TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 11, November 1995

$\mathbb{R}$-TREES, SMALL CANCELLATION, AND CONVERGENCE

ANDREW CHERMAK

ABSTRACT. The "metric small cancellation hypotheses" of combinatorial group theory imply, when satisfied, that a given presentation has a solvable Word Problem via Dehn's Algorithm. The present work both unifies and generalizes the small cancellation hypotheses, and analyzes them by means of group actions on trees, leading to the strengthening of some classical results.

INTRODUCTION

The aim of this paper is to develop an arboreal version of part of metric small cancellation theory, and to generalize that theory, with some specific applications in mind. The applications will be developed in [2], and in [3].

Here is a simple example of the kind of thing that we will be looking at. Let F be a free group with free basis $\mathcal{S}$, let $\mathcal{R}$ be a subset of F with $1 \notin \mathcal{R} = \mathcal{R}^{-1}$, and let $\mathcal{H}$ be the set of all conjugates in F of elements of $\mathbf{R}$. Let $\mathbf{X}$ denote the Cayley graph of F with respect to the generating set $\mathcal{S}$. Then $\mathbf{X}$ is a tree and F acts freely on $\mathbf{X}$. (In this paper, we take all actions to be from the right.) Identify $\mathbf{X}$ with its natural realization as a metric space, so that edges become segments of length 1, and so that $\mathcal{H}$ becomes a set of isometries. Every element of $\mathcal{H}$ (and, indeed, in this example, every non-identity element of F) is then a **hyperbolic** isometry of $\mathbf{X}$. This means that for any $h \in \mathcal{H}$ there is a uniquely determined linear subtree $A(h)$ of $\mathbf{X}$, invariant under h, and on which h acts as a translation of some positive **amplitude** $a(h)$. We refer to $A(h)$ as the **axis** of h.

Suppose now that Dehn's Algorithm solves the Word Problem for the presentation $\langle \mathcal{S} \mid \mathcal{R} \rangle$. One observes that this is equivalent to supposing $\mathcal{H}$ to have the following "convergence property".

(cp) For any g in the subgroup $\langle \mathcal{H} \rangle$ of Isom($\mathbf{X}$) generated by $\mathcal{H}$, and for any point P of $\mathbf{X}$, there is a sequence $(h_0, \dots, h_n)$ of elements of $\mathcal{H}$ such that, upon setting $P_0 = P$ and $P_i = P_{i-1}h_{i-1}$ for $i = 1, \dots, n+1$, we have

$$d(P_0, Pg) > d(P_1, Pg) > \cdots > d(P_{n+1}, Pg) = 0.$$

Now, it is well known (cf. Chapter 6 of [4]) that the solvability of the Word Problem by Dehn's Algorithm is implied by the metric small cancellation hypothesis $C'(1/6)$, or by the combination of the hypotheses $C'(1/4)$ and $T(4)$. These hypotheses can be stated as follows.

Received by the editors May 24, 1993 and, in revised form, January 20, 1995.
1991 *Mathematics Subject Classification*. Primary 20E08, 20F05, 20F06.

$C'(\lambda)$: We have $hh' = 1$ whenever h, h' are elements of $\mathscr{H}$ such that
(1) $a(hh') < a(h) + a(h')$ and
(2) $\ell(A(h) \cap A(h')) \geq \lambda \cdot a(h)$.
(Here ℓ is the length function associated with the metric space $\mathbf{X}$.)

$T(4)$: There do not exist elements h_1, h_2, h_3 of $\mathscr{H}$ such that, for all i and j with $1 \leq i < j \leq 3$, we have

$$a(h_i h_j) < a(h_i) + a(h_j).$$

We should remark on the geometric meaning of the condition (1) in $C'(\lambda)$. Namely, one has $a(hh') < a(h) + a(h')$ if and only if $A(h) \cap A(h')$ is a **non-degenerate segment** (i.e., a geodesic path in $\mathbf{X}$ of positive length) which is oriented in opposite directions by the translations h and h' on $A(h)$ and $A(h')$, respectively. (See Lemma 1.2 of [2] for a proof of this result. For a proof of the corresponding result for actions on $\mathbb{R}$-trees or, more generally, on Λ-trees, see section 8 of [1].)

All of the conditions (cp), $C'(\lambda)$, and $T(4)$ are without any reference to the group F with which the discussion began. Indeed, in all that follows we require only the following:

Basic Hypothesis. $\mathbf{X}$ is a tree and $\mathscr{H}$ is a set of hyperbolic isometries of $\mathbf{X}$, such that
(a) $\mathscr{H}$ is **symmetric** (i.e., $\mathscr{H} = \mathscr{H}^{-1}$) and
(b) $\mathscr{H}$ is **self-normalizing** (i.e., $h^{-1}\mathscr{H}h = \mathscr{H}$ for all $h \in \mathscr{H}$).

Let us now consider ways in which to improve on the conclusion (cp), above. First, for $n \geq 0$ let $\mathscr{H}_n$ denote the set of sequences $\mathbf{h} = (h_0, \ldots, h_n)$ of elements of $\mathscr{H}$, of length $n+1$. There is a natural action on $\mathscr{H}_n$ by the Artin braid group B_{n+1} on $n+1$ strings. This action is generated by n "simple braidings" of the following form:

$$\mathbf{h} \mapsto \mathbf{h}' = (h'_0, \ldots, h'_n),$$

where $h'_j = h_j$ for all $j \notin \{i, i+1\}$, and where $h'_i = h_i h_{i+1} h_i^{-1}$ and $h'_{i+1} = h_i$. One observes that the product $h_0 \ldots h_n$ remains unaltered by these operations. The image of $\mathbf{h}$ under an element of B_{n+1} will be called a **braiding** of $\mathbf{h}$.

Let $\mathscr{H}^*$ denote the set of all $\mathbf{h}$ such that, for any braiding $\mathbf{h}'$ of $\mathbf{h}$, we have $h'_i h'_{i+1} \notin \mathscr{H} \cup \{1\}$ for all i. It follows, by (1.2)(c) below, that $\mathscr{H}^*$ includes the set of all sequences $\mathbf{h} = (h_0, \ldots, h_n)$ such that $\mathbf{h}$ is a word for $h_0 \ldots h_n$ of minimal length in terms of $\mathscr{H}$.

Now, in place of (cp) we consider the following stronger Convergence Property.

(CP) For any $\mathbf{h}' \in \mathscr{H}^*$ and any $P \in \mathbf{X}$ there exists a braiding $\mathbf{h} = (h_0, \ldots, h_n)$ of $\mathbf{h}'$ such that, upon setting $P_0 = P$ and $P_i = P_{i-1}h_{i-1}$ for $i = 1, \ldots, n+1$, we have

$$d(P_0, P_{n+1}) > d(P_1, P_{n+1}) > \cdots > d(P_n, P_{n+1}).$$

For the applications, it will be convenient to give consideration also to a Weak Convergence Property:

(WCP) Same as (CP), but with the weaker conclusion:

$$d(P_0, P_{n+1}) \geq d(P_1, P_{n+1}) \geq \cdots \geq d(P_n, P_{n+1}).$$

Having now strengthened (cp) to (CP) and introduced the variation (WCP), we consider some ways in which to weaken the hypotheses $C'(\lambda)$ and $T(4)$.

Let λ be a mapping of $\mathscr{H}$ into the real interval $[0, 1/4]$. Assume that λ is $\mathscr{H}$**-equivariant** in the sense that

$$\lambda(g^{-1}hg) = \lambda(h) \quad \text{for all} \quad g, h \in \mathscr{H}.$$

We introduce hypotheses $C_1(\lambda)$, $C_2(\lambda)$, and $\Delta(\lambda)$, as follows.

$C_1(\lambda)$: We have $hh' \in \mathscr{H} \cup \{1\}$ whenever h, h' are elements of $\mathscr{H}$ such that

(i) $a(hh') < a(h) + a(h')$ and

(ii) $\ell(A(h) \cap A(h')) > a(h)\lambda(h)$.

$C_2(\lambda)$: The same as $C_1(\lambda)$, but with (ii) replaced by

(ii)′ $\ell(A(h) \cap A(h')) \geq a(h)\lambda(h)$.

$\Delta(\lambda)$: We have $\{h_1h_2, h_1h_3, h_2h_3\} \cap \mathscr{H} \neq \varnothing$ whenever h_1, h_2, h_3 are elements of $\mathscr{H}$ such that

(i) $a(h_ih_j) < a(h_i) + a(h_j)$ for all i and j, $1 \leq i < j \leq 3$, and

(ii) one of the following holds.

(a) For at least two of the three even permutations (i, j, k) of $(1, 2, 3)$ we have

$$\ell(A(h_i) \cap A(h_j)) + \ell(A(h_j) \cap A(h_k)) > (\frac{1}{2} - \lambda(h_j))a(h_j).$$

(b) For some permutation (i, j, k) of $(1, 2, 3)$ we have $\lambda(h_j) = \frac{1}{4}$ and

$$\ell(A(h_i) \cap A(h_j)) = \ell(A(h_j) \cap A(h_k)) = \frac{1}{4}a(h_j).$$

We shall prove the following result.

Theorem (3.1). *Let $\mathscr{H}$ be a symmetric, self-normalizing set of hyperbolic isometries of a tree, and let λ be an $\mathscr{H}$-equivariant mapping of $\mathscr{H}$ into* $[0, 1/4]$.

