TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 347, Number 12, December 1995

THE VARIATIONS OF HODGE STRUCTURE OF MAXIMAL DIMENSION WITH ASSOCIATED HODGE NUMBERS $h^{2,0} > 2$ AND $h^{1,1} = 2q+1$ DO NOT ARISE FROM GEOMETRY

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ABSTRACT. The specified variations are proved to be covered by a bounded contractible domain Ω. After classifying the analytic boundary components of Ω with respect to a fixed realization, the group of the biholomorphic automorphisms $\operatorname{Aut}\Omega$ and the $\operatorname{Aut}\Omega$-orbit structure of Ω are found explicitly. Then Ω is shown to admit no quasiprojective arithmetic quotients, whereas the lack of geometrically arising variations, covered by Ω.

The weight 1 period domain $Sp(n, \mathbf{R})/U_n$ arises from geometry, in as much as it classifies the first cohomologies of the n-dimensional abelian varieties with fixed polarization.

The geometric locus on the weight 2 period domain is shown to satisfy the Griffiths' horizontality (cf. [9])

$$\frac{\partial H_s^{2,0}}{\partial s_i} \subset (H_s^{2,0})^{\perp}.$$

The nonintegrability of the horizontal distribution raises the problem of describing the locally liftable holomorphic integral manifolds, called variations of Hodge structure. In particular, one can ask about the maximal dimension of a variation of Hodge structure and try to uniformize the universal covers of the variations of that dimension.

In the case of $h^{2,0} = p > 2$ and even $h^{1,1} = 2q$, Carlson has proved in [3] that the corresponding variations of Hodge structure have maximal dimension pq. The ones attaining that dimension are shown to be covered by the generalized ball

$$B_{p,q} = \{Z \in \operatorname{Mat}_{p,q}(\mathbf{C}) \mid {}^t\overline{Z}Z < I_q\} = SU(p, q)/S(U_p \times U_q),$$

equivariantly embedded in the period domain $D = SO(2p, 2q)/S(U_p \times O_{2q})$. More precisely, if Q denotes the polarization form and h stands for its associated Hermitian form, then the choice of a maximal Q-isotropic subspace $H^{2,0} \subset V \subset H_{\mathbf{C}}$ determines the embedding of the Q-orthogonal and h-unitary groups

$$U(p, q) = \operatorname{Aut}(V, h) = \operatorname{Aut}(V, Q, h) \hookrightarrow \operatorname{Aut}(H_{\mathbf{C}}, Q, h) = O(2p, 2q).$$

Received by the editors November 23, 1994 and, in revised form, January 20, 1995.
1991 *Mathematics Subject Classification*. Primary 32J20, Secondary 14D07, 32M05.

Descending to the quotient spaces, that situates the generalized ball

$$B_{p,q}(V) = \{\Lambda \in \mathrm{Grass}(p, V) | h|_\Lambda > 0\}$$

in the period domain

$$D = \{\Lambda \in \mathrm{Grass}(p, H_{\mathbf{C}}) | Q(x, y) = 0 \ \forall x, y \in \Lambda, \ \ h|_\Lambda > 0\}.$$

Carlson and Simpson have constructed in [6] a family of special abelian varieties, parametrized by $B_{p,q}$. It exhibits the considered variations of maximal dimension as arising from geometry.

For $h^{2,0} = p > 2$ and odd $h^{1,1} = 2q+1$, the joint article [5] with Carlson and Toledo obtains the maximal dimension $pq+1$ of an associated variation of Hodge structure. An example of a connected simply connected variation, attaining that dimension, is constructed as a pencil $\Omega := \bigcup_{t \in B_{1,1}} B_{p,q}(t)$ of generalized balls

$$B_{pq,}(t) = \left\{ X \in \mathrm{Mat}_{p-1,q}(\mathbf{C}),\ Y \in \mathrm{Mat}_{1,q}(\mathbf{C}) \mid \begin{pmatrix} X \\ \frac{Y}{1-|t|^2} \end{pmatrix} \in B_{p,q} \right\}$$

over the unit disk $B_{1,1}$. The same article establishes that the bounded domain Ω is not an equivariantly embedded Hermitian symmetric subspace of the period domain $D = SO(2p, 2q+1)/S(U_p \times O_{2q+1})$.

The present note shows that all the variations of maximal dimension with associated $h^{2,0} = p > 2$ and $h^{1,1} = 2q+1$ are covered by Ω. The proof goes along the lines of Carlson's argument from [3].

Tracing out the relation between the geometry and the group $\mathrm{Aut}\,\Omega$ of the biholomorphic automorphisms of Ω, the work classifies the analytic boundary components of Ω, up to the action of an a priori guessed subgroup $G \subset \mathrm{Aut}\,\Omega$, extendable over the boundary. Though pertaining only to the specific realization, the analytic boundary structure enables us to calculate for $p > 2$ the effectively acting group $\mathrm{Aut}_{\check{o}}\,\Omega$ of the biholomorphic automorphisms, fixing the origin $\check{o}$ $(t = 0, X = 0, Y = 0)$. Then the analyticity and the Hermitian symmetry of the orbit $\mathrm{Aut}\,\Omega(\check{o})$ (cf. [2]), together with the presence of subgroups $\mathrm{Aut}_{\check{o}}\,\Omega \subset G \subset \mathrm{Aut}\,\Omega$, allow to determine that $\mathrm{Aut}\,\Omega = S(U(1,1) \times U(p-1, q))$ in the case of $p > 2$. Moreover, $\mathrm{Aut}\,\Omega$ acts linearly on the directions, transversal to the orbit $\mathrm{Aut}\,\Omega(\check{o})$. Therefore, an arbitrary discrete quotient $\Gamma\backslash\Omega$ can be embedded in a rank q vector bundle over $\Gamma\backslash \mathrm{Aut}\,\Omega(\check{o})$. If $\Gamma\backslash\Omega$ is a quasiprojective arithmetic quotient of the domain Ω, then its projective closure fibers over the Baily-Borel compactification of $\Gamma\backslash \mathrm{Aut}\,\Omega(\check{o})$ and the fiber over the reference point appears to be a finite quotient of a q-dimensional compact complex analytic subvariety of $\mathbf{C}^q \subset \mathbf{CP}^q$. The contradiction with Chow's theorem reveals the nonexistence of quasiprojective arithmetic quotients $\Gamma\backslash\Omega$ for $p > 2$, and justifies that the variations of Hodge structure of maximal dimension with associated $h^{2,0} = p > 2$ and $h^{1,1} = 2q+1$ do not arise from geometry.

The uniformization result and the nonexistence of compact variations of maximal dimension with $h^{2,0} > 2$ and $h^{1,1} = 2q+1$ can be found in the author's Ph.D. thesis [15] with graduate advisor Professor James Carlson. For the discussions on the problem and related topics, the author is extremely grateful to James Carlson, Domingo Toledo, Rolf-Peter Holzapfel, Vasil Kanev and Vasil Tsanov.

1. The holomorphic tangent bundle to the lifting of a variation of maximal dimension

Let $f\colon S \to \Gamma\backslash D$ be a variation of maximal dimension with $h^{2,0} = p > 2$, $h^{1,1} = 2q+1$, monodromy representation $\pi_1(S) \to \Gamma$ in the freely and properly discontinuously acting group $\Gamma \subset \mathrm{Aut}(H_{\mathbf{Z}}, Q)$, and a lifting $\tilde{f}\colon \widetilde{S} \to D$. The present section aims a fibrewise coincidence of the holomorphic tangent bundle to $\tilde{f}(\widetilde{S})$ with the holomorphic tangent spaces to a field of domains Ω over $\tilde{f}(\widetilde{S})$.

The argument starts with a conjugacy result for the tangent spaces $\mathscr{A} := T^{1,0}\tilde{f}(\widetilde{S})_{\check{o}}$ at the reference point $\check{o} \in D$. Making use of [9], let us identify the holomorphic tangent space $T^{1,0}D_{\check{o}}$ of the period domain D with the subspace

$$\mathscr{G}_- := \{\varphi \in \mathrm{Lie}\, SO_{\mathbf{C}}(2p, 2q+1) \mid \varphi(H^{2-k,k}) \subset H^{1-k,1+k} + H^{-k,2+k}\ \forall k\}$$

of the complexified infinitesimal isometries $\mathscr{G} := \mathrm{Lie}\, SO_{\mathbf{C}}(2p, 2q+1)$. Then the variations of Hodge structure are tangent to the so-called horizontal subspace

$$\mathscr{G}^{-1,1} := \{\varphi \in \mathscr{G} \mid \varphi(H^{2-k,k}) \subset H^{1-k,1+k}\ \forall k\}.$$

According to [3], [7] and other articles, $\mathscr{A}$ is an abelian Lie algebra, contained in $\mathscr{G}^{-1,1}$. In order to describe the $S(U_p \times O_{2q+1})$-conjugacy classes of the abelian subspaces $\mathscr{A} \subset \mathscr{G}^{-1,1}$, one needs to study the adjoint action of $G^{0,0} := S(GL_{\mathbf{C}}(p) \times O_{\mathbf{C}}(2q+1))$. Observe that $\mathrm{Lie}\, G^{0,0}$ can be identified with the complex infinitesimal isometries

$$\mathscr{G}^{0,0} = \{\varphi \in \mathscr{G} \mid \varphi(H^{2-k,k}) \subset H^{2-k,k}\ \forall k\}.$$

Therefore, the horizontal subspace $\mathscr{G}^{-1,1}$ is $\mathrm{ad}\,\mathscr{G}^{0,0}$-invariant and the set of the abelian subspaces $\mathscr{A} \subset \mathscr{G}^{-1,1}$ splits into a disjoint union of $G^{0,0}$-conjugacy classes. The abelian subspaces of maximal dimension will be proved to constitute a single $G^{0,0}$-conjugacy class, so that the holomorphic tangent spaces to variations of maximal dimension will appear to be $\mathrm{Ad}\, G^{0,0}$-equivalent.

Let us fix an adapted Hodge frame of the reference structure, i.e., a basis

$$\begin{gathered}\phi_1, \dots, \phi_p \in H^{2,0},\\ \psi_1, \dots, \psi_q, \overline{\psi_1}, \dots, \overline{\psi_q}, \psi_{2q+1} = \overline{\psi_{2q+1}} \in H^{1,1},\\ \overline{\phi_1}, \dots, \overline{\phi_p} \in H^{0,2},\end{gathered}$$

with respect to which the polarization form Q has a matrix

$$Q = \begin{pmatrix} & & & & I_p \\ & 0_q & -I_q & 0 & \\ & -I_q & 0_q & 0 & \\ & 0 & 0 & -1 & \\ I_p & & & & \end{pmatrix}.$$

Then denote by superscripts the corresponding dual vectors and consider the Cartan subalgebra $\mathscr{H} \subset \mathscr{G}^{0,0}$ generated by

$$\begin{aligned}\mathscr{E}_i &:= \phi_i \otimes \phi^i - \overline{\phi_i} \otimes \overline{\phi^i}, & 1 \le i \le p,\\ \mathscr{E}_\alpha &:= \psi_\alpha \otimes \psi^\alpha - \overline{\psi_\alpha} \otimes \overline{\psi^\alpha}, & p+1 \le \alpha \le p+q.\end{aligned}$$

Observe that $\mathcal{H}$ is diagonally represented with respect to the adapted Hodge frame and constitutes a Cartan subalgebra of $\mathcal{G}$, as well. The subspaces $\mathcal{G}^{-1,1}$ and $\mathcal{G}^{0,0}$ are invariant under the adjoint action $\operatorname{ad}\mathcal{H}$ and split in a sum of 1-dimensional root spaces. If e_i stand for the duals of $\mathcal{E}_i$, then the horizontal space $\mathcal{G}^{-1,1} = \sum_{\rho\in\Delta^{-1,1}} \mathcal{G}^{\rho}$ is associated with the root system

$$\Delta^{-1,1} = \{-e_i + e_\alpha\,,\, -e_i - e_\alpha\,,\, -e_i \mid 1 \le i \le p\,,\, p+1 \le \alpha \le p+q\}.$$

Let us fix the corresponding root vectors

$$\begin{aligned} X_{-e_i+e_\alpha} &= \psi_\alpha \otimes \phi^i + \overline{\phi_i} \otimes \overline{\psi^\alpha}, && 1 \le i \le p\,,\, p+1 \le \alpha \le p+q\,,\\ X_{-e_i-e_\alpha} &= \overline{\psi_\alpha} \otimes \phi^i + \overline{\phi_i} \otimes \psi^\alpha, && 1 \le i \le p\,,\, p+1 \le \alpha \le p+q\,,\\ X_{-e_i} &= \sqrt{2}(\psi_{2q+1} \otimes \phi^i + \overline{\phi_i} \otimes \psi^{2q+1}), && 1 \le i \le p. \end{aligned}$$

