.*

The organization of the rest of this note is as follows. In section 1, we exactly describe two quotients $N(\pi_1)/\pi_1$ and $N(\pi_2)/\pi_2$ as finite index subgroups of Γ_1^2 and Γ_0^5. In section 2, we calculate $H^*(\Gamma_1^2)$. In sections 3 and 4, we compute $H^*(N(\pi_1)/\pi_1)$ and $H^*(N(\pi_2)/\pi_2)$ respectively. In section 5, we obtain $H^*(N(\pi_1))$, $H^*(N(\pi_2))$ and prove the main result, Theorem 5.4. In last section, we finish the proof of Theorem 6.2.

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1. The $N(\mathbb{Z}/3)/\mathbb{Z}/3$'s of Γ_3

Recall that for x an orientation-preserving periodic diffeomorphism of a closed orientable surface F_g of prime period p, the fixed point data of x are a set (unordered) $\delta(x) = \langle \beta_1, \beta_2, \dots, \beta_q \rangle$, where q is the number of fixed points of x and β_i is the integer (mod p) such that x^{β_i} acts as multiplication by $e^{2\pi i/p}$ in the local invariant complex structure at the ith fixed point. The fixed point data are well defined for an element $\bar{x} \in \Gamma_g$ of period p too. According to a classical theorem of Nielsen, the conjugacy classes of elements of Γ_g of period p are exactly given by all possible fixed point data. It is easy to check that there are exactly two conjugacy classes of order 3 subgroups of Γ_3, the one with the fixed point data of a generator $\langle 1, 2\rangle$ is denoted as π_1 and the other with the fixed point data of a generator $\langle 1, 1, 1, 1, 2\rangle$ is denoted as π_2. The structure of quotients $N(\pi_1)/\pi_1$ and $N(\pi_2)/\pi_2$ are described as follows.

A result of MacLachlan and Harvey [MH] states that for a finite subgroup $G \subset \Gamma_g$ the quotient $N(G)/G$ maps injectively into the mapping class group Γ_h^q, where h is the genus of orbit space F_g/G, and q the number of singular points. It is clear in our cases that the quotients $N(\pi_1)/\pi_1$ and $N(\pi_2)/\pi_2$ are isomorphic to subgroups of mapping class groups Γ_1^2 and Γ_0^5 respectively. We give a more precise description now.

Consider a natural homomorphism

$$\lambda : \Gamma_h^n \to GL(n-1+2h, \mathbb{Z})$$

that is given by mapping a diffeomorphism $f \in \mathrm{Diff}_+(F_h; \{n\})$ to its action on $H_1(F_h - \{n\}; \mathbb{Z})$ with a base $\langle x_1, x_2, \dots, x_{n-1}, a_1, \dots, a_h, b_1, \dots, b_h\rangle$ in the obvious notation. The map λ is clearly not a surjection. An element of $\mathrm{Im}(\lambda)$ must be in the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

where $A \in G$ ($\cong \Sigma_n$, the symmetric group of n letters), $D \in Sp(2g, \mathbb{Z})$, the symplectic group. Reducing the group $GL(n-1+2h, \mathbb{Z})$ to a finite group $GL(n-1+2h, \mathbb{Z}/p)$ with coefficient in the field $\mathbb{Z}/p$, one gets a map $\tilde{\lambda} : \Gamma_h^n \to GL(n-1+2h, \mathbb{Z}/p)$. Actually, for any elementary abelian p subgroup $E \subset \Gamma_g$, the quotient $N(E)/E$ is isomorphic to a finite index subgroup of Γ_h^n, which is a preimage of a subgroup $K_E \subset GL(n-1+2h, \mathbb{Z}/p)$ under the map $\tilde{\lambda}$. The group K_E is specifically determined by some geometric data, for example, the fixed point data of E. The details of this general result will appear somewhere else. Here, only special cases of the quotients $N(\pi_1)/\pi_1$ and $N(\pi_2)/\pi_2$ are illustrated for the purpose of the calculation of $\widehat{H}^*(\Gamma_3; \mathbb{Z})_{(3)}$.

Consider the natural map

$$\tilde{\lambda} : \Gamma_1^2 \to GL(3, \mathbb{Z}/3)$$

defined as above for $h = 1$ and $n = 2$. Let K_1 denote a subgroup of $\mathrm{Im}(\tilde{\lambda})$ consisting of all elements of $GL(3, \mathbb{Z}/3)$ in the form of

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

with $A \in \{1, -1\}$, and $D \in SL(2, \mathbb{Z}/3)$.

Proposition 1.1. *The quotient* $N(\pi_1)/\pi_1$ *is isomorphic to* $\tilde{\lambda}^{-1}(K_1) \subset \Gamma_1^2$.

The following well-known lemma is needed in the proof of Proposition 1.1 above.