(a) *If $\mathscr{H}$ and λ satisfy both $C_1(\lambda)$ and $\Delta(\lambda)$, then $\mathscr{H}$ has the weak convergence property.*

(b) *If $\mathscr{H}$ and λ satisfy both $C_2(\lambda)$ and $\Delta(\lambda)$, then $\mathscr{H}$ has the convergence property.*

Moreover, if λ maps $\mathscr{H}$ into $[0, 1/6]$, *then $\Delta(\lambda)$ is implied by $C_1(\lambda)$ (and hence also by $C_2(\lambda)$).*

Theorem (3.1) is not the main result of this paper. Rather, Theorem (3.1) comes as a corollary to a result (Theorem (1.7)) which bears a much less obvious resemblance to classical small cancellation theorems. Readers who are interested only in the applications in [2] can, in fact, dispense with all but Theorem (1.7). These readers can also ignore another aspect of this paper: our choosing to interpret "trees" in the broader sense of ℝ-trees rather than the usual simplicial trees. My feeling is that everything works just as easily with ℝ-trees, and that the arguments sometimes display their essential features more clearly when ℝ-trees are employed. The situation for Λ-trees, Λ an arbitrary ordered abelian group, becomes a bit more complicated, though perhaps not by too much. We do make essential use of the Archimedean property of ℝ exactly once (in the proof of (1.8)(b)). If he likes, the devotee of Λ-trees can surmount this problem by adding the hypothesis that, for every $h \in \mathscr{H}$, the amplitude $a(h)$ of h as a Λ-isometry of the Λ-tree $\mathbf{X}$ is contained in no proper convex subgroup of Λ.

1. Convergence

Let $\mathbf{X}$ be an $\mathbb{R}$-tree and let $\mathscr{H}$ be a set of hyperbolic isometries of $\mathbf{X}$. We will assume throughout that $\mathscr{H}$ is *symmetric* and *self-normalizing*; i.e. $\mathscr{H} = \mathscr{H}^{-1}$ and $h^{-1}\mathscr{H}h = \mathscr{H}$ for all $h \in \mathscr{H}$. Among other things, we are looking for criteria that will guarantee that the group $\langle\mathscr{H}\rangle$ generated by $\mathscr{H}$ will act freely on $\mathbf{X}$.

For each $h \in \mathscr{H}$ there is a unique, isometrically embedded copy of $\mathbb{R}$ in $\mathbf{X}$ on which h acts as a translation. (See [1, Theorem 6.6, p. 329].) We call this copy of $\mathbb{R}$ the *axis* of h, and it will be denoted $A(h)$. The *amplitude* of the translation induced by h on $A(h)$ is denoted here by $a(h)$. One has

$$a(h) = \operatorname{Inf}\{d(P, Ph)\}_{P\in\mathbf{X}}, \quad \text{and}$$
$$A(h) = \{P \in \mathbf{X} : d(P, Ph) = a(h)\}.$$

If E and F are points of $\mathbf{X}$, not necessarily distinct, then we have the various *directed segments* from E to F. These are the closed segment $[E, F]$, open segment (E, F), and the two half-open/half-closed segments $(E, F]$ and $[E, F)$. If J is a segment, then J^{-1} denotes the *opposite segment* corresponding to J. Thus, if $J = [E, F]$, then $J^{-1} = [F, E]$. When convenient, we identify a directed segment with its underlying point-set. Thus, we may write $A \in (B, C)$, for example.

If $J = [E, F]$ is a closed segment contained in $A(h)$ for some $h \in \mathscr{H}$, and if P is a point such that $F \in [E, P]$, we say that J is *oriented towards* (resp. *away from*) P by h if $d(Eh, F) \leq d(E, Fh)$ (resp. $d(Eh, F) \geq d(E, Fh)$). It follows that J is oriented both towards and away from P by h if and only if $E = F$.

By an *$\mathscr{H}$-sequence* we will always mean a finite sequence of elements of $\mathscr{H}$. If $\mathbf{h}$ and $\mathbf{h}'$ are $\mathscr{H}$-sequences, then $\mathbf{h} \circ \mathbf{h}'$ denotes the $\mathscr{H}$-sequence obtained by concatenation of $\mathbf{h}$ and $\mathbf{h}'$ in the given order. If $\mathbf{h} = (h_0, \dots, h_k)$, then we write $\mathbf{h}^{-1}$ for $\left(h_k^{-1}, \dots, h_0^{-1}\right)$, and if $g \in \langle\mathscr{H}\rangle$, then we write $\mathbf{h}^g$ for $\left(g^{-1}h_0 g, \dots, g^{-1}h_k g\right)$. An $\mathscr{H}$-sequence $(h_0, \dots, h_k)$ is *reduced* if for all i, $0 \leq i < k$, we have $h_i h_{i+1} \notin \mathscr{H} \cup \{1\}$.

1.1 **Definition.** Let $\mathbf{h} = (h_0, \dots, h_k)$ and $\mathbf{h}' = \left(h_0', \dots, h_k'\right)$ be two $\mathscr{H}$-sequences of length $k+1$. We say that $\mathbf{h}'$ is a *simple braiding* of $\mathbf{h}$ if there exists i, $0 \leq i < k$, such that $h_j = h_j'$ whenever $j \notin \{i, i+1\}$, and such that either

$$\left(h_i', h_{i+1}'\right) = \left(h_{i+1}, h_{i+1}^{-1}h_i h_{i+1}\right), \quad \text{or}$$
$$\left(h_i', h_{i+1}'\right) = \left(h_i h_{i+1} h_i^{-1}, h_i\right).$$

We say that $\mathbf{h}'$ is a *braiding* of $\mathbf{h}$ if there exists a finite sequence

$$\mathbf{h} = \mathbf{s}_0, \dots, \mathbf{s}_\ell = \mathbf{h}'$$

of $\mathscr{H}$-sequences, such that $\mathbf{s}_j$ is a simple braiding of $\mathbf{s}_{j-1}$ for all j, $1 \leq j \leq \ell$.

The set of all braidings of $\mathbf{h} = (h_0, \dots, h_k)$ will be denoted by $[\mathbf{h}]$, or $[h_0, \dots, h_k]$.

1.2 **Lemma.** *Let* $\mathbf{h} = (h_0, \dots, h_k)$ *be an* $\mathscr{H}$*-sequence, and let* $\mathbf{h}' = (h'_0, \dots, h'_k)$ *be a braiding of* $\mathbf{h}$. *Then the following hold.*

(a) $h_0 \cdots h_k = h'_0 \cdots h'_k$.

(b) $\{a(h_i)\}_{0 \le i \le k} = \{a(h'_i)\}_{0 \le i \le k}$.

(c) *If* $g \in \langle \mathscr{H} \rangle$ *and if* $\mathbf{h}$ *is an* $\mathscr{H}$*-sequence of shortest length such that* $g = h_0 \cdots h_k$, *then* $\mathbf{h}'$ *is reduced.*

Proof. An obvious induction argument reduces (a) and (b) to the case where $\mathbf{h}'$ is a simple braiding of $\mathbf{h}$, where the desired results are immediate. Part (c) follows from (a).

Henceforth, $\mathscr{H}^*$ will denote the set of all $\mathscr{H}$-sequences $\mathbf{h}$ such that every braiding of $\mathbf{h}$ is reduced. Thus, 1.2(c) says that $\mathscr{H}^*$ contains all "words of minimal length" for elements of $\langle \mathscr{H} \rangle$.

1.3 **Definition.** We say that $\mathscr{H}$ has the *convergence property* (resp. the *weak convergence property*) if whenever $P \in \mathbf{X}$ and $\mathbf{h}' = (h'_0, \dots, h'_k) \in \mathscr{H}^*$, with $g = h'_0 \cdots h'_k$, there exists a braiding $\mathbf{h} = (h_0, \dots, h_k)$ of $\mathbf{h}'$ such that the sequence of distances

$$d(P, Pg), d(Ph_0, Pg), d(Ph_0h_1, Pg), \dots, d(Ph_0 \dots h_k, Pg) = 0$$

is monotonically decreasing (resp. non-increasing). We then say also that P *converges to* Pg *via* $\mathbf{h}$ (resp., P *converges weakly to* Pg *via* $\mathbf{h}$).

Of course, the convergence property implies the weak convergence property. Notice also that if $\mathscr{H}$ has the weak convergence property, then $\langle \mathscr{H} \rangle$ acts freely on $\mathbf{X}$. Indeed, let $g \in \langle \mathscr{H} \rangle$ and let $\mathbf{h}'$ be a word of minimal length for g in terms of the generating set $\mathscr{H}$. Then $\mathbf{h}' \in \mathscr{H}^*$ by 1.2(c). If g fixes a point P, then the weak convergence property implies that P converges weakly to P via $\mathbf{h}$ for some braiding $\mathbf{h}$ of $\mathbf{h}'$. But then $\mathbf{h}$ is the empty sequence, and $g = 1$.

It turns out that, for some applications, it is best to work with a somewhat more refined notion of convergence than that given above.