The Lie algebra $\mathcal{G}^{0,0} = \mathcal{H} + \sum_{\rho\in\Delta^{0,0}} \mathcal{G}^{\rho}$ consists of the above described Cartan subalgebra and the span of the root vectors

$$\begin{aligned} X_{e_i-e_j} &= \phi_i \otimes \phi^j - \overline{\phi_j} \otimes \overline{\phi^i}, && 1 \le i \ne j \le p\,,\\ X_{e_\alpha-e_\beta} &= \psi_\alpha \otimes \psi^\beta - \overline{\psi_\beta} \otimes \overline{\psi^\alpha}, && p+1 \le \alpha \ne \beta \le p+q\,,\\ X_{e_\alpha+e_\beta} &= \psi_\alpha \otimes \overline{\psi^\beta} - \psi_\beta \otimes \overline{\psi^\alpha}, && p+1 \le \alpha < \beta \le p+q\,,\\ X_{-e_\alpha-e_\beta} &= \overline{\psi_\beta} \otimes \psi^\alpha - \overline{\psi_\alpha} \otimes \psi^\beta, && p+1 \le \alpha < \beta \le p+q\,,\\ X_{e_\alpha} &= \sqrt{2}(\psi_\alpha \otimes \psi^{2q+1} - \psi_{2q+1} \otimes \overline{\psi^\alpha}), && p+1 \le \alpha \le p+q\,,\\ X_{-e_\alpha} &= \sqrt{2}(\psi_{2q+1} \otimes \psi^\alpha - \overline{\psi_\alpha} \otimes \psi^{2q+1}), && p+1 \le \alpha \le p+q. \end{aligned}$$

The study of the abelian subspaces $\mathcal{A} \subset \mathcal{G}^{-1,1}$ of maximal dimension, consisting entirely of nilpotent elements, is initiated by means of Malčev's method of the leading root vectors (cf. [18]) and its Hodge theory application from [5].

To introduce an ordering of the roots $\rho \in \mathcal{H}^*$, let us pick a generic $\mathcal{E} = \sum_{i=1}^{p+q} a_i \mathcal{E}_i$ with real coefficients $a_1 > \cdots > a_{p+q}$, and set

$$\rho > 0 \text{ if and only if } \rho(\mathcal{E}) > 0.$$

This ordering extends accordingly to the root vectors.

Any basis of $\mathcal{A}$ can be transformed by Gauss-Jordan reduction to the form

$$\begin{array}{llllll} U_1 = & X_{\alpha_1} + & & & & \sum_{\beta>\alpha_1\,,\,\beta\ne\alpha_i} A_1^\beta X_\beta\,,\\ U_2 = & & X_{\alpha_2} + & & & \sum_{\beta>\alpha_2\,,\,\beta\ne\alpha_i} A_2^\beta X_\beta\,,\\ \dots & \dots & \dots & \dots & \dots & \dots\\ U_N = & & & & X_{\alpha_N} + & \sum_{\beta>\alpha_N\,,\,\beta\ne\alpha_i} A_N^\beta X_\beta \end{array}$$

with $\alpha_1 < \alpha_2 < \cdots < \alpha_N$. Due to the compatibility of the ordering with the Lie bracket of the root vectors, the commutations of U_i imply the commutations of their leading root vectors X_{α_i}. In such a way, to any abelian subspace $\mathcal{A} \subset \mathcal{G}^{-1,1}$ there corresponds a root system $C = \{\alpha_1\,,\,\dots\,,\,\alpha_N\}$ with the property

$$\forall \alpha_i\,,\, \alpha_j \in C \Rightarrow \alpha_i + \alpha_j \text{ is not a root.}$$

Such root systems are called commutative (cf. [5]).

As far as the Weyl group of $G^{0,0}$ can be considered as the normalizer of $\mathscr{H}$ in $\mathscr{G}^{0,0} = \operatorname{Lie} G^{0,0}$ (cf. [14]), the $\operatorname{Ad}(G^{0,0})$-orbits of abelian subspaces $\mathscr{A} \subset \mathscr{G}^{-1,1}$ turn out to be associated with the $\operatorname{Weyl}(G^{0,0})$-orbits of commutative root systems $C \subset \Delta^{-1,1}$.

Lemma 1. *For any commutative root system $C \subset \Delta^{-1,1}$ of maximal cardinality $pq+1$, there exists $\sigma \in \operatorname{Weyl}(G^{0,0})$, such that*

$$\sigma C = \{-e_i + e_\alpha, -e_p \mid 1 \leq i \leq p, p+1 \leq \alpha \leq p+q\}.$$

Proof. Let $C_\alpha := \{-e_i + e_\alpha \text{ or } -e_i - e_\alpha \in C\}$, $C_0 := \{-e_i \subset C\}$ and decompose into a disjoint union $C = \bigcup_{\alpha=p+1}^{p+q} C_\alpha \cup C_0$. No sum of roots from different subsets is a root, so that C is commutative, provided $C_{p+1}, \dots, C_{p+q}, C_0$ are such.

Obviously, C_0 consists of a single element $-e_i$, which can be transformed by $\operatorname{Weyl}(G^{0,0})$ to $-e_p$.

If $-e_i \pm e_\alpha$ belong simultaneously to C_α, then they are the only roots there and $\operatorname{card} C_\alpha = 2$. Otherwise C_α equals either $\{-e_i + e_\alpha \mid 1 \leq i \leq p\}$ or $C_\alpha = \{-e_i - e_\alpha \mid 1 \leq i \leq p\}$, which are $\operatorname{Weyl}(G^{0,0})$-symmetric with respect to the hyperplane of $\sum_{k=1}^{p+q} \mathbf{R} e_k$ orthogonal to e_α. The assumption $p > 2$ determines the choice of $C_\alpha = \{-e_i + e_\alpha \mid 1 \leq i \leq p\}$ for any $p+1 \leq \alpha \leq p+q$, q.e.d. Lemma 1.

The precise description of $\mathscr{A}$ is obtained along the lines of Carlson's conjugacy result for an even $h^{1,1} = 2q$ (cf. [3]).

Proposition 1. *The abelian subspaces $\mathscr{A} \subset \mathscr{G}^{-1,1}$ of maximal dimension $pq + 1$, associated with $h^{2,0} = p > 2$ and $h^{1,1} = 2q+1$, constitute the $\operatorname{Ad}(G^{0,0})$-orbit of the Cartan invariant one*

$$\mathscr{A}_0 := \operatorname{Span}\{X_{-e_i+e_\alpha}, X_{-e_p} \mid 1 \leq i \leq p, p+1 \leq \alpha \leq p+q\}.$$

Proof. According to Lemma 1, any abelian Lie algebra $\mathscr{A} \subset \mathscr{G}^{-1,1}$ of dimension $pq+1$ is $G^{0,0}$-conjugate to the span of

$$U_0 = X_{-e_p} + \sum_{j=1}^{p} \sum_{\beta=p+1}^{p+q} A^{j\beta} X_{-e_j-e_\beta} + \sum_{k=1}^{p-1} B^k X_{-e_k},$$

$$U_{i\alpha} = X_{-e_i+e_\alpha} + \sum_{j=1}^{p} \sum_{\beta=p+1}^{p+q} C_{i\alpha}^{j\beta} X_{-e_j-e_\beta} + \sum_{k=1}^{p-1} D_{i\alpha}^{k} X_{-e_k}$$

for $1 \leq i \leq p$ and $p+1 \leq \alpha \leq p+q$.

To avoid the presence of B^k, $1 \leq k < p$, one applies

$$g_1 := \operatorname{Ad}\exp\left(\sum_{k=1}^{p-1} B^k X_{-e_k+e_p}\right)$$

to the above generators and then eliminates $X_{-e_k+e_\alpha}$ from $g_1(U_{p\alpha})$.

Without any confusion, let us keep the same notations for the new generators.

For the further annihilation of $A^{p\alpha}$, $p+1 \leq \alpha \leq p+q$, one acts by

$$g_2 := \operatorname{Ad}\exp\left(\frac{1}{2} \sum_{\beta=p+1}^{p+q} A^{p\beta} X_{-e_\beta}\right)$$

and eliminates X_{-e_p} from $g_2(U_{p\alpha})$.

Then the vanishing of $[U_{i\alpha}, U_0]$ and $[U_{i\alpha}, U_{k\gamma}]$, under the assumption $p > 2$, implies the existence of a basis

$$U_0 = X_{-e_p}, \quad U_{i\alpha} = X_{-e_i+e_\alpha} + \sum_{\beta=p+1}^{p+q} k_\alpha^\beta X_{-e_i-e_\beta},$$

with skew-symmetric $k_\alpha^\beta = -k_\beta^\alpha$, for the current $G^{0,0}$-conjugate of $\mathscr{A}$.

Finally,

$$g_3 = \operatorname{Ad}\exp\left(\sum_{\beta<\gamma} k_\gamma^\beta X_{-e_\beta-e_\gamma}\right)$$

transforms the above U_0, $U_{i\alpha}$ into the generators of $\mathscr{A}_0$. Thus, any abelian subspace $\mathscr{A} \subset \mathscr{G}^{-1,1}$ of maximal dimension can be modified in its $\operatorname{Ad}(G^{0,0})$-orbit to the standard model $\mathscr{A}_0$. As far as the adjoint representation of $G^{0,0}$ acts by Lie algebra automorphisms, the entire $\operatorname{Ad}(G^{0,0})$-orbit of $\mathscr{A}_0$ consists of abelian subspaces $\mathscr{A} \subset \mathscr{G}^{-1,1}$ of $\dim \mathscr{A} = pq+1$, q.e.d. Proposition 1.

The $G^{0,0}$-conjugacy of the abelian subspaces $\mathscr{A} \subset \mathscr{G}^{-1,1}$ of $\dim \mathscr{A} = pq+1$ implies their $S(U_p \times O_{2q+1})$-conjugacy, according to the coincidence of the $\operatorname{Ad} G^{0,0}$- and $\operatorname{Ad} S(U_p + O_{2q+1})$-orbits of $\mathscr{A}_0$. Indeed, the stabilizer of $\mathscr{A}_0$ with respect to the adjoint action of $S(U_p \times O_{2q+1})$ is $S(U_{p-1} \times U_q)$, so that the $\operatorname{Ad} S(U_p \times O_{2q+1})$-orbit is of real dimension $2p - 1 + q^2 + q$. The $\operatorname{Ad} G^{0,0}$-orbit is of the same dimension, as far as the corresponding stabilizer is the parabolic subgroup of $G^{0,0}$, intersecting its opposite in $S(GL_{\mathbf{C}}(p-1) \times GL_{\mathbf{C}}(q))$. Bearing in mind that the $\operatorname{Ad} S(U_p \times O_{2q+1})$-orbit is contained in the $\operatorname{Ad} G^{0,0}$-orbit of $\mathscr{A}_0$, one concludes their coincidence.

In order to identify the holomorphic tangent space $T^{1,0}\tilde{f}(\widetilde{S})_{\tilde{o}}$ with the holomorphic tangent space to a domain Ω, one needs a global description of Ω as an open subset of a Q-isotropic Schubert cell. Recall that the generalized ball

$$B_{m,n} = \{Z \in \operatorname{Mat}_{m,n}(\mathbf{C}) \mid {}^t\overline{Z}Z < I_n\} = \{Z \in \operatorname{Mat}_{m,n}(\mathbf{C}) \mid Z\,{}^t\overline{Z} < I_m\}$$

can be regarded as the set of the m-planes $\lambda = (I_m\ Z)$ in $\mathbf{C}^{m+n}$, positive definite $\lambda\chi{}^t\overline{\lambda} > 0$ with respect to the indefinite Hermitian form $\chi = \begin{pmatrix} I_m & \\ & -I_n \end{pmatrix}$. Similarly, the points of

$$\Omega = \left\{ t \in B_{1,1},\ X \in \operatorname{Mat}_{p-1,q}(\mathbf{C}),\ Y \in \operatorname{Mat}_{1,q}(\mathbf{C}) \;\middle|\; \begin{pmatrix} X \\ \frac{Y}{1-|t|^2} \end{pmatrix} \in B_{p,q} \right\}$$

can be viewed as p-planes in $\mathbf{C}^{p+q+2}$, generated by the row-vectors of

$$\Lambda = \begin{pmatrix} I_{p-1} & 0 & X & 0 & 0 \\ 0 & 1 & Y & \sqrt{2}t & t^2 \end{pmatrix},$$

and positive definite with respect to the Hermitian form

$$h = \begin{pmatrix} I_{p-1} & & & & \\ & 1 & & & \\ & & -I_q & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix}.$$

Indeed, $(t, X, Y) \in \Omega$ exactly when $\Lambda h {}^t\overline{\Lambda} > 0$, i.e.,

$$\begin{pmatrix} I_{p-1} & \\ & 1-|t|^2 \end{pmatrix} \left[I_p - \begin{pmatrix} X \\ \frac{Y}{1-|t|^2} \end{pmatrix} \begin{pmatrix} {}^t\overline{X} & \frac{{}^t\overline{Y}}{1-|t|^2} \end{pmatrix} \right] \begin{pmatrix} I_{p-1} & \\ & 1-|t|^2 \end{pmatrix} > 0.$$

According to the explicit construction of Ω from [5], the polarization form Q restricts to the degenerate bilinear form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

on the ambient space $V \simeq \mathbf{C}^{p+q+2}$. In other words, V contains $H^{2,0}$, intersects $H^{1,1}$ in a sum $K + L$ of a maximal Q-isotropic subspace K and a line $L = (K + \overline{K})^\perp \cap H^{1,1}$, while $V \cap H^{0,2} = \overline{N}$ is a line in $H^{0,2}$.