Lemma 1.2. *Let* $p : F_g \to F_h$ *be a* p*-sheeted branched covering map with* n *ramification points. Then a diffeomorphism* $w \in \mathrm{Diff}_+(F_h, \{n\})$ *lifts to a diffeomorphism* $w \in \mathrm{Diff}_+(F_g, \{n\})$ *if and only if every closed curve which lifts to a closed curve maps (via* w*) to a closed curve which lifts to a closed curve.*

Proof (of Proposition 1.1). Let $p : F_3 \to F_1$ be the 3-sheeted branched covering map with ramification points x_1 and x_2 induced by a generator of π_1

(strictly speaking, some lift of π_1 to $\text{Diff}_+(F_3, \{2\})$). We show that $w \in \text{Diff}_+(F_1, \{2\})$ lifts if and only if $\tilde{\lambda}(w) \in K_1$ (we abuse the notation w here). Let $f : \pi_1(F_1 - \{x_1, x_2\}) \to \pi_1$ be the surjective map determined by the map p. Up to conjugation of π_1, one could choose

$$f : \pi_1(F_1 - \{x_1, x_2\}) = \langle a, b, x_1, x_2 \mid [a; b]x_1x_2 = 1\rangle \to \pi_1 = \langle y\rangle$$

as $f(a) = f(b) = 1$, $f(x_1) = y$ and $f(x_2) = y^2$. The basic covering space theory says that a closed curve $\gamma \in F_1 - \{x_1, x_2\}$ lifts to a closed curve $\gamma' \in F_3 - \{\bar{x}_1, \bar{x}_2\}$ if and only if $f([\gamma]) = 1$, where $[-]$ stands for homotopy class here. Note that the set of surjective homomorphisms $\text{epi}(\pi_1(F_1 - \{x_1, x_2\}) \to \pi_1)$ is in one-to-one correspondence to the set of surjective homomorphisms $\text{epi}(H_1(F_1 - \{x_1, x_2\}; \mathbb{Z}) \to \pi_1)$ since the group π_1 is abelian. Let $\bar{\gamma} \in H_1(F_1 - \{x_1, x_2\}; \mathbb{Z})$ be the homology class of γ. Suppose $\bar{\gamma} = x_1^m a^{l_1} b^{l_2}$ and $\bar{f} : H_1(F_1 - \{x_1, x_2\}; \mathbb{Z}) \to \pi_1$ is induced by $f : \pi_1(F_1 - \{x_1, x_2\}) \to \pi_1$. It is easy to see that $\bar{f}(\bar{\gamma}) = 1$ is equivalent to $m \equiv 0 \pmod 3$. Let $\tilde{\lambda}(w)$ be denoted by

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

Then $\bar{f}(\bar{w}\bar{\gamma}) = 1$ is equivalent to $Am + BL = 0 \pmod 3$, where $\begin{pmatrix} m \\ L \end{pmatrix}$ is a 3-vector of $H_1(F_1 - \{x_1, x_2\}; \mathbb{Z})$ with

$$L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}.$$

Lemma 1.2 above says that w lifts is equivalent to the statement $\bar{f}(\bar{w}\bar{\gamma}) = 1$ if $\bar{f}(\bar{\gamma}) = 1$; i.e., $B = 0 \pmod 3$ because L could be an arbitrary two vector. We complete the proof.

Consider, for any n, the well-known map $\mu : \Gamma_0^n \to \Sigma_n$ defined via the permutation of $f \in \text{Diff}_+(S^2, \{n\})$ on n punctures. Recall that the quotient $N(\pi_2)/\pi_2$ is isomorphic to a subgroup of Γ_0^5. Then, one has

Proposition 1.3. *The quotient $N(\pi_2)/\pi_2$ is isomorphic to $\mu^{-1}(\Sigma_4) \subset \Gamma_0^5$.*

This proposition is a special case of Lemma 1.1 of [X2].

2. Cohomology of Γ_1^2

Let $P\Gamma_g^n$ denote the pure mapping class group of genus g with n punctures, i.e., the group of path components of orientation-preserving diffeomorphisms of a connected oriented surface F_g^n with n punctures which fix n punctures. Consider the group extension (see [Bi])

$$(1) \qquad 1 \to F(2) = \pi_1(F_1 - \{x_1\}) \to P\Gamma_1^2 \to P\Gamma_1^1 = SL(2, \mathbb{Z}) \to 1$$

given by forgetting one puncture, where $F(2)$ is the free group of 2 generators. The Lyndon-Hochschild-Serre spectral sequence (LHS^3) for the extension above is given by

$$E_2^{p,q} = H^p(SL(2, \mathbb{Z}); H^q(F(2); \mathbb{Z})) \Rightarrow H^{p+q}(P\Gamma_1^2; \mathbb{Z})$$

where $H^0(F(2); \mathbb{Z}) = \mathbb{Z}$ as a trivial $SL(2, \mathbb{Z})$ module; $H^1(F(2); \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ as the $SL(2, \mathbb{Z})$ module is obtained by the usual $SL(2, \mathbb{Z})$ action on $\mathbb{Z} \oplus \mathbb{Z}$.