1.4 **Definition.** Let $\mathscr{H} = \mathscr{H}_1 \cup \mathscr{H}_2$ be a decomposition of $\mathscr{H}$ into two disjoint subsets $\mathscr{H}_1$ and $\mathscr{H}_2$, such that each $\mathscr{H}_i$ is symmetric, and such that each $\mathscr{H}_i$ is normalized by $\mathscr{H}$. (Such a decomposition will be said to be a *splitting* of $\mathscr{H}$.) We say that $\mathscr{H}$ has the $(\mathscr{H}_1, \mathscr{H}_2)$*-relative convergence property* if whenever $P \in \mathbf{X}$ and $\mathbf{h}' = (h'_0, \dots, h'_k) \in \mathscr{H}^*$, with $g = h'_0 \cdots h'_k$, there exists a braiding $\mathbf{h} = (h_0, \dots, h_k)$ of $\mathbf{h}'$ such that the following condition holds.

(RCP) Setting $P_0 = P$ and, inductively, setting $P_{i+1} = P_i h_i$, $0 \le i \le k$, we have

$$d(P_i, Pg) \ge d(P_{i+1}, Pg) \quad \text{if} \quad h_i \in \mathscr{H}_1, \quad \text{and}$$
$$d(P_i, Pg) > d(P_{i+1}, Pg) \quad \text{if} \quad h_i \in \mathscr{H}_2.$$

We say that P *converges relatively to* Pg *via* $\mathbf{h}$, if (RCP) holds.

Notice that convergence and weak convergence are special cases of relative convergence, obtained by taking $\mathscr{H}_1$ or $\mathscr{H}_2$ to be empty.

We now turn to conditions on $\mathscr{H}$ which will guarantee the relative convergence property.

1.5 **Definition.** A *polarization* $\mathscr{P}$ (*of* $\mathbf{X}$ *over* $\mathscr{H}$) consists of a collection $\mathrm{Seg}(\mathscr{P})$ of non-degenerate, oriented closed segments of $\mathbf{X}$, together with a relation $\sim$ from $\mathrm{Seg}(\mathscr{P})$ to the set $\mathscr{H}^*$ /(braiding) of braiding classes of elements of $\mathscr{H}^*$, such that whenever $J \in \mathrm{Seg}(\mathscr{P})$, $g \in \langle \mathscr{H} \rangle$, and $\mathbf{h} \in \mathscr{H}^*$ with $J \sim [\mathbf{h}]$, the following hold.

(a) $J^{-1} \in \mathrm{Seg}(\mathscr{P})$ and $J^{-1} \sim [\mathbf{h}^{-1}]$.

(b) $Jg \in \mathrm{Seg}(\mathscr{P})$ and $Jg \sim [\mathbf{h}^g]$.

We shall write $J \in \mathscr{P}(\mathbf{h})$ to express that $J \in \mathrm{Seg}(\mathscr{P})$, $\mathbf{h} \in \mathscr{H}^*$, and $J \sim [\mathbf{h}]$.

Whenever $\mathscr{P}$ is a fixed polarization and $[E, F] = J \in \mathscr{P}(\mathbf{h})$, $\mathscr{E}(J, \mathbf{h})$ will denote the set of all $h \in \mathscr{H}$ such that $\mathbf{h} \circ (h) \in \mathscr{H}^*$, and such that $A(h) \cap J$ is a non-degenerate segment, oriented away from E by h. One may think of $\mathscr{E}(J, h)$ as the set of possible *extensions* of J and $\mathbf{h}$.

1.6 **Definition.** Let $\mathscr{P}$ be a polarization of $\mathbf{X}$ over $\mathscr{H}$, and let $(\mathscr{H}_1, \mathscr{H}_2)$ be a splitting of $\mathscr{H}$. We say that $\mathscr{P}$ *controls convergence relative to* $(\mathscr{H}_1, \mathscr{H}_2)$ if the following hold.

(a) (Existence). Whenever $h \in \mathscr{H}$ and $J = [E, F] \subseteq A(h)$ with J oriented towards E by h, we have

(i) $J \in \mathscr{P}(h)$ if $\frac{1}{2}a(h) < \ell(J) \leq a(h)$ and $h \in \mathscr{H}_1$.

(ii) $J \in \mathscr{P}(h)$ if $\frac{1}{2}a(h) \leq \ell(J) \leq a(h)$ and $h \in \mathscr{H}_2$.

(b) (Exclusion). Whenever $J \in \mathscr{P}(\mathbf{h})$ and $h \in \mathscr{E}(J, \mathbf{h})$, we have the following.

(i) $J \not\subseteq A(h)$.

(ii) If $h \in \mathscr{H}_1$ (resp. $h \in \mathscr{H}_2$), then

$$\ell(A(h) \cap J) < \tfrac{1}{2}a(h) \quad (\text{resp. } \leq),$$

and if $A(h)$ contains an end-point of J, then

$$\ell(A(h) \cap J) \leq \tfrac{1}{3}a(h).$$

(c) (Extension). Whenever $J = [E, F] \in \mathscr{P}(\mathbf{h})$, $h \in \mathscr{E}(J, \mathbf{h})$, and Q is a point of $A(h)$ with $E \in (Q, F)$, then $[Qh, F] \in \mathscr{P}(\mathbf{h} \circ (h))$ provided that

$$d(Q, E) < \tfrac{1}{2}a(h) \leq d(Q, E) + \ell(A(h) \cap J) \quad \text{if } h \in \mathscr{H}_1,$$

$$d(Q, E) \leq \tfrac{1}{2}a(h) < d(Q, E) + \ell(A(h) \cap J) \quad \text{if } h \in \mathscr{H}_2.$$

Remarks on 1.6. Part (a) of the definition says that $\mathscr{P}$ is non-empty, by displaying a subset of $\mathscr{P}(\mathbf{h})$ when $\mathbf{h}$ is an $\mathscr{H}$-sequence of length 1. Parts (b) and (c) concern $\mathscr{P}(\mathbf{h})$ and $\mathscr{P}(\mathbf{h} \circ (h))$ for $h \in \mathscr{E}(J, \mathbf{h})$. Part (b) says that $\mathscr{E}(J, \mathbf{h})$ should imply some restrictions on the structure of $\mathscr{P}(\mathbf{h})$, and it is here that we have our first glimpse of something like small cancellation hypotheses. However, the number $\frac{1}{3}$ appearing in (b)(ii) is chosen somewhat arbitrarily. In fact, we could just as well use any fixed number r with $0 < r < \frac{1}{2}$. (One has to alter the proof of 1.8(b) if one replaces $\frac{1}{3}$ by r, but that can be easily done.) Part (c) tells us that if $h \in \mathscr{E}(J, \mathbf{h})$ and $A(h) \cap J$ is non-degenerate (but not too large, either, by 1.6 (b)), then we get a member of $\mathscr{P}(\mathbf{h} \circ (h))$ in the manner indicated by Figure 1.

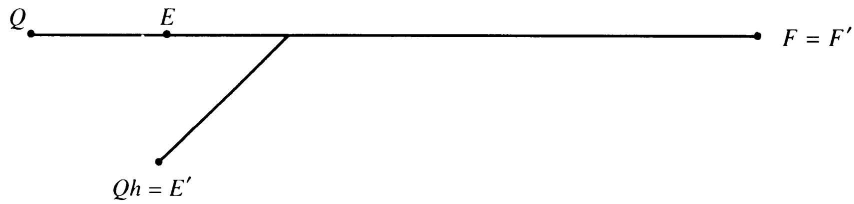

FIGURE 1

Our first main result may now be stated.

1.7 **Theorem.** *Let* $(\mathscr{H}_1, \mathscr{H}_2)$ *be a splitting of* $\mathscr{H}$. *Suppose that there exists a polarization* $\mathscr{P}$ *of* $\mathbf{X}$ *over* $\mathscr{H}$ *such that* $\mathscr{P}$ *controls convergence relative to* $(\mathscr{H}_1, \mathscr{H}_2)$. *Then* $\mathscr{H}$ *has the* $(\mathscr{H}_1, \mathscr{H}_2)$*-relative convergence property.*

The proof of Theorem 1.7 will proceed by contradiction. Thus, we assume that we are given a splitting $(\mathscr{H}_1, \mathscr{H}_2)$ of $\mathscr{H}$, and a polarization $\mathscr{P}$ of $\mathbf{X}$ over $\mathscr{H}$, such that $\mathscr{P}$ controls convergence relative to $(\mathscr{H}_1, \mathscr{H}_2)$; and we assume that $\mathscr{H}$ does not have the $(\mathscr{H}_1, \mathscr{H}_2)$-relative convergence property. There then exists an element $\mathbf{h} = (h_0, \dots, h_n)$ of $\mathscr{H}^*$ and a point P of $\mathbf{X}$ such that, setting $g = h_0 \cdots h_n$, P does not converge relatively to Pg via $\mathbf{h}'$ for any braiding $\mathbf{h}'$ of $\mathbf{h}$. Among all such "bad" pairs, choose $\mathbf{h}$ and P so that the length $n+1$ of $\mathbf{h}$ is as small as possible, and put $Z = Pg$. By the minimality of $n+1$ there then exists a braiding $(h'_1, \dots, h'_n)$ of $(h_1, \dots, h_n)$ such that Ph_0 converges relatively to Z via $(h'_1, \dots, h'_n)$. But $(h_0, h'_1, \dots, h'_n)$ is then a braiding of $\mathbf{h}$; so we may assume to begin with that Ph_0 converges relatively to Z via $(h_1, \dots, h_n)$.