By definition, the p-planes $\Lambda \in \Omega$ are Q-isotropic, i.e., $Q(x, y) = 0\, \forall x, y \in \Lambda$. If $E = \overline{N}^\perp \cap H^{2,0}$ is the Q-orthogonal to the $\overline{N}$ hyperplane of $H^{2,0}$, then the Q-orthogonal $V^\perp = E + K$, and any $\Lambda \in \Omega$ has a $(p-1)$-dimensional intersection with $V^\perp$. Conversely, any Q-isotropic and h-positive $\Lambda \in \mathrm{Grass}(p, V)$ with $\dim(\Lambda \cap V^\perp) = p - 1$ belongs to Ω, if it is in the connected component of the reference point $\Lambda = H^{2,0}$. Thus,

$$\Omega = \Omega(V) = \{\Lambda \in \mathrm{Grass}^Q(p, V) | \dim(\Lambda \cap V^\perp) = p-1,\ h|_\Lambda > 0\}_{\check{o}},$$

where $\mathrm{Grass}^Q(p, V) := \{\Lambda \in \mathrm{Grass}(p, V) \mid Q(x, y) = 0\ \forall x, y \in \Lambda\}$ denotes the Q-isotropic Grassmannian.

Theorem 1. *Let $\tilde{f}\colon \widetilde{S} \to D$ be a lifting of a variation of maximal dimension with associated Hodge numbers $h^{2,0} = p > 2$ and $h^{1,1} = 2q+1$. As an abelian Lie algebra, the tangent space $T_{\check{o}} := T^{1,0}\tilde{f}(\widetilde{S})_{\check{o}}$ at the reference point $\check{o} \in D$ is associated with an abelian complex Lie group $\exp(T_{\check{o}})$ and determines*

$$V = \exp(T_{\check{o}})H^{2,0} := \mathrm{Span}\left\{ \left(\mathrm{Id} + \tau + \frac{\tau^2}{2}\right)\phi \middle| \tau \in T_{\check{o}},\ \phi \in H^{2,0} \right\}.$$

Then the vector space V bears construction of a domain $\Omega(V)$, which is tangent to the variation at the reference point, i.e.,

$$T_{\check{o}} = T^{1,0}\Omega(V)_{\check{o}}.$$

Proof. The remark after Proposition 1 has established that the tangent space $T_{\check{o}}$ is of the form

$$g^{-1}\,\mathrm{Span}\{\psi_\alpha \otimes \phi^i + \overline{\phi_i} \otimes \overline{\psi^\alpha},\ \psi_{2q+1} \otimes \phi^p + \overline{\phi_p} \otimes \psi^{2q+1} \mid 1 \le i \le p,\ p+1 \le \alpha \le p+q\} g$$

for some $g \in S(U_p \times O_{2q+1})$. Therefore, one can express

$$\begin{aligned} V &= \exp(T_{\check{o}})H^{2,0} \\ &= H^{2,0} + g^{-1}\,\mathrm{Span}\{\psi_1, \dots, \psi_q\} + g^{-1}\,\mathrm{Span}\{\psi_{2q+1}\} + g^{-1}\,\mathrm{Span}\{\overline{\phi_p}\}. \end{aligned}$$

Observe that V contains $H^{2,0}$, intersects $H^{1,1}$ in a sum of a maximal Q-isotropic subspace $K := g^{-1}\,\mathrm{Span}\{\psi_1, \dots, \psi_q\}$ and a line

$$L := g^{-1}\,\mathrm{Span}\{\psi_{2q+1}\} = (K + \overline{K})^\perp \cap H^{1,1},$$

and overlaps with $H^{0,2}$ in a line

$$\overline{N} := V \cap H^{0,2} = g^{-1}\operatorname{Span}\{\overline{\phi_p}\}.$$

Thus, V admits construction of a domain $\Omega(V)$.

Moreover V, and therefore $T_{\check{o}}$, determine invariantly not only $\overline{N} \subset H^{0,2}$ and its Q-orthogonal $E := \overline{N}^{\perp} \cap H^{2,0}$, but $K = T_{\check{o}}(E)$ and $L = T_{\check{o}}(N)$, as well.

In terms of the above introduced subspaces of $H_{\mathbf{C}}$, the restriction of $T_{\check{o}}$ on $H^{2,0}$ produces an isomorphism

$$T_{\check{o}} \simeq \operatorname{Hom}(E, K) + \operatorname{Hom}(N, K + L).$$

On the other hand, $\Omega(V) \subset \operatorname{Grass}(p, V)$ induces an inclusion $T^{1,0}\Omega(V)_{\check{o}} \subset T^{1,0}\operatorname{Grass}(p, V)_{\check{o}} = \operatorname{Hom}(H^{2,0}, V/H^{2,0})$. By means of an h-orthogonal lifting, let us identify $T^{1,0}\operatorname{Grass}(p, V)_{\check{o}} = \operatorname{Hom}(H^{2,0}, K+L+\overline{N})$. The extraction of the $O(1)$-term from the defining equation of the Q-isotropic Grassmannian yields

$$\begin{aligned} T^{1,0}\operatorname{Grass}^{Q}(p, V)_{\check{o}} &= \operatorname{Hom}^{Q}(H^{2,0}, K + L + \overline{N}) \\ &:= \{\tau \in \operatorname{Hom}(H^{2,0}, K + L + \overline{N}) \mid Q(\tau(x), y) + Q(x, \tau(y)) = 0 \\ &\qquad\qquad \forall x, y \in H^{2,0}\}. \end{aligned}$$

Because of the total isotropy of $V^{\perp} = E + K$, the entire Grassmannian $\operatorname{Grass}(p-1, V^{\perp})$ is Q-isotropic and

$$B_{p-1,q}(V^{\perp}) = \{\lambda \in \operatorname{Grass}(p-1, V^{\perp}) | h|_{\lambda} > 0\}$$

embeds in $\Omega(V)$ as a zero section $t = 0$, $Y = 0$. Consequently, its holomorphic tangent space $T^{1,0}B_{p-1,q}(V^{\perp})_{\check{o}} = \operatorname{Hom}(E, K) \hookrightarrow T^{1,0}\Omega(V)_{\check{o}}$. However, the correspondence $\Lambda \mapsto \Lambda \cap V^{\perp}$ defines a surjective map $\Omega(V) \to B_{p-1,q}(V^{\perp})$, so that $\operatorname{Hom}(E, K)$ splits $T^{1,0}\Omega(V)_{\check{o}}$ into a direct sum

$$T^{1,0}\Omega(V)_{\check{o}} = \operatorname{Hom}(E, K) + (T^{1,0}\Omega(V)_{\check{o}} \cap \operatorname{Hom}^{Q}(N, K + L + \overline{N})).$$

For a line N, it is straightforward to show that $\operatorname{Hom}^{Q}(N, K + L + \overline{N}) = \operatorname{Hom}(N, K + L)$. Moreover, the entire $\operatorname{Hom}(N, K + L)$ is contained in $T^{1,0}\Omega(V)_{\check{o}}$, as far as any h-positive line in $N+K+L$ extends to a Q-isotropic and h-positive line in $N+K+L+\overline{N}$, which together with the reference point $E \in B_{p-1,q}(V^{\perp})$ generates some $\Lambda \in \Omega(V)$. Therefore,

$$T^{1,0}\Omega(V)_{\check{o}} = \operatorname{Hom}(E, K) + \operatorname{Hom}(N, K + L) = T_{\check{o}},$$

q.e.d. Theorem 1.

The above description will be extended to the entire tangent bundle $T := T^{1,0}\tilde{f}(\widetilde{S})$. For an arbitrary $s \in \tilde{f}(\widetilde{S}) \subset D$, let us choose a representative $g \in SO(2p, 2q+1)$ of the left coset class

$$s = g\check{o} \in D = SO(2p, 2q+1)/S(U_p \times O_{2q+1}).$$

Then the left translation $g^{-1}: \tilde{f}(\widetilde{S}) \to g^{-1}\tilde{f}(\widetilde{S})$ induces a linear isomorphism $g_*^{-1}: T^{1,0}\tilde{f}(\widetilde{S})_s \to T^{1,0}g^{-1}\tilde{f}(\widetilde{S})_{\check{o}}$. According to the Theorem 1, the vector space $V_g := \exp(T^{1,0}g^{-1}\tilde{f}(\widetilde{S})_{\check{o}})H^{2,0}$ bears construction of a domain Ω with

$T^{1,0}\Omega(V_g)_{\check{o}} = T^{1,0}g^{-1}\tilde{f}(\tilde{S})_{\check{o}}$. It is straightforward that the left translate $g\Omega(V_g)$ of such a domain coincides with the domain $\Omega(gV_g)$, constructed on the left translate gV_g of the ambient space. Thus

$$g_*^{-1}T_s = T^{1,0}g^{-1}g\Omega(V_g)_{\check{o}} = g_*^{-1}T^{1,0}\Omega(gV_g)_s,$$

whereas $T_s = T^{1,0}\Omega(gV_g)_s$ for any $s \in \tilde{f}(\tilde{S})$. As far as the exponential map commutes with the conjugation by $g \in SO(2p, 2q+1)$, and the action of T_s on $H_s^{2,0}$ is invariant under the left translation by g^{-1}, one can identify

$$\begin{aligned} gV_g &= g\exp(T^{1,0}g^{-1}\tilde{f}(\tilde{S})_{\check{o}})g^{-1}(gH^{2,0}) = \exp(gT^{1,0}g^{-1}\tilde{f}(\tilde{S})_{\check{o}}g^{-1})H_s^{2,0} \\ &= \exp(gg^{-1}T^{1,0}\tilde{f}(\tilde{S})_s)H_s^{2,0} = \exp(T^{1,0}\tilde{f}(\tilde{S})_s)H_s^{2,0}. \end{aligned}$$

Consequently, $T_s = T^{1,0}\Omega(\exp(T_s)H_s^{2,0})_s$ and we have established

Corollary 1. *If $\tilde{f}\colon \tilde{S} \to D$ is a lifting of a variation of maximal dimension with associated $h^{2,0} = p > 2$ and $h^{1,1} = 2q+1$, then the holomorphic tangent bundle $T := T^{1,0}\tilde{f}(\tilde{S})$ determines a vector bundle $\mathscr{V} := \exp(T)\mathbf{H}^{2,0}$, bearing construction of field of domains $\Omega(\mathscr{V}) \to \tilde{f}(\tilde{S})$, such that*

$$T_s = T^{1,0}\Omega(\mathscr{V}_s)_s \quad \forall s \in \tilde{f}(\tilde{S}).$$

2. The domain Ω uniformizes the liftings of the variations of maximal dimension

The goal of the present section is to integrate the description of the holomorphic tangent bundle to a description of the lifting itself.

Theorem 2. *An arbitrary variation of Hodge structure $f\colon S \to \Gamma\backslash D$, $f(0) = \check{o}$ of maximal dimension with associated $h^{2,0} = p > 2$ and $h^{1,1} = 2q+1$ factors through a quotient $\Gamma'\backslash\Omega$ of the domain*

$$\Omega = \{\Lambda \in \mathrm{Grass}^Q(p, V) \mid \dim(\Lambda \cap V^\perp) = p-1,\, h|_\Lambda > 0\}_{\check{o}}$$

with $V = \exp(T^{1,0}f(S)_{\check{o}})H^{2,0}$. Namely,

$$f\colon S \to \Gamma'\backslash\Omega \to \Gamma\backslash D$$

is a composition of a locally liftable holomorphic map $g\colon S \to \Gamma'\backslash\Omega$ and an immersion $\Gamma'\backslash\Omega \to \Gamma\backslash D$, where Γ' is the normalizer of Ω in Γ.