It is well known that there is an amalgamated product decomposition $SL(2, \mathbb{Z}) = \mathbb{Z}/6 *_{\mathbb{Z}/2} \mathbb{Z}/4$. Choose generators $x \in \mathbb{Z}/6$, $y \in \mathbb{Z}/4$ and $z \in \mathbb{Z}/2$ as

$$x = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$$

and

$$z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A direct calculation gives

$$H^1(F(2); \mathbb{Z})^{\mathbb{Z}/6} = 0, \quad H^1(F(2); \mathbb{Z})_{\mathbb{Z}/6} = 0, \quad H^1(F(2); \mathbb{Z})^{\mathbb{Z}/4} = 0,$$
$$H^1(F(2); \mathbb{Z})_{\mathbb{Z}/4} = H^1(F(2); \mathbb{Z})/M_4 = \mathbb{Z}/2$$

where M_4 is a submodule consisting of all elements $\langle -2b, a-2b\rangle^T$ (a and b are integers);

$$H^1(F(2); \mathbb{Z})^{\mathbb{Z}/2} = 0$$

and

$$H^1(F(2); \mathbb{Z})_{\mathbb{Z}/2} = H^1(F(2); \mathbb{Z})/M_2 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

where M_2 is a submodule consisting of all elements $\langle -2a, -2b\rangle^T$. This implies

$$H^n(\mathbb{Z}/6; H^1(F(2); \mathbb{Z})) = 0$$

for any n;

$$H^{\mathrm{odd}}(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) = \mathbb{Z}/2,$$
$$H^{\mathrm{even}}(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) = 0$$

and

$$H^{\mathrm{odd}}(\mathbb{Z}/2; H^1(F(2); \mathbb{Z})) = \mathbb{Z}/2 \oplus \mathbb{Z}/2,$$
$$H^{\mathrm{even}}(\mathbb{Z}/2; H^1(F(2); \mathbb{Z})) = 0.$$

Applying the M-V sequence to the group $SL(2, \mathbb{Z})$ with module $H^1(F(2); \mathbb{Z})$, one gets a long exact sequence

$$\begin{aligned} &\to H^n(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) \oplus H^n(\mathbb{Z}/6; H^1(F(2); \mathbb{Z})) \\ &\quad \to H^n(\mathbb{Z}/2; H^1(F(2); \mathbb{Z})) \to H^{n+1}(SL(2, \mathbb{Z}); H^1(F(2); \mathbb{Z})) \\ &\quad \to H^{n+1}(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) \oplus H^{n+1}(\mathbb{Z}/6; H^1(F(2); \mathbb{Z})) \\ &\quad \to H^{n+1}(\mathbb{Z}/2; H^1(F(2); \mathbb{Z})) \to . \end{aligned}$$

Note that the restriction map

$$H^n(\mathbb{Z}/4; H^1(F(2); \mathbb{Z})) \to H^n(\mathbb{Z}/2; H^1(F(2); \mathbb{Z}))$$

is an injection. It follows that

$$H^n(SL(2, \mathbb{Z}); H^1(F(2); \mathbb{Z})) = 0$$

if $n = 0$ or odd; and

$$H^n(SL(2, \mathbb{Z}); H^1(F(2); \mathbb{Z})) = \mathbb{Z}/2$$

if $n > 0$ even. Recall $H^*(SL(2, \mathbb{Z}); \mathbb{Z}) = \mathbb{Z}[u]/\langle 12u\rangle$. One claims that the LHS[3] for (1) collapses by dimension reason. We conclude now

Proposition 2.1.

$$H^0(P\Gamma_1^2;\mathbb{Z})=\mathbb{Z},\quad H^{\mathrm{odd}}(P\Gamma_1^2;\mathbb{Z})=\mathbb{Z}/2,\quad H^{\mathrm{even}}(P\Gamma_1^2;\mathbb{Z})=\mathbb{Z}/12.$$

It is a routine to construct a $\mathbb{Z}/3$ action on Torus F_1 with three fixed points. This gives an order 3 subgroup $\pi\subset P\Gamma_1^2\subset\Gamma_1^2$. Proposition 2.1 tells that the restriction map $H^*(P\Gamma_1^2;\mathbb{Z})_{(3)}\to H^*(\pi;\mathbb{Z})_{(3)}$ is an isomorphism. Furthermore, the universal coefficient theorem implies that the restriction map $H^*(P\Gamma_1^2;M)_{(3)}\to H^*(\pi;M)_{(3)}$ is an isomorphism for any trivial $P\Gamma_1^2$-module M. Note $H^*(\Gamma_1^2;\mathbb{Z})_{(3)}=H^*(P\Gamma_1^2;\mathbb{Z})_{(3)}^{\Sigma_2}$. In order to show the restriction map $H^*(\Gamma_1^2;\mathbb{Z})_{(3)}\to H^*(\pi;\mathbb{Z})_{(3)}$ is an isomorphism too, we only need to show the 3-period of Γ_1^2 is 2. The general form of the 3-period of a group is $\mathrm{LCM}\{2\,|\,N(\pi)/C(\pi)|\}p^\alpha$ (see [GMX] for details). We know that $\alpha=0$ above from Proposition 2.1. Therefore, we only need to see the order $|N_{\Gamma_1^2}(\pi)/C_{\Gamma_1^2}(\pi)|=1$ in this case. Let $x\in\mathrm{Diff}_+(F_1,\{2\})$ denote a period 3 element with three fixed points. It is obvious that x is not conjugate to x^2 because they are not conjugate even mapping to $SL(2,\mathbb{Z})$. In summary, one obtains