Set $a = \operatorname{Inf}\{a(h_i)\}_{0 \le i \le n}$. By the Archimedean property of $\mathbb{R}$ (which we now make use of for the first and last time), there exists a natural number N such that $N \cdot a > 3 \cdot d(P, Z)$. It then follows from 1.2(b) that we may assume $\mathbf{h}$ and P chosen so that, for any braiding $(h'_0, \dots, h'_n)$ of $\mathbf{h}$, we have $d(Ph'_0, Z) > d(Ph_0, Z) - \frac{1}{3}a(h_0)$. Indeed, the alternative is that after composing N successive braidings we obtain $d(Ph'_0, Z) < 0$, which is absurd.

Summing up, we now have:

1.8. (a) Ph_0 converges relatively to Z via $(h_1, \dots, h_n)$.

(b) For any braiding $(h'_0, \dots, h'_n)$ of $\mathbf{h}$, we have $d(Ph'_0, Z) > d(Ph_0, Z) - \frac{1}{3}a(h_0)$.

1.9 **Lemma.** *Let* $\mathbf{h}' = (h'_0, \dots, h'_n)$ *be a braiding of* $\mathbf{h}$, *and suppose that* $h'_0 \in \mathscr{H}_1$ *(resp.* $h'_0 \in \mathscr{H}_2$*). Then* $d(Ph'_0, Z) > d(P, Z)$ *(resp.* $\ge$*).*

Proof. Suppose false, and put $(h'_1, \dots, h'_n) = \mathbf{h}'_0$. The length-minimality condition on our choice of $\mathbf{h}$ implies that there exists a braiding $\mathbf{h}''_0$ of $\mathbf{h}'_0$ such that Ph'_0 converges relatively to Z via $\mathbf{h}''_0$. Put $\mathbf{h}'' = (h'_0) \circ \mathbf{h}''_0$. Then $\mathbf{h}''$ is a braiding of $\mathbf{h}'$, hence also of $\mathbf{h}$, and P converges relatively to Z via $\mathbf{h}''$. This contradicts our choice of $\mathbf{h}$ and P, so 1.9 is proved.

We require some further notation at this point. Set $P_0 = P$ and, inductively, set $P_{i+1} = P_i h_i$ for all i, $0 \le i \le n$. Thus, we have $Z = P_{n+1}$. Now for any i as above, put

$$T_i = [P_i, P_{i+1}] \cap A(h_i).$$

This is a closed segment of length $a(h_i)$, by [1, Theorem 6.6(c)]. We write

$$T_i = [Q_i, R_i] \quad \text{with} \quad R_i = Q_i h_i.$$

Let M_i denote the mid-point of T_i, which is then also the mid-point of $[P_i, P_{i+1}]$. Put

$$Y_i = [P_i, P_{i+1}] \cap [P_i, Z] \cap [P_{i+1}, Z].$$

Then Y_i is a single point, as follows from whatever definition one has of $\mathbb{R}$-tree.

We now provide an outline of the remaining steps in the proof of Theorem 1.7. Set

$$J_0 = T_0 \cap [P_1, Z].$$

We shall see that $J_0 \in \mathscr{P}(h_0)$ and that J_0 is oriented away from Z by h_0. Inductively, whenever J_{i-1} is in $\mathscr{P}(h_0, \dots, h_{i-1})$, $1 \leq i \leq n$, with J_{i-1} contained in $[P_i, Z]$, we will write

$$J_{i-1} = [E_{i-1}, F_{i-1}], \quad \text{with} \quad F_{i-1} \in (E_{i-1}, Z],$$

and (it will be shown that) we will then be able to define a segment J_i in precisely one of the following two ways.

1.10 (A) If $Q_i \in [P_i, E_{i-1})$, $M_i \in [E_{i-1}, F_{i-1})$, and $Y_i \in T_i \cap (E_{i-1}, F_{i-1})$, put $J_i = [R_i, F_{i-1}]$.

(B) If $Q_i \in (E_{i-1}, F_{i-1})$, both M_i and Y_i lie in $T_i \cap (F_{i-1}, Z]$, and $d(Q_i, F_{i-1}) \geq d(M_i, Y_i)$, put $J_i = [(E_{i-1})h_i, Y_i]$.

We shall see, moreover, that if $h_i \in \mathscr{H}_1$, then either $M_i \in (E_{i-1}, F_{i-1})$ (in 1.10(A)), or $d(Q_i, F_{i-1}) > d(M_i, Y_i)$ (in 1.10(B)).

We shall argue that $J_0, \dots, J_n$ are all well-defined by the above scheme, and that $J_i \in \mathscr{P}(h_0, \dots, h_i)$. This will imply, in particular, that J_n is a non-degenerate segment contained in $[P_{n+1}, Z]$. But since P_{n+1} is equal to Z, by definition, we will have the desired contradiction, proving 1.7.

The proof of Theorem 1.7 will now be carried out in a sequence of six "steps".

Step 1. We have $F_0 \in [P, Z]$.

Proof. Suppose false. We then have Figure 2.

Suppose further that $Q_0 \notin [P_1, M_1]$, so that (in particular) $M_1 \notin [Q_0, Z]$. But $M_1 \in [P_1, Z]$ by 1.8(a), so we get $M_1 \in [P_1, Q_0)$. If now $Q_1 \in T_0$, then $[Q_1, M_1] \subseteq A(h_1) \cap T_0$ and 1.6(b)(ii) implies that $h_1 \in \mathscr{H}_2$ and $[Q_1, M_1] = A(h_1) \cap T_0$. But with $h_1 \in \mathscr{H}_2$ we get $d(P_1, Z) < d(P_2, Z)$ by 1.8(a), and since $M_1 \neq Q_0$ we then have $[Q_1, M_1]$ properly contained in $A(h_1) \cap T_0$. This contradiction shows that, in fact, $Q_1 \notin T_0$, and so $Q_1 \in [P_1, R_0)$.

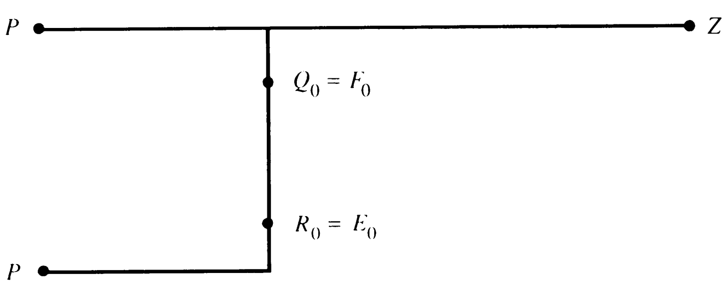

FIGURE 2

Let X be the point in T_0 at distance $d = \frac{1}{3}\operatorname{Inf}\{a(h_0), a(h_1)\}$ from R_0. Applying 1.6(b)(ii) to both h_0 and h_1 we then find that $M_1 \in [P_1, X]$. It follows that $d(P_2, R_0) \le d(P_1, R_0) + 2d$. Set $P_1' = P_2 h_0^{-1}$. Then

$$\begin{aligned} d(P_1, Q_0) &= d(P_1, R_0) + a(h_0) \\ &\ge d(P_2, R_0) - 2d + a(h_0) \\ &= d\left(P_1', Q_0\right) - 2d + a(h_0), \end{aligned}$$

and hence

$$\begin{aligned} d\left(P_1', Q_0\right) &\le d(P_1, Q_0) - a(h_0) + 2d \\ &\le d(P_1, Q_0) - \tfrac{1}{3}a(h_0). \end{aligned}$$

As $Q_0 \in [P_1, Z]$ we then have

$$d\left(P_1', Z\right) \le d(P_1, Z) - \tfrac{1}{3}a(h_0).$$

But we now observe that $\left(h_0 h_1 h_0^{-1}, h_0, h_2, \dots, h_n\right)$ is a (simple) braiding of $\mathbf{h}$, and that $P h_0 h_1 h_0^{-1} = P_1'$. This contradicts 1.8(b), so we conclude that $Q_0 \in [P_1, M_1]$.

We now switch notation and set $P_1' = P_0 h_1$, and we take X to be the point in J_0 at distance d from Q_0, where d is as above. Then 1.6 implies that $Q_1 \in [X, Z]$. Also, we have $M_1 \in [Q_1, Z]$ by 1.8(a), and $M_1 \notin [X, Q_0]$ since $\frac{1}{2}a(h_1) > d$. Now

$$\begin{aligned} d(P_1, M_1) &= d(P_1, R_0) + a(h_0) + d(Q_0, M_1) \\ &= d(P, Q_0) + a(h_0) - d + d(X, M_1) \\ &= d(P, Q_0) + a(h_0) - d + \tfrac{1}{2}a(h_1) + d(X, Q_1). \end{aligned}$$

But we also have

$$\begin{aligned} d\left(P_1', M_1\right) &= d\left(P, M_1 h_1^{-1}\right) \\ &\le d(P, Q_1) + d\left(Q_1, M_1 h_1^{-1}\right) \\ &= d(P, Q_1) + \tfrac{1}{2}a(h_1) \\ &\le d(P, Q_0) + d(Q_0, Q_1) + \tfrac{1}{2}a(h_1) \\ &\le d(P, Q_0) + d + d(X, Q_1) + \tfrac{1}{2}a(h_1). \end{aligned}$$

Thus,

$$d(P_1, M_1) - d\left(P_1', M_1\right) \ge a(h_0) - 2d \ge \tfrac{1}{3}a(h_0).$$

Since $M_1 \in [P_1, Z]$ we then have

$$d\left(P_1', Z\right) \le d(P_1, Z) - \tfrac{1}{3}a(h_0).$$

But this contradicts 1.8(b) since $\left(h_1, h_1^{-1} h_0 h_1, h_2, \dots, h_n\right)$ is a braiding of $\mathbf{h}$. This completes Step 1.