If the vector bundle $\mathscr{V}$ from Corollary 1 is constant, then the tangent bundle to the lifting of the variation is contained in the tangent bundle to the domain Ω, constructed on $\mathscr{V}_{\check{o}}$. In fact, it suffices for the bundle $\mathscr{V}$ to be constant on a neighborhood of the reference point in $f(S)$. Then the coincidence $\mathscr{V}_s = \mathscr{V}_{\check{o}}$ follows after connecting $s \in \tilde{f}(\tilde{S})$ with $\check{o}$ by a chain of regular neighborhoods for the covering $\tilde{f}(\tilde{S}) \to f(S)$, pushing $\mathscr{V}$ down to $f(S)$ and applying several times the local statement.

Thus, if the pull-back $f^{-1}(\exp(T^{1,0}f(S))\mathbf{H}^{2,0})$ is locally constant around the origin $0 \in S$ (local rigidity), then the lifting $\tilde{f}\colon \tilde{S} \to \Omega \subset D$ of a variation of maximal dimension factors through the domain Ω.

Under the above mentioned circumstances, observe that the monodromy representation $\rho\colon \pi_1(S) \to \Gamma$ maps into $\Gamma' := \Gamma \cap \mathrm{Aut}\,\Omega$. To this end, it suffices to check that $\gamma\Omega \subseteq \Omega$ for any $\gamma \in \rho(\pi_1(S))$. The points $\tilde{f}(s)$ and $\gamma\tilde{f}(s) = \tilde{f}(\rho^{-1}(\gamma)s)$ from $\tilde{f}(\tilde{S}) \subseteq \Omega$ are contained in Ω together with their neighborhoods $\mathscr{U}_1$, respectively $\mathscr{U}_2$. That is why the open subset $\mathscr{U} := \mathscr{U}_1 \cap \gamma^{-1}(\mathscr{U}_2)$ of Ω is mapped biholomorphically onto $\gamma(\mathscr{U}) \subset \Omega$. Connecting $\omega \in \Omega$ with $\tilde{f}(s)$ by a chain of such consecutively overlapping neighborhoods, one concludes that $\gamma(\omega) \in \Omega$. Thus, the lifting $\tilde{f}: \tilde{S} \to \Omega$ and the monodromy representation $\rho: \pi_1(S) \to \Gamma'$ in $\operatorname{Aut}\Omega$ produce a locally liftable holomorphic map $g: S \to \Gamma'\backslash\Omega$. Obviously, the free and properly discontinuous action of Γ on D restricts to a free and properly discontinuous action of Γ' on Ω, so that the inclusion $\Omega \subset D$ induces an immersion of the manifold $\Gamma'\backslash\Omega$ into the manifold $\Gamma\backslash D$.

The proof of Theorem 2 will be completed by deriving the local rigidity from the properties of the Gauss-Manin connection, corresponding to a variation of maximal dimension.

To this end, let us start with some general relations among the second fundamental forms of chains of subbundles.

Lemma 2. *Let $\nabla: \mathbf{A}^0(E) \to \mathbf{A}^1(E)$ be a flat metric connection on the Hermitian vector bundle E. Given subbundles $E_1 \subset E_2 \subset E$ with second fundamental forms*

$$\sigma_1: \mathbf{A}^0(E_1) \to \mathbf{A}^1(E_2/E_1), \quad \sigma_2: \mathbf{A}^0(E_2) \to \mathbf{A}^1(E/E_2),$$

let us decompose $E_2 = E_1 + (E_1^{\perp} \cap E_2)$ by the means of the Hermitian orthogonal complement $E_1^{\perp}$ and represent accordingly $\sigma_2 = \sigma_2' + \sigma_2''$, where

$$\sigma_2': \mathbf{A}^0(E_1) \to \mathbf{A}^1(E/E_2), \quad \sigma_2'': \mathbf{A}^0(E_1^{\perp} \cap E_2) \to \mathbf{A}^1(E/E_2).$$

Then for an arbitrary sheaf of differential ideals $\mathscr{I} \subseteq \mathbf{A}^*$, there hold:

(i) *If $\sigma_2' \in \mathscr{I}$, then $\sigma_2'' \wedge \sigma_1 \in \mathscr{I}$.*

(ii) *If $\sigma_2'' \in \mathscr{I}$, then $\sigma_2' \wedge {}^t\overline{\sigma_1} \in \mathscr{I}$.*

Proof. A metric compatible connection ∇ can be given by a skew-Hermitian matrix of 1-forms $(\theta_j^i)_{i,j=1}^n$, specifying the covariant derivatives $\nabla e_j = e_i \theta_j^i$ of a unitary frame $e_1, \ldots, e_n$ for E. With respect to a union of unitary frames for E_1, $E_1^{\perp} \cap E_2$ and $E_2^{\perp}$, the holomorphic metric connection of E has a matrix

$$\theta = \begin{pmatrix} \theta_1 & -{}^t\overline{\sigma_1} & -{}^t\overline{\sigma_2'} \\ \sigma_1 & {}_1\theta & -{}^t\overline{\sigma_2''} \\ \sigma_2' & \sigma_2'' & {}_2\theta \end{pmatrix},$$

where θ_1, ${}_1\theta$ and ${}_2\theta$ stand for the holomorphic metric connections of E_1, $E_1^{\perp} \cap E_2$ and $E_2^{\perp}$, respectively. The vanishing of the entry from the third row and the first column of the flatness equation $\nabla^2 = d\theta + \theta \wedge \theta = 0$ yields the assertion (i). The annihilation of the entry $(3, 2)$ of the matrix ∇^2 implies the statement (ii), q.e.d. Lemma 2.

Recall from [11], [12] that the variations of Hodge structure $f: S \to \Gamma\backslash D$, $f(0) = \check{o}$, are in one-to-one correspondence with the so-called Gauss-Manin connections

$$\nabla: \mathbf{A}^0(\mathbf{H}) \to \mathbf{A}^1(\mathbf{H})$$

on the Hodge bundle $\mathbf{H} := f^{-1}(\mathbf{H}_{\mathbf{Z}} \otimes \mathbf{C})$. These are flat $\nabla^2 = 0$, horizontal

$$\nabla: \mathbf{A}^0(\mathbf{H}^{2,0}) \to \mathbf{A}^1(\mathbf{H}^{2,0} + \mathbf{H}^{-1,1}),$$

and compatible with the Hermitian form h and the conjugation of $\mathbf{H}$. Moreover, appropriate restrictions of the infinitesimal variation $(f_*)_0 \in \operatorname{Hom}(T^{1,0}S_0, \mathscr{G}^{-1,1}) = \Omega_S^{1,0}(\mathscr{G}^{-1,1})_0$ coincide with the second fundamental forms of $\mathbf{H}^{p,q}$ with respect to ∇.

For a variation of maximal dimension $f: S \to \Gamma\backslash D$, $f(0) = \check{o}$ with associated $h^{2,0} = p > 2$ and $h^{1,1} = 2q+1$, Theorem 1 has established the existence of an adapted Hodge frame, i.e., $S(U_p \times O_{2q+1})$-conjugation of $H_{\mathbf{C}}$, such that

$$\{\psi_\alpha \otimes \phi^i + \overline{\phi_i} \otimes \overline{\psi^\alpha},\ \psi_{2q+1} \otimes \phi^p + \overline{\phi_p} \otimes \psi^{2q+1} \mid 1 \leq i \leq p,\ p+1 \leq \alpha \leq p+q\}$$

span the tangent space $T^{1,0}f(S)_{\check{o}}$ at the reference point. Suppose that s^0, $s^{i\alpha}$ with $1 \leq i \leq p$, $p+1 \leq \alpha \leq p+q$ form (eventually a part of) a local coordinate system on S around $0 \in S$. Let us denote, as before, $E := \operatorname{Span}\{\phi_1, \ldots, \phi_{p-1}\}$, $N := \operatorname{Span}\{\phi_p\}$, $K := \operatorname{Span}\{\psi_1, \ldots, \psi_q\}$, $L := \operatorname{Span}\{\psi_{2q+1}\}$ and $\overline{N} := \operatorname{Span}\{\overline{\phi_p}\}$. Then the infinitesimal variation $\tau := (f_*)_0 \in \Omega_S^{1,0}(\mathscr{G}^{-1,1})_0$ decomposes into a sum $\tau = \tau_0 + \tau_1 + \tau_2$, where

$$\tau_0 := ds^0(\psi_{2q+1} \otimes \phi^p + \overline{\phi_p} \otimes \psi^{2q+1}) \in \mathbf{A}^1(\operatorname{Hom}(N, L) + \operatorname{Hom}(L, \overline{N})),$$

$$\tau_1 := \sum_{i=1}^{p-1} \sum_{\alpha=p+1}^{p+q} ds^{i\alpha}(\psi_\alpha \otimes \phi^i + \overline{\phi_i} \otimes \overline{\psi^\alpha}) \in \mathbf{A}^1(\operatorname{Hom}(E, K) + \operatorname{Hom}(\overline{K}, \overline{E})),$$

$$\tau_2 := \sum_{\alpha=p+1}^{p+q} ds^{p\alpha}(\psi_\alpha \otimes \phi^p + \overline{\phi_p} \otimes \overline{\psi^\alpha}) \in \mathbf{A}^1(\operatorname{Hom}(N, K) + \operatorname{Hom}(\overline{K}, \overline{N})).$$

Comparing the coefficients of the endomorphism-valued differential forms, one checks straightforward

Lemma 3. *For the infinitesimal variation $\tau \in \Omega_S^{1,0}(T^{1,0}f(S)_{\check{o}})$, derived from a variation of maximal dimension with associated $h^{2,0} = p > 2$ and $h^{1,1} = 2q+1$, there hold the following implications*:

(a) *If $\rho \wedge \tau_1 = 0$ for $\rho \in \Omega_S^{1,0}(\operatorname{Hom}(K, \overline{K}))$, then $\rho = 0$.*

(b) *If $\tau_0 \wedge \rho \in I$ for the differential ideal*

$$I := \langle ds^{i\alpha} \operatorname{Hom}(H_{\mathbf{C}}, H_{\mathbf{C}}) \mid 1 \leq i \leq p,\ p+1 \leq \alpha \leq p+q\rangle$$

given by its generators and $\rho \in \Omega_S^{0,1}(\operatorname{Hom}(\overline{K}, L))$, then $\rho = 0$.

(c) *If $\rho \wedge \tau_0 = 0$ for $\rho \in \Omega_S^{1,0}(\operatorname{Hom}(\overline{N}, \overline{E}))$, and $I_0 := \langle ds^0 \operatorname{Hom}(H_{\mathbf{C}}, H_{\mathbf{C}})\rangle$ is the differential ideal generated by ds^0, then $\rho \in I_0$.*

(d) *If $\rho \wedge \tau_2 \in I_1$ for the differential ideal*

$$I_1 := \langle ds^{i\alpha} \operatorname{Hom}(H_{\mathbf{C}}, H_{\mathbf{C}}) \mid 1 \leq i \leq p-1,\ p+1 \leq \alpha \leq p+q\rangle$$

and $\rho \in \Omega_S^{1,0}(\operatorname{Hom}(\overline{N}, \overline{E}))$, then $\rho \in I_1$.

The proof of (a) and (d) relies strongly on the assumption $p > 2$.

Now we are ready to justify the local rigidity.

Proposition 2. *Let $\mathbf{H} \to S$ be the Hodge bundle, corresponding to the variation of maximal dimension $f: S \to \Gamma\backslash D$ with $h^{2,0} = p > 2$ and $h^{1,1} = 2q+1$. The pull-back $T := f^{-1}T^{1,0}f(S)$ constitutes a field of abelian Lie algebras, associated with the field of Lie groups $\exp(T) \to S$. Then the vector bundle $\mathscr{V} := \exp(T)\mathbf{H}^{2,0}$, generated by the $\exp(T)$-images of $\mathbf{H}^{2,0}$, is locally constant* *around* $0 \in S$. *In other words,* $\mathscr{V}$ *is a locally flat, locally geodesic subbundle of* $\mathbf{H}$ *with respect to the Gauss-Manin connection.*

Proof. One needs to establish the vanishing of the second fundamental form

$$\sigma_{\mathscr{V}/\mathbf{H}}: \mathbf{A}^0(\mathscr{V}) \to \mathbf{A}^1(\mathbf{H}/\mathscr{V})$$

at $0 \in S$. Similarly to Carlson and Hernandez's treatment from [4], let us decompose the Gauss-Manin connection $\nabla = \widetilde{\nabla} + \tau$ into a sum of the infinitesimal variation τ and an isotropy connection

$$\widetilde{\nabla}: \mathbf{A}^0(\mathbf{H}^{p,q}) \to \mathbf{A}^1(\mathbf{H}^{p,q}).$$

According to Theorem 1, $\mathscr{V}_0 = H^{2,0} + K + L + \overline{N}$ is invariant $\tau(\mathscr{V}_0) \subset \mathscr{V}_0$ under the infinitesimal variation. Therefore $\sigma_{\mathscr{V}/\mathbf{H}}$ can be calculated with respect to $\widetilde{\nabla}$.