Theorem 2.2. *The restriction map*

$$R:H^*(\Gamma_1^2;M)_{(3)}\to H^*(P\Gamma_1^2;M)_{(3)}\to H^*(\pi;M)_{(3)}$$

is an isomorphism for any trivial Γ_1^2-module M.

3. Cohomology of $N(\pi_1)/\pi_1$

Recall that we defined the map $\tilde{\lambda}:\Gamma_1^2\to GL(3,\mathbb{Z}/3)$ and a subgroup $K_1\subset GL(3,\mathbb{Z}/3)$ in section 1. Proposition 1.1 says the quotient $N(\pi_1)/\pi_1$ is isomorphic to $\tilde{\lambda}^{-1}(K_1)$. Let G denote the image of $\tilde{\lambda}$. Recall that any element of G must be in the form

$$\begin{pmatrix}A & B\\ 0 & D\end{pmatrix}$$

(see section 1 for details). We remark here that in our case G is exactly the group consisting of all such matrices. In fact, one can see from geometry that $\tilde{\lambda}(F(2))$ contains matrices

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}.$$

Thus, the index of K_1 in G is 9 and $F(2)$ acts on G/K_1 via the map $\tilde{\lambda}$ transitively. It is clear that $\Gamma_1^2/N(\pi_1)/\pi_1$ is in one-one correspondence to G/K_1 as cosets. By the well-known Shapiro lemma, one has $H^*(N(\pi_1)/\pi_1;\mathbb{Z})=H^*(\Gamma_1^2;\mathbb{Z}[G/K_1])$, where Γ_1^2 acts on the permutation module $\mathbb{Z}[G/K_1]$ via the map $\tilde{\lambda}$.

We have seen that Γ_1^2 contains a subgroup π of order 3 in section 2. However, one can show

Proposition 3.1. *The group $N(\pi_1)/\pi_1$ does not contain any subgroup of order* 3.

Proof. It is obvious from the Riemann-Hurwitz formula that Γ_3 does not contain $\mathbb{Z}/3\times\mathbb{Z}/3$. We only need to show that the third power 3 of any order 9 diffeomorphism of F_3 has five fixed points, not two fixed points like a lift of π_1. This again follows directly from the Riemann-Hurwitz formula.

Proposition 3.1 above implies that the permutation module $\mathbb{Z}[G/K_1]$ is not the trivial module $\mathbb{Z}$ and π_1 acts on $\mathbb{Z}[G/K_1]$ (by multiplication) nontrivially. It is elementary to observe that

Lemma 3.2. *The group π_1 acts on the coset G/K_1 freely.*

Proof. If not, assume that $x \in \pi_1$ fixes $\bar{g} \in G/K_1$; i.e., $xgk = gk'$, or $g^{-1}xg = k'k^{-1} \in K_1$. This contradicts Proposition 3.1.

Therefore, one has the invariant $\mathbb{Z}[G/K_1]^{\pi_1} = \bigoplus \mathbb{Z}\langle \bar{n}_i \rangle$, where $\bar{n}_i = \bar{g}_i + x\bar{g}_i + x^2\bar{g}_i$ for some g_i $(1 \le i \le 3)$ in this case. The co-invariant $\mathbb{Z}[G/K_1]_{\pi_1} = \mathbb{Z}[G/K_1]/M_1 = \bigoplus \mathbb{Z}$ spanned by $\bar{g}_i$'s. A direct computation implies the normal map

$$N : \mathbb{Z}[G/K_1]_{\pi_1} \to \mathbb{Z}[G/K_1]^{\pi_1}$$

is an isomorphism. So, one gets

Proposition 3.3. $H^n(\pi_1; \mathbb{Z}[G/K_1]) = 0$ *for* $n > 0$.

Consider the LHS[3] given by

$$E_2^{p,q} = H^p(SL(2,\mathbb{Z}); H^q(F(2); \mathbb{Z}[G/K_1])) \Rightarrow H^{p+q}(P\Gamma_1^2; \mathbb{Z}[G/K_1])$$

for the extension (1) with coefficient $\mathbb{Z}[G/K_1]$.