Step. 2. We have $J_0 = [R_0, Y_0]$, $J_0 \in \mathscr{P}(h_0)$, and J_0 is oriented away from Z by h_0. Moreover, if $h_0 \in \mathscr{H}_1$ (resp. $h_0 \in \mathscr{H}_2$), then $\ell(J_0) > \frac{1}{2}a(h_0)$ (resp. $\ge$).

Proof. We have $d(P, Z) < d(P_1, Z)$ if $h_0 \in \mathscr{H}_1$, and $d(P, Z) \leq d(P_1, Z)$ if $h_0 \in \mathscr{H}_2$, by 1.9. Step 2 then follows from Step 1 and from 1.6(a).

Assume henceforth that we are given k with $1 \leq k \leq n$, such that for each i with $0 \leq i < k$, J_i is a well-defined member of $\mathscr{P}(h_0, \ldots, h_i)$ contained in $[P_{i+1}, Z]$, and satisfying the various conditions given by 1.10 if $i > 0$. Our goal will now be to show that also J_k is well-defined by 1.10, and is a member of $\mathscr{P}(h_0, \ldots, h_k)$ contained in $[P_{k+1}, Z]$.

Step 3. We have $F_{k-1} \in [P, Z]$.

Proof. By 1.10 we have $F_i \in [F_{i-1}, Z]$ whenever J_i is defined, with $i > 0$. But also $F_0 \in [P, Z]$ by Step 1. An obvious induction argument then yields $F_0, F_1, \ldots, F_{k-1} \in [P, Z]$.

Step 4. We have $M_k \in [E_{k-1}, Z]$ and $Y_k \in (E_{k-1}, Z]$. Moreover, if $h_k \in \mathscr{H}_1$, then $M_k \in (E_{k-1}, Z]$.

Proof. The statement concerning Y_k follows immediately from the statements about M_k, by way of 1.8(a).

Assume by way of contradiction that $h_k \in \mathscr{H}_2$ (resp. $h_k \in \mathscr{H}_1$) and $M_k \notin [E_{k-1}, Z]$ (resp. $M_k \notin (E_{k-1}, Z]$). Then $M_k \in [P_k, E_{k-1})$ (resp. $M_k \in [P_k, E_{k-1}]$) by 1.8(a). Then 1.8(a) further implies:

$$(*)_0 \quad d(P_{k-1}, E_{k-1}) < d(P_k, E_{k-1}) \quad (\text{resp. } d(P_{k-1}, E_{k-1}) \leq d(P_k, E_{k-1})).$$

Now set $P'_{k+1} = P_{k+1}$, $P'_k = P_{k+1}h_{k-1}^{-1}$ and, for all i with $0 \leq i < k$, set

$$P'_{k-i} = P_{k+1}(h_{k-i-1} \cdots h_{k-1})^{-1}.$$

We will show that for all such i we have

$$(*)_i \quad d(P'_{k-i+1}, E_{k-i-1}) < d(P_{k-i}, E_{k-i-1}) \quad (\text{resp. } d(P'_{k-i+1}, E_{k-i-1}) \leq d(P_{k-i}, E_{k-i-1})).$$

We prove $(*)_i$ by induction on i, observing that we already have $(*)_0$. So assume that we have $(*)_i$ for $i = 0, \ldots, j$ for some j, $j < k-1$. By 1.10 we have $J_{k-j-2} \subseteq [P_{k-j-1}, Z]$, and we may further employ 1.10 in order to locate J_{k-j-1} and P_{k-j}. Suppose first that 1.10(A) applies in this connection. We then have Figure 3.

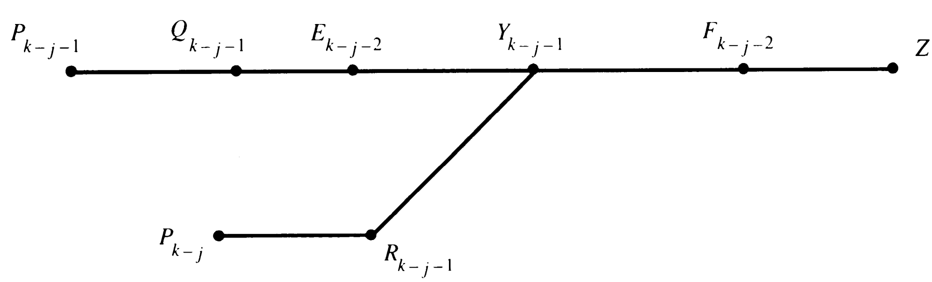

FIGURE 3

Here $R_{k-j-1} = E_{k-j-1}$. Then $(*)_j$ and an application of $(h_{k-j-1})^{-1}$ yields

$$d\left(P'_{k-j}, Q_{k-j-1}\right) < d\left(P_{k-j-1}, Q_{k-j-1}\right) \quad (\text{resp. } \leq).$$

Since $Q_{k-j-1} \in \left[P_{k-j-1}, E_{k-j-2}\right]$ we then obtain $(*)_{j+1}$, as desired.

Suppose instead that 1.10(B) applies. Then $E_{k-j-1} = \left(E_{k-j-2}\right) h_{k-j-1}$, and we immediately obtain $(*)_{j+1}$ by applying $\left(h_{k-j-1}\right)^{-1}$ to $(*)_j$. Thus $(*)_i$ holds for all i, $0 \leq i < k$.

Taking $i = k-1$, we now have

$$d\left(P'_2, E_0\right) < d\left(P_1, E_0\right) \quad (\text{resp. } \leq).$$

But $E_0 = R_0$ by Step 2, so we may apply h_0^{-1} to $(*)_{k-1}$ and obtain

$$d\left(P'_1, Q_0\right) < d\left(P, Q_0\right) \quad (\text{resp. } \leq).$$

We will then have a contradiction via 1.9 if we can show that the $\mathscr{H}$-sequence $\mathbf{h}'$ given by

$$\left((h_k)^{(h_0\cdots h_{k-1})^{-1}}, (h_{k-1})^{(h_0\cdots h_{k-2})^{-1}}, \ldots, h_0 h_1 h_0^{-1}, h_0, h_{k+1}, \ldots, h_n\right)$$

is a braiding of $\mathbf{h}$.

It happens that $\mathbf{h}'$ really is a braiding of $\mathbf{h}$, and one may "perform" this braiding in the following way. First, move h_0 from the 0-position of $\mathbf{h}$ into the k-position, via k simple braidings. Then $h_0 h_1 h_0^{-1}$ is the new occupier of the 0-position, and we move it into the $(k-1)$-position via $k-1$ simple braidings. Continuing on in this way we obtain $\mathbf{h}'$, thus completing Step 4.

Step 5. If $M_k \notin \left[E_{k-1}, F_{k-1}\right)$, then $Q_k \in \left(E_{k-1}, F_{k-1}\right)$, $Y_k \in [M_k, R_k)$, and $d(M_k, Y_k) \leq d(Q_k, F_{k-1})$. Moreover, if $h_k \in \mathscr{H}_1$, then $d(M_k, Y_k) < d(Q_k, F_{k-1})$, and if $h_k \in \mathscr{H}_2$, then $Y_k \in (M_k, R_k)$.

Proof. Set $\mathbf{h}' = \left(h_k, h_k^{-1} h_0 h_k, \ldots, h_k^{-1} h_{k-1} h_k, h_{k+1}, \ldots, h_n\right)$ and set $P'_1 = P h_k$. Then $\mathbf{h}'$ is a braiding of $\mathbf{h}$, so it follows from 1.9 that

$$d(P, Z) - d\left(P'_1, Z\right) \leq 0 \quad (\text{resp. } d(P, Z) - d\left(P'_1, Z\right) < 0)$$

according as $h_k \in \mathscr{H}_2$ (resp. $h_k \in \mathscr{H}_1$).

We have $d(P_k, Z) > d(P_{k+1}, Z)$ (resp. $\geq$) by 1.8(a), so $M_k \in [P_k, Z]$, and if $h_k \in \mathscr{H}_2$, then $M_k \neq Y_k$. Assume that $M_k \notin \left[E_{k-1}, F_{k-1}\right)$. Then $M_k \in \left[F_{k-1}, Z\right]$ by Step 4. Suppose that also $Q_k \in \left[F_{k-1}, Z\right]$. Then Step 3 yields

$$\begin{aligned} d(P, Z) &= d(P, Q_k) + d(Q_k, Z) \\ &= d\left(P'_1, R_k\right) + d(Q_k, Z). \end{aligned}$$

But $d\left(P'_1, R_k\right) \geq d\left(P'_1, Z\right) - d(R_k, Z)$ by the triangle inequality, so

$$\begin{aligned} 0 \geq d(P, Z) - d\left(P'_1, Z\right) &\geq d(Q_k, Z) - d(R_k, Z) \\ (\text{resp. } 0 > d(P, Z) - d\left(P'_1, Z\right) &\geq d(Q_k, Z) - d(R_k, Z)). \end{aligned}$$

That is, we have $d(Q_k, Z) \leq d(R_k, Z)$ (resp. $<$), and this is contrary to 1.8(a). Thus, $Q_k \notin \left[F_{k-1}, Z\right]$.