The first aim is to specify $\widetilde{\nabla}$ in terms of the second fundamental forms of some invariantly defined subbundles of $\mathbf{H}^{p,q}$, $p+q=2$. Let us identify the bundles with their associated sheaves of sections, and consider

$$\operatorname{Ker}(T^2) \cap \mathbf{H}^{2,0} := \{\phi \in \mathbf{H}^{2,0} \mid \tau_2\tau_1(\phi) = 0 \ \forall \tau_1, \tau_2 \in T\}.$$

Applying twice the infinitesimal Q-isotropy of T, observe that

$$\operatorname{Ker}(T^2) \cap \mathbf{H}^{2,0} = (\operatorname{Im} T^2)^{\perp} \cap \mathbf{H}^{2,0}.$$

If $\overline{\mathbf{N}} := \mathscr{V} \cap \mathbf{H}^{0,2}$ and $\mathbf{E} := \overline{\mathbf{N}}^{\perp} \cap \mathbf{H}^{2,0}$, then the explicit knowledge of $T^2 = \operatorname{Hom}(\mathbf{N}, \mathbf{H}^{2,0} + \mathbf{H}^{1,1} + \overline{\mathbf{N}}/\mathbf{H}^{2,0} + \mathbf{H}^{1,1})$ reveals that $(\operatorname{Im} T^2)^{\perp} \cap \mathbf{H}^{2,0} = \mathbf{E}$. Due to the holomorphy of $T \subset \operatorname{Hom}(\mathbf{H}^{2,0}, \mathbf{H}^{2,0} + \mathbf{H}^{1,1}/\mathbf{H}^{2,0})$ and $\mathbf{H}^{2,0}$, the subbundle $\mathbf{E} \subset \mathbf{H}^{2,0}$ is holomorphic as well as the subbundles $T(\mathbf{E}) \subset T(\mathbf{H}^{2,0})$ of $\mathbf{H}^{2,0} + \mathbf{H}^{1,1}/\mathbf{H}^{2,0}$ and $T^2(\mathbf{H}^{2,0})$ of $\mathbf{H}/\mathbf{H}^{2,0} + \mathbf{H}^{1,1}$.

By means of h-orthogonal complements, the restriction of $T(\mathbf{E}) \subset T(\mathbf{H}^{2,0}) \subset \mathbf{H}^{2,0} + \mathbf{H}^{1,1}/\mathbf{H}^{2,0}$ at $0 \in S$ can be identified with the filtration $K \subset K + L \subset H^{1,1}$. Then the skew-Hermitian matrix of $\widetilde{\nabla}|_0: \mathbf{A}^0(H^{1,1}) \to \mathbf{A}^1(H^{1,1})$ takes the form

$$\begin{pmatrix} * & * & -{}^t\overline{\rho_1} \\ * & * & -{}^t\overline{\rho_0} \\ \rho_1 & \rho_0 & * \end{pmatrix},$$

where $\rho_1 \in \Omega_S^{1,0}(\operatorname{Hom}(K, \overline{K}))$, $\rho_0 \in \Omega_S^{1,0}(\operatorname{Hom}(L, \overline{K}))$.

With respect to the filtration $T^2(\mathbf{H}^{2,0})|_0 = \overline{N} \subset H^{0,2}$, the isotropy connection $\widetilde{\nabla}|_0: \mathbf{A}^0(H^{0,2}) \to \mathbf{A}^1(H^{0,2})$ is represented by

$$\begin{pmatrix} * & -{}^t\overline{\rho_2} \\ \rho_2 & * \end{pmatrix}$$

for $\rho_2 \in \Omega_S^{1,0}(\operatorname{Hom}(\overline{N}, \overline{E}))$.

Thus, in terms of h-orthogonal liftings, the second fundamental form

$$\sigma_{\mathscr{V}/\mathbf{H}}|_0: \mathbf{A}^0(H^{2,0} + K + L + \overline{N}) \to \mathbf{A}^1(\overline{K} + \overline{E})$$

splits into a sum $\sigma_{\mathscr{V}/\mathbf{H}}|_0 = \rho_0 + \rho_1 + \rho_2$.

In order to show the vanishing of ρ_1, let us apply Lemma 2(i) to the Gauss-Manin connection on $\mathbf{H}$ and the chain of subbundles

$$\mathbf{H}^{2,0} \subset \mathbf{H}^{2,0} + T(\mathbf{H}^{2,0}) \subset \mathbf{H}.$$

As far as $\sigma_2'|_0 = 0, \sigma_1|_0 = \tau|_{H^{2,0}}, \sigma_2''|_0 = \tau_0|_L + \rho_0 + \rho_1$, it follows $\sigma_2'' \wedge \sigma_1|_{H^{2,0}} = 0$ and, in particular, $\sigma_2'' \wedge \sigma_1|_E = \rho_1 \wedge \tau_1 = 0$. Then Lemma 3(a) reveals that $\rho_1 = 0$.

For the vanishing of ρ_0 consider the second fundamental forms of

$$\mathbf{H}^{2,0} + T(\mathbf{H}^{2,0}) \subset \mathbf{H}^{2,0} + \mathbf{H}^{1,1} \subset \mathbf{H}$$

with respect to the Gauss-Manin connection. Lemma 2(ii) with $\sigma_1|_0 = \rho_0$, $\sigma_2'|_0 = \tau_0|_L$, $\sigma_2''|_0 = (\tau_1 + \tau_2)|_{\overline{K}} \in I$ forces $\sigma_2' \wedge {}^t\overline{\sigma_1}|_0 = \tau_0 \wedge {}^t\overline{\rho_0} \in I$, whereas $\rho_0 = 0$, according to Lemma 3(b).

In order to examine ρ_2, let us make use of Lemma 2(i) for the chain of subbundles

$$\mathbf{H}^{2,0} + T(\mathbf{H}^{2,0}) \subset \mathscr{V} + \mathbf{H}^{1,1} \subset \mathbf{H}.$$

As far as $\sigma_1|_0 = \tau_0|_L$, $\sigma_2'|_0 = 0$ and $\sigma_2''|_0 = \tau_1|_{\overline{K}} + \rho_2$, it follows $\sigma_2'' \wedge \sigma_1|_0 = \rho_2 \wedge \tau_0|_L = 0$. Lemma 3(c) asserts that it is sufficient for the conclusion $\rho_2 \in I_0$.

For the complete annihilation of ρ_2, one applies Lemma 2(i) to the subbundles

$$\mathbf{H}^{2,0} + \mathbf{H}^{1,1} \subset \mathscr{V} + \mathbf{H}^{1,1} \subset \mathbf{H}.$$

More precisely, from $\sigma_1|_0 = \tau_0|_L + \tau_2|_{\overline{K}}$, $\sigma_2'|_0 = \tau_1|_{\overline{K}} \in I_1$, $\sigma_2''|_0 = \rho_2$, it follows that $\sigma_2'' \wedge \sigma_1|_0 = \rho_2 \wedge \tau_2|_{\overline{K}} \in I_1$. Then Lemma 3(d) specifies that $\rho_2 \in I_1$, i.e., $\rho_2 \in I_0 \cap I_1$, which for the 1-form ρ_2 is equivalent to $\rho_2 = 0$. That concludes the proof of the local rigidity $\sigma_{\mathscr{V}/\mathbf{H}}|_0 = 0$, q.e.d. Proposition 2.

3. The automorphism group of Ω

In order to guess a subgroup G of the effectively acting biholomorphic automorphisms $\operatorname{Aut}\Omega$ of Ω, let us represent

$$\Omega = \{t \in B_{1,1},\ X \in B_{p-1,q},\ Y \in \mathrm{Mat}_{1,q}(\mathbf{C}) \mid {}^t\overline{Y}Y < (1-|t|^2)^2(I_q - {}^t\overline{X}X)\},$$

where the generalized balls are regarded in Cartan's realization

$$B_{m,n} = \{Z \in \mathrm{Mat}_{m,n}(\mathbf{C}) \mid {}^t\overline{Z}Z < I_n\}.$$

Under the presence of a fibering

$$\varphi\colon \Omega \to B_{1,1} \times B_{p-1,q}, \quad \varphi(t, X, Y) = (t, X),$$

one looks for an extension of the automorphism group of $B_{1,1} \times B_{p-1,q}$ to the entire Ω. Recall that $g = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in U(m, n)$ acts by fractional-linear transformations $g(Z) = (AZ+B)(CZ+D)^{-1}$ on $B_{m,n}$, and the matrix-valued function $I_n - {}^t\overline{Z}Z$ behaves "automorphically", i.e.,

$$I_n - {}^t\overline{g(Z)}g(Z) = {}^t\overline{(CZ+D)^{-1}}(I_n - {}^t\overline{Z}Z)(CZ+D)^{-1}.$$

Then for any $(t, X, Y) \in \Omega$ and

$$(g_1, g_2) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) \in U(1,1) \times U(p-1, q),$$

introduce an action

$$(g_1, g_2)(t, X, Y) := ((at+b)(ct+d)^{-1}, (AX+B)(CX+D)^{-1}, Y(CX+D)^{-1}(ct+d)^{-2}).$$

Observe that the ineffective kernel $\{(e^{i\theta}I_2, e^{-2i\theta}I_{p-1+q}) \in U(1,1) \times U(p-1,q)\}$ is isomorphic to U_1 and denote by $G := S(U(1,1)\times U(p-1,q))$ the automorphisms of $B_{1,1}\times B_{p-1,q}$ which act effectively on Ω. It will be established that the entire $\operatorname{Aut}\Omega$ is depleted by G.

Following Pyatetskii-Shapiro's ideas for studying the bounded symmetric domains (cf. [20]), let us classify the analytic boundary components of Ω up to the action of G. Certainly, the notions of an analytic closure $\overline{\Omega}$ and a boundary $\partial\Omega = \overline{\Omega}\backslash\Omega$ depend on the particular realization of Ω. By virtue of Theorem 1, one can interpret $t \in \operatorname{Hom}(N, L)$, $X \in \operatorname{Hom}(E, K)$, $Y \in \operatorname{Hom}(N, K)$, so that the aforementioned realization of Ω takes the form of a bounded circular domain in its tangent space $T^{1,0}\Omega_{\check{o}}$ at the reference point, i.e., $\Omega = \{\Lambda = (t, X, Y) \in T^{1,0}\Omega_{\check{o}} | h|_\Lambda > 0\}_{\check{o}}$. The resemblance with the Harish-Chandra realization of a bounded symmetric domain justifies the efforts for describing the analytic structure of

$$\partial\Omega = \{t, X, Y \mid |t| \leq 1,\ {}^t\overline{X}X \leq I_q,\ {}^t\overline{Y}Y \leq (1-|t|^2)^2(I_q - {}^t\overline{X}X)$$
$$\text{with at least one of the equalities } |t| = 1,\ \det({}^t\overline{X}X - I_q) = 0 \text{ or}$$
$$\det[{}^t\overline{Y}Y - (1-|t|^2)^2(I_q - {}^t\overline{X}X)] = 0\}.$$

Recall from [20] that an analytic subset $\mathscr{F} \subset \partial\Omega$ is said to be a boundary component if it is piecewise analytically connected and any analytic curve $C(s)$, $|s| < \varepsilon$, contained in $\partial\Omega$ and intersecting $\mathscr{F}$ stays entirely in $\mathscr{F}$.

Let us consider the distinguished section $\mathscr{B} := \Omega \cap \{Y = 0\}$, isomorphic to the base of the fibering $\Omega \to B_{1,1}\times B_{p-1,q}$. According to Pyatetskii-Shapiro (cf. [20]),

$$B^r := \left\{t = 1,\ \begin{pmatrix} I_r & 0 \\ 0 & X' \end{pmatrix} \in \operatorname{Mat}_{p-1,q}(\mathbf{C}),\ Y = 0 \mid {}^t\overline{X'}X' < I_{q-r}\right\} \simeq B_{p-1-r,q-r}$$

for $0 \leq r \leq \min(p-1, q)$ constitute analytic components of $\partial\mathscr{B} = \partial(B_{1,1}\times B_{p-1,q}\times 0)$. Suppose that B^r extends to a boundary component $\mathscr{F}$ of Ω. Then the projection of $\mathscr{F}$ in the closure

$$\overline{\mathscr{B}} = \overline{B_{1,1}} \times \overline{B_{p-1,q}} \times 0 = \{t \in \mathbf{C},\ X \in \operatorname{Mat}_{p-1,q}(\mathbf{C}) \mid |t| \leq 1,\ {}^t\overline{X}X \leq I_q\}$$

coincides with B^r, according to the analytic maximality of B^r in $\mathscr{B}$. Furthermore, the inequality ${}^t\overline{Y}Y \leq (1-|t|^2)^2(I_q - {}^t\overline{X}X)$ for $(1, X, Y) \in \partial\Omega$ forces $Y = 0$. Consequently, $\mathscr{F} \subset \partial\Omega \cap \{Y = 0\} = \partial\mathscr{B}$ and $B^r = \mathscr{F}$ turns out to be an analytic boundary component of the domain Ω.