It is immediate from Proposition 3.3 and the M-V sequence that

Proposition 3.4. $H^n(SL(2,\mathbb{Z}); \mathbb{Z}[G/K_1]^{F(2)})_{(3)} = 0$ *for* $n > 0$.

Note that the $SL(2,\mathbb{Z})$ acts on

$$H^1(F(2); \mathbb{Z}[G/K_1]) = H^1(\mathbb{Z}; \mathbb{Z}[G/K_1]) \oplus H^1(\mathbb{Z}; \mathbb{Z}[G/K_1])$$

as matrix multiplications given in Section 2. One obtains

Proposition 3.5. $H^n(SL(2,\mathbb{Z}); H^1(F(2); \mathbb{Z}[G/K_1]))_{(3)} = 0$ *for* $n > 0$.

Combining Propositions 3.4 and 3.5, one concludes

Proposition 3.6. $H^n(N(\pi_1)/\pi_1; \mathbb{Z})_{(3)} = 0$ *for any* $n \ge 0$.

Repeating the argument above with $\mathbb{Z}/3$ coefficient, one gets

Proposition 3.7. $H^n(N(\pi_1)/\pi_1; \mathbb{Z}/3) = 0$ *for* $n > 0$.

A similar proof of Proposition 2.1 and the Shapiro lemma give

Proposition 3.8. $H^n(N(\pi_1)/\pi_1; \mathbb{Z})$ *does not contain any copy of* $\mathbb{Z}$ *for* $n > 0$.

4. Cohomology of $N(\pi_2)/\pi_2$

Consider the group extension

$$(2) \qquad 1 \to P\Gamma_0^5 \to N(\pi_2)/\pi_2 \to \Sigma_4 \to 1$$

described in Proposition 1.3. The LHS[3] for the extension above is given by

$$E_2^{p,q} = H^p(\Sigma_4; H^q(P\Gamma_0^5; \mathbb{Z}/3)) \Rightarrow H^{p+q}(N(\pi_2)/\pi_2; \mathbb{Z}/3)$$

where Σ_4 acts on $H^q(P\Gamma_0^5; \mathbb{Z}/3)$ as shown in work of Cohen (the $P\Gamma_0^5$ is denoted by K_5 in [BC]). Recall that $H^*(P\Gamma_0^5; \mathbb{Z}/3)$ is generated by one-dimensional elements B_{42}, B_{43}, B_{52}, B_{53} and B_{54} subject to some relations specifically given in [BC]. Let $x = (123) \in \Sigma_4$ be a generator of a Sylow 3-subgroup. It is a routine to have

$$H^1(P\Gamma_0^5; \mathbb{Z}/3)^{\langle x \rangle} = \mathbb{Z}/3 \oplus \mathbb{Z}/3 = \langle 2B_{42} + B_{43}, B_{52} + 2B_{53} \rangle$$

and

$$H^1(P\Gamma_0^5; \mathbb{Z}/3)_{\langle x \rangle} = H^1(P\Gamma_0^5; \mathbb{Z}/3)/M_5$$

where the submodule M_5 consists of all elements in the form

$$(m_1 + m_2 + m_5)B_{42} + (2m_2 - m_1)B_{43} + (m_3 + m_4 - m_5)B_{52} + (2m_4 - m_3 - m_5)B_{53}$$

with $m_i \in \mathbb{Z}/3$. Let $b_1 = m_1 + m_2 + m_5$, $b_2 = 2m_2 - m_1$, $b_3 = m_3 + m_4 - m_5$ and $b_4 = 2m_4 - m_3 - m_5$. Elementary linear algebra implies $3m_1 = 2b_1 - b_2 - 2m_5 = 0$, $3m_2 = b_1 + b_2 - m_5 = 0$, $3m_3 = 2b_3 - b_4 + m_5$ and $3m_4 = b_3 + b_4 + 2m_5 = 0$. Thus, the equation $b_1 + b_2 + 2b_3 + 2b_4 = 0$ holds. This amounts to showing

$$H^1(P\Gamma_0^5; \mathbb{Z}/3)_{\langle x \rangle} = \mathbb{Z}/3 \oplus \mathbb{Z}/3$$

generated by $\langle \overline{B}_{54}, \overline{B}_{42} \rangle$. It is easy to check that the normal map

$$N : H^1(P\Gamma_0^5; \mathbb{Z})_{\langle x \rangle} \to H^1(P\Gamma_0^5; \mathbb{Z})^{\langle x \rangle}$$

is given by $N(\overline{B}_{54}) = B_{42} + 2B_{43} + B_{52} + 2B_{53}$ and $N(\overline{B}_{42}) = 0$. Thus, one obtains

Lemma 4.1.

