**L^p SPECTRA OF PSEUDODIFFERENTIAL OPERATORS GENERATING INTEGRATED SEMIGROUPS**

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**Abstract.** Consider the $L^p$-realization $\text{Op}_p(a)$ of a pseudodifferential operator with symbol $a \in S^m_{\rho,0}$ having constant coefficients. We show that for a certain class of symbols the spectrum of $\text{Op}_p(a)$ is independent of $p$. This implies that $\text{Op}_p(a)$ generates an $N$-times integrated semigroup on $L^p(\mathbb{R}^n)$ for a certain $N$ if and only if $\rho(\text{Op}_p(a)) \neq \emptyset$ and the numerical range of $a$ is contained in a left half-plane. Our method allows us also to construct examples of operators generating integrated semigroups on $L^p(\mathbb{R}^n)$ if and only if $p$ is sufficiently close to 2.

**1. Introduction**

Let $\text{Op}_p(a)$ be the $L^p$-realization of a pseudodifferential operator $\text{Op}(a)$ with symbol $a \in S^m_{\rho,0}$ having constant coefficients. Consider the initial value problem

\begin{equation}
    u'(t) = \text{Op}_p(a)u(t), \quad u(0) = u_0,
\end{equation}

in the space $L^p(\mathbb{R}^n)$, where $1 \leq p < \infty$. We are interested in the question how the location of the spectrum $\sigma(\text{Op}_p(a))$ of $\text{Op}_p(a)$ or the numerical range $a(\mathbb{R}^n)$ of the symbol $a$ influences the existence and regularity theory of a solution of (1.1) for $u_0 \in D(\text{Op}_p(a))^N$, $N \in \mathbb{N}$. If for all $t \geq 0$ the function $\xi \mapsto e^{it\xi}$ is a Fourier multiplier for $L^p(\mathbb{R}^n)$ (with exponentially bounded norm), then a complete answer of the above question is obtained via the classic theory of strongly continuous semigroups (cf. [F], [G], [P]). Notice that, by results of Hörmander [Höl], there exist however many examples of operators $\text{Op}_p(a)$ which generate a $C_0$-semigroup on an $L^p$-space only for certain values of $p$. In fact, for simplicity let $\text{Op}_p(a)$ be a differential operator of order $m > 1$ such that the real part of the principal part of its symbol $a$ vanishes for all $\xi \in \mathbb{R}^n$. Then $\text{Op}_p(a)$ generates a $C_0$-semigroup on $L^p(\mathbb{R}^n)$ only if $p = 2$. In particular, this holds for the operator $i\Delta$, where $\Delta$ denotes the Laplacian.

In this paper we examine the initial value problem (1.1) by means of integrated semigroups (cf. [AJ]). The relationship between (1.1) and integrated semigroups may be described as follows: a linear operator $A$ generates an $N$-times integrated semigroup on $E$ if and only if $\rho(A) \neq \emptyset$ and (1.1) admits a unique, exponentially bounded solution for all $u_0 \in D(A^{N+1})$. By estimating the order $N$ of integration we thus obtain precise information on the regularity
of the initial data needed in order to obtain a unique classical solution of (1.1). We carry out this approach for a certain class of symbols $a \in S_{p,0}^m$ having constant coefficients. The estimates obtained turn out to be optimal for a large class of symbols. Moreover, they illustrate the special role of the case $p = 1$ and show in particular the different regularity behavior of the solution for the cases $p = 1$ and $p \in (1, \infty)$. As an immediate consequence we obtain $L^p$-resolvent estimates for $\text{Op}_p(a)$ in a right half-plane. Former results in this direction are contained in [AK], [BE], [Hi1] and [dEl].

Similarly to the case of semigroups, there exist operators $\text{Op}_p(a)$ generating integrated semigroups on $L^p(\mathbb{R}^n)$ for some but not for all values of $p$. The method to construct such examples is inextricably entangled with the spectral theory of the operators under consideration (cf. [Hi2], [IS], [KT]). It is shown that, for a certain class of symbols, the spectrum, being contained in a left half-plane for some value of $p$, may “explode” to be all of the complex plane for other values of $p$.

On the other hand it is natural to ask for conditions under which $\sigma(\text{Op}_p(a))$ of $\text{Op}_p(a)$ coincides with the numerical range $a(\mathbb{R}^n)$ of $a$. We show that

$$\sigma(\text{Op}_p(a)) = \sigma(\text{Op}_2(a)) = a(\mathbb{R}^n)$$

provided the symbol $a$ and its derivatives satisfy certain growth conditions. In particular, in this case $\sigma(\text{Op}_p(a))$ is independent of $p$. Since our assumptions are satisfied above all for elliptic polynomials, assertion (1.2) extends results of Balslev [Ba] and Iha and Schubert [IS] to our situation.

Finally, we illustrate our results by means of Dirac’s equation on $L^p(\mathbb{R}^3)$.

2. Preliminaries and notations

Let $\mathcal{S} := \mathcal{S}(\mathbb{R}^n)$ be the space of all rapidly decreasing functions and $\mathcal{S}'$ its dual, the space of all tempered distributions. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, let $\langle x, \xi \rangle = \sum_{i=1}^{n} x_i \xi_i$ and $\|x\| = \langle x, x \rangle^{1/2}$. The Fourier transform on $\mathcal{S}$ and its inverse transform are defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} f(x) \, dx, \quad \xi \in \mathbb{R}^n,$$

and

$$(\mathcal{F}^{-1}f)(x) := \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{i\langle x, \xi \rangle} \hat{f}(\xi) \, d\xi.$$

Throughout this paper, $\alpha$, $\beta$, $\gamma$ will denote multi-indices and $D^\alpha$ is defined by $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$, where $|\alpha| := \sum_{i=1}^{n} \alpha_i$.

We denote by $\mathcal{M}_p$ $(1 \leq p \leq \infty)$ the set of all functions $u \in L^\infty(\mathbb{R}^n)$ such that $\mathcal{F}^{-1}(u\phi) \in L^p(\mathbb{R}^n)$ for all $\phi \in \mathcal{S}$ and

$$\|u\|_{\mathcal{M}_p} := \sup\{\|\mathcal{F}^{-1}(u\phi)\|_{L^p} : \phi \in \mathcal{S}, \|\phi\|_{L^p} \leq 1\} < \infty.$$

We give this space the norm $\|\cdot\|_{\mathcal{M}_p}$ so that it becomes a Banach space. Then $\mathcal{M}_p = \mathcal{M}_q\left(\frac{1}{p} + \frac{1}{q} = 1; 1 \leq p \leq \infty\right)$ and we have

$$\sup_{\xi} \|u(\xi)\| = \|u\|_{\mathcal{M}_2} \leq \|u\|_{\mathcal{M}_p} \leq \|u\|_{\mathcal{M}_1}.$$
In order to determine whether or not a given function belongs to $\mathcal{M}_p$, the following fact is useful: there exists a function $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\text{supp } \psi \subseteq \{\frac{1}{2} < |\xi| < 2\}$ such that

$$\sum_{l=-\infty}^{\infty} \psi(2^{-l}\xi) = 1 \quad (\xi \neq 0).$$

A very efficient sufficient criterion for a function $u$ to belong to $\mathcal{M}_p$, $1 < p < \infty$, is given by the Mikhlin multiplier theorem (cf. [S, p. 96]). For the case $p = 1$ the following elementary bound for the $\mathcal{F}L^1(\mathbb{R}^n)$-norm is useful. Here we consider $\mathcal{F}L^1(\mathbb{R}^n)$ as a Banach space for the norm inherited by $L^1(\mathbb{R}^n)$.

**Lemma 2.1.** Let $u \in H^j(\mathbb{R}^n)$ for some $j > \frac{n}{2}$. Then $u \in \mathcal{F}L^1(\mathbb{R}^n)$ and

$$\|u\|_{\mathcal{F}L^1} \leq C_n\|u\|_{L^2}^{1-n/2j}\|u\|_{j,2}^{n/2j}$$

for some constant $C_n$ depending only on $n$.

For a proof we refer to [Hil, Lemma 2.1]. We call a function $a \in C(\mathbb{R}^n, \mathbb{C})$ a symbol if there exist constants $M > 0$, $m \in \mathbb{R}$ such that

$$|a(\xi)| \leq M (1 + |\xi|)^m \quad \text{for all } \xi \in \mathbb{R}^n.$$

Then we define the pseudodifferential operator $\text{Op}(a) : \mathcal{S} \to \mathcal{S}^\prime$ with symbol $a$ by

$$\text{Op}(a)u(x) := \int_{\mathbb{R}^n} e^{i(x,\xi)}a(\xi)\hat{u}(\xi) \, d\xi.$$

Moreover, we denote by $\mathcal{S}$ the class of all symbols $a$ such that $\text{Op}(a)$ maps $\mathcal{S}$ into $\mathcal{S}$.

For $m \in \mathbb{R}$ and $\rho \in [0, 1]$, we define $S^{m,\rho}_{p,0}$ to be the set of all functions $a \in C^\infty(\mathbb{R}^n)$ such that for each multi-index $\alpha$ there exists a constant $C_\alpha$ such that

$$|D^\alpha a(\xi)| \leq C_\alpha (1 + |\xi|)^{m-\rho|\alpha|} \quad (\xi \in \mathbb{R}^n).$$

Obviously, $S^{m,\rho}_{p,0} \subset S$ and a polynomial of order $m$ is of class $S^{m,0}_{1,0}$. Furthermore, for $a \in S^{m,0}_{p,0}$ we put $a(\mathbb{R}^n) := \{a(\xi) : \xi \in \mathbb{R}^n\}$.

For the time being, let $a \in S$. Then we associate with $a$ a linear operator $\text{Op}_p(a)$ on $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) as follows. Set

$$D(\text{Op}_p(a)) := \{f \in L^p(\mathbb{R}^n) : \mathcal{F}^{-1}(a\hat{f}) \in L^p(\mathbb{R}^n)\} \quad \text{and define}$$

$$\text{Op}_p(a)f := \mathcal{F}^{-1}(a\hat{f}) \quad \text{for all } f \in D(\text{Op}_p(a)).$$

Then it is not difficult to verify that $\text{Op}_p(a)$ is closed.

We call a polynomial $a$ of degree $m$ elliptic if its principal part $a_m$ given by $a_m(\xi) := \sum_{|\alpha|=m} a_\alpha(i\xi)^\alpha$ vanishes only at $\xi = 0$. Moreover, $a$ is called hypoelliptic if

$$\frac{D^\alpha a(\xi)}{a(\xi)} \to 0 \quad \text{as } |\xi| \to \infty \text{ and } \alpha \neq 0.$$

Finally, if $A$ is a linear operator acting on a Banach space $E$, we denote its resolvent set by $\rho(A)$ and its spectrum by $\sigma(A)$.

For the time being, let $A$ be a linear operator on a Banach space $E$ and $k \in \mathbb{N} \cup \{0\}$. Then $A$ is called the generator of a $k$-times integrated semigroup.
if and only if \((\omega, \infty) \subset \rho(A)\) for some \(\omega \in \mathbb{R}\) and there exists a strongly
continuous mapping \(S : [0, \infty) \to \mathcal{L}(E)\) satisfying \(\|S(t)\| \leq Me^{\omega t}\) \((t \geq 0)\)
for some \(M \geq 0\) such that
\[
R(\lambda, A) = \lambda^k \int_0^\infty e^{-\lambda t} S(t) \, dt \quad (\lambda > \omega).
\]

In this case \((S(t))_{t \geq 0}\) is called the \(k\)-times integrated semigroup generated by \(A\). In particular, a 0-times integrated semigroup is a \(C_0\)-semigroup. For more
detailed information on semigroups and integrated semigroups we refer to [F],
[G], [P] and [A], [deL], [Hi2] and [L]. The connection between integrated semi-
groups and the Cauchy problem

\[
(2.4) \quad u'(t) = Au(t) , \quad u(0) = 0
\]
is given by the following fact: Let \(A\) be a linear operator on a Banach space
\(E\) and let \(k \in \mathbb{N} \cup \{0\}\). Then \(A\) generates a \(k\)-times integrated semigroup on \(E\) if and only if \(\rho(A) \neq \emptyset\) and there exists a unique, classical solution \(u\) of

\[(2.4)\]

for all \(u_0 \in D(A^{k+1})\) satisfying \(\|u(t)\| \leq Me^{\omega t}\) for all \(t \geq 0\) and some
\(M, \omega \geq 0\).