Now 1.6(b)(ii) implies that $Q_k \in (E_{k-1}, F_{k-1})$ and that $M_k \in (F_{k-1}, Z]$. Suppose next that $R_k \in [M_k, Z]$. We then have

$$\begin{aligned} d(P, Z) &= d(P, F) + d(F, R_k) + d(R_k, Z) \\ &> d(P, F) + \tfrac{1}{2}a(h) + d(R_k, Z), \end{aligned}$$

while also

$$\begin{aligned} d(P_1', Z) &\le d(P_1', R_k) + d(R_k, Z) \\ &= d(P, Q_k) + d(R_k, Z) \\ &\le d(P, F) + d(F, Q_k) + d(R_k, Z) \\ &\le d(P, F) + \tfrac{1}{3}a(h) + d(R_k, Z) \end{aligned}$$

by 1.6(b)(ii). But then $d(P, Z) - d(P_1', Z) > 0$, for a contradiction. Thus $R_k \notin [M_k, Z]$, and hence $Y_k \in (M_k, R_k)$ (resp. $Y_k \in [M_k, R_k)$) by 1.8(a).

Set $d = d(Q_k, F_{k-1})$. Then

$$\begin{aligned} d(P, Z) &\ge d(P, Q_k) - 2d + d(Q_k, Z) \\ &= d(P_1', R_k) - 2d + d(Q_k, Z). \end{aligned}$$

Also, we have $d(P_1', Z) \le d(P_1', R_k) + d(R_k, Z)$, so

$$\begin{aligned} 0 &\ge d(P, Z) - d(P_1', Z) \quad \text{(resp. >)} \\ &\ge d(Q_k, Z) - d(R_k, Z) - 2d. \end{aligned}$$

As $Y_k \in [M_k, Z]$ we have

$$d(Q_k, Z) = d(Q_k, M_k) + d(M_k, Y_k) + d(Y_k, Z),$$

and since $Y_k \in [R_k, Z]$ by the foregoing, we have

$$\begin{aligned} d(R_k, Z) &\le d(R_k, M_k) - d(M_k, Y_k) + d(Y_k, Z) \\ &= d(Q_k, M_k) - d(M_k, Y_k) + d(Y_k, Z). \end{aligned}$$

Thus,

$$\begin{aligned} 2d &\ge d(Q_k, Z) - d(R_k, Z) \quad \text{(resp. >)} \\ &\ge 2d(M_k, Y_k), \end{aligned}$$

and this completes Step 5.

Step 6. We have $J_k \in \mathscr{P}(h_0, \dots, h_k)$, and J_k satisfies the conditions given by 1.10(A) or 1.10(B).

Proof. Put $J = J_{k-1}$, $E = E_{k-1}$, $F = F_{k-1}$, $\mathbf{u} = (h_0, \dots, h_{k-1})$, and $h = h_k$. Suppose first that $M_k \in [E, F)$. Then 1.6(b) and 1.8(a) imply that $Q_k \in [P_k, E)$ and, with Step 4, $Y_k \in A(h) \cap (E, F)$. Moreover, Step 4 shows that $M_k \in (E, F)$ if $h_k \in \mathscr{H}_1$, and 1.8(a) shows that $M_k \neq Y_k$ if $h \in \mathscr{H}_2$. Thus

$$\begin{aligned} d(Q_k, E) &\le \tfrac{1}{2}a(h) < d(Q_k, E) + \ell(A(h) \cap J) \quad \text{if } h \in \mathscr{H}_2, \\ d(Q_k, E) &< \tfrac{1}{2}a(h) \le d(Q_k, E) + \ell(A(h) \cap J) \quad \text{if } h \in \mathscr{H}_1. \end{aligned}$$

Now 1.6(c) implies that $[R_k, F] \in \mathscr{P}(\mathbf{u} \circ (h))$. That is, $J_k \in \mathscr{P}(h_0, \dots, h_k)$ and J_k satisfies the conditions given by 1.10(A).

Suppose next that $M_k \notin [E, F)$. Then $M_k \in [F, Z]$ by Step 4. Then Step 5 yields

$$d(Y_k, F) \le \tfrac{1}{2}a(h) < d(Y_k, F) + \ell(A(h) \cap J) \quad \text{if } h \in \mathscr{H}_2,$$
$$d(Y_k, F) < \tfrac{1}{2}a(h) \le d(Y_k, F) + \ell(A(h) \cap J) \quad \text{if } h \in \mathscr{H}_1.$$

Now 1.6(c) gives $[Y_k h^{-1}, E] \in \mathscr{P}(\mathbf{u}^{-1} \circ (h^{-1}))$, and 1.5 then gives $[E, Y_k h^{-1}] \in \mathscr{P}((h) \circ \mathbf{u})$, and $[Eh, Y_k] \in \mathscr{P}((h) \circ \mathbf{u}^h)$. But

$$(h) \circ \mathbf{u}^h = \left(h, h^{-1}h_0h, \dots, h^{-1}h_{k-1}h\right)$$

is a braiding of $(h_0, \dots, h_{k-1}, h)$, so we also have $[Eh, Y_k] \in \mathscr{P}(\mathbf{u} \circ (h))$. This result, along with the information given by Step 5, shows that 1.10(B) defines J_k as a member of $\mathscr{P}(\mathbf{u} \circ (h))$. This completes Step 6.

With Step 6, we now conclude that $J_n \in \mathscr{P}(\mathbf{h})$ and that $J_n \subseteq [P_{n+1}, Z]$. As remarked earlier, this contradicts the solid fact that P_{n+1} is equal to Z, and completes the proof of Theorem 1.7.

2. Control of convergence

Let $\mathbf{X}$, $\mathscr{H}$, and $\mathscr{H}^*$ be as in the preceding section, and let $(\mathscr{H}_1, \mathscr{H}_2)$ be a splitting of $\mathscr{H}$.

For any discrete subset $\mathscr{D}$ of $\mathbf{X}$, put

$$\Sigma(\mathscr{D}) = \bigcup_{P, Q \in \mathscr{D}} [P, Q].$$

We say that $\mathscr{D}$ is in **general position** if $P \notin \Sigma(\mathscr{D} - \{P\})$ for any $P \in \mathscr{D}$.

Let $J = [E, F]$ be a non-degenerate, directed closed segment of $\mathbf{X}$, and let $\mathbf{h} = (h_0, \dots, h_k)$ be a member of $\mathscr{H}^*$. We write

$$J \sim \mathbf{h}$$

if there exists a permutation π of $\{0, \dots, k\}$ and a set $\mathscr{D} = \{Q_0, \dots, Q_{k+1}\}$ of points in general position, with $E = Q_0$ and $Q_{k+1} = F$, such that the following three conditions hold for all i, $0 \le i \le k$.

($\mathscr{P}$1) $\Sigma(\mathscr{D}) \cap A(h_i) = [Q_{\pi(i)}, Q_{\pi(i)+1}]$, and

$$a(h_i) \ge \ell(\Sigma(\mathscr{D}) \cap A(h_i)) \begin{cases} > \frac{1}{2}a(h_i) & \text{if } h_i \in \mathscr{H}_1, \\ \ge \frac{1}{2}a(h_i) & \text{if } h_i \in \mathscr{H}_2. \end{cases}$$

($\mathscr{P}$2) $J \cap A(h_i)$ is non-degenerate, and is oriented towards E by h_i.

($\mathscr{P}$3) $A(h_i) \cap A(h_j) \cap J$ is at most one point for all j, $j \ne i$. (See Figure 4.)

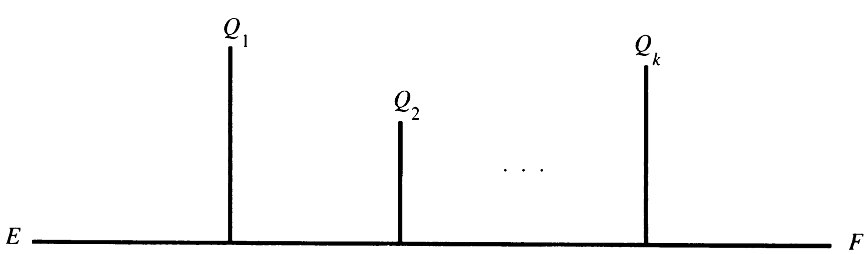

Figure 4

Extend the relation $\sim$ to a relation from segments of $\mathbf{X}$ to braiding classes of $\mathscr{H}^*$, by writing $J \sim [\mathbf{h}]$ if $J \sim \mathbf{h}$ for some representative $\mathbf{h}$ of $[\mathbf{h}]$. It is then routine to check that $\sim$ defines a polarization, in the sense of definition 1.5. We denote this polarization by $\mathscr{P}^*$. As in section 1, write $J \in \mathscr{P}^*(\mathbf{h})$ if $J \sim [\mathbf{h}]$. Also, as in section 1, if $J \in \mathscr{P}^*(\mathbf{h})$, then $\mathscr{E}^*(J, \mathbf{h})$ denotes the set of all $h \in \mathscr{H}$ such that $\mathbf{h} \circ (h) \in \mathscr{H}^*$ (for some, and hence every, representative $\mathbf{h}$ of $[\mathbf{h}]$), and such that $A(h) \cap J$ is oriented by h towards the terminal point of the oriented segment J.

We shall need to assume the following small cancellation hypothesis.