$$\begin{aligned} H^0(\langle x \rangle; H^1(P\Gamma_0^5; \mathbb{Z}/3)) &= \mathbb{Z}/3 \oplus \mathbb{Z}/3, \\ H^{\text{odd}}(\langle x \rangle; H^1(P\Gamma_0^5; \mathbb{Z}/3)) &= \mathbb{Z}/3, \\ H^{\text{even}}(\langle x \rangle; H^1(P\Gamma_0^5; \mathbb{Z}/3)) &= \mathbb{Z}/3. \end{aligned}$$

Consider the x action on $H^2(P\Gamma_0^5; \mathbb{Z}/3)$; one gets the invariant

$$H^2(P\Gamma_0^5; \mathbb{Z}/3)^{\langle x \rangle} = \mathbb{Z}/3 \oplus \mathbb{Z}/3 = \langle B_{42}B_{53} + 2B_{43}B_{52}, B_{42}B_{52} + B_{43}B_{52} + B_{43}B_{53} \rangle$$

and the co-invariant

$$H^2(P\Gamma_0^5; \mathbb{Z})_{\langle x \rangle} = H^2(P\Gamma_0^5; \mathbb{Z})/M_5$$

where the submodule M_5 consists of all elements in the form

$$\begin{aligned} (m_1 - m_5 + m_6)B_{42}B_{52} &+ (m_2 + m_4 - m_5 + m_6)B_{42}B_{53} \\ &+ (m_3 + m_6)B_{42}B_{54} + (m_2 - m_3 + m_4 - m_5 + m_6)B_{43}B_{52} \\ &+ (m_2 - m_1 - m_3 + m_4 + m_6)B_{43}B_{53} + (-m_3 + 2m_6)B_{43}B_{54} \end{aligned}$$

with $m_i \in \mathbb{Z}/3$. Let $b_1 = m_1 - m_2 + m_6$, $b_2 = m_2 + m_4 - m_5 + m_6$, $b_3 = m_3 + m_6$, $b_4 = m_2 - m_3 + m_4 - m_5 + m_6$, $b_5 = -m_1 + m_2 - m_3 + m_4 + m_6$ and $b_6 = -m_3 + 2m_6$. It is easy to have from linear algebra that $-2b_3 + b_6 = 0$, $b_1 + 2b_2 - b_3 - 2b_4 + b_5 = 0$, $m_1 = b_1 + b_2 - b_3 - b_4 + m_5$, $m_2 = 2b_2 - b_3 - b_4 - m_4 + m_5$, $m_3 = b_2 - b_4$ and $m_6 = -b_2 + b_3 + b_4$. Thus, one gets

$$H^2(P\Gamma_0^5; \mathbb{Z}/3)_{\langle x \rangle} = \mathbb{Z}/3 \oplus \mathbb{Z}/3$$

generated by $\langle \overline{B}_{42}\overline{B}_{52}, \overline{B}_{43}\overline{B}_{54} \rangle$. Also, it is straightforward to check that the normal map

$$N : H^2(P\Gamma_0^5; \mathbb{Z}/3)_{\langle x \rangle} \to H^2(P\Gamma_0^5; \mathbb{Z}/3)^{\langle x \rangle}$$

given by

$$N(\overline{B}_{42}\overline{B}_{52}) = 2B_{42}B_{52} + B_{42}B_{53} + B_{43}B_{52} + 2B_{43}B_{52}$$

and

$$N(\overline{B}_{43}\overline{B}_{54}) = -B_{42}B_{52} - B_{43}B_{52} - B_{43}B_{53}$$

is an isomorphism. This implies

Lemma 4.2. *$H^0(\langle x\rangle; H^2(P\Gamma_0^5; \mathbb{Z}/3)) = \mathbb{Z}/3\oplus\mathbb{Z}/3$ and $H^n(\langle x\rangle; H^2(P\Gamma_0^5; \mathbb{Z}/3)) = 0$ for $n > 0$.*

Recall that $H^1(P\Gamma_0^5; \mathbb{Z}/3)_{\langle x\rangle}$ is generated by $\overline{B}_{54}$ and $\overline{B}_{42}$. We can check directly that $(12) \in \Sigma_4$ permutes $\overline{B}_{54}$ to $\overline{B}_{54} - \overline{B}_{42}$ and $\overline{B}_{42}$ to $-\overline{B}_{42}$; that is, $H^1(P\Gamma_0^5; \mathbb{Z}/3)_{\langle x\rangle}^{(12)} = 0$. It is also straightforward to verify $(12) \in \Sigma_4$ acts on generators $2B_{42} + B_{43}$ and $B_{52} + 2B_{53}$ of $H^1(P\Gamma_0^5; \mathbb{Z}/3)^{\langle x\rangle}$ trivially and acts on the one-dimensional space generated by

$$2B_{42}B_{52} + B_{43}B_{52} + B_{42}B_{53} + 2B_{43}B_{53} \in H^2(P\Gamma_0^5; \mathbb{Z}/3)^{\langle x\rangle}$$

trivially. These calculations imply

Lemma 4.3.

$$H^0(\Sigma_4; H^1(P\Gamma_0^5; \mathbb{Z}/3)) = \mathbb{Z}/3 \oplus \mathbb{Z}/3,$$
$$H^0(\Sigma_4; H^2(P\Gamma_0^5; \mathbb{Z})) = \mathbb{Z}/3,$$
$$H^n(\Sigma_4; H^1(P\Gamma_0^5; \mathbb{Z}/3)) = \mathbb{Z}/3$$

for $n \equiv 0, 1 \pmod 4$;

$$H^n(\Sigma_4; H^1(P\Gamma_0^5; \mathbb{Z}/3)) = 0$$

for $n \equiv 2, 3 \pmod 4$.