We claim that

$$\Omega^r := \left\{t \in B_{1,1}\ \begin{pmatrix} I_r & 0 \\ 0 & X' \end{pmatrix} \in \operatorname{Mat}_{p-1,q}(\mathbf{C}),\ (0 \quad Y') \in \operatorname{Mat}_{1,q}(\mathbf{C}) \mid \right.$$
$$\left. {}^t\overline{X'}X' < I_{q-r},\ {}^t\overline{Y'}Y' < (1-|t|^2)^2(I_{q-r} - {}^t\overline{X'}X')\right\}$$

for $1 \leq r \leq \min(p-1, q)$, where in the case of $p-1 < q$

$$\Omega^{p-1} := \{t \in B_{1,1},\ (I_{p-1} \quad 0) \in \operatorname{Mat}_{p-1,q}(\mathbf{C}),\ (0 \quad Y') \in \operatorname{Mat}_{1,q}(\mathbf{C}) \mid$$
$$t\overline{Y'}Y' < (1-|t|^2)^2 I_{q-p+1}\}$$

are also analytic components of $\partial\Omega$. Observe that Ω^r are analytically equivalent to the domains Ω with sizes $p-r$, $q-r$. The analytic connectedness of a domain Ω follows from the analytic connectedness of the generalized balls and the presence of a projection $\varphi\colon \Omega \to \mathscr{B}$ whose fibers $\varphi^{-1}(t, X, 0)$ are isomorphic to $B_{1,q}$. For the analytic maximality, assume that $\{(t(s), X(s), Y(s)) \in \partial\Omega \,|\, |s| < \varepsilon\}$ in an arc, intersecting Ω^r at $s = 0$. If $t(0) \in B_{1,1}$, and

$$X(0) \in B^r_{p-1,q} := \left\{ \begin{pmatrix} I_r & 0 \\ 0 & X' \end{pmatrix} \in \mathrm{Mat}_{p-1,q}(\mathbf{C}) \mid {}^t\overline{X'}X' < I_{q-r} \right\},$$

then according to the analytic maximality of the unit disk $B_{1,1}$ and the canonical component $B^r_{p-1,q} \subset \partial B_{p-1,q}$, the entire curves $t(s) \in B_{1,1}$ and

$$X(s) = \begin{pmatrix} I_r & 0 \\ 0 & X'(s) \end{pmatrix} \in B^r_{p-1,q}$$

stay in these sets. Let us represent $Y(s) = (Y''(s) \quad Y'(s))$ with $Y''(s) \in \mathrm{Mat}_{1,r}(\mathbf{C})$, $Y'(s) \in \mathrm{Mat}_{1,q-r}(\mathbf{C})$. Then the matrix-valued inequality ${}^t\overline{Y(s)}Y(s) \le (1-|t(s)|^2)^2(I_q - {}^t\overline{X(s)}X(s))$ implies the vanishing $Y''(s) \equiv 0$ and the semidefiniteness condition

$${}^t\overline{Y'(s)}Y'(s) \le (1 - |t(s)|^2)^2(I_{q-r} - {}^t\overline{X'(s)}X'(s)).$$

The last inequality is strict for all s, $|s| < \varepsilon$, if it is strict for $s = 0$. Assume the opposite, i.e., $\det[{}^t\overline{Y'(s_0)}Y'(s_0) - (1 - |t(s_0)|^2)^2(I_{q-r} - {}^t\overline{X'(s_0)}X'(s_0))] = 0$ for some s_0, $|s_0| < \varepsilon$. Choose $g = (g_1, g_2) \in S(U(1,1) \times U(p-1-r, q-r))$ such that $g_1(t(s_0)) = 0 \in B_{1,1}$, $g_2(X'(s_0)) = 0 \in B_{p-1-r,q-r}$, and transform the analytic arc $T(s) = (t(s), X'(s), Y'(s))$ into the analytic arc $T_1(s) = (t_1(s), X_1'(s), Y_1'(s))$, where

$$\begin{aligned} t_1(s) &= g_1(t(s)) = (at(s) + b)(ct(s) + d)^{-1}, \\ X_1'(s) &= g_2(X'(s)) = (AX'(s) + B)(CX'(s) + D)^{-1}, \\ Y_1'(s) &= Y'(s)(CX'(s) + D)^{-1}(ct(s) + d)^{-2}. \end{aligned}$$

As far as g leaves invariant the defining inequalities of the domain Ω of sizes $p - r$ and $q - r$, there holds ${}^t\overline{Y_1'(s)}Y_1'(s) \le (1 - |t_1(s)|^2)^2(I_{q-r} - {}^t\overline{X_1'(s)}X_1'(s))$, whose strict satisfaction is simultaneous with the corresponding inequality for $T(s)$. Thus, $\det[{}^t\overline{Y_1'(s_0)}Y_1'(s_0) - I_{q-r}] = 0$, so that $Y_1'(s_0) \in \partial B_{1,q-r}$ is disconnected from any other point of the closed ball

$$\overline{B_{1,q-r}} = \{Z \in \mathrm{Mat}_{1,q-r}(\mathbf{C}) | {}^t\overline{Z}Z \le I_{q-r}\}.$$

However, the arc $Y_1'(s)$, contained in $\overline{B_{1,q-r}}$, starts from $Y_1'(0) \in B_{1,q-r}$. The contradiction reveals that any analytic arc from $\partial\Omega$, intersecting Ω^r, is contained entirely in Ω^r.

Observe that the fractional-linear action of G on Ω has poles in the exterior of $\partial\Omega$, so that extends biholomorphically over $\partial\Omega$. All the analytic boundary components of Ω are claimed to be G-equivalent to some of the above listed types, or to points.

Proposition 3. *For any analytic boundary component $\mathscr{F} \subset \partial\Omega$ there is a biholomorphic automorphism $g \in G = S(U(1,1) \times U(p-1,q))$ of the domain Ω,* *such that* $g\mathscr{F}$ *is among the following canonical boundary components*:

$$B^r \quad \textit{for } 0 \le r \le \min(p-1, q),$$
$$\Omega^r \quad \textit{for } 1 \le r \le \min(p-1, q), \ \textit{or}$$
$$\left(0, \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, (0, \dots, 0, 1)\right) \quad \textit{for } 0 \le r \le \min(p-1, q).$$

Proof. If $Y \equiv 0$ on the analytic component $\mathscr{F} \subset \partial\Omega$, then $\mathscr{F} \subset \partial\mathscr{B}$. Suppose that $\mathscr{F}$ contains a point $P_1(t_1, X_1, 0)$ with $|t_1| < 1$. According to [20], there exists some $g_2 \in SU(p-1, q)$ such that

$$g_2(X_1) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

for some $1 \le r \le \min(p-1, q)$. It is straightforward that $g = (1, g_2) \in G$ transforms P_1 to

$$g(P_1) = \left(t_1, \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, 0\right) \in \Omega^r.$$

Since each boundary point belongs to exactly one analytic component, it follows $g\mathscr{F} = \Omega^r$. However, G acts linearly on Y, implying $g\mathscr{F} \subset \partial\mathscr{B}$ which is an absurd. Thus, $\mathscr{F} \subset \partial B_{1,1} \times \overline{B_{p-1,q}} \times 0$ and Pyatetskii-Shapiro's description of the analytic structure of $\partial B_{1,1} \times \overline{B_{p-1,q}}$ implies that $\mathscr{F}$ is G-equivalent to B^r for some $0 \le r \le \min(p-1, q)$.

Let $P_1(t_1, X_1, Y_1) \in \mathscr{F}$ be a point with $Y_1 \ne 0$. The inequalities of $\partial\Omega$ require $|t_1| < 1$ and ${}^t\overline{X_1}X_1 \le I_q$. Due to the transitivity of G on $\overline{\mathscr{B}}$, there exists $g \in G$ transforming P_1 to a point

$$g(P_1) = \left(0, \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, Y_0\right) \in g\mathscr{F}$$

for some integer $0 \le r \le \min(p-1, q)$. According to [20] that implies the inclusion

$$g\mathscr{F} \subseteq \left\{\left(t, \begin{pmatrix} I_r & 0 \\ 0 & X' \end{pmatrix}, Y\right) \in \partial\Omega \mid t \in B_{1,1}, X' \in B_{p-1-r, q-r}\right\}.$$

If $Y = (Y'' \quad Y')$ for $Y'' \in \mathrm{Mat}_{1,r}(\mathbf{C})$, $Y' \in \mathrm{Mat}_{1,q-r}(\mathbf{C})$, then the inequality

$${}^t\overline{Y}Y \le (1 - |t|^2)^2 \begin{pmatrix} 0 & 0 \\ 0 & I_{q-r} - {}^t\overline{X'}X' \end{pmatrix}$$

forces $Y'' = 0$ and, respectively, ${}^t\overline{Y'}Y' \le (1 - |t|^2)^2(I_{q-r} - {}^t\overline{X'}X')$.

In the case of ${}^t\overline{Y'}_0Y'_0 < I_{q-r}$, the boundary point $g(P_1)$ belongs to Ω^r for some $r > 0$ (due to $\Omega^0 \equiv \Omega$). Bearing in mind the uniqueness of the analytic boundary component $g\mathscr{F}$ through $g(P_1)$, one concludes that $g\mathscr{F} = \Omega^r$.

In the case of $\det({}^t\overline{Y'}_0Y'_0 - I_{q-r}) = 0$, the point $g(P_1)$ with $Y'_0 \in \partial B_{1,q-r}$ will be shown to constitute a 0-dimensional boundary component of Ω. To this end, it suffices to prove that the t- and X-coordinates are constant along $g\mathscr{F}$, since then $g\mathscr{F}$ is an analytically connected subset of the closed ball $\overline{B_{1,q-r}}$, intersecting the boundary $\partial B_{1,q-r}$. Assume that there exists an analytic arc $\{(t(s), X(s), Y(s)) \in g\mathscr{F} \mid |s| < \varepsilon\}$ through

$$\left(t(0) = 0, X(0) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, Y(0) = (0 \quad Y'_0)\right)$$

with nonconstant $t(s)$ or

$$X(s) = \begin{pmatrix} I_r & 0 \\ 0 & X'(s) \end{pmatrix}.$$

If $t(s) \not\equiv 0$, then $s = 0$ is an isolated zero of $t(s)$, and there exists some $\delta > 0$ such that $1 - |t(s)|^2 < 1$ for all s, $0 < |s| \leq \delta$. If $t(s) \equiv 0$ but $X'(s)$ is not identically 0, then $I_{q-r} - {}^t\overline{X'(s)}X'(s) < I_{q-r}$ over a punctured s-disk of radius δ. In either case there holds

$${}^t\overline{Y'(s)}Y'(s) \leq (1 - |t(s)|^2)^2 (I_{q-r} - {}^t\overline{X'(s)}X'(s)) < I_{q-r},$$

which is equivalent to $Y'(s){}^t\overline{Y'(s)} < 1 = Y_0'{}^t\overline{Y_0'}$ for all s, $0 < |s| \leq \delta$. That contradicts the maximum principle for $Y'(s)$ on the subarc $\{s \in \mathbf{C} \mid |s| \leq \delta\}$ and implies that $t(s) \equiv 0$, and $X'(s) \equiv 0$ on $g\mathscr{F}$. Consequently,

$$g\mathscr{F} \subseteq \left\{ \left(0, \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, (0 \quad Y')\right) \mid Y' \in \overline{B_{1,q-r}} \right\}$$

is a boundary component of Ω through the point

$$g(P_1) = \left(0, \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, (0 \quad Y_0')\right)$$

with $Y_0' \in \partial B_{1,q-r}$. According to the analytic disconnectedness of Y_0' from the other points of $\overline{B_{1,q-r}}$, there follows $g\mathscr{F} = g(P_1)$. Moreover, the unit vector Y_0' can be transformed by some unitary transformation $A \in SU_{q-r}$ into $Y_0'A = (0, \ldots, 0, 1) \in \mathrm{Mat}_{1,q-r}(\mathbf{C})$. Putting

$$u = (1, 1, 1, A) \in S(U(1, 1) \times U_{p-1} \times U_r \times U_{q-r}) \subset G$$

one observes that

$$ug\mathscr{F} = \left(0, \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, (0, \ldots, 0, 1)\right),$$

q.e.d. Proposition 3.