3. Fourier multipliers

We start this section with a sufficient criterion for a function \(a\) to belong to \(\mathcal{M}_p\).

**Theorem 3.1.** Let \(a \in C^j(\mathbb{R}^n), j > \frac{n}{2}\), and suppose that \(a(\xi) = 0\) for all \(\xi \in \mathbb{R}^n\)
with \(|\xi| \leq 1\). Let \(\varepsilon \geq 0\) and \(\rho \in (-\infty, 1]\). Assume that there exist constants
\(M_0 > 0, M \geq 1\) such that

\[\sup_{0 < |\alpha| \leq j} \left( \sup_{|\xi| \geq 1} |D^\alpha a(\xi)| |\xi|^{\varepsilon + \rho |\alpha|} \right)^{1/|\alpha|} \leq M \quad \text{and} \quad \sup_{\xi \in \mathbb{R}^n} |a(\xi)| |\xi|^\varepsilon \leq M_0.\]

(a) Let \(1 \leq p \leq \infty\). If \(\varepsilon > n \frac{1}{2} - \frac{1}{\rho} (1 - \rho)\), then \(a \in \mathcal{M}_p\) and there exists a constant \(C_{n,p,\rho}\) such that

\[\|a\|_{\mathcal{M}_p} \leq C_{n,p,\rho} \max(M_0, 1) M^n |1/2 - 1/p|^{1/p}.\]

(b) (Miyachi) Let \(1 < p < \infty\). Assume that \(M_0 = 1 \text{ and } \rho \neq 1\). If
\(\varepsilon > n \frac{1}{2} - \frac{1}{\rho} (1 - \rho)\), then \(a \in \mathcal{M}_p\) and there exists a constant \(C_{p}\) such that

\[\|a\|_{\mathcal{M}_p} \leq C_{p} M^n |1/2 - 1/p|^{1/p}.\]

**Proof.** (a) Without loss of generality we may assume that \(1 \leq p \leq 2\). Let \(\psi \in C_0^\infty(\mathbb{R}^n)\) be the function defined in (2.2). For \(l \in \mathbb{N}\) put \(a_l := a \psi_l\), where
\(\psi_l(x) := \psi(2^{-l}x)\) for all \(x \in \mathbb{R}^n\). We claim that \(\|a_l\|_{\mathcal{M}_p} \leq \sum_{l=1}^{\infty} \|a_l\|_{\mathcal{M}_p} < \infty\).

To this end, observe that by Leibniz’s rule

\[|D^\alpha a_l(\xi)| = \left| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^{\alpha - \beta} a(\xi) 2^{-l|\beta|} (D^\beta \psi)(2^{-l} \xi) \right| \leq \left\{ \begin{array}{ll}
C_0 M_0 2^{-l \varepsilon} & \text{if } \alpha = 0, \\
C_\alpha M^{|\alpha|} 2^{l(-\varepsilon - \rho |\alpha|)} & \text{if } \alpha \neq 0
\end{array} \right.\]
for some constants $C_0, C_\alpha > 0$. Consequently, there exist constants $C_{\alpha, n}$ such that

$$\|D^n a_l\|_2 \leq \begin{cases} C_{0, n} 2^{-l} e^{ln/2} & (\alpha = 0), \\ C_{\alpha, n} 2^{l(-\varepsilon - \rho|\alpha|)} e^{ln/2} & (\alpha \neq 0). \end{cases}$$

Now, choosing $j > \frac{\varepsilon}{2}$, we conclude by Lemma 2.1 that

$$\|a_l\|_{\mathcal{L}^1} \leq C_n (M_0 2^{-l} e^{ln/2})^{1-\varepsilon l/2} (M_j 2^{l(-\varepsilon - \rho)l/2})^{n/2j}$$

$$\leq C_n M_0^{1-\varepsilon l/2} M^{n/2} 2^{l(-\varepsilon - \rho)(1-l/2)}.$$

Setting $\theta := 2(1 - \frac{1}{2})$ for some $p \in (1, 2)$ it follows from the Riesz-Thorin theorem that

$$\|a_l\|_{\mathcal{L}^p} \leq \|a_l\|_{\mathcal{L}^\infty}^{1-\theta} \|a_l\|_{\mathcal{L}^p}^\theta \leq C_{n, p} \max\{M_0, 1\} M^{n/2-1/p} 2^{l(-\varepsilon - \rho)(1-1/p)}.$$

Finally, since $\|a_l\|_{\mathcal{L}^p} \leq \sum_{l=1}^{\infty} \|a_l\|_{\mathcal{L}^p} < \infty$, the proof of assertion (a) is complete. The assertion (b) follows immediately from [M, Theorem G].

Assume that the symbol $a$ belongs to $S_{\rho, 0}$. We consider the following hypothesis:

(1) $\sup_{\xi \in \mathbb{R}^n} \text{Re} a(\xi) \leq \omega$ for some $\omega \in \mathbb{R}$.

(2) There exist constants $C, L, r > 0$ such that $|a(\xi)| \geq C|\xi|^r$ for all $\xi \in \mathbb{R}^n$ with $|\xi| \geq L$.

Remark 3.2. We note that by the Seidenberg-Tarski theorem Hypothesis (H2) above is in particular satisfied for all polynomials $a$ satisfying $|a(\xi)| \to \infty$ as $|\xi| \to \infty$ (cf. [Hö2, Theorem 11.1.3]). Hence assumption (H2) holds especially for hypoelliptic polynomials.