It is easy to see a $\mathbb{Z}/3 \subset N(\pi_2)/\pi_2 \subset \Gamma_0^5$ by constructing a $\mathbb{Z}/3$ action on S^2 with two fixed points and permuting three points. The following lemma is needed for the study of LHS^3 associated to the extension (2) in the beginning of this section.

Lemma 4.4. *The group $N(\pi_2)/\pi_2$ has the $\mathbb{Z}/3$ as a retract.*

Proof. Recall the group $N(\pi_2)/\pi_2$ is an extension of $P\Gamma_0^5$ over Σ_4. There is a surjective map by forgetting the fifth puncture from $N(\pi_2)/\pi_2$ to Γ_0^4, therefore, to $H_1(\Gamma_4; \mathbb{Z}) = \mathbb{Z}/2 \oplus \mathbb{Z}/3$, to $\mathbb{Z}/3$. Note the $\mathbb{Z}/3 \subset N(\pi_2)/\pi_2$ is compatible with the $\mathbb{Z}/3 \subset \Gamma_0^4$. The lemma follows since Γ_0^4 has the $\mathbb{Z}/3$ as a retract.

Now, one can conclude the LHS^3 collapses by Lemma 4.4 and

Proposition 4.5.

$$H^0(N(\pi_2)/\pi_2; \mathbb{Z}/3) = \mathbb{Z}/3,$$
$$H^n(N(\pi_2)/\pi_2; \mathbb{Z}/3) = \mathbb{Z}/3 \oplus \mathbb{Z}/3$$

if $n = 1, 2$;

$$H^n(N(\pi_2)/\pi_2; \mathbb{Z}/3) = \mathbb{Z}/3$$

if $n \geq 3$.

Repeat the calculation in this section above with coefficient $\mathbb{Z}$ and consider LHS^3 for the extension (2) with coefficient $\mathbb{Z}$; one gets

Proposition 4.6. *The restriction map*

$$R : H^n(N(\pi_2)/\pi_2\,;\,\mathbb{Z})_{(3)} \to H^n(\mathbb{Z}/3\,;\,\mathbb{Z})_{(3)}$$

induces an isomorphism; the group $H^n(N(\pi_2)/\pi_2\,;\,\mathbb{Z})$ *contains exactly one copy of* $\mathbb{Z}$ *for* $n = 0, 1, 2$ *and contains no copy of* $\mathbb{Z}$ *for* $n \geq 3$.

5. Farrell cohomology of Γ_3

We actually calculate not only the 3-components of

$$H^*(N(\pi_1)\,;\,\mathbb{Z}) \quad \text{and} \quad H^*(N(\pi_2)\,;\,\mathbb{Z}),$$

but also their free parts. Consider the group extensions

$$1 \to \pi_1 \to N(\pi_1) \to N(\pi_1)/\pi_1 \to 1$$

and

$$1 \to \pi_2 \to N(\pi_2) \to N(\pi_2)/\pi_2 \to 1.$$

One has the LHS[3] for the extensions above giving as

$$E_2^{p,q} = H^p(N(\pi_i)/\pi_i\,;\,H^q(\pi_i\,;\,\mathbb{Z})) \Rightarrow H^{p+q}(N(\pi_i)\,;\,\mathbb{Z}).$$

Note that the group $N(\pi_1)$ acts on π_1 nontrivially and the group $N(\pi_2)$ acts on π_2 trivially from the observation of the fixed point data of generators of π_1 and π_2.

It is easy to see a dihedral subgroup $D_6 \subset \Gamma_3$ of order 6 containing the π_1 by realizing a D_6 action on F_3 with four singular points of order 2 and one singular point of order 3 in the orbit space $F_3/D_6 = S^2$ (2 sphere). The following proposition is immediate.

Proposition 5.1.

(1) *The restriction map*

$$R : H^n(N(\pi_1)\,;\,\mathbb{Z})_{(3)} \to H^n(D_6\,;\,\mathbb{Z})_{(3)}$$

is an isomorphism for any $n \geq 0$.

(2) $H^n(N(\pi_1)\,;\,\mathbb{Z})$ *does not contain any* $\mathbb{Z}$ *for* $n > 0$.

Again, it is clear that the π_2 is contained in a $\mathbb{Z}/9 \subset \Gamma_3$ if one notices that there is a $\mathbb{Z}/9$ action on F_3 with two singular points of order 9 and one singular point on the orbit space $F_3/\mathbb{Z}/9 = S^2$ (2 sphere). Comparing the LHS[3] for the extension

$$1 \to \pi_2 \to \mathbb{Z}/9 \to \mathbb{Z}/3 \to 1$$

with Proposition 4.5, one obtains

Proposition 5.2.

$$H^n(N(\pi_2)\,;\,\mathbb{Z})_{(3)} = 0$$

for $n = 0, 1$;

$$H^2(N(\pi_2)\,;\,\mathbb{Z})_{(3)} = \mathbb{Z}/9, \quad H^n(N(\pi_2)\,;\,\mathbb{Z})_{(3)} = \mathbb{Z}/3$$

for $n \geq 3$ *odd*;

$$H^n(N(\pi_2)\,;\,\mathbb{Z})_{(3)} = \mathbb{Z}/3 \oplus \mathbb{Z}/9$$

for $n \geq 4$ *even.*

Proposition 5.3. *$H^n(N(\pi_2); \mathbb{Z})$ contains exactly one copy of $\mathbb{Z}$ for $n = 0, 1, 2$ and contains no $\mathbb{Z}$ for $n \geq 3$.*

The main result about Farrell cohomology now follows readily since Γ_3 is 3-periodic.

Theorem 5.4.

$$\widehat{H}^n(\Gamma_3; \mathbb{Z})_{(3)} = \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/9$$

for $n \equiv 0 \pmod 4$;

$$\widehat{H}^n(\Gamma_3; \mathbb{Z})_{(3)} = \mathbb{Z}/3$$

for n odd;

$$H^n(\Gamma_3; \mathbb{Z})_{(3)} = \mathbb{Z}/3 \oplus \mathbb{Z}/9$$

for $n \equiv 2 \pmod 4$.

6. The p-component of Farrell cohomology of Γ_p for $p > 3$

For any prime $p > 3$, it is easy to check from possible fixed point data that there is one and only one conjugacy class of order p subgroup of Γ_p, denoted as $\pi \subset \Gamma_p$. The fixed point data of a generator of π is $\langle 1, p-1 \rangle$. Thus, the cyclic group $N(\pi)/C(\pi)$ is $\mathbb{Z}/2$. Actually, it is not difficult to observe a dihedral subgroup $D_{2p} \subset \Gamma_p$ by constructing a surjective map from $\pi_1(F_1 - \{x_1, x_2\})$ onto D_{2p}.

Let K_1 denote a subgroup of $\mathrm{Im}(\tilde{\lambda})$ consisting of all elements of $GL(3, \mathbb{Z}/p)$ in the form of

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

with $A \in \{1, -1\}$ and $D \in SL(2, \mathbb{Z}/p)$, where

$$\tilde{\lambda} : \Gamma_1^2 \to GL(3, \mathbb{Z}/p)$$

is defined as in section 1.

Proposition 6.1. *The quotient $N(\pi)/\pi$ is isomorphic to $\tilde{\lambda}^{-1}(K_1) \subset \Gamma_1^2$.*

The proof is the same as Proposition 1.1.

Proposition 6.2. *$H^n(\Gamma_1^2; M)_{(p)} = 0$ for any prime $p > 3$, $n > 0$ and $\mathbb{Z}\Gamma_1^2$-module M.*

Repeat the argument in section 2 with any coefficient $\mathbb{Z}\Gamma_1^2$-module M; the proof follows immediately.

By using the Shapiro lemma again, one gets

Proposition 6.3. *$H^n(N(\pi)/\pi; \mathbb{Z})_{(p)} = 0$ for any $p > 3$ and $n > 0$.*

Finally, comparing two short exact sequences

$$1 \to \pi \to N(\pi) \to N(\pi)/\pi \to 1,$$
$$1 \to \pi \to D_{2p} \to \mathbb{Z}/2 \to 1$$

and considering two LHS[3] associated to them, one obtains

Proposition 6.4. *The restriction map*

$$R : H^n(N(\pi)\,;\,\mathbb{Z})_{(p)} \to H^n(D_{2p}\,;\,\mathbb{Z})_{(p)}$$

is an isomorphism for any $p > 3$ *and* $n \geq 0$.

Theorem 6.5. *For a prime* $p > 3$, *the restriction map*

$$R : \widehat{H}^n(\Gamma_p\,;\,\mathbb{Z})_{(p)} \to \widehat{H}^n(D_{2p}\,;\,\mathbb{Z})_{(p)}$$

is an isomorphism for any n. *Namely,*

$$\widehat{H}^n(\Gamma_p\,;\,\mathbb{Z})_{(p)} = \mathbb{Z}/p$$

for any $n \equiv 0 \pmod 4$;

$$\widehat{H}^n(\Gamma_p\,;\,\mathbb{Z})_{(p)} = 0$$

for any $n \not\equiv 0 \pmod 4$.

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