Towards the justification of $\mathrm{Aut}\,\Omega = G$, the automorphisms $\mathrm{Aut}_o\,\Omega$ fixing the origin of Ω will be shown to coincide with

$$G \cap \mathrm{Aut}_o\,\Omega = S(U_1 \times U_1 \times U_{p-1} \times U_q).$$

Theorem 3. *For $p > 2$, the biholomorphic automorphisms $\mathrm{Aut}_o\,\Omega$ fixing the origin of the domain Ω constitute the compact Lie group*

$$S(U_1 \times U_1 \times U_{p-1} \times U_q).$$

Proof. Observe that the domain $\Omega \subset B_{1,1} \times B_{p,q}$ is bounded and circular, i.e., invariant under the S^1-action

$$(t, X, Y) \mapsto (e^{i\theta}t, e^{i\theta}X, e^{i\theta}Y) \quad \text{for } e^{i\theta} \in S^1.$$

A theorem of Cartan asserts that the automorphisms fixing the origin of a bounded circular domain Ω act linearly on Ω (cf. [19] or [17]). In particular, they extend over the boundary $\partial\Omega$. Observe that, for $p > 2$, the only boundary components of maximal dimension $(p-1)q$ are gB^0 for $g \in G$. Their $\mathbf{C}$-span, characterized by the linear equation $Y = 0$, is mapped into itself by an arbitrary $\lambda \in \mathrm{Aut}_o\,\Omega$. Consequently, λ restricts to a nonsingular linear transformation of B onto itself. That produces a well-defined group homomorphism

$$\rho\colon \operatorname{Aut}_{\check{o}} \Omega \to \operatorname{Aut}_{\check{o}} \mathscr{B} = U_1 \times S(U_{p-1} \times U_q).$$

As far as the entire $\operatorname{Aut}\mathscr{B}$ extends to $\operatorname{Aut}\Omega$, the map ρ is surjective and split, so that $\operatorname{Aut}_{\check{o}} \Omega = \operatorname{Ker}\rho \times U_1 \times S(U_{p-1} \times U_q)$.

In order to complete the proof of the theorem, one has to establish that $\operatorname{Ker}\rho = U_1$.

An arbitrary $\lambda \in \operatorname{Aut}_{\check{o}} \Omega$ has the form

$$\lambda(t, X, Y) = (\lambda_{00}(t) + \lambda_{01}(X) + \lambda_{02}(Y), \lambda_{10}(t) + \lambda_{11}(X) + \lambda_{12}(Y), \lambda_{20}(t) + \lambda_{21}(X) + \lambda_{22}(Y)),$$

for appropriate linear maps λ_{ij}. If λ restricts to the identity of $\mathscr{B}$, then

$$\lambda(t, X, Y) = (t + \lambda_{02}(Y), X + \lambda_{12}(Y), \lambda_{22}(Y)).$$

As far as the bounded domain Ω is a Kobayashi hyperbolic manifold, an arbitrary $\lambda \in \operatorname{Aut}_{\check{o}} \Omega$ is a unitary transformation for the Kobayashi metric of Ω (cf. [16]). Therefore, λ can be diagonalized. It is straightforward that the Y-components of the eigenvectors of $\lambda \in \operatorname{Ker}\rho$ are eigenvectors for λ_{22}. Therefore, λ_{22} can be diagonalized, as well. According to [16], the eigenvalues of $d\lambda|_{\check{o}} = \lambda$ have absolute value 1. This property is inherited from λ_{22}. Let $\lambda_{22}(Y_0) = e^{i\theta} Y_0$ for $Y_0 \in \operatorname{Mat}_{1,q}(\mathbf{C})$ with $Y_0{}^t\overline{Y}_0 < 1$. Modifying λ by $g = (1, e^{-i\theta} I_{p-1+q}) \in \check{G} \cap \operatorname{Ker}\rho$, one obtains $\mu := g \circ \lambda \in \operatorname{Ker}\rho$, subject to $\mu(0, 0, Y_0) = (\lambda_{02}(Y_0), \lambda_{12}(Y_0), Y_0)$. By induction on $n \in \mathbf{N}$, it follows that the nth iterate $\mu^n(0, 0, Y_0) = (n\lambda_{02}(Y_0), n\lambda_{12}(Y_0), Y_0)$.

If $\lambda_{02}(Y_0) \neq 0$, then $|n\lambda_{02}(Y_0)| \geq 1$ for a sufficiently large $n \in \mathbf{N}$, which contradicts $\mu^n(0, 0, Y_0) \in \Omega$. Similarly, $\lambda_{12}(Y_0) \neq 0$ implies the existence of $n \in \mathbf{N}$, such that $n\lambda_{12}(Y_0) \notin B_{p-1,q}$ and $\mu^n(0, 0, Y_0) \notin \Omega$. Therefore, Y_0 belongs to $\operatorname{Ker}\lambda_{02}$ and $\operatorname{Ker}\lambda_{12}$. Bearing in mind that $\operatorname{Mat}_{1,q}(\mathbf{C})$ admits a basis, consisting of eigenvectors for λ_{22}, one concludes that $\lambda_{02} \equiv 0$ and $\lambda_{12} \equiv 0$.

Consequently,

$$\lambda(t, X, Y) = (t, X, \lambda_{22}(Y))$$

acts identically on the t-component and restricts to a linear map of the generalized ball $\Omega \cap \{t = 0\} \simeq B_{p,q}$ into itself. The above-mentioned expression for λ makes clear the invertibility of the restriction $\lambda|_{t=0}$. Thus $\lambda|_{t=0} \in U_p \times U_q$ is an automorphism, fixing the origin of $B_{p,q}$ and

$$\begin{pmatrix} X \\ \lambda_{22}(Y) \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix} B$$

for some unitary matrices $A \in U_p$ and $B \in U_q$. Decomposing

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

into blocks $A_{11} \in \operatorname{Mat}_{p-1,p-1}(\mathbf{C})$, $A_{12} \in \operatorname{Mat}_{p-1,1}(\mathbf{C})$, $A_{21} \in \operatorname{Mat}_{1,p-1}(\mathbf{C})$, $A_{22} \in \mathbf{C}$, one obtains

$$A_{11}XB + A_{12}YB = X \quad \text{and} \quad A_{21}XB + A_{22}YB = \lambda_{22}(Y)$$

for all $\binom{X}{Y} \in B_{p,q}$. Therefore, $A_{12} = 0$ and $A_{21} = 0$. Bearing in mind that the ineffective kernel of the $U_{p-1} \times U_q$-action on $B_{p-1,q}$ equals U_1, one concludes that $A_{11} = e^{i\theta} I_{p-1}$ and $B = e^{-i\theta} I_q$. Finally, $A \in U_p$ reveals that $A_{22} = e^{i\tau}$, so that $\lambda \in \operatorname{Ker} \rho$ acts according to $\lambda(t, X, Y) = (t, X, e^{i(\tau-\theta)} Y)$, q.e.d Theorem 3.

Last preparation for recognizing $\operatorname{Aut}\Omega$ is the description of the G-orbits of Ω.

Lemma 4. *The domain Ω decomposes into a disjoint union of orbits $\coprod_{r\in[0,1)} GP_r$ where $G = S(U(1,1) \times U(p-1, q))$, $P_r(t = 0, X = 0, Y = (0, \dots, 0, r))$. As homogeneous spaces,these orbits are of two different types: $GP_0 = G/G_0$ for $r = 0$ has an isotropy group $G_0 = S(U_1 \times U_1 \times U_{p-1} \times U_q)$, while $GP_r = G/G_r$ for $r \in (0, 1)$ are quotients by subgroups $G_r = S(U_1 \times U_1 \times U_{p-1} \times U_{q-1})$ of G_0 and constitute fiberings $\varphi: GP_r \to GP_0$ of $(2q-1)$-spheres over the distinguished section $\mathscr{B}$.*

Proof. Due to the homogenity of the generalized balls, for any $P(t_0, X_0, Y_0) \in \Omega$ there exists some $g_1 \in G$ such that $g_1(P) = (t = 0, X = 0, Y_1)$ with ${}^t\overline{Y_1} Y_1 < I_q$. An appropriate unitary transformation $A \in U_q$ rotates Y_1 with $Y_1 {}^t\overline{Y_1} = r^2$ in the sphere of radius r to $Y_1 A = (0, \dots, 0, r)$,and extends to $g_2 \in 1 \times 1 \times 1 \times U_q \subset \operatorname{Aut}_{\check{o}} \Omega \subset G$, such that $g_2(t = 0, X = 0, Y_1) = (t = 0, X = 0, Y = (0, \dots, 0, r))$. In other words, $g_2 g_1(P) = P_r$ and the point P is from the orbit GP_r.

To establish the disjointness of the orbits GP_r in the union $\bigcup_{r\in[0,1)} GP_r = \Omega$, assume that $g(P_{r_1}) = P_{r_2}$ for some $r_1 \neq r_2$,

$$g \in S(U(1,1) \times U(p-1, q)).$$

As far as g preserves the origin $(t = 0, X = 0)$ of $B_{1,1} \times B_{p-1,q}$, it should be of the form $g(t, X, Y) = (e^{i\tau} t, AX\dot{B}, YBe^{i\theta})$ for some $e^{i\tau}, e^{i\theta} \in U_1$, $A \in U_{p-1}$, $B \in U_q$. In particular, for $Y(r) := (0, \dots, 0, r)$ there holds $Y(r_1)Be^{i\theta} = Y(r_2)$, whereas $r_2^2 = Y(r_2) {}^t\overline{Y(r_2)} = Y(r_1) B {}^t\overline{B} {}^t\overline{Y(r_1)} = r_1^2$. The contradiction justifies that Ω is a disjoint union of the orbits GP_r, $r \in [0, 1)$.

The stabilizer of the reference point $\check{o} = P_0$ was calculated to be G_0. According to the transitiveness of G on $\mathscr{B} = \Omega \cap \{Y = 0\}$ and the linearity of the G-action with respect to the Y-component, the orbit $GP_0 = \mathscr{B}$.

According to the disjointness argument, the automorphisms $g \in G$ fixing some P_r for $0 < r < 1$ restrict on $Y(r)$ to rotations $Y(r)Be^{i\theta} = Y(r)$. Let us decompose

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in U_q$$

into blocks $B_{11} \in \operatorname{Mat}_{q-1,q-1}(\mathbf{C})$, $B_{12} \in \operatorname{Mat}_{q-1,1}(\mathbf{C})$, $B_{21} \in \operatorname{Mat}_{1,q-1}(\mathbf{C})$, $B_{22} \in \mathbf{C}$. The equation $(rB_{21}e^{i\theta}, rB_{22}e^{i\theta}) = (0, r)$ reveals that $B_{21} = 0$ and $B_{22} = e^{-i\theta}$. For a unitary B that implies $B_{12} = 0$ and $B_{11} \in U_{q-1}$. Therefore, the effectively acting automorphisms $g \in G$ stabilizing P_r, $0 < r < 1$, are given by the formula

$$g(t, X, Y) = \left(e^{i\tau} t, AX \begin{pmatrix} B_{11} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, Y \begin{pmatrix} B_{11} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} e^{i\theta} \right),$$

where $e^{i\tau}$, $e^{i\theta} \in U_1$, $A \in U_{p-1}$, and $B_{11} \in U_{q-1}$. In other words, the stabilizers G_r of P_r, $r \neq 0$, are isomorphic to the subgroup $S(U_1 \times U_1 \times U_{p-1} \times U_{q-1})$ of G_0.

To reveal the shape of $GP_r = G/G_r$, $0 < r < 1$, let us consider the projection $\varphi\colon G/G_r \to G/G_0$ onto the distinguished section $\mathscr{B} = G/G_0$. Its fibers $G_0/G_r = U_q/U_{q-1}$ are diffeomorphic to $(2q-1)$-spheres on the Y-component, q.e.d. Lemma 4.

The above described G-orbit structure of Ω is, actually, the $\operatorname{Aut}\Omega$-orbit structure for $p > 2$, according to the following

Theorem 4. *The effectively acting biholomorphic automorphisms of the domain Ω with $p > 2$, $q \geq 1$ constitute the subgroup $\operatorname{Aut}\Omega = S(U(1,1) \times U(p-1, q))$ of $SO(2p, 2q+1)$.*

Proof. Recall that $G := S(U(1,1) \times U(p-1,q))$ was defined as a subgroup of $\operatorname{Aut}\Omega$. The theorem requires the proof of the opposite inclusion $\operatorname{Aut}\Omega \subseteq G$.

Assume that $\alpha \in \operatorname{Aut}\Omega$ maps the reference point P_0 into $Q_0(t_0, X_0, Y_0)$. According to Lemma 4, one can transform Q_0 in its G-orbit to $P_r = g(Q_0)$, $g \in G$. If $r = 0$, then $g\alpha(P_0) = P_0$, i.e., $g\alpha \in \operatorname{Aut}_{\breve{o}}\Omega \subset G$, whereas $\alpha \in G$.

Assume that $g\alpha(P_0) = P_r$ for some $r \neq 0$. The inclusion of the orbits $GP_r \subseteq \operatorname{Aut}\Omega(P_0)$ forces the inequality $\dim_{\mathbf{R}} \operatorname{Aut}\Omega(P_0) \geq \dim_{\mathbf{R}} GP_r = 2pq + 1$. Braun, Kaup and Upmeier have shown in [2] that the $\operatorname{Aut}\Omega$-orbit of the origin of a bounded circular domain Ω is a Hermitian symmetric space, embedded as a closed complex submanifold of Ω. The fact that $\operatorname{Aut}\Omega(P_0)$ is of real codimension ≤ 1 in Ω requires $\operatorname{Aut}\Omega(P_0) = \Omega$ to be a Hermitian symmetric space. Since the center of the effectively acting

$$\operatorname{Aut}_{\breve{o}}\Omega = S(U_1 \times U_1 \times U_{p-1} \times U_q)$$

is 3-dimensional, Ω should consist of 3 irreducible components (cf. [14]). However, $\Omega \cap \{t = 0\} \simeq B_{p,q}$ reveals that Ω can have at most 2 irreducible components. The contradiction implies that $g\alpha(P_0) = P_0$, whereas $\alpha \in G$ and $\operatorname{Aut}\Omega = G$.

Recall that

$$\Omega = \Omega(V) = \{\Lambda \in \operatorname{Grass}^Q(p, V) \mid \dim(\Lambda \cap V^\perp) = p-1,\ h|_\Lambda > 0\}_{\breve{o}}$$

where $V^\perp = E + K$ is the Q-orthogonal to V, and put V^o for the h-orthogonal complement of $V^\perp$ to V, i.e., $V^o = N + L + \overline{N}$. Obviously, the Q-orthogonal and h-unitary group

$$\operatorname{Aut}(V^o, Q, h) \times \operatorname{Aut}(V^\perp, Q, h) \simeq O(2,1) \times U(p-1, q)$$

induces a subgroup of biholomorphic automorphisms of Ω. Bearing in mind the isomorphism of the Lie groups $O(2,1) \simeq U(1,1)$ (cf. [14]), one observes that the entire $\operatorname{Aut}\Omega$ arises in that way. Consequently, $\operatorname{Aut}\Omega$ turns out to be a subgroup of the effectively acting $\operatorname{Aut}(H_{\mathbf{C}}, Q, h)$, or $\operatorname{Aut}\Omega \subset SO(2p, 2q+1)$ extends to the period domain D, q.e.d. Theorem 4.

For a variation of Hodge structure

$$f\colon S \to \Gamma\backslash SO(2p, 2q+1)/S(U_p \times O_{2q+1}), \quad \Gamma \subset SO_{\mathbf{Z}}(2p, 2q+1),$$

of constant $\operatorname{rank}_{\mathbf{C}} f = pq + 1$, $p > 2$, we have constructed a holomorphic map $g\colon S \to M$ of constant $\operatorname{rank}_{\mathbf{C}} g = pq + 1$ into a discrete quotient $M =$

$\Gamma \cap \operatorname{Aut}\Omega \backslash \Omega$ of the domain Ω. One can enlarge Γ to an arithmetic subgroup of $SO(2p, 2q+1)$. As far as $\operatorname{Aut}\Omega \subset SO(2p, 2q+1)$ implies that $\operatorname{Aut}_{\mathbf{Z}}\Omega = SO_{\mathbf{Z}}(2p, 2q+1) \cap \operatorname{Aut}\Omega$, the arithmetic Γ corresponds to an arithmetic $\Gamma \cap \operatorname{Aut}\Omega$. Thus, a variation of maximal dimension with associated $h^{2,0} = p > 2$ and $h^{1,1} = 2q+1$ gives rise to a holomorphic map $g: S \to M$ of constant $\operatorname{rank}_{\mathbf{C}} g = pq+1$ in an arithmetic quotient M of the domain Ω. The next theorem proves the nonexistence of a compact complex analytic closure S^* of S under the above-mentioned circumstances. However, a geometrically arising variation of Hodge structure is associated with a quasiprojective complex analytic variety S (cf. [11]), which always admits a projective closure S^*. That justifies the nonexistence of geometrically arising variations of maximal dimension with $h^{2,0} = p > 2$ and $h^{1,1} = 2q+1$.

Theorem 5. *If there exists a holomorphic map $g: S \to M$ of constant $\operatorname{rank}_{\mathbf{C}} g = \dim_{\mathbf{C}} M$ from the complex analytic variety S in the arithmetic quotient M of the domain Ω with $p > 2$, then the analytic structure of S does not extend to a compact complex analytic variety S^*, containing S as an everywhere dense subset.*

Proof. Recall that the holomorphic projection

$$\varphi: \Omega \to \mathscr{B} = B_{1,1} \times B_{p-1,q},$$
$$\varphi(t, X, Y) = (t, X)$$

exhibits the domain Ω as a fibering of q-balls

$$\varphi^{-1}(t, X) = \{Y \in \mathbf{C}^q \mid {}^t\overline{Y}Y < (1-|t|^2)^2(I_q - {}^t\overline{X}X)\}$$

over the Hermitian symmetric space $\mathscr{B}$. The biholomorphic automorphisms of Ω arise from the biholomorphic automorphisms of $\mathscr{B}$, so that $\pi_1(M)$ is an (eventually noneffective) arithmetic subgroup of $\operatorname{Aut}\mathscr{B}$ with a complex analytic quotient $B = \pi_1(M)\backslash\mathscr{B}$. The projection φ descends to $\varphi: M \to B$, as far as it commutes with the $\operatorname{Aut}\Omega$-action. According to Baily-Borel (cf. [1]), B is quasiprojective and if $\mathscr{B}^*$ stands for the union of $\mathscr{B}$ with its rational boundary components, then $B^* = \pi_1(M)\backslash\mathscr{B}^*$ is the projective closure of B. Observe that $\pi_1(M) \subset \operatorname{Aut}\Omega$ acts linearly on the fibers of φ and extends properly discontinuously to $\mathscr{B}^* \times \mathbf{C}^q$, providing a complex analytic variety $F^* = \pi_1(M)\backslash\mathscr{B}^* \times \mathbf{C}^q$ containing M. As far as the fibering $\varphi: F^* \to B^*$ is locally trivial, one can regard F^* as a rank q vector bundle over B^*. Let $\overline{B_{1,q}} = \{Y \in \mathbf{C}^q \mid {}^t\overline{Y}Y \leq I_q\}$ be the closed q-ball and $D^* = \pi_1(M)\backslash \bigcup_{\gamma\in\pi_1(M)} \gamma(\mathscr{B}^* \times \overline{B_{1,q}})$ be the locally trivial bundle of closed q-balls over B^*. According to the compactness of the base B^* and the fiber $\overline{B_{1,q}}$, D^* is a compact subset of F^*, containing M.

Suppose that the complex analytic structure of S extends to a compact complex analytic variety $S^* \supset S$, in which S is everywhere dense. Then the holomorphic map $g: S \to M$ admits a continuous extension $g^*: S^* \to D^*$ which is totally bounded and, therefore, holomorphic. According to the compactness of S^*, the holomorphic map g^* is proper, so that Remmert's mapping theorem (cf. [8]) implies that $g^*(S^*)$ is a compact complex analytic subvariety of F^*. In particular, the zero set $Z_1 = \{z \in g^*(S^*) \mid \varphi(z) = \pi_1(M)(0,0)\}$ of the holomorphic map $\varphi: g^*(S^*) \to B^*$ is a q-dimensional compact complex analytic subspace of $Z_2 = \{z \in F^* \mid \varphi(z) = \pi_1(M)(0,0)\} = \{\pi_1(M)(0,0,Y) \in F^*\}$. It is straightforward that the subgroup of $\operatorname{Aut}\Omega$ normalizing $0 \times 0 \times \mathbf{C}^q$ coincides with the isotropy group $\mathrm{Aut}_{\bar{o}}\,\Omega$. The intersection $\pi_1(M) \cap \mathrm{Aut}_{\bar{o}}\,\Omega$ is discrete and compact, whereas, finite subgroup of $\mathrm{Aut}\,\Omega$. Consequently, Z_2 is a finite quotient $\mathbf{C}^q$, or there is a finite map $\nu\colon \mathbf{C}^q \to Z_2$ such that $\nu^{-1}(Z_1)$ is a q-dimensional compact complex analytic subvariety of $\mathbf{C}^q \subset \mathbf{CP}^q$. According to Chow's theorem (cf. [13]), $\nu^{-1}(Z_1) \neq \mathbf{CP}^q$ is defined by nontrivial polynomial equations. That contradicts $\dim_{\mathbf{C}} \nu^{-1}(Z_1) = q$ and reveals the nonexistence of a compact complex analytic closure S^* of S, q.e.d. Theorem 5.

The lack of a variation of Hodge structure

$$f\colon S \to \Gamma\backslash SO(2p\,,\,2q+1)/S(U_p \times O_{2q+1})$$

of constant $\mathrm{rank}_{\mathbf{C}} f = pq + 1$ with a quasiprojective S implies that the geometrically arising variations of Hodge structure with associated $h^{2,0} = p > 2$ and $h^{1,1} = 2q + 1$ are of $\mathrm{rank}_{\mathbf{C}} f \leq pq = h^{2,0}[\frac{h^{1,1}}{2}]$.

References

1. W. Baily and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 422–528.
2. R. Braun, W. Kaup, and H. Upmeier, *On the automorphisms of circular and Reinhardt domains in complex Banach spaces*, Manuscripta Math. **133** (1978), 25–97.
3. J. Carlson, *Bounds on the dimension of variations of Hodge structure*, Trans. Amer. Math. Soc. **294** (1986), 45–64.
4. J. Carlson and L. Hernandez, *Harmonic maps from compact Kahler manifolds to exceptional hyperbolic spaces*, J. Geom. Analysis **1** (1991), 339–357.
5. J. Carlson, A. Kasparian, and D. Toledo, *Variations of Hodge structure of maximal dimension*, Duke Math. J. **58** (1989), 669–694.
6. J. Carlson and C. Simpson, *Shimura varieties of weight two Hodge structures*, Lecture Notes in Math., vol. 1246, Springer-Verlag, Berlin-Heidelberg-New York, 1987, pp. 1–15.
7. J. Carlson and D. Toledo, *Integral manifolds, harmonic mappings, and the abelian subspace problem*, Lecture Notes in Math., vol. 1352, Springer-Verlag, Berlin-Heidelberg-New York, 1988, pp. 60–74.
8. G. Fisher, *Complex analytic geometry*, Lecture Notes in Math., vol. 538, Berlin-Heidelberg-New York, 1976.
9. P. Griffiths, *Periods of integrals on algebraic manifolds.* I, II, Amer. J. Math. **90** (1968), 568–626; 805–865.
10. ——, *Periods of integrals on algebraic manifolds—summary of main results and discussion of open problems*, Bull Amer. Math Soc. **75** (1970), 228–296.
11. ——, *Periods of integrals on algebraic manifolds.* III, Publ. Math. Inst. Hautes Ètudes Sci. **38** (1970), 125–180.
12. ——, *Topics in transcendental algebraic geometry*, Ann. of Math. Stud., no. 106, Princeton Univ. Press, Princeton, NJ, 1984.
13. P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley-Interscience, New York, 1978.
14. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, Orlando, FL, 1978.
15. A. Kasparian, *Variations of Hodge structure of maximal dimension with associated Hodge numbers $h^{2,0} > 2$ and $h^{1,1} = 2q + 1$*, Ph.D. Thesis, Univ. of Utah, 1993.
16. S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Marcel Dekker, New York, 1970.
17. S. Krantz, *Function theory of several complex variables*, Wiley Interscience, New York, 1982.
18. A. Malčev, *Commutative subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl. Ser. 2, vol. 40, Amer. Math. Soc., Providence, RI, 1951.

19. R. Naramsinhan, *Several complex variables*, Chicago Lectures in Math, Chicago-London, 1971.

20. I. Pyatetskii-Shapiro, *Automorphic functions and the geometry of classical domains*, