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ON SPECTRAL GEOMETRY OF MINIMAL SURFACES IN CP^n

YI-BING SHEN

ABSTRACT. By employing the standard isometric imbedding of CP^n into the Euclidean space, a classification theorem for full, minimal, 2-type surfaces in CP^n that are not $\pm$ holomorphic is given. All such compact minimal surfaces are either totally real minimal surfaces in CP^2 or totally real superminimal surfaces in CP^3 and CP^4. In the latter case, they are locally unique. Moreover, some eigenvalue inequalities for compact minimal surfaces of CP^n with constant Kaehler angle are shown.

1. INTRODUCTION

Let M be a connected compact real two-dimensional Riemannian manifold and $\tilde{x} : M \to CP^n$ an isometric minimal immersion of M into a complex projective n-space CP^n. By [9] we may associate $\tilde{x}$ with an isometric immersion $x = A \circ \tilde{x} : M \to HM(n+1)$, where $HM(n+1) = \{A \in \mathrm{gl}(n+1, \mathbb{C}) : \overline{A} = A^t\}$ viewed as a Euclidean space and A: $CP^n \to HM(n+1)$ denotes the standard isometric imbedding of CP^n into $HM(n+1)$ with parallel second fundamental form. Let Δ be the Laplace-Beltrami operator on M and $\mathrm{Spec}(M) = \{0 = \lambda_0 < \lambda_1 < \cdots\}$ the spectrum of Δ acting on differentiable functions, where each eigenvalue may be repeated as many times as its multiplicity. We then have the L^2-decomposition $x = x_0 + \sum_{u \geq 1} x_u$, $u \in \mathbb{N}$, where x_0 is a constant-valued map which is called the centre of gravity of M, and $\{x_u\}$ are $HM(n+1)$-valued maps of M such that $\Delta x_u = \lambda_u x_u$.

We shall say that the immersion x is of *k-type* if there exists k $(< \infty)$ natural numbers $u_1, \ldots, u_k$ such that $x = x_0 + \sum_{t=1}^{k} x_{u_t}$, which is also called of *order* $(u_1, \ldots, u_k)$. Such immersions are usually called of *finite type* [2]. A. Ros et al. obtained a lot of information about the spectral geometry of CR-minimal submanifolds, Kaehler submanifolds and minimal real hypersurfaces in CP^n [6, 9, 10]. Recently, we have studied the spectral geometry of totally real minimal submanifolds in CP^n [11].

In this paper we want to study the spectral geometry of real minimal surfaces of CP^n in terms of the complex version of the theory of minimal surfaces in a Kaehler manifold. We prove that if $\tilde{x} : M \to CP^n$ is a full minimal immersion which is not $\pm$ holomorphic and if $x = A \circ \tilde{x} : M \to HM(n+1)$ is at most

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of 2-type, then the Kaehler angle of $\tilde{x}$ must be constant. We also prove that all minimal 2-type surfaces in CP^2 that are not $\pm$ holomorphic are actually of 1-type, so that they are totally real minimal surfaces in CP^2. Moreover, we obtain the following

Classification theorem. *All full, minimal, 2-type surfaces in CP^n that are neither $\pm$ holomorphic nor of 1-type are either the totally real Veronese surface in CP^4, or totally real, flat, superminimal surfaces in CP^3 and CP^4. Moreover, they are locally unique.*

A brief survey of the complex version of minimal surfaces in CP^n will be offered in §2. The description of the immersion $x = A \circ \tilde{x} : M \to HM(n+1)$ and some eigenvalue inequalities of $\mathrm{Spec}(M)$ will be given in §3. In §4 we shall show a characteristic for the immersion $x = A \circ \tilde{x} : M \to HM(n+1)$ to be at most of 2-type. The proof of the classification theorem will be completed in the last section.

The manifolds considered here are assumed to be connected and smooth. For the necessary knowledge and notations of the geometry of submanifolds see [2, 9]. For the complex version of real minimal surfaces in a Kaehler manifold see [3, 13].

2. Preliminaries

Let $\widetilde{M}^n$ be a Kaehler manifold of complex dimension n of constant holomorphic sectional curvature $4\tilde{c}$ and $\{\omega_\alpha\}$ a local field of unitary coframes on $\widetilde{M}^n$ so that the Kaehler metric $\langle\ ,\ \rangle$ of $\widetilde{M}^n$ may be written by

$$\sum_\alpha \omega_\alpha \bar{\omega}_\alpha = \sum_\alpha \omega_\alpha \omega_{\bar{\alpha}}.$$

Here and later on we will agree with the following convention on the ranges of indices unless otherwise stated:

$$\alpha, \beta, \gamma, \ldots = 1, \ldots, n, \qquad i, j, k, \ldots = 1, 2,$$
$$\lambda, \mu = 3, \ldots, n, \qquad \tau, \zeta, \xi, \eta = 3, \ldots, n, 3^*, \ldots, n^*.$$

Let $\{\omega_{\alpha\bar{\beta}}\}$ be the unitary connection forms with respect to $\{\omega_\alpha\}$. Then we have

$$\begin{aligned} d\omega_a &= \sum_\beta \omega_{\alpha\bar{\beta}} \wedge \omega_\beta, \quad \omega_{\alpha\bar{\beta}} + \omega_{\bar{\beta}\alpha} = 0 \quad (\omega_{\bar{\beta}\alpha} = \bar{\omega}_{\beta\bar{\alpha}}), \\ d\omega_{\alpha\bar{\beta}} &= \sum_\gamma \omega_{\alpha\bar{\gamma}} \wedge \omega_{\gamma\bar{\beta}} + \Omega_{\alpha\bar{\beta}}, \\ \Omega_{\alpha\bar{\beta}} &= -\tilde{c}\left(\omega_\alpha \wedge \omega_{\bar{\beta}} + \delta_{\alpha\beta} \sum_\gamma \omega_\gamma \wedge \omega_{\bar{\gamma}}\right). \end{aligned} \tag{2.1}$$

If we put $\omega_\alpha = \tilde{\theta}_\alpha + i\tilde{\theta}_{\alpha^*}$ where $i = \sqrt{-1}$, then $\{\tilde{\theta}_\alpha, \tilde{\theta}_{\alpha^*}\}$ is a canonical basis of the underlying Riemannian structure of $\widetilde{M}^n$, whose dual basis $\{\tilde{e}_\alpha, \tilde{e}_{\alpha^*}\}$ satisfies $\tilde{e}_{\alpha^*} = J\tilde{e}_\alpha$ where J stands for the complex structure of $\widetilde{M}^n$.

Let M be an oriented real Riemannian 2-manifold and $\tilde{x} : M \to \widetilde{M}^n$ an isometric minimal immersion. The Kaehler angle α of $\tilde{x}$ is defined by $\cos\alpha = \langle Je_1, e_2\rangle$, where $\{e_i\}$ is an orthonormal basis of M [3]. We denote by $\{\theta_i\}$

the dual field with respect to $\{e_i\}$ and put $\varphi = \theta_1 + i\theta_2$. Then the metric of M can be expressed by

$$ds_M^2 = \varphi\bar{\varphi}.$$

The structure equations of M are

$$d\varphi = -i\theta_{12} \wedge \varphi, \quad d\theta_{12} = -\frac{i}{2}K\varphi \wedge \bar{\varphi}, \tag{2.2}$$

where K is the Gauss curvature of M and θ_{12} is the real connection 1-form.

It is known by [3] that the Kaehler angle α plays an important role in the study of minimal surfaces of $\widetilde{M}^n$. Indeed, the immersion $\tilde{x}$ is $\pm$ holomorphic if and only if $\cos\alpha = \pm 1$, while $\tilde{x}$ is totally real if and only if $\cos\alpha = 0$. From now on we assume that $\tilde{x}$ is not $\pm$ holomorphic, so that points on M where $\sin\alpha = 0$ are isolated. Thus, in the open subset where $\sin\alpha \neq 0$, we may extend $\{e_i, Je_i\}$ to a neighbourhood U of $\widetilde{M}^n$ such that $\mathfrak{D}_U = \operatorname{span}\{e_i, Je_i\}$ is a real 4-dimensional distribution on U. We now put [7]

$$(\sin\alpha)e_{1^*} = -(\cos\alpha)e_1 - Je_2, \quad (\sin\alpha)e_{2^*} = Je_1 - (\cos\alpha)e_2 \tag{2.3}$$

and

$$\begin{aligned} \tilde{e}_1 &= \left(\cos\frac{\alpha}{2}\right)e_1 + \left(\sin\frac{\alpha}{2}\right)e_{1^*}, \quad \tilde{e}_{1^*} = \left(\cos\frac{\alpha}{2}\right)e_2 + \left(\sin\frac{\alpha}{2}\right)e_{2^*}, \\ \tilde{e}_2 &= \left(\sin\frac{\alpha}{2}\right)e_1 - \left(\cos\frac{\alpha}{2}\right)e_{1^*}, \quad \tilde{e}_{2^*} = -\left(\sin\frac{\alpha}{2}\right)e_2 + \left(\cos\frac{\alpha}{2}\right)e_{2^*}. \end{aligned} \tag{2.4}$$

Clearly, both $\{e_i, e_{i^*}\}$ and $\{\tilde{e}_i, \tilde{e}_{i^*}\}$ are the orthonormal bases of $\mathfrak{D}_U$ and $J\tilde{e}_i = \tilde{e}_{i^*}$. Let $\{e_\alpha, e_{\alpha^*}\}$ be a local orthonormal basis in $\widetilde{M}^n$ which extends $\{e_i, e_{i^*}\}$ and satisfies $e_{\lambda^*} = Je_\lambda$ for $\lambda \geq 3$. We denote its dual basis by $\{\theta_\alpha, \theta_{\alpha^*}\}$. Then, $\{v_j, v_\lambda\}$ is a local field of unitary frames in $\widetilde{M}^n$ where $v_j = \frac{1}{2}(\tilde{e}_j - i\tilde{e}_{j^*})$ and $v_\lambda = \frac{1}{2}(e_\lambda - ie_{\lambda^*})$. Let $\{\omega_\alpha\}$ be the dual field of unitary coframes with respect to $\{v_\alpha\}$. By virtue of (2.3) and (2.4) we have

$$\begin{aligned} \theta_1 + i\theta_2 &= \left(\cos\frac{\alpha}{2}\right)\omega_1 + \left(\sin\frac{\alpha}{2}\right)\omega_{\bar{2}}, \\ \theta_{1^*} + i\theta_{2^*} &= \left(\sin\frac{\alpha}{2}\right)\omega_1 - \left(\cos\frac{\alpha}{2}\right)\omega_{\bar{2}}, \\ \theta_\lambda + i\theta_{\lambda^*} &= \omega_\lambda. \end{aligned} \tag{2.5}$$

By restricting these forms to M, we get

$$\omega_1 = \left(\cos\frac{\alpha}{2}\right)\varphi, \quad \omega_2 = \left(\sin\frac{\alpha}{2}\right)\bar{\varphi}, \quad \omega_\lambda = 0, \tag{2.6}$$

which satisfy [3]

$$\frac{1}{2}\{d\alpha + (\sin\alpha)(\omega_{1\bar{1}} + \omega_{2\bar{2}})\} = a\varphi, \tag{2.7}$$

$$\omega_{1\bar{2}} = c\bar{\varphi}, \tag{2.8}$$

$$\left(\cos\frac{\alpha}{2}\right)\omega_{\lambda\bar{1}} = a_\lambda\varphi, \quad \left(\sin\frac{\alpha}{2}\right)\omega_{\lambda\bar{2}} = c_\lambda\bar{\varphi}, \tag{2.9}$$

for some complex-valued smooth functions a, c, a_λ and c_λ on M.

$\tilde{x}(M)$ is said to be *superminimal* if $c \equiv 0$.

Let σ be the second fundamental form of $\tilde{x}$. The Gauss equation of $\tilde{x}$ is

$$\|\sigma\|^2 = 2(1 + 3\cos^2\alpha)\tilde{c} - 2K, \tag{2.10}$$

where

$$\|\sigma\|^2 = 4\left(|a|^2 + |c|^2 + \sum_\lambda |a_\lambda|^2 + \sum_\lambda |c_\lambda|^2\right). \tag{2.11}$$

Taking the exterior derivative of each equation in (2.5) and using (2.1)–(2.9), we have

$$\theta_{12} = i\left\{\left(\cos\frac{\alpha}{2}\right)^2 \omega_{1\bar{1}} - \left(\sin\frac{\alpha}{2}\right)^2 \omega_{2\bar{2}}\right\}, \tag{2.12}$$

$$\theta_{11^*} + i\theta_{21^*} = -(\bar{a} + c)\bar{\varphi}, \quad \theta_{12^*} + i\theta_{22^*} = -i(\bar{a} - c)\bar{\varphi}, \tag{2.13}$$

$$\theta_{1\lambda} + i\theta_{2\lambda} = -(\bar{a}_\lambda + c_\lambda)\bar{\varphi}, \quad \theta_{1\lambda^*} + i\theta_{2\lambda^*} = -i(\bar{a}_\lambda - c_\lambda)\bar{\varphi}. \tag{2.14}$$

For convenience, the basis $\{e_\alpha, e_{\alpha^*}\}$ chosen as above for minimal immersion $\tilde{x} : M \to \widetilde{M}^n$ will be called an *adapted Darboux basis* of $\tilde{x}$.

3. Minimal surfaces of CP^n in $HM(n+1)$

Let $HM(m) = \{A \in \mathrm{gl}(m, \mathbb{C}) : \overline{A} = A^t\}$ be the set of $(m \times m)$-Hermitian matrices with the metric $\langle\ ,\ \rangle$ defined by

$$\langle A, B\rangle = \mathrm{trace}(AB)/2\tilde{c}$$

for any $A, B \in HM(m)$, where $\tilde{c}$ is a positive real number. Then,

$$CP^n = \{A \in HM(n+1) : AA = A, \mathrm{trace}\, A = 1\},$$

with the metric induced by one of $HM(n+1)$, is isometric to the complex projective n-space with the Fubini-Study metric of constant holomorphic sectional curvature $4\tilde{c}$ (for details see [9]). Let $A : CP^n \to HM(n+1)$ denote this isometry. The curvature tensor $\widetilde{R}$ of CP^n can be expressed by

$$\begin{aligned}\widetilde{R}(X, Y)Z = \tilde{c}(&\langle Y, Z\rangle X - \langle X, Z\rangle Y \\ &+ \langle JY, Z\rangle JX - \langle JX, Z\rangle JY + 2\langle X, JY\rangle JZ)\end{aligned} \tag{3.1}$$

for any $X, Y, Z \in TCP^n$, where J is the complex structure of CP^n.

Let D be the Riemannian connection of $HM(n+1)$, $\widetilde{\nabla}$ the connection induced on CP^n, and let $\tilde{\sigma}$, $\widetilde{\nabla}^\perp$, $\widetilde{\Lambda}$ and $\widetilde{H}$ denote, respectively, the second fundamental form, the normal connection, the Weingarten endomorphism and the mean curvature vector of CP^n in $HM(n+1)$. We then have [9]

$$\tilde{\sigma}(JX, Y) = -\tilde{\sigma}(X, JY), \quad \widetilde{\nabla}\tilde{\sigma} = 0, \tag{3.2}$$

$$\begin{aligned}\widetilde{\Lambda}_{\tilde{\sigma}(X,Y)}Z = \tilde{c}(&2\langle X, Y\rangle Z + \langle X, Z\rangle Y + \langle Y, Z\rangle X \\ &+ \langle JX, Z\rangle JY + \langle JY, Z\rangle JX), \\ \langle\tilde{\sigma}(X, Y), \tilde{\sigma}(Z, W)\rangle &= \langle\widetilde{\Lambda}_{\tilde{\sigma}(X,Y)}Z, W\rangle,\end{aligned} \tag{3.3}$$

$$\langle\tilde{\sigma}(X, Y), A\rangle = -\langle X, Y\rangle, \quad \langle\tilde{\sigma}(X, Y), I\rangle = 0, \tag{3.4}$$

$$\widetilde{H} = 2\tilde{c}(I - (n-1)A)/n, \tag{3.5}$$

where $X, Y, Z, W \in TCP^n$, $A \in CP^n$ and I is the $(n+1)\times(n+1)$-identity matrix.

We now consider an isometric minimal immersion $\tilde{x} : M \to CP^n$ of a compact real surface M into CP^n, and let $x = A \circ \tilde{x} : M \to HM(n+1)$ be the composition of $\tilde{x}$ and A, which is an isometric immersion of M into $HM(n+1)$. We denote the mean curvature vector of the immersion x by H. Let ∇ be the induced connection on M, and let σ, $\nabla^\perp$ and Λ denote, respectively, the second fundamental form, the normal connection and the Weingarten endomorphism of $\tilde{x}$. Since $\tilde{x}$ is minimal, then

$$H = \frac{1}{2}\sum_i \tilde{\sigma}(e_i, e_i), \tag{3.6}$$

where $\{e_i\}$ is a local field of orthonormal frames on M.

Lemma 3.1. *Let Δ be the Laplace-Beltrami operator on M. Then, for any $A \in x(M)$ we have*

$$\begin{aligned}\Delta H(A) = &-\sum_{i,j} \tilde{\sigma}(\sigma(e_i, e_j), \sigma(e_i, e_j)) + 8\tilde{c}(\cos\alpha)\tilde{\sigma}(Je_1, e_2)\\ &+ 2(4\tilde{c} - K)H + 2\tilde{c}(\nabla_{e_2}\cos\alpha)Je_1 - 2\tilde{c}(\nabla_{e_1}\cos\alpha)Je_2,\end{aligned} \tag{3.7}$$

where α is the Kaehler angle of $\tilde{x}$ and K is the Gauss curvature of M.

Proof. Without loss of generality, we may assume that $\nabla_{e_i}e_j|_A = 0$ at a point $A \in x(M)$. By means of (3.2), (3.3) and the minimality of $\tilde{x}$, we have

$$\begin{aligned}dH(e_j) &= \frac{1}{2}\sum_i D_{e_j}\tilde{\sigma}(e_i, e_i)\\ &= \sum_i \tilde{\sigma}(\sigma(e_i, e_j), e_i) - 3\tilde{c}e_j - \tilde{c}\sum_i \langle Je_i, e_j\rangle Je_i,\end{aligned}$$

and thus

$$\begin{aligned}\Delta H(A) &= -\sum_j D_{e_j}(dH(e_j))\\ &= \sum_{i,j} \widetilde{\Lambda}_{\tilde{\sigma}(\sigma(e_i, e_j), e_i)}e_j - \sum_{i,j} \widetilde{\nabla}^\perp_{e_j}\tilde{\sigma}(\sigma(e_i, e_j), e_i)\\ &\quad + 6\tilde{c}H + \tilde{c}\sum_{i,j} D_{e_j}(\langle Je_i, e_j\rangle Je_i)\\ &= -\sum_{i,j} \widetilde{\nabla}^\perp_{e_j}\tilde{\sigma}(\sigma(e_i, e_j), e_i) + 2\tilde{c}\sum_{i,j}\langle J\sigma(e_i, e_j), e_j\rangle Je_i\\ &\quad + 2\tilde{c}(\cos\alpha)\tilde{\sigma}(Je_1, e_2) + 6\tilde{c}H.\end{aligned} \tag{3.8}$$

By (3.1) and the Codazzi equation of $\tilde{x}$, a direct calculation gives

$$\begin{aligned}\sum_{i,j} \widetilde{\nabla}^\perp_{e_j}\tilde{\sigma}(\sigma(e_i, e_j), e_i) &= \sum_{i,j} \tilde{\sigma}(\sigma(e_i, e_j), \sigma(e_i, e_j))\\ &\quad - \sum_{i,j} \tilde{\sigma}(\Lambda_{\sigma(e_i, e_j)}e_j, e_i) - 6\tilde{c}(\cos\alpha)\tilde{\sigma}(Je_1, e_2) + 6\tilde{c}(\cos^2\alpha)H,\end{aligned}$$

which together with (3.8) yields

(3.9)
$$\Delta H(A) = -\sum_{i,j} \tilde{\sigma}(\sigma(e_i, e_j), \sigma(e_i, e_j)) + \sum_{i,j} \tilde{\sigma}(\Lambda_{\sigma(e_i, e_j)} e_j, e_i) + 6\tilde{c}(\sin^2\alpha)H + 8\tilde{c}(\cos\alpha)\tilde{\sigma}(Je_1, e_2) + 2\tilde{c}\sum_{i,j}\langle J\sigma(e_i, e_j), e_j\rangle Je_i.$$

From the minimality of $\tilde{x}$ we have

$$\sum_{i,j}\langle J\sigma(e_i, e_j), e_j\rangle Je_i = (\nabla_{e_2}\cos\alpha)Je_1 - (\nabla_{e_1}\cos\alpha)Je_2. \tag{3.10}$$

From the Gauss equation of $\tilde{x}$ it follows that

$$\sum_{i,j} \tilde{\sigma}(\Lambda_{\sigma(e_i, e_j)} e_j, e_i) = 2[\tilde{c}(1 + 3\cos^2\alpha) - K]H. \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9) gives (3.7) immediately. □

The normal space to M in CP^n at a point $\tilde{x}$ is denoted by $T_{\tilde{x}}^{\perp}M$. As in [10], the tensor $T : T_{\tilde{x}}^{\perp}M \times T_{\tilde{x}}^{\perp}M \to \mathbb{R}$ is defined by

$$T(\xi, \eta) = \operatorname{trace}(\Lambda_\xi \Lambda_\eta) = \sum_{i,j}\langle\sigma(e_i, e_j), \xi\rangle\langle\sigma(e_i, e_j), \eta\rangle \tag{3.12}$$

for all $\xi, \eta \in T_{\tilde{x}}^{\perp}M$. A subspace $\mathscr{H}_{\tilde{x}}^{\perp}M$ of $T_{\tilde{x}}^{\perp}M$ is said to be *holomorphic* if $J\xi \in \mathscr{H}_{\tilde{x}}^{\perp}M$ for any $\xi \in \mathscr{H}_{\tilde{x}}^{\perp}M$.

As in §2, from an orthonormal basis $\{e_i\}$ of M we can deduce an adapted Darboux basis $\{e_\alpha, e_{\alpha^*}\}$ of $\tilde{x}$ in CP^n. Then, by (2.3) and $(3.2)_1$, we have

$$(\sin^2\alpha)\sum_k \tilde{\sigma}(e_{k^*}, e_{k^*}) = 2(1 + \cos^2\alpha)H - 4(\cos\alpha)\tilde{\sigma}(Je_1, e_2). \tag{3.13}$$

Substituting (3.13) into (3.7), we get

$$\begin{aligned}\Delta H = &-\sum_{i,j} \tilde{\sigma}(\sigma(e_i, e_j), \sigma(e_i, e_j)) + 2[2\tilde{c}(3 + \cos^2\alpha) - K]H \\ &- 2\tilde{c}(\sin^2\alpha)\sum_k \tilde{\sigma}(e_{k^*}, e_{k^*}) + 2\tilde{c}[(\nabla_{e_2}\cos\alpha)Je_1 - (\nabla_{e_1}\cos\alpha)Je_2].\end{aligned} \tag{3.14}$$

By virtue of (2.10), (3.2)–(3.4), (3.12) and (3.14), a simple computation gives the following

Lemma 3.2. *For the immersion* $x = A \circ \tilde{x} : M \to HM(n+1)$ *we have the relations*:

$$\begin{gathered}\langle H, I\rangle = 0, \quad \langle H, A\rangle = -1, \\ \langle H, H\rangle = \tilde{c}(3 + \cos^2\alpha), \quad \langle \Delta H, A\rangle = -2\tilde{c}(3 + \cos^2\alpha), \\ \langle \Delta H, H\rangle = 4\tilde{c}^2(5 + 7\cos^2\alpha) - 2\tilde{c}(1 + \cos^2\alpha)K \\ - \tilde{c}(\sin^2\alpha)\sum_k T(e_{k^*}, e_{k^*}).\end{gathered} \tag{3.15}$$

We now obtain the following theorem directly from Lemma 3.2 using the methods of [10].

Theorem 3.3. *Let M be a compact minimal surface in CP^n with constant Kaehler angle α. Then we have*

$$\lambda_1 \leq 2\tilde{c}(3+\cos^2\alpha),$$

$$2\pi(1+\cos^2\alpha)\chi(M)/\mathrm{Area}(M)$$
$$\leq [\tilde{c}(3+\cos^2\alpha)-\tfrac{1}{2}\lambda_1][\tilde{c}(3+\cos^2\alpha)-\tfrac{1}{2}\lambda_2]-\tilde{c}^2(\cos^4\alpha-8\cos^2\alpha-1),$$

where $\lambda_1, \lambda_2 \in \mathrm{Spec}(M)$, $\chi(M)$ is the Euler-Poincaré characteristic of M. Moreover, the first equality holds if and only if M is of order 1 in $HM(n+1)$, and the second equality holds if and only if M is of order $(1, 2)$ in $HM(n+1)$ and the first normal space to M in CP^n is holomorphic.

Remark. When $\cos^2\alpha = 1$, i.e., M is a compact Kaehler submanifold of complex dimension 1, the above eigenvalue inequalities were shown in Theorem 5.1 and 5.2 of [10] with $\tilde{c} = 1/4$. A similar result for totally real minimal submanifolds of CP^n was obtained in [11].

Before concluding this section we state the following proposition used below.

Proposition 3.4. *Let $\tilde{x} : M \to CP^n$ be a minimal immersion which is not $\pm$ holomorphic. Then, the Kaehler angle α of $\tilde{x}$ is constant if and only if*

$$\langle\sigma(e_i, e_j), e_{k^*}\rangle = \langle\sigma(e_i, e_k), e_{j^*}\rangle \tag{3.16}$$

for an adapted Darboux basis $\{e_\alpha, e_{\alpha^*}\}$ of $\tilde{x}$.

Proof. From (3.10) we see that $\cos\alpha$ is constant if and only if

$$\sum_{i,j}\langle J\sigma(e_i, e_j), e_j\rangle Je_i = 0,$$

which is equivalent to

$$\sum_j\langle\sigma(e_i, e_j), Je_j\rangle = 0.$$

Since $\sin\alpha \neq 0$, then it is known easily from (2.3) that the above equations are just (3.16). □

4. 2-TYPE MINIMAL SURFACES

Let M be a compact Riemannian 2-manifold. As is known by [2, 10], an isometric immersion $x : M \to HM(n+1)$ is at most of 2-type if and only if the mean curvature vector H of x satisfies

$$\Delta H = pH + q(A - A_0) \tag{4.1}$$

for some real numbers p and q, where $A \in x(M)$ and $A_0 = x_0$ is the centre of the gravity of M. Particularly, x is of 1-type if and only if $q = 0$ in (4.1).

By virtue of (3.7) and (4.1) we have the following

Proposition 4.1. *Let $\tilde{x} : M \to CP^n$ be a full minimal immersion which is not $\pm$ holomorphic. If $x = A \circ \tilde{x} : M \to HM(n+1)$ is at most of 2-type, then the Kaehler angle α of $\tilde{x}$ is constant and*

$$A_0 = (n+1)^{-1}I \quad \text{for } q \neq 0, \tag{4.2}$$

where I is the $(n+1)\times(n+1)$-identity matrix.

Proof. Let $\{e_\alpha, e_{\alpha^*}\}$ be an adapted Darboux basis of $\tilde{x}$. Applying $\langle X, -\rangle$ to both sides of (4.1) for any $X \in TM$ and using (2.3), (3.6) and (3.7), we get

$$\tilde{c}\nabla_X(\cos^2\alpha) = -q\langle A_0, X\rangle. \tag{4.3}$$

On the other hand, applying $\langle A, -\rangle$ to both sides of (4.1) and using (3.15), we can see that

$$2\tilde{c}(3+\cos^2\alpha) = p - 2q + q\langle A, A_0\rangle.$$

From this it follows that

$$2\tilde{c}\nabla_X(\cos^2\alpha) = q\langle A_0, X\rangle \tag{4.4}$$

Thus, (4.3) and (4.4) yield $\nabla_X(\cos^2\alpha) = 0$ for any $X \in TM$; i.e., α is constant.

Since (3.7) implies that $\Delta H \in T^\perp CP^n$, then we now obtain (4.2) directly using the methods of [10]. □

The main result of this section is the following

Theorem 4.2. *Let $\tilde{x} : M \to CP^n$ $(n \geq 3)$ be a full minimal immersion which is not $\pm$ holomorphic. Then the immersion $x = A \circ \tilde{x} : M \to HM(n+1)$ is at most of 2-type if and only if the following conditions are satisfied*:

(i) *Both the Kaehler angle α of $\tilde{x}$ and the Gauss curvature K of M are constant and $\tilde{x}(M)$ is superminimal*;

(ii) *$T(\xi, \eta) + T(J\xi, J\eta) = \tilde{k}\langle\xi, \eta\rangle$ for all $\xi, \eta \in \mathscr{H}^\perp M$, where T is defined by* (3.12), *$\mathscr{H}^\perp M$ denotes the holomorphic subspace of the normal space to $\tilde{x}(M)$, $\tilde{k}$ is a real constant factor and $\tilde{k} = 4\tilde{c}\sin^2\alpha$ when $\alpha \neq \pi/2$.*

To prove this theorem we need to establish some lemmas. In the following, we always assume that $\tilde{x} : M \to CP^n$ is a full minimal immersion which is not $\pm$ holomorphic, and $\{e_\alpha, e_{\alpha^*}\}$ is an adapted Darboux basis of $\tilde{x}$ mentioned as in §2. Notice that $\mathscr{H}^\perp M = \operatorname{span}\{e_\lambda, e_{\lambda^*}\}$.

Lemma 4.3. *If $x = A \circ \tilde{x} : M \to HM(n+1)$ is at most of 2-type, then*

$$\|\sigma\|^2 - \|\sigma_r\|^2 = \textit{constant},$$

where $\|\sigma_r\|^2 = \sum_k T(e_{k^*}, e_{k^*})$.

Proof. By (3.14), (4.1) and Proposition 4.1, we have

$$\begin{aligned} &-\sum_{i,j}\tilde{\sigma}(\sigma(e_i, e_j), \sigma(e_i, e_j)) + 2[2\tilde{c}(3+\cos^2\alpha) - K]H \\ &\qquad - 2\tilde{c}(\sin^2\alpha)\sum_k \tilde{\sigma}(e_{k^*}, e_{k^*}) \\ &= pH + q[A - (n+1)^{-1}I]. \end{aligned} \tag{4.5}$$

Applying $\langle H, -\rangle$ to both sides of (4.5) and using (3.3), (3.4) and (3.15), we obtain

$$p(3+\cos^2\alpha) - \frac{q}{\tilde{c}} - 4\tilde{c}(5+7\cos^2\alpha) + 2(1+\cos^2\alpha)K + (\sin^2\alpha)\|\sigma_r\|^2 = 0. \tag{4.6}$$

Similarly, applying $\langle \sum_k \tilde{\sigma}(e_{k^*}, e_{k^*}), -\rangle$ to both sides of (4.5), we have

$$p(2+\sin^2\alpha) - \frac{q}{\tilde{c}} + 4\tilde{c}(\cos^2\alpha - 5) + 2(\sin^2\alpha)K + (1+\cos^2\alpha)\|\sigma_r\|^2 = 0. \tag{4.7}$$

Adding (4.6) to (4.7) and using (2.10), we find

$$\|\sigma\|^2 - \|\sigma_r\|^2 = 3p - \frac{q}{\tilde{c}} - 6\tilde{c}(3+\cos^2\alpha). \tag{4.8}$$

Clearly, by Proposition 4.1, the right-hand side of (4.8) is a constant. □

Lemma 4.4. *Under the same hypothesis as in Lemma* 4.3, *if* $n \geq 3$, *then we have* $\|\sigma_r\|^2 \equiv 0$.

Proof. Applying $\langle \tilde{\sigma}(e_k, Je_\tau), -\rangle$ to both sides of (4.5) and using (3.3), (3.4) and (2.3), we get

$$\sum_{i,j} \langle \sigma(e_i, e_j), e_{k^*}\rangle \langle \sigma(e_i, e_j), e_\tau\rangle = 0. \tag{4.9}$$

From this, the minimality of $\tilde{x}$ and (3.16) we have

$$\begin{aligned} &\langle \sigma(e_1, e_1), e_{1^*}\rangle \langle \sigma(e_1, e_1), e_\tau\rangle + \langle \sigma(e_1, e_1), e_{2^*}\rangle \langle \sigma(e_1, e_2), e_\tau\rangle = 0, \\ &\langle \sigma(e_1, e_1), e_{2^*}\rangle \langle \sigma(e_1, e_1), e_\tau\rangle - \langle \sigma(e_1, e_1), e_{1^*}\rangle \langle \sigma(e_1, e_2), e_\tau\rangle = 0. \end{aligned} \tag{4.10}$$

If $\langle \sigma(e_1, e_1), e_\tau\rangle = \langle \sigma(e_1, e_2), e_\tau\rangle = 0$, namely, $\langle \sigma(e_i, e_j), e_\tau\rangle = 0$, then from Lemma 4.3 we would have $\langle \sigma(e_i, e_j), e_\tau\rangle = 0$ everywhere, i.e., $\theta_{i\tau} \equiv 0$. Thus, from (2.6), (2.9) and (2.14) it follows that

$$\omega_\lambda = 0, \quad \omega_{1\bar{\lambda}} = 0, \quad \omega_{2\bar{\lambda}} = 0 \quad \text{along } \tilde{x}(M),$$

which imply that there exists a totally geodesic CP^2 in CP^n containing $\tilde{x}(M)$. This contradicts the fullness of $\tilde{x}$. Hence, either $\langle \sigma(e_1, e_1), e_\tau\rangle$ or $\langle \sigma(e_1, e_2), e_\tau\rangle$ is nonzero. Thus, from (4.10) we find

$$\langle \sigma(e_1, e_1), e_{1^*}\rangle = \langle \sigma(e_1, e_1), e_{2^*}\rangle = 0 \tag{4.11}$$

everywhere. By (3.16) and the minimality of $\tilde{x}$, (4.11) implies that $\|\sigma_r\|^2 \equiv 0$. □

Lemma 4.5. *Under the same hypothesis as in Lemmas* 4.3 *and* 4.4, *the Gauss curvature* K *is constant and the immersion* $\tilde{x}$ *is superminimal.*

Proof. From (4.8), (2.10) and Lemma 4.4 we have

$$2K = 4\tilde{c}(5+3\cos^2\alpha) - 3p + \frac{q}{\tilde{c}}. \tag{4.12}$$

This is a constant because α is constant. Also, from (2.7) and its complex conjugation it follows that

$$a \equiv 0. \tag{4.13}$$

On the other hand, from Lemma 4.4 it follows that $\langle \sigma(e_i, e_j), e_{k^*}\rangle = 0$, which implies that $\theta_{ij^*} \equiv 0$ according to the definitions as in §2. Hence, from (2.13) and (4.13) we obtain $c \equiv 0$; i.e., $\tilde{x}$ is superminimal. □

We now complete the proof of Theorem 4.2.

Proof of the necessity. Condition (i) of Theorem 4.2 has been shown by Proposition 4.1 and Lemma 4.5.

By Lemma 4.4, we may write

$$\sum_{i,j} \tilde{\sigma}(\sigma(e_i, e_j), \sigma(e_i, e_j)) = \sum_{\tau,\zeta} T(e_\tau, e_\zeta)\tilde{\sigma}(e_\tau, e_\zeta). \tag{4.14}$$

Applying $\langle \tilde{\sigma}(e_\xi, e_\eta), -\rangle$ to both sides of (4.5) and using (3.3), (3.4), (4.8), (4.12) and (4.14), we get

$$T(e_\xi, e_\eta) + T(Je_\xi, Je_\eta) = \tilde{k}\langle e_\xi, e_\eta\rangle, \tag{4.15}$$

where

$$\tilde{k} = 2\tilde{c}(3 + \cos^2\alpha) - p + \frac{q}{2\tilde{c}}. \tag{4.16}$$

Since T defined by (3.12) is bilinear, then (4.15) implies that the part of condition (ii) of Theorem 4.2 has been proved.

Subtracting (4.7) from (4.6) and using Lemma 4.4, we have $K = 8\tilde{c} - p/2$ for $\alpha \neq \pi/2$. From this and (4.12) it follows that

$$p - \frac{q}{2\tilde{c}} = 2\tilde{c}(1 + 3\cos^2\alpha). \tag{4.17}$$

Substituting (4.17) into (4.16) gives $\tilde{k} = 4\tilde{c}(\sin^2\alpha)$ for $\alpha \neq \pi/2$. □

Proof of the sufficiency. From condition (i), (2.7) and (2.13) we see that $\theta_{ij^*} \equiv 0$; i.e., $\langle \sigma(e_i, e_j), e_{k^*}\rangle \equiv 0$. Thus, by condition (ii), the formula (3.14) may be rewritten

$$\begin{aligned} \Delta H = &-\tilde{k}\sum_{\lambda} \tilde{\sigma}(e_\lambda, e_\lambda) + 2[2\tilde{c}(3 + \cos^2\alpha) - K]H \\ &- 2\tilde{c}(\sin^2\alpha)\sum_{k} \tilde{\sigma}(e_{k^*}, e_{k^*}). \end{aligned} \tag{4.18}$$

On the other hand, from (3.5) we have

$$2H + \sum_{k} \tilde{\sigma}(e_{k^*}, e_{k^*}) + 2\sum_{\lambda} \tilde{\sigma}(e_\lambda, e_\lambda) = 2n\tilde{H} = 4\tilde{c}[I - (n+1)A] \tag{4.19}$$

for $A \in x(M)$. Thus, if $\alpha \neq \pi/2$ so that $\tilde{k} = 4\tilde{c}(\sin^2\alpha)$, then (4.18) together with (4.19) yields

$$\Delta H = 2(8\tilde{c} - K)H + 8\tilde{c}^2(n+1)(\sin^2\alpha)[A - (n+1)^{-1}I]. \tag{4.20}$$

Since both K and α are constant, (4.20) shows $x = A \circ \tilde{x} : M \to HM(n+1)$ is of 2-type.

Now consider the case that $\alpha = \pi/2$. From (2.3) and (3.2) we have

$$\sum_{k} \tilde{\sigma}(e_{k^*}, e_{k^*}) = \sum_{k} \tilde{\sigma}(e_k, e_k).$$

Thus, (4.18) together with (4.19) yields

$$\Delta H = 2(4\tilde{c} - K + \tilde{k})H + 2\tilde{c}(n+1)\tilde{k}[A - (n+1)^{-1}I], \tag{4.21}$$

which shows also $x = A \circ \tilde{x}$ is of 2-type. □

Thus, Theorem 4.2 is proved completely.

Remark 4.1. From (4.15) and Lemma 4.4 we have $\tilde{k} = \|\sigma\|^2/(n-2)$ for $n \geq 3$.

Remark 4.2. A 2-type immersion $x : M \to HM(n+1)$ is said to be of order (u_1, u_2) for some natural numbers u_1 and u_2 if $x = x_0 + x_{u_1} + x_{u_2}$ where $\Delta x_{u_i} = \lambda_{u_i} x_{u_i}$ for $i = 1, 2$. Thus, we have [2, 10]

$$\Delta H = (\lambda_{u_1} + \lambda_{u_2})H + \frac{1}{2}\lambda_{u_1}\lambda_{u_2}(x - x_0), \tag{4.22}$$

so that

$$\lambda_{u_1} + \lambda_{u_2} = p, \quad \lambda_{u_1}\lambda_{u_2} = 2q. \tag{4.23}$$

5. Classification of 2-type minimal surfaces

In this section, we assume that CP^n is the complex projective n-space with the Fubini-Study metric of constant holomorphic sectional curvature 4, i.e., $\tilde{c} = 1$.

First of all, we consider the case that $n = 2$.

Proposition 5.1. *Let $\tilde{x} : M \to CP^2$ be a minimal immersion of a compact real surface M into CP^2 that is not $\pm$ holomorphic. If the Kaehler angle α of $\tilde{x}$ is constant, then $\alpha = \pi/2$, namely, $\tilde{x}(M)$ is totally real, so that $x = A \circ \tilde{x} : M \to HM(3)$ is of 1-type.*

Proof. Since $\sin\alpha \neq 0$ and α is constant, then we have (4.13) and

$$\omega_{1\bar{1}} + \omega_{2\bar{2}} = 0. \tag{5.1}$$

Taking the exterior derivative of (5.1) and using (2.1) and (2.8), we obtain $\cos\alpha = 0$, i.e., $\alpha = \pi/2$. Thus, $\tilde{x}(M)$ is totally real in CP^2. By Theorem 2.8 of [9], $x(M)$ is minimal in some hypersphere of $HM(3)$, so that x is of 1-type according to the Takahashi theorem [12]. Moreover, in such a case we have $\|\sigma\|^2 = \|\sigma_r\|^2$ and $q = 0$. Thus, from (4.8) it follows that $p = 6$. Hence, x is of order u_1 with the corresponding eigenvalue $\lambda_{u_1} = 6$. □

Combining Proposition 5.1 and Proposition 4.1, we have the following

Corollary 5.2. *In CP^2, all minimal 2-type surfaces that are not $\pm$ holomorphic are actually of 1-type.*

Remark 5.1. It is known by Theorem 2.8 of [9] that a totally real minimal surface in CP^2 is of 1-type and vice versa. For general minimal surfaces in CP^2 see [4].

We now consider the case that $n \geq 3$.

Proposition 5.3. *Let $\tilde{x} : M \to CP^n$ $(n \geq 3)$ be a full minimal immersion which is not $\pm$ holomorphic, such that $x = A \circ \tilde{x} : M \to HM(n+1)$ is of 2-type. Then, we have $n \leq 4$.*

Proof. Let $\{e_\alpha, e_{\alpha^*}\}$ be an adapted Darboux basis of $\tilde{x}$. According to notations of §2, we have $\theta_{i\lambda} = \sum_j \langle \sigma(e_i, e_j), e_\lambda \rangle \theta_j$, so that

$$\theta_{1\lambda} \wedge \theta_{2\mu} = -\frac{1}{2}T(e_\lambda, e_\mu)\theta_1 \wedge \theta_2, \tag{5.2}$$

where T is defined by (3.12).

On the other hand, from $(2.14)_1$ we have

$$\theta_{1\lambda} \wedge \theta_{2\mu} = -\frac{1}{2}\{(a_\lambda + \bar{c}_\lambda)(\bar{a}_\mu + c_\mu) + (\bar{a}_\lambda + c_\lambda)(a_\mu + \bar{c}_\mu)\}\theta_1 \wedge \theta_2 ,$$

from which and (5.2) it follows that

$$T(e_\lambda , e_\mu) = (a_\lambda + \bar{c}_\lambda)(\bar{a}_\mu + c_\mu) + (\bar{a}_\lambda + c_\lambda)(a_\mu + \bar{c}_\mu). \tag{5.3}$$

Similarly, we have

$$T(Je_\lambda , Je_\mu) = T(e_{\lambda^*} , e_{\mu^*}) = (a_\lambda - \bar{c}_\lambda)(\bar{a}_\mu - c_\mu) + (\bar{a}_\lambda - c_\lambda)(a_\mu - \bar{c}_\mu) , \tag{5.4}$$

$$T(e_\lambda , Je_\mu) = a_\lambda \bar{a}_\mu - \bar{a}_\lambda a_\mu , \quad T(Je_\lambda , e_\mu) = \bar{c}_\lambda c_\mu - c_\lambda \bar{c}_\mu . \tag{5.5}$$

From (5.3)–(5.5) we can see that condition (ii) of Theorem 4.2 is equivalent to

$$a_\lambda \bar{a}_\mu + c_\lambda \bar{c}_\mu = \frac{\tilde{k}}{4}\delta_{\lambda\mu}. \tag{5.6}$$

Suppose now that $n \geq 5$. From (5.6) we have

$$a_3\bar{a}_4 + c_3\bar{c}_4 = 0 , \quad a_3\bar{a}_5 + c_3\bar{c}_5 = 0 ,$$

which has a nonzero solution (a_3 , c_3) if and only if

$$a_4c_5 - c_4a_5 = 0.$$

By (5.6) we have also

$$a_4\bar{a}_5 + c_4\bar{c}_5 = 0.$$

These equations imply that

$$|a_5|^2 + |c_5|^2 = 0 ,$$

i.e., $a_5 = c_5 = 0$. Since $\tilde{k}$ is constant, then (5.6) would yield $\tilde{k} = 0$, namely, $\|\sigma\|^2 \equiv 0$ (see Remark 4.1). This contradicts the fullness of $\tilde{x}$. Therefore, $n \leq 4$. □

As this proposition shows, the only cases that remain to be considered are $n = 3$ and $n = 4$.

Theorem 5.4. *Let $\tilde{x} : M \to CP^3$ be a full minimal immersion of a compact real surface M into CP^3 that is not $\pm$ holomorphic. Then, the immersion $x = A \circ \tilde{x} : M \to HM(4)$ is of 2-type if and only if $\tilde{x}(M)$ is a totally real, flat, superminimal surface, x is of order (u_1 , u_2) with corresponding eigenvalues $\lambda_{u_1} = 4$ and $\lambda_{u_2} = 8$, and, with respect to an adapted local field of unitary coframes $\{\omega_\sigma\}$ in CP^3, the unitary connection forms $\{\omega_{\alpha\bar{\beta}}\}$ of CP^3 restricted to M are given locally by*

$$\begin{pmatrix} 0 & 0 & -\frac{\sqrt{2}}{2}d\bar{z} \\ 0 & 0 & -\frac{\sqrt{2}}{2}e^{-i\rho}\,dz \\ \frac{\sqrt{2}}{2}dz & \frac{\sqrt{2}}{2}e^{i\rho}\,d\bar{z} & 0 \end{pmatrix} , \tag{5.7}$$

where ρ is a real constant and z is the local complex coordinate on M.

Proof. Suppose that $x = A \circ \tilde{x} : M \to HM(4)$ is of 2-type. From Theorem 4.2, (2.12) and (5.1) we have

$$\omega_{1\bar{1}} = -\omega_{2\bar{2}} = -i\theta_{12}, \tag{5.8}$$

whose exterior derivatives, by formulas of §2, are

$$\begin{aligned} \left(\cos\frac{\alpha}{2}\right)^{-2} |a_3|^2 &= -\frac{1}{2}K + 3\left(\cos\frac{\alpha}{2}\right)^2 - 1, \\ \left(\sin\frac{\alpha}{2}\right)^{-2} |c_3|^2 &= -\frac{1}{2}K + 3\left(\sin\frac{\alpha}{2}\right)^2 - 1. \end{aligned} \tag{5.9}$$

On the other hand, taking the exterior derivatives of (2.9), we have [7]

$$\begin{aligned} da_3 - 2ia_3\theta_{12} - a_3\omega_{3\bar{3}} &= 0, \\ dc_3 + 2ic_3\theta_{12} - c_3\omega_{3\bar{3}} &= 0. \end{aligned} \tag{5.10}$$

Here we have used the fact that $d|a_3|^2 = d|c_3|^2 = 0$ because of (5.9).

Taking the exterior derivatives of (5.10) gives

$$a_3(K - 4\cos\alpha) = 0, \quad c_3(K + 4\cos\alpha) = 0. \tag{5.11}$$

We now claim that $a_3 \not\equiv 0$ and $c_3 \not\equiv 0$ on M. In fact, if $a_3 \equiv 0$ and $c_3 \not\equiv 0$ (the case that $a_3 \not\equiv 0$ and $c \equiv 0$ is similar), then (5.11) would show that $K = -4\cos\alpha$, which together with $(5.9)_1$ yields that $(\cos\frac{\alpha}{2})^2 = 3/7$ and $\cos\alpha = -1/7$. Thus, it would follow from (5.6), $(5.9)_2$ and Theorem 4.2 that

$$\tilde{k} = 4|c_3|^2 = 12/49 \neq \sin^2\alpha.$$

This contradiction proves our assertion.

Then, (5.11) implies that $K = 0$ and $\cos\alpha = 0$; namely, $\tilde{x}(M)$ is a totally real, flat, superminimal surface in CP^3. From (5.9) we have

$$|a_3|^2 = |c_3|^2 = \frac{1}{4}, \tag{5.12}$$

so that $\tilde{k} = 2$ according to (5.6). Thus, from (4.12) and (4.16) we find that $p = 12$ and $q = 16$. By means of (4.23) we see that $\lambda_{u_1} = 4$ and $\lambda_{u_2} = 8$.

Since $K = 0$, we can use the local complex coordinate z on M such that

$$\varphi = dz \quad \text{and} \quad \theta_{12} = 0. \tag{5.13}$$

Then, from (5.8) and (5.13) we have

$$\omega_{1\bar{1}} = \omega_{2\bar{2}} = 0. \tag{5.14}$$

Moreover, from (2.8) and (2.9) we have

$$\omega_{1\bar{2}} = 0, \quad \omega_{3\bar{1}} = \sqrt{2}a_3\,dz, \quad \omega_{3\bar{2}} = \sqrt{2}c_3\,d\bar{z}. \tag{5.15}$$

Let $\{v_\alpha\}$ be the dual frames of $\{\omega_\alpha\}$. We put $v_3' = 2a_3v_3$. Clearly, from (5.12) it follows that $\{v_1, v_2, v_3'\}$ is also a unitary basis. By the formula $\tilde{\nabla} v_\alpha = \sum_\beta \omega_{\bar{\alpha}\beta} v_\beta$ (cf., for example, [13]), we obtain

$$\omega_{1\bar{3}}' = 2a_3\omega_{1\bar{3}},$$

from which together with (5.12) and (5.15) it follows that

$$\omega'_{3\bar{1}} = \frac{\sqrt{2}}{2}\,dz \quad \text{and} \quad a'_3 = \frac{1}{2}. \tag{5.16}$$

From this and $(5.10)_1$ with (5.13) we find $\omega'_{3\bar{3}} = 0$, from which and $(5.10)_2$ we have $dc'_3 = 0$, so that $c'_3 = \frac{1}{2}\exp\{i\rho\}$ for a certain real constant ρ according to (5.12). Hence, with respect to the new unitary basis $\{v_1, v_2, v'_3\}$, the unitary connection forms of CP^3 restricted to M are given locally by (5.7).

Conversely, by Theorem 4.2, it is easy to check that such a totally real, flat, superminimal surface M in CP^3 is of 2-type in $HM(4)$ and has corresponding eigenvalues 4 and 8. □

Remark 5.2. (5.7) and (5.13) provide an example of totally real, flat, superminimal surfaces in CP^3. For general totally real, flat, minimal surfaces in CP^n see [5].

Theorem 5.5. *Let $\tilde{x} : M \to CP^4$ be a full minimal immersion of a compact real surface M into CP^4 that is not $\pm$ holomorphic. Then, the immersion $x = A \circ \tilde{x} : M \to HM(5)$ is of 2-type if and only if either*

(i) *$M = S^2(1/3)$, the immersion $\tilde{x}$ is the following composition:*

$$S^2(1/3) \xrightarrow[\text{immersion}]{\text{minimal}} S^4(1) \xrightarrow[\text{covering}]{\text{double}} RP^4 \xrightarrow[\text{geodesic}]{\text{totally}} CP^4,$$

and x is of order $(2, 4)$ with corresponding eigenvalues $\lambda_2 = 2$ and $\lambda_4 = 20/3$; or

(ii) *$\tilde{x}(M)$ is a totally real, flat, superminimal surface, x is of order (u_1, u_2) with corresponding eigenvalues $\lambda_{u_1} = 5 - \sqrt{5}$ and $\lambda_{u_2} = 5 + \sqrt{5}$, and, with respect to an adapted local field of unitary coframes $\{\omega_\alpha\}$ in CP^4, the unitary connection forms $\{\omega_{\alpha\bar{\beta}}\}$ of CP^4 restricted to M are given locally by*

$$\begin{pmatrix} 0 & 0 & -\frac{\sqrt{2}}{2}\,d\bar{z} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{2}\,dz \\ \frac{\sqrt{2}}{2}\,dz & 0 & 0 & -\frac{\sqrt{2}}{2}\,d\bar{z} \\ 0 & \frac{\sqrt{2}}{2}\,d\bar{z} & \frac{\sqrt{2}}{2}\,dz & 0 \end{pmatrix}, \tag{5.17}$$

where z is the local complex coordinate on M.

Proof. Let $\{\omega_\alpha\}$ be an adapted local field of unitary coframes in CP^4 as in §2 and $\{v_\alpha\}$ the dual frames with respect to $\{\omega_\alpha\}$. Suppose that $x = A \circ \tilde{x} : M \to HM(5)$ is of 2-type. From Theorem 4.2 and (5.6) we have

$$a_3\bar{a}_4 + c_3\bar{c}_4 = 0 \quad \text{and} \quad |a_4|^2 + |c_4|^2 = \frac{\tilde{k}}{4}, \tag{5.18}$$

where $\tilde{k} = \|\sigma\|^2/2 \neq 0$ (cf. Remark 4.1). From the above system (5.18) of nonhomogeneous equations it follows that $a_3c_4 - a_4c_3 \neq 0$, so that

$$(a_3v_3 + a_4v_4) \wedge (c_3v_3 + c_4v_4) \neq 0. \tag{5.19}$$

On the other hand, from (5.8) we can get the following relations analogous to (5.9):

$$\begin{aligned} \left(\cos\frac{\alpha}{2}\right)^{-2}(|a_3|^2+|a_4|^2) &= -\frac{1}{2}K+3\left(\cos\frac{\alpha}{2}\right)^2-1, \\ \left(\sin\frac{\alpha}{2}\right)^{-2}(|c_3|^2+|c_4|^2) &= -\frac{1}{2}K+3\left(\sin\frac{\alpha}{2}\right)^2-1. \end{aligned} \tag{5.20}$$

Since both α and K are constant, from (5.19) and (5.20) we see that

$$|a_3|^2+|a_4|^2 = \text{constant} \neq 0.$$

We now perform a transformation on $\{v_3, v_4\}$ as follows:

$$\begin{aligned} v_3' &= (a_3v_3+a_4v_4)/(|a_3|^2+|a_4|^2)^{1/2}, \\ v_4' &= (c_3v_3+c_4v_4-b_{34}v_3')/\|c_3v_3+c_4v_4-b_{34}v_3'\|, \end{aligned} \tag{5.21}$$

where

$$b_{34} = (\bar{a}_3c_3+\bar{a}_4c_4)/(|a_3|^2+|a_4|^2)^{1/2}.$$

Then, with respect to the new frames $\{v_1, v_2, v_3', v_4'\}$, from (2.9) and (5.21) we have

$$\omega_{\bar{1}4}' = \langle\tilde{\nabla}v_1, v_4'\rangle = \langle\omega_{\bar{1}3}v_3+\omega_{\bar{1}4}v_4, v_4'\rangle = 0,$$

so that $a_4' = 0$. Since the immersion $\tilde{x}$ is full, then it follows from the equations analogous to (5.18) that

$$|a_4'|^2 = |c_3'|^2 = 0, \quad |a_3'|^2 = |c_4'|^2 = \frac{\tilde{k}}{4}\ (\neq 0). \tag{5.22}$$

Taking the exterior derivative of (5.1) and using (5.22), we have

$$(\cos\alpha)(3\sin^2\alpha+4|a_3'|^2) = 0,$$

from which it follows that $\cos\alpha = 0$; i.e., $\tilde{x}(M)$ is totally real.

Recall that it is impossible that $K<0$ when both α and K are constant on M [8]. So, we consider two cases separately.

Case (i): $K>0$. Since $\cos\alpha = 0$, then, by [1], we conclude that $K = 1/3$, $M = S^2(1/3)$ is the Veronese minimal surface in $S^4(1)$, and $\tilde{x} : S^2(1/3) \to CP^4$ is a totally real superminimal immersion composed of the standard double covering $S^4 \to RP^4$ and the totally geodesic imbedding $RP^4 \to CP^4$. Thus, from (5.20) and (5.22) we find that

$$|a_3'|^2 = |c_3'|^2 = \frac{1}{6} \quad \text{and} \quad \tilde{k} = \frac{2}{3}.$$

Then, by means of (4.12) and (4.16), we obtain $p = 26/3$ and $q = 20/3$. By Remark 4.2, it follows that $\lambda_{u_1} = 2$ and $\lambda_{u_2} = 20/3$. From the spectrum of $S^2(1/3)$ we see that $u_1 = 2$ and $u_2 = 4$.

Conversely, by Theorem 4.2, such a Veronese surface in CP^4 is of order $(2, 4)$ in $HM(5)$.

Case (ii): $K = 0$. $\tilde{x}(M)$ is a totally real, flat, superminimal surface in CP^4. By the same manner as above, it is not hard to see that

$$|a_3'|^2 = |c_4'|^2 = \frac{1}{4}, \quad \tilde{k} = 1, \quad \lambda_{u_i} = 5\pm\sqrt{5} \quad (i = 1, 2). \tag{5.23}$$

With respect to the unitary basis $\{v_1, v_2, v_3', v_4'\}$, from (5.21) and (5.23) we have

$$\omega'_{\bar{1}3} = \langle \tilde{\nabla} v_1, v_3' \rangle = -\frac{\sqrt{2}}{2}\varphi. \tag{5.24}$$

Thus, together with the similar computation for $\omega'_{\bar{2}4}$, we obtain

$$\omega'_{1\bar{3}} = -\frac{\sqrt{2}}{2}\bar{\varphi}, \quad \omega'_{2\bar{4}} = -\frac{\sqrt{2}}{2}\varphi. \tag{5.25}$$

Taking the exterior derivatives of (5.25), we have

$$\omega'_{3\bar{3}} = -\omega'_{4\bar{4}} = i\theta_{12}. \tag{5.26}$$

Moreover, (5.22) implies that

$$\omega'_{1\bar{4}} = \omega'_{2\bar{3}} = 0, \tag{5.27}$$

whose exterior derivative gives

$$\omega'_{3\bar{4}} = f\bar{\varphi} \tag{5.28}$$

for a certain complex-valued function f on M. Taking the exterior derivative of (5.26) and using (5.28), we find

$$|f|^2 = \frac{1}{2}. \tag{5.29}$$

Now we put

$$v_1'' = v_1, \quad v_2'' = -\sqrt{2}\bar{f}v_2, \quad v_3'' = v_3', \quad v_4'' = -\sqrt{2}\bar{f}v_4'.$$

Then, with respect to the new unitary basis $\{v_\alpha''\}$, from (5.24)–(5.29) we have

$$\begin{gathered} \omega''_{1\bar{1}} = -\omega''_{2\bar{2}} = -\omega''_{3\bar{3}} = \omega''_{4\bar{4}} = -i\theta_{12}, \\ \omega''_{1\bar{2}} = \omega''_{1\bar{4}} = \omega''_{2\bar{3}} = 0, \\ \omega''_{1\bar{3}} = \omega''_{3\bar{4}} = -\frac{\sqrt{2}}{2}\bar{\varphi}, \quad \omega''_{2\bar{4}} = -\frac{\sqrt{2}}{2}\varphi. \end{gathered} \tag{5.30}$$

Since $K = 0$, then we can use the local complex coordinate z on M so that (5.13) holds. Thus, from (5.30) and (5.13) we obtain (5.17).

Conversely, by Theorem 4.2, we see that such a totally real, flat, superminimal surface M in CP^4 is of 2-type in $HM(5)$ and has corresponding eigenvalues $5 \pm \sqrt{5}$.

Theorem 5.5 is proved completely. □

Finally, the classification theorem in §1 follows directly from Corollary 5.2, Proposition 5.3, and Theorems 5.4 and 5.5.

Remark 5.3. As is well known, minimal immersions are isometrically harmonic maps. A harmonic map from a 2-dimensional surface M is defined up to a conformal change of the metric of M. Hence, totally real, flat, superminimal surfaces of CP^3 and CP^4 described as in Theorems 5.4 and 5.5 provide a class of examples of harmonic tori in CP^3 and CP^4.

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DEPARTMENT OF MATHEMATICS, HANGZHOU UNIVERSITY, HANGZHOU, 310028 P. R. CHINA