(SCH) Whenever h and h' are elements of $\mathscr{H}$, with $a(hh') < a(h) + a(h')$ and with $hh' \notin \mathscr{H} \cup \{1\}$, we have

$$\ell\left(A(h) \cap A(h')\right) \begin{cases} \leq \frac{1}{4} a(h) & \text{if } h \in \mathscr{H}_1, \\ < \frac{1}{4} a(h) & \text{if } h \in \mathscr{H}_2. \end{cases}$$

Our goal in this section is to prove the following result, which is in fact the main result of this paper.

2.1 **Theorem.** *Assume (SCH), and assume also that whenever $J \in \mathscr{P}^*(\mathbf{h})$ and $h \in \mathscr{E}^*(J, \mathbf{h})$ we have*

$$\ell\left(A(h) \cap J\right) \begin{cases} < \frac{1}{2} a(h) & \text{if } h \in \mathscr{H}_1, \\ \leq \frac{1}{2} a(h) & \text{if } h \in \mathscr{H}_2. \end{cases}$$

Then $\mathscr{H}$ has the convergence property relative to $(\mathscr{H}_1, \mathscr{H}_2)$.

Before undertaking the proof of 2.1, we first prove two lemmas.

2.2 **Lemma.** *Let $\mathbf{h} = (h_0, \dots, h_k) \in \mathscr{H}^*$. Then for any i and j with $0 \leq i \neq j \leq k$ we have $h_i h_j \notin \mathscr{H} \cup \{1\}$.*

Proof. Suppose that $i < j$. Then the $\mathscr{H}$-sequence

$$\left(h_0, \dots, h_i, h_j, h_j^{-1} h_{i+1} h_j, \dots, h_j^{-1} h_{j-1} h_j, h_{j+1}, \dots, h_k\right)$$

is a braiding of $\mathbf{h}$, and so $h_i h_j \notin \mathscr{H} \cup \{1\}$ by definition of $\mathscr{H}^*$. As $\mathscr{H}$ is self-normalizing, we also have $h_j h_i \notin \mathscr{H} \cup \{1\}$, and this proves the lemma.

2.3 **Lemma.** *Assume (SCH), and let $J = [E, F]$ be a segment, and $\mathbf{h} = (h_0, \dots, h_k) \in \mathscr{H}^*$, with $J \in \mathscr{P}^*(\mathbf{h})$. Let $\mathscr{D} = \{Q_0, \dots, Q_{k+1}\}$ be a set of points in general position, with $E = Q_0$ and $Q_{k+1} = F$, such that the conditions $(\mathscr{P}1)$ through $(\mathscr{P}3)$ hold with respect to J, $\mathbf{h}$, and $\mathscr{D}$. Set $Y_0 = E$, $Y_{k+1} = F$, and for any i with $1 \leq i \leq k$ set $Y_i = [Q_{i-1}, Q_i] \cap [Q_{i-1}, Q_{i+1}] \cap [Q_i, Q_{i+1}]$. Let $\mathbf{h}' = (h_0', \dots, h_k')$ be the permutation of $\mathbf{h}$ such that $A(h_i') \cap \Sigma(\mathscr{D}) = [Q_i, Q_{i+1}]$ $(0 \leq i \leq k)$. Then the following hold:*

(a) *$A(h_i') \cap J = [Y_i, Y_{i+1}]$.*

(b) *For any $h \in \mathscr{E}^*(J, \mathbf{h})$, we have*

(i) *$J \not\subseteq A(h)$, and*

(ii) *if E (resp. F) is in $A(h)$, then $A(h) \cap J \subseteq [E, Y_1)$ (resp. $(Y_k, F]$).*

Proof. Part (a) follows immediately from $(\mathscr{P}1)$ through $(\mathscr{P}3)$. Also, (b)(i) follows from (b)(ii). In order to prove (b)(ii) we assume that $E \in A(h) \cap J$. By $(\mathscr{P}2)$, $A(h_0') \cap J$ is non-degenerate, so $E \neq Y_1$. Assuming that $A(h) \cap J \not\subseteq [E, Y_1)$ we have $[E, Y_1] \subseteq A(h) \cap J$, and $a(hh_0') < a(h) + a(h_0')$ since $A(h) \cap J$

is oriented towards F by h (by the definition of $\mathscr{E}^*(J, \mathbf{h})$) and since $[E, Y_1]$ is oriented towards E by h_0' (by $(\mathscr{P}2)$). It then follows from (SCH) and 2.2 that $d(E, Y_1) \leq \frac{1}{4}a(h_0')$, with strict inequality if $h_0' \in \mathscr{H}_2$. On the other hand, $(\mathscr{P}1)$ tells us that $d(E, Q_1) \geq \frac{1}{2}a(h_0')$, with strict inequality if $h_0' \in \mathscr{H}_1$. Hence $d(Y_1, Q_1) > \frac{1}{4}a(h_0')$ in any case. But $[Y_1, Q_1] \subseteq A(h_0') \cap A(h_1')$ (in fact, equality holds here, by $(\mathscr{P}3)$), and this contradicts (SCH) and 2.2 as applied to h_0' and h_1'. A symmetric argument disposes of the case where $F \in A(h)$, and the proof of the lemma is thereby complete.

We now proceed to the proof of Theorem 2.1, where we need only show that $\mathscr{P}^*$ controls convergence, in the sense of 1.6, and by appeal to Theorem 1.7. Here 1.6(a) is immediate from $(\mathscr{P}1)$ and $(\mathscr{P}2)$, taking $k = 0$. In order to check the other parts of 1.6 let $\mathbf{h} = (h_0, \dots, h_k) \in \mathscr{H}^*$, let $J = [E, F]$ with $J \in \mathscr{P}^*(\mathbf{h})$, and let $h \in \mathscr{E}^*(J, \mathbf{h})$. By 2.3 and (SCH), and the hypothesis of 2.1, we have 1.6(b), and it only remains to verify 1.6(c).

Let Q_i and Y_i be as in the statement of 2.3, and suppose that we are given a point Q of $A(h)$ with $E \in (Q, F)$, and such that

$$(*) \qquad \begin{aligned} &d(Q, E) < \frac{1}{2}a(h) \leq d(Q, E) + \ell\,(A(h) \cap J) \quad \text{if } h \in \mathscr{H}_1, \\ &d(Q, E) \leq \frac{1}{2}a(h) < d(Q, E) + \ell\,(A(h) \cap J) \quad \text{if } h \in \mathscr{H}_2. \end{aligned}$$

Put $Q_0' = Qh$, $Q_1' = E$, $Q_i' = Q_{i-1}$ $(1 \leq i \leq k+1)$, and $Q_{k+2}' = F$. We have $A(h) \cap J \subseteq [E, Y_1)$ by 2.3, and then $(*)$ implies that the set $\mathscr{D}' = \{Q_0', \dots, Q_{k+2}'\}$ is in general position. Set $J' = \left[Q_0', Q_{k+2}'\right]$. Then $(\mathscr{P}1)$ through $(\mathscr{P}3)$ hold with respect to J', $\mathbf{h} \circ (h)$, and $\mathscr{D}'$, as follows from $(*)$ and from the corresponding statements with respect to J, $\mathbf{h}$, and $\{Q_0, \dots, Q_k\}$. With this we conclude that $J' \in \mathscr{P}^*(\mathbf{h} \circ (h))$, and that $\mathscr{P}^*$ controls convergence relative to $(\mathscr{H}_1, \mathscr{H}_2)$. This completes the proof of 2.1.

3. Small cancellation

Let $\mathbf{X}$ and $\mathscr{H}$ be as in the preceding sections, and let λ be an $\langle\mathscr{H}\rangle$-equivariant mapping from $\mathscr{H}$ into the real interval $\left[0, \frac{1}{4}\right]$. That is,

$$\lambda\left(g^{-1}hg\right) = \lambda(h) \quad \text{for all } g \in \langle\mathscr{H}\rangle, \quad h \in \mathscr{H}.$$

For example, if $\mathscr{H}$ satisfies the condition (SCH) of the preceding section, then a suitable such mapping λ is obtained by taking

$$\lambda(h) = \frac{1}{a(h)} \operatorname{Sup}\ \{\ell\,(A(h) \cap A(h'))\},$$

where the supremum is taken over all $h' \in \mathscr{H}$ such that $a(hh') < a(h) + a(h')$ and such that $hh' \notin \mathscr{H} \cup \{1\}$.

For any λ as above, we define three conditions on λ, as follows.

$C_1(\lambda)$: We have $hh' \in \mathscr{H} \cup \{1\}$ whenever h and h' are elements of $\mathscr{H}$ such that

(i) $a(hh') < a(h) + a(h')$, and

(ii) $\ell\,(A(h) \cap A(h')) > \lambda(h)a(h)$.

$C_2(\lambda)$: (The same as $C_1(\lambda)$, but with a "$\geq$-sign" in place of the strict inequality of $C_1(\lambda)$ (ii)).

$\Delta(\lambda)$: We have $\{h_1h_2, h_1h_3, h_2h_3\} \cap \mathscr{H} \neq \varnothing$ whenever h_1, h_2, h_3 are elements of $\mathscr{H}$ such that

(i) $a(h_ih_j) < a(h_i) + a(h_j)$ for all i and j with $1 \leq i < j \leq 3$, and

(ii) one of the following holds.

(a) For at least two of the three even permutations (i, j, k) of $(1, 2, 3)$ we have

$$\ell\,(A(h_i) \cap A(h_j)) + \ell\,(A(h_j) \cap A(h_k)) > \left(\tfrac{1}{2} - \lambda(h_j)\right) a(h_j).$$

(b) For some permutation (i, j, k) of $(1, 2, 3)$ we have $\lambda(h_j) = \frac{1}{4}$ and

$$\ell\,(A(h_i) \cap A(h_j)) = \ell\,(A(h_j) \cap A(h_k)) = \tfrac{1}{4}a(h_j).$$

3.1 **Theorem.** *Let $\mathscr{H}$ be a symmetric, self-normalizing set of hyperbolic isometries of an $\mathbb{R}$-tree, and let λ be an $\langle\mathscr{H}\rangle$-equivariant mapping from $\mathscr{H}$ into the interval $\left[0, \frac{1}{4}\right]$.*

(a) *If $\mathscr{H}$ and λ satisfy both $C_1(\lambda)$ and $\Delta(\lambda)$, then $\mathscr{H}$ has the weak convergence property.*

(b) *If $\mathscr{H}$ and λ satisfy both $C_2(\lambda)$ and $\Delta(\lambda)$, then $\mathscr{H}$ has the convergence property.*

Moreover, if λ maps $\mathscr{H}$ into $\left[0, \frac{1}{6}\right]$, then $\Delta(\lambda)$ is already implied by $C_1(\lambda)$ and by $C_2(\lambda)$.

Proof. Assume that $\mathscr{H}$ and λ satisfy both $C_j(\lambda)$ and $\Delta(\lambda)$ for some $j = 1$ or 2. If $j = 1$ set $\mathscr{H}_1 = \mathscr{H}$ and $\mathscr{H}_2 = \varnothing$, while if $j = 2$ set $\mathscr{H}_2 = \mathscr{H}$ and $\mathscr{H}_1 = \varnothing$. Thus $(\mathscr{H}_1, \mathscr{H}_2)$ is a splitting of $\mathscr{H}$, and in order to prove (a) and (b) it suffices (by definition) to show that $\mathscr{H}$ has the convergence property relative to $(\mathscr{H}_1, \mathscr{H}_2)$.

We aim to apply Theorem 2.1, and we therefore ask the reader to recall the relation $\sim$, the polarization $\mathscr{P}^*$, the conditions $(\mathscr{P}1)$ through $(\mathscr{P}3)$, and the notion of $\mathscr{E}^*(J, \mathbf{h})$, from section 2. Evidently $C_j(\lambda)$ implies the condition (SCH) of section 2, so Lemma 2.3 is available to us.

Let $\mathbf{h} = (h_0, \dots, h_k) \in \mathscr{H}^*$, let $[E, F] = J$ be a non-degenerate directed segment with $J \sim \mathbf{h}$, and let $\mathscr{D} = \{Q_0, \dots, Q_{k+1}\}$ be a set of points in general position, with $E = Q_0$ and $Q_{k+1} = F$, and such that $(\mathscr{P}1)$ through $(\mathscr{P}3)$ hold with respect to J, $\mathbf{h}$, and $\mathscr{D}$. Define points Y_i, $0 \leq i \leq k+1$, as in the statement of 2.3, and let π be the permutation of $\{0, \dots, k\}$ such that $A(h_i) \cap \Sigma(\mathscr{D}) = \left[Q_{\pi(i)}, Q_{\pi(i)+1}\right]$. Then π^{-1} defines a permutation $\mathbf{h}' = \left(h'_0, \dots, h'_k\right)$ of $\mathbf{h}$ such that $A(h'_i) \cap \Sigma(\mathscr{D}) = [Q_i, Q_{i+1}]$, and 2.3(a) tells us that $A(h'_i) \cap J = [Y_i, Y_{i+1}]$.

Let $h \in \mathscr{E}^*(J, \mathbf{h})$. In order to apply 2.1 we need only show that $\ell\,(A(h) \cap J) \leq \frac{1}{2}a(h)$, with strict inequality if $h \in \mathscr{H}_1$. By 2.3(b) we may assume without loss that neither E nor F is in $A(h) \cap J$. Also, by (SCH) we may assume that $A(h) \cap J \not\subseteq [Y_i, Y_{i+1}]$ for any i. This leaves three cases to consider; as follows.

Case (i). $A(h) \supseteq [Y_{i-1}, Y_{i+1}]$ for some i, $1 < i < k$.

Here the orientations given by $(\mathscr{P}2)$ and by the definition of $\mathscr{E}^*(J, \mathbf{h})$ imply that part (i) of $\Delta(\lambda)$ holds with respect to h'_{i-1}, h'_i, and h. Further, we have $d\,(Y_{i-1}, Q_{i-1}) \leq \ell\,\left(A\left(h'_{i-2}\right) \cap A\left(h'_{i-1}\right)\right) \leq \lambda\left(h'_{i-1}\right) a\left(h'_{i-1}\right)$ (with the second of these inequalities being strict if $\mathscr{H} = \mathscr{H}_2$) by $(\mathscr{P}1)$, $(\mathscr{P}2)$, 2.2, and

$C_j(\lambda)$. Then $(\mathscr{P}1)$ implies that $d(Y_{i-1}, Q_i) > \left(\frac{1}{2} - \lambda(h'_{i-1})\right) a(h'_{i-1})$. But

$$\ell(A(h) \cap A(h'_{i-1})) + \ell(A(h'_{i-1}) \cap A(h'_i)) \geq d(Y_{i-1}, Y_i) + d(Y_i, Q_i) = d(Y_{i-1}, Q_i),$$

so

$$\ell(A(h) \cap A(h'_{i-1})) + \ell(A(h'_{i-1}) \cap A(h'_i)) > \left(\tfrac{1}{2} - \lambda(h'_{i-1})\right) a(h'_{i-1}).$$

A symmetric argument yields the same result with (h, h'_{i-1}, h'_i) replaced by (h'_{i-1}, h'_i, h), and then $\Delta(\lambda)$ and 2.2 yield a contradiction.

Case (ii). $A(h) \supseteq [Y_i, Y_{i+1}]$ for some unique i, $1 \leq i < k$.

Here $A(h) \cap J$ is the union of the segments $A(h) \cap (Y_{i-1}, Y_{i+1}]$ and $A(h) \cap [Y_i, Y_{i+2})$, and by $C_j(\lambda)$ each of the segments $A(h) \cap (Y_{i-1}, Y_i]$ and $A(h) \cap [Y_{i+1}, Y_{i+2})$ is of length at most $\lambda(h)a(h)$, and less than $\lambda(h)a(h)$ if $j = 2$. Since $\ell(A(h) \cap J) \geq \frac{1}{2}a(h)$ we then have $\ell(A(h) \cap [Y_i, Y_{i+2})) \geq \left(\frac{1}{2} - \lambda(h)\right) a(h)$, with strict inequality if $j = 2$. Moreover, since $[Y_i, Y_{i+1}]$ has positive length by $(\mathscr{P}2)$, at least one of the segments $A(h) \cap (Y_{i-1}, Y_{i+1}]$ and $A(h) \cap [Y_i, Y_{i+2})$ has length exceeding $\left(\frac{1}{2} - \lambda(h)\right) a(h)$ if $\lambda(h) = \frac{1}{4}$.

Without loss, we may take $A(h) \cap [Y_i, Y_{i+2})$ to be at least as long as $A(h) \cap (Y_{i-1}, Y_{i+1}]$. Then the foregoing yields

$$\ell(A(h'_i) \cap A(h)) + \ell(A(h) \cap A(h'_{i+1})) \geq \left(\tfrac{1}{2} - \lambda(h)\right) a(h),$$

with strict inequality if $j = 2$ or if $\lambda(h) = \frac{1}{4}$. But we also have $d(Y_i, Q_{i+1}) > \left(\frac{1}{2} - \lambda(h'_i)\right) a(h'_i)$ by an argument as in case (i), above, and hence

$$\ell(A(h) \cap A(h'_i)) + \ell(A(h'_i) \cap A(h'_{i+1})) > \left(\tfrac{1}{2} - \lambda(h'_i)\right) a(h'_i).$$

Once again, $\Delta(\lambda)$ and 2.2 yield a contradiction.

Case (iii). $Y_i \in A(h)$ for some unique i, $0 < i \leq k$.

Here $A(h) \cap J$ is the union of $A(h) \cap (Y_{i-1}, Y_i]$ and $A(h) \cap [Y_i, Y_{i+1})$. Since $\ell(A(h) \cap J) \geq \frac{1}{2}a(h)$ and (SCH) holds, it then follows from 2.2 that $\lambda(h) = \frac{1}{4}$ and each of the segments $A(h) \cap A(h'_{i-1})$ and $A(h) \cap A(h'_i)$ has length precisely $\frac{1}{4}a(h)$. Again, 2.2 and $\Delta(\lambda)$ yield a contradiction, and this completes the proof of Theorem 3.1.

References

1. R. Alperin and H. Bass, *Length functions of group actions on Λ-trees*, Combinatorial Group Theory and Topology (Alta, Utah, 1984), Ann. of Math. Studies, no. 111, Princeton Univ. Press, Princeton, NJ, 1987, pp. 265–378.
2. Andrew Chermak, *Triangles of groups*, preprint.
3. ———, *Colimits of sporadic triangles of groups*, preprint.
4. R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer-Verlag, Berlin and New York, 1977.

Department of Mathematics, Kansas State University, Manhattan, Kansas 66506
E-mail address: chermak@math.ksu.edu