Lemma 3.3. Let $N \in \mathbb{N}$, $m \in (0, \infty)$, $\rho \in [0, 1]$. Suppose that $a \in S_{\rho, 0}$ satisfies (H2) and that $0 \not\in a(\mathbb{R}^n)$.

(a) If $N > \frac{\varepsilon}{2} (\frac{m-\rho-r-1}{r})$, then $a^{-N} \in \mathcal{M}_1$.

(b) If $1 < \rho < \infty$ and $N \geq n\left(\frac{r}{2} - \frac{1}{\rho}\right)(\frac{m-\rho-1}{r})$, then $a^{-N} \in \mathcal{M}_p$.

Proof. Let $\phi \in C_c^\infty$ such that

$$\phi(\xi) := \begin{cases} 1 & \text{for } |\xi| \leq L, \\ 0 & \text{for } |\xi| \geq L + 1. \end{cases}$$

Then, writing $a^{-N} = \phi a^{-N} + (1 - \phi) a^{-N}$, we conclude by Lemma 2.1 that it suffices to prove that $(1 - \phi) a^{-N} \in \mathcal{M}_p$. Observe that by assumption

$$\|D^n (a^{-N})(\xi)\| \leq C_n |\xi|^{-r N + (m-r-\rho)|\alpha|} (|\xi| \geq \max(L, 1)).$$

Hence the assertion follows from Theorem 3.1 provided $\rho \neq 1$. If $\rho = 1$, then the assertion follows from Mikhlin's theorem.

Lemma 3.4. Let $N \in \mathbb{N}$, $m \in (0, \infty)$, $\rho \in [0, 1]$ and let $a \in S_{\rho, 0}$. Assume that (H1) and (H2) are satisfied and that $0 \not\in a(\mathbb{R}^n)$.

(a) If $N > \frac{\varepsilon}{2} (\frac{1+m-\rho}{r})$, then $e^{it a}/a^{-N} \in \mathcal{M}_1$ and there exists a constant $C_{N, n}$ such that

$$\left\| \frac{e^{it a}}{a^{-N}} \right\|_{\mathcal{M}_1} \leq C_{N, n} (1 + t)^{n/2} e^{\omega t} \quad (t \geq 0).$$
(b) If $1 < p < \infty$ and $N \geq n \left( \frac{1}{2} - \frac{1}{p} \right) (1 + m - \rho)$, then $e^{ia} / a^N \in \mathcal{M}_p$ and there exists a constant $C_{N, n, p}$ such that

$$\left\| \frac{e^{ia}}{a^N} \right\|_{\mathcal{M}_p} \leq C_{N, n, p} (1 + t)^{n/2 - 1/p} e^{\omega t} \quad (t \geq 0).$$

Proof. Note first that after rescaling we may assume that $\omega = 0$. Moreover, thanks to (2.1), we may restrict ourselves to the case $1 \leq p \leq 2$. Now, let $\varphi \in C_c^\infty$ such that $0 \leq \varphi(\xi) \leq 1$ ($\xi \in \mathbb{R}^n$) and

$$\varphi(\xi) := \begin{cases} 1 & \text{for } |\xi| \leq L_1, \\ 0 & \text{for } |\xi| \geq L_1 + 1, \end{cases}$$

where $L_1 := \max(L, C^{-1/r})$. We put $v_i^N := e^{ia} / a^N$. Then Lemma 2.1 implies that $\varphi v_i^N \in \mathcal{M}_p$ and that

$$\left\| \frac{e^{ia}}{a^N} \right\|_{\mathcal{M}_1} \leq C_n (1 + t)^{n/2}$$

for some constant $C_n$. Writing $v_i^N = \varphi v_i^N + (1 - \varphi) v_i^N$, we conclude that it remains to prove the assertion for $(1 - \varphi) v_i^N$. Now, by Leibniz’s rule

$$D^\alpha (v_i^N) = \sum_{\beta + \gamma = \alpha} D^\beta (e^{ia}) D^\gamma (a^{-N}).$$

Since $|(D^\gamma a^{-N})(\xi)| \leq C_\gamma |\xi|^{-rN + |\gamma|/(m - p)}$ for all $\xi$ with $|\xi| \geq L$ (see (3.1)) and since $|(D^\beta e^{ia})(\xi)| \leq C_\beta (1 + t)^{\beta |\xi|/(m - p)}$ it follows that there exists a constant $C > 0$ such that

$$\sup_{0<|\alpha| \leq j} \sup_{|\xi| \geq 1} \left| (D^\alpha (1 - \varphi)(\xi) v_i^N)(\xi) \right| |\xi|^{rN + |\alpha|/(m - p)} \leq C (1 + t)$$

and

$$\sup_{|\xi| \geq 1} \left( (1 - \varphi) v_i^N(\xi) |\xi|^{rN} \leq 1 \right)$$

for all $t \geq 0$. Hence the assertion follows from Theorem 3.1. □

4. $L^p$ SPECTRA OF PSEUDODIFFERENTIAL OPERATORS

We start this section with a result illustrating the close relationship between $L^p$ multipliers and the $L^p$ spectra of the pseudodifferential operators under consideration.

**Lemma 4.1.** Let $1 \leq p < \infty$ and $a \in S$. Then $\lambda \in \rho(\text{Op}_p(a))$ if and only if $(\lambda - a)^{-1} \in \mathcal{M}_p$.

We note that if the symbol $a$ is a polynomial, then Lemma 4.1 was first proved by Schechter [Sch, Theorem 4.4.1]. The generalization to symbols $a$ belonging to $S$ is a straightforward modification of Schechter’s proof. We therefore omit the details.

In order to obtain a precise description of $\sigma(\text{Op}_p(a))$ we need to decide whether or not the function $(\lambda - a)^{-1}$ is an $L^p$ multiplier. In general this is not an easy matter; however if the symbol $a$ satisfies the growth condition (H2), then the situation is fairly easy to describe. Indeed, in this case we obtain the following result.
Proposition 4.2. Let \( 1 \leq p < \infty \), \( m \in (0, \infty) \) and \( \rho \in [0,1] \). Suppose that \( a \in S_{p,0}^m \) satisfies (H2). If \( p(\text{Op}_p(a)) \neq \emptyset \), then \( \sigma(\text{Op}_p(a)) = \sigma(\text{Op}_2(a)) = a(\mathbb{R}^n) \).

Related results on the \( p \)-independence of the spectrum of differential operators on \( L^p(\mathbb{R}^n) \) are contained in [Ba] and [IS].

Remarks 4.3. (a) If \( a \) is a polynomial, then the Seidenberg-Tarski theorem implies that Hypothesis (H2) is fulfilled provided \( |a(\xi)| \to \infty \) as \( |\xi| \to \infty \).

(b) We emphasize that Hypothesis (H2) is essential for obtaining the above assertion. In fact, consider the example of the symbol \( a \) given by

\[
a(\xi) := -i(\xi_1^2 + \xi_2^2 + \xi_3^2 - i).
\]

Then \( \sigma(\text{Op}_2(a)) = \{ z \in \mathbb{C}; \text{Re} \, z = -1 \} \), but Kenig and Tomas [KT] showed that \( a^{-1} \notin \mathcal{M}_p \) if \( p \neq 2 \). Hence, by Lemma 4.1, \( 0 \notin \sigma(\text{Op}_p(a)) \) whenever \( p \neq 2 \).

Proof. We note first that Lemma 4.1 together with Hypothesis (H2) and the fact that \( \mathcal{M}_p = L^\infty \) implies that \( \sigma(\text{Op}_p(a)) \) coincides with \( a(\mathbb{R}^n) \). Therefore and in view of Lemma 4.1 we only have to prove that \( \sigma(\text{Op}_p(a)) \subset a(\mathbb{R}^n) \). By assumption we have \( \mathbb{C}(a(\mathbb{R}^n)) \neq \emptyset \). Let \( \lambda \in \mathbb{C}(a(\mathbb{R}^n)) \). By Lemma 3.3 there exists an integer \( N \) such that the function \( r_\lambda^N := (\lambda - a)^{-N} \) belongs to \( \mathcal{M}_p \) (\( 1 \leq p < \infty \)). Hence, by Lemma 4.1, \( 0 \in \rho(\text{Op}_p(\lambda - a)^N) \). We claim that \( 0 \in \rho((\lambda - \text{Op}_p(a))^N) \). Since \( (\lambda - \text{Op}_p(a))^N f = \text{Op}_p((\lambda - a)^N) f \) for all \( f \in \mathcal{S} \), we conclude by [HP, Theorem 2.16.4] that \( (\lambda - \text{Op}_p(a))^N \) is an extension of \( \text{Op}_p((\lambda - a)^N) \). Furthermore, \( \text{ker}(\lambda - \text{Op}_p(a))^N = \{ 0 \} \). In fact, assume that \( (\lambda - \text{Op}_p(a))^N u = 0 \). Then \( 0 = ((\lambda - \text{Op}_p(a))^Nu, g) = (u, (\lambda - \text{Op}_p(a))^Ng) \) for all \( g \in \mathcal{S} \). Hypothesis (H2) implies that, given \( f \in \mathcal{S} \), we find \( g \in \mathcal{S} \) such that \( (\lambda - \text{Op}_p(a))^Ng = f \). Consequently \( 0 = (u, f) \) for all \( f \in \mathcal{S} \) and hence \( u = 0 \). It follows that \( \text{Op}_p((\lambda - a)^N) = (\lambda - \text{Op}_p(a))^N \) and hence \( 0 \in \rho((\lambda - \text{Op}_p(a))^N) \). In a second step, we claim that \( 0 \in \rho(\lambda - \text{Op}_p(a)) \). Suppose the contrary. Then the spectral mapping theorem for closed operators (cf. [DS, p. 604]) implies that \( \sigma((\lambda - \text{Op}_p(a))^N) = (\sigma(\lambda - \text{Op}_p(a)))^N \) which yields a contradiction. Hence \( \sigma(\text{Op}_p(a)) \subset a(\mathbb{R}^n) \). The proof is complete.

We now give a quantitative version of Proposition 4.2.

Theorem 4.4. Let \( 1 \leq p < \infty \), \( m \in (0, \infty) \) and \( \rho \in [0,1] \). Suppose that \( a \in S_{p,0}^m \) satisfies (H2).

(a) Then the following assertions hold.

(i) If \( 1 < p < \infty \) and \( n \frac{1}{2} - \frac{1}{p} (\frac{m-p-r-1}{r}) \leq 1 \), then \( \sigma(\text{Op}_p(a)) = \sigma(\text{Op}_2(a)) \).

(ii) If \( \frac{p}{2} (\frac{m-p-r-1}{r}) < 1 \), then \( \sigma(\text{Op}_1(a)) = \sigma(\text{Op}_2(a)) \).

(b) If \( p \neq 0 \), then the bounds in assertions (i) and (ii) are optimal; i.e. given \( p \in [1, \infty) \), there exists \( a \in S_{p,0}^m \) (\( m > 0 \), \( \rho \in (0,1) \)) such that \( \sigma(\text{Op}_p(a)) \neq \sigma(\text{Op}_2(a)) \) whenever \( n \frac{1}{2} - \frac{1}{p} (\frac{m-p-r-1}{r}) > 1 \) or \( \frac{p}{2} (\frac{m-p-r-1}{r}) \geq 1 \), respectively.

Proof. The assertion (a) follows by combining Lemma 3.3 and Proposition 4.2. In order to prove (b) let \( \alpha \in (0,1) \), \( m \in (0, \frac{n\alpha}{2}) \) and let \( a : \mathbb{R}^n \to \mathbb{C} \) be a
C^\infty\)-function such that
\[
a(\xi) := \begin{cases} 
\frac{|\xi|^m}{e^{i|\xi|^\alpha}}, & |\xi| \geq 2, \\
0, & |\xi| \leq 1.
\end{cases}
\]

Then \(a \in S_{m-\alpha,0}^m\) and (H2) is satisfied with \(r = m\). Hence in this case \(m - \frac{p - r + 1}{r} = \frac{a}{m}\). It follows from the results in [FS, p. 160] that \(a^{-1} \in \mathcal{M}_p (\mathcal{M}_1)\) if and only if \(n \frac{1}{2} - \frac{1}{p} \leq \frac{m}{\alpha} (\frac{q}{2} < \frac{m}{\alpha})\). Therefore, by Lemma 4.1, \(0 \in \sigma(\text{Op}_p(a))\) if and only if \(n \frac{1}{2} - \frac{1}{p} \frac{a}{m} > 1 (\frac{n}{2} \frac{a}{m} \geq 1)\). On the other hand, \(0 \not\in a(\mathbb{R}^n) = \sigma(\text{Op}_2(a))\), which proves the assertion. □

**Remark 4.5.** The above assumptions are in particular satisfied for elliptic polynomials \(a\), in which case we have \(\rho = 1\) and \(m = r\).

Suppose that \(a \in S_{m,0}^m\) satisfies Hypothesis (H2) and let \(N_p\) be the smallest integer such that

\[
(4.1)
\]

\[
N_p := \begin{cases} 
\geq n \frac{1}{2} - \frac{1}{p} \left(\frac{1 + m - \rho}{r}\right) & \text{if } 1 < p < \infty, \\
\geq n \frac{2}{m} \left(\frac{1 + m - \rho}{r}\right) & \text{if } p = 1.
\end{cases}
\]

**Theorem 4.6.** Let \(1 < p < \infty\), \(m \in (0, \infty)\), \(N \in \mathbb{N} \cup \{0\}\) and \(\rho \in [0, 1]\). Suppose that \(a \in S_{m,0}^m\) satisfies (H2). Then the following assertions are equivalent.

(a) \(\rho(\text{Op}_p(a)) \neq \emptyset\) and \(\sup_{\xi \in \mathbb{R}^n} \text{Re } a(\xi) \leq \omega\) for some \(\omega \in \mathbb{R}\).

(b) The operator \(\text{Op}_p(a)\) generates an \(N_p\)-times integrated semigroup \((S(t))_{t \geq 0}\) on \(L^p(\mathbb{R}^n)\).

(c) \(\sigma(\text{Op}_p(a)) \subset \{ z \in \mathbb{C} ; \text{Re } z \leq \omega \}\) for some \(\omega \in \mathbb{R}\).

**Proof.** (a) \(\Rightarrow\) (b). A rescaling argument shows that we may assume that \(\omega = -1\). Hence, it follows from Proposition 4.2 that \(0 \in \rho(\text{Op}_p(a))\). For \(t \geq 0\) and \(k \in \mathbb{N}\) define the function \(u^k_t : \mathbb{R}^n \to \mathbb{C}\) by

\[
u^k_t(\xi) := \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} e^{sa(\xi)} ds.
\]

Then, integrating by parts we obtain

\[
u^k_t = \frac{e^{ta}}{a^k} - \sum_{j=1}^{k} \frac{1}{(k-j)!} \frac{t^{k-j}}{a^j}.
\]

Since \(\mathcal{M}_p\) is a Banach algebra, we conclude from Lemma 4.1 that there exists a constant \(C_{k,p}\) such that

\[
(4.2) \quad \left\| \sum_{j=1}^{k} \frac{1}{(k-j)!} \frac{t^{k-j}}{a^j} \right\|_{\mathcal{M}_p} \leq C_{k,p}(1 + t)^{k-1} \quad (t \geq 0).
\]

By assumption, the symbol \(a\) satisfies Hypothesis (H2). Therefore, choosing \(N_p\) as in (4.1), it follows from Lemma 3.4 that

\[
(4.3) \quad \left\| \frac{e^{ta}}{a^N} \right\|_{\mathcal{M}_p} \leq C_{N,p,n}(1 + t)^{n(1/2 - 1/p)} e^{-t} \quad (t \geq 0).
\]
for some constant $C_{N,p,n}$. Combining (4.2) with (4.3) it follows that $u_t^{N_p} \in \mathcal{M}_p$ for all $t \geq 0$ and that
\[ \|u_t^{N_p}\|_{\mathcal{M}_p} \leq C_{N,p,n}(1 + t)^{\max\{n|1/2 - 1/p|, N_p - 1\}} \quad (t \geq 0) \]
for some constant $C_{N,p,n}$. Following the proof of [Hi2, Theorem 5.1] it is now not difficult to verify that the mapping $S : [0, \infty) \to \mathcal{L}(L^p(\mathbb{R}^n)), t \mapsto \mathcal{F}^{-1}(u_t^{N_p})$ is strongly continuous and to prove that $\mathcal{O}_{p}(a)$ is the generator of the integrated semigroup $(S(t))_{t \geq 0}$. Finally, a well-known perturbation argument completes the proof of this assertion.

The assertion (b) $\Rightarrow$ (c) follows from the definition of the integrated semigroup and assertion (c) $\Rightarrow$ (a) is a consequence of Proposition 4.2. □

**Remarks 4.7.** Suppose that the assumptions of Theorem 4.6 are fulfilled and that assertion (a) or (c) of Theorem 4.6 is satisfied for some $\omega \in \mathbb{R}$.

(a) It follows from the above proof that the $N_p$-times integrated semigroup $(S(t))_{t \geq 0}$ on $L^p(\mathbb{R}^n)$ satisfies an estimate of the form
\[ \|S(t)\|_{\mathcal{L}(L^p)} \leq M(1 + t)^{\max\{n|1/2 - 1/p|, N_p - 1\}} e^{\max\{\omega', 0\} t} \quad (t \geq 0) \]
for some constants $M > 0$ and $\omega' > \omega$.

(b) If in addition $a$ is homogeneous, then
\[ \|S(t)\|_{\mathcal{L}(L^p)} \leq Mt^k \quad (t \geq 0) \]
for some constant $M > 0$ and some integer
\[ k \begin{cases} \geq n \left\lfloor \frac{1}{2} - \frac{1}{p} \right\rfloor & \text{if } 1 < p < \infty, \\ \geq \frac{n}{2} & \text{if } p = 1. \end{cases} \]

In order to prove (b) note that $\mathcal{M}_p$ is isometrically invariant under affine transformations of $\mathbb{R}^n$. Thus
\[ \|S(t)\|_{\mathcal{L}(L^p)} = \left\| \int_0^t (t-s)^{k-1} e^{sa} \right\|_{\mathcal{M}_p} = t^k \left\| \int_0^1 \frac{(1-s)^{k-1}}{(k-1)!} e^{sa} \right\|_{\mathcal{M}_p} \quad \text{for all } t \geq 0. \]

Recalling the fact that $\mathcal{M}_1 \subset \mathcal{M}_p$, $1 \leq p \leq \infty$, we immediately obtain the following result.

**Corollary 4.8.** Let $m \in (0, \infty)$ and $\rho \in [0, 1]$. Suppose that $a \in S^m_{\rho,0}$ satisfies (H2). Then the following assertions are equivalent.

(a) $\mathcal{O}_{p}(a)$ generates a $k$-times integrated semigroup on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ for some integer $k$ and $\rho(\mathcal{O}_1(a)) \neq \emptyset$.

(b) $\mathcal{O}_1(a)$ generates an $l$-times integrated semigroup on $L^1(\mathbb{R}^n)$ for some integer $l$.

The numbers $k$ and $l$ in Corollary 4.8 are related to each other in the following manner.

**Corollary 4.9.** Assume that the assumptions of Corollary 4.8 are satisfied. Then the following hold.

(i) If assertion (a) of Corollary 4.8 holds for some $k \in \mathbb{N} \cup \{0\}$, then assertion (b) is true for any integer $l > \frac{n}{2} \left\lfloor \frac{1}{p} - m - \rho \right\rfloor$. 

(bi) If assertion (b) of Corollary 4.8 holds for some $l > \frac{n}{2} \left\lfloor \frac{1}{p} - m - \rho \right\rfloor$, then assertion (a) is true for any integer $k \geq n \left\lfloor \frac{1}{2} - \frac{1}{p} \right\rfloor$. 


(ii) If assertion (b) of Corollary 4.8 holds for some \( l \in \mathbb{N} \cup \{0\} \), then assertion (a) is true for any integer \( k \geq n\left|\frac{1}{2} - \frac{1}{p}\right|(1+m/p) \).

**Remark 4.10.** It follows from Theorem 4.3 in [Hil] that the orders of integration in Theorem 4.6 and Corollary 4.9, respectively, are optimal for a large class of operators including the operator \( \text{Op}_p(a) = i\Delta \). Indeed, in this case \( \rho = 1 \) and \( m = r \). Thus \( i\Delta \) generates an \( N \)-times integrated semigroup on \( L^p(\mathbb{R}^n) \), \( 1 < p < \infty \), if and only if \( N \geq n\left|\frac{1}{2} - \frac{1}{p}\right| \) and on \( L^1(\mathbb{R}^n) \) if and only if \( N > \frac{n}{2} \). For more general results in this direction, see [Hil].

**Corollary 4.11.** Let \( 1 \leq p < \infty \), \( m \in (0, \infty) \), \( N \in \mathbb{N} \) and \( \rho \in [0, 1] \). Assume that \( a \in S^m_{\rho,0} \) satisfies (H2). If \( \sup_{\xi \in \mathbb{R}^n} \text{Re} a(\xi) \leq \omega \) for some \( \omega \in \mathbb{R} \), then there exists a constant \( \delta_N > 0 \) such that \( \text{Op}_p(a) \) generates an \( N \)-times integrated semigroup on \( L^p(\mathbb{R}^n) \) provided \( |1/2 - 1/p| < \delta_N \).

**Example 4.12 (see [Hi2]).** The example of the symbol \( a \) given by
\[
a(\xi) := (-i)(\xi_1 - \xi_2^2 - \xi_3^2 - i)(\xi_1 + \xi_2^2 + \xi_3^2 + i)
\]
shows that \( \text{Op}_p(a) \) generates an integrated semigroup on \( L^p(\mathbb{R}^3) \) only for certain values of \( p \). Indeed, we verify that \( \sup_\mathbb{R} \text{Re} a(\xi) \leq 0 \) and that \( r = 1 \). Hence, by Theorem 4.4 we see that \( \rho(\text{Op}_p(a)) \neq \emptyset \) provided \( |1/2 - 1/p| < 1/2 \). Therefore \( \text{Op}_p(a) \) generates a once integrated semigroup on \( L^p(\mathbb{R}^3) \) provided \( |1/2 - 1/p| \leq 1/12 \). However, it follows from the results in [IS] that \( \sigma(\text{Op}_p(a)) \neq a(\mathbb{R}^n) \) if \( |1/2 - 1/p| > 3/8 \). Hence by Proposition 4.2, \( \rho(\text{Op}_p(a)) = \emptyset \) if \( |1/2 - 1/p| > 3/8 \). Consequently, in this case \( \text{Op}_p(a) \) does not generate an \( N \)-times integrated semigroup on \( L^p(\mathbb{R}^3) \) for any \( N \).

Recall that the Laplace transform of an exponentially bounded, strongly continuous function exists in a right half-plane of \( C \). Hence, as a consequence of Theorem 4.6 and Remark 4.7, we obtain the following \( L^p \) resolvent estimates for pseudodifferential operators with symbol \( a \in S^m_{\rho,0} \) having constant coefficients.

**Corollary 4.13.** Let \( 1 \leq p < \infty \), \( m \in (0, \infty) \) and \( \rho \in [0, 1] \). Assume that \( a \in S^m_{\rho,0} \) satisfies (H2) and that \( \sup_\mathbb{R} \text{Re} a(\xi) \leq 0 \).

(a) If \( \rho(\text{Op}_p(a)) \neq \emptyset \), then \( (\lambda - \text{Op}_p(a)) \) is invertible for all \( \lambda \in \mathbb{C} \setminus a(\mathbb{R}^n) \) and for \( \varepsilon > 0 \) and \( N > N_\rho \) there exists a constant \( C_{N,\rho,n} > 0 \) such that
\[
\| (\lambda - \text{Op}_p(a))^{-1} \| \leq C_{N,\rho,n} |\lambda|^N \left( \frac{1}{\text{Re} \lambda - \varepsilon} + \frac{1}{(\text{Re} \lambda - \varepsilon)^2 N+1} \right) \quad (\text{Re} \lambda > \varepsilon).
\]

(b) If in addition \( a \) is homogeneous, then \( (\lambda - \text{Op}_p(a)) \) is invertible for all \( \lambda \in \mathbb{C} \setminus a(\mathbb{R}^n) \) and for
\[
N \begin{cases} 
\geq n \left|\frac{1}{2} - \frac{1}{p}\right| & \text{if } 1 < p < \infty, \\
> \frac{n}{2} & \text{if } p = 1,
\end{cases}
\]
there exists a constant \( C_{N,\rho,n} > 0 \) such that
\[
\| (\lambda - \text{Op}_p(a))^{-1} \| \leq C_{N,\rho,n} |\lambda|^N \left( \frac{1}{(\text{Re} \lambda)^{N+1}} \right) \quad (\text{Re} \lambda > 0).
\]
5. An Application: Dirac’s Equation on $L^p(\mathbb{R}^3)$

The relativistic description of the motion of a particle of mass $m$ with spin $1/2$ is provided by the Dirac equation (see [G] or [F])

$$\frac{\partial}{\partial t} u(x, t) = c^2 \sum_{j=1}^{3} A_j D_j u(x, t) - A_4 \frac{mc^2}{i\hbar} u(x, t) + Vu(x, t), \quad x \in \mathbb{R}^3, \quad t \geq 0.$$ 

Here $u$ is a function defined on $\mathbb{R}^3 \times \mathbb{R}_+$ which takes values in $\mathbb{C}^4$, $c$ is the speed of light, $\hbar$ is Planck’s constant and $A_j$ ($j = 1, 2, 3, 4$) are $4 \times 4$ matrices given by

$$A_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad (j = 1, 2, 3) \quad \text{and} \quad A_4 = \begin{pmatrix} \sigma_4 & 0 \\ 0 & -\sigma_4 \end{pmatrix},$$

where $\sigma_j$ are the Pauli spin matrices defined by

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Suppose that $V = 0$ and all units are chosen so that all constants are equal to 1. Then Dirac’s equation can be rewritten as a symmetric, hyperbolic system of the form

$$v'(t) = Dv(t), \quad v(0) = v_0$$

where

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad D := \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} + i \begin{pmatrix} \sigma_4 & 0 \\ 0 & -\sigma_4 \end{pmatrix}$$

and

$$A := \begin{pmatrix} D_3 & D_1 - iD_2 \\ D_1 + iD_2 & -D_3 \end{pmatrix}.$$ 

Here $D_j := \frac{\partial}{\partial x_j}$ ($j = 1, 2, 3$). Let $E := L^p(\mathbb{R}^3, \mathbb{C})^4$ ($1 \leq p < \infty$). We define the $L_p$-realization $D_p$ of $D$ by

$$D(D_p) := D(D_p) \times D(D_p) \quad \text{and} \quad D_p f := Df \quad \text{for all } f \in D(D_p),$$

where $D(D_p) := \{ f \in L^p(\mathbb{R}^3)^2; Af \in L^p(\mathbb{R}^3)^2 \}$. Then it is well known that the Dirac operator $D_p$ generates a $C_0$-semigroup on $L^p(\mathbb{R}^3)^4$ if and only if $p = 2$ (cf. [Br]).

The symbol $a$ if the Dirac operator $D_p$ is similar (in the sense of matrices) to the function $b : \mathbb{R}^3 \rightarrow GL_4$ given by

$$b(\xi) := \text{diag}(\lambda_1(\xi), \lambda_2(\xi), \lambda_3(\xi), \lambda_4(\xi)).$$

The values $\lambda_j(\xi)$ ($j = 1, 2, 3, 4$) are the eigenvalues of $a(\xi)$ and are determined by

$$\lambda_1(\xi) = \lambda_2(\xi) = -\lambda_3(\xi) = -\lambda_4(\xi) = i(\xi^2 + 1)^{1/2}.$$ 

We immediately verify that for $\lambda_j$ ($j = 1, 2, 3, 4$) we have $\rho = 1$ and $m = r$. Thus, by Proposition 4.2

$$\sigma(\text{Op}_p(\lambda_1)) = i[1, \infty) \quad \text{and} \quad \sigma(\text{Op}_p(\lambda_3)) = i(-\infty, -1]$$

for all $p$ satisfying $1 \leq p < \infty$. Moreover, it follows from Theorem 4.6 that $\text{Op}_p(\lambda_j)$ ($j = 1, 2, 3, 4$) generates a once integrated semigroup on $L^p(\mathbb{R}^3)$.
provided $\frac{1}{2} - \frac{1}{p} \leq \frac{1}{3}$ and a twice integrated semigroup for all other values of $p$. Finally, we may conclude from Remark 4.7 and Corollary 4.13 that

$$\| (\lambda - \text{Op}_p (\lambda_j))^{-1} \| \leq \begin{cases} C |\lambda| \left( \frac{1}{\text{Re} \lambda} + \frac{1}{\text{Re} \lambda^2} \right) & \text{if } \left| \frac{1}{2} - \frac{1}{p} \right| \leq \frac{1}{3}, \\ C |\lambda|^2 \left( \frac{1}{\text{Re} \lambda} + \frac{1}{\text{Re} \lambda^2} \right) & \text{if } \left| \frac{1}{2} - \frac{1}{p} \right| > \frac{1}{3}, \end{cases}$$

for all $j = 1, 2, 3, 4$ and all $\lambda \in \mathbb{C}$ with $\text{Re} \lambda > 0$.

**References**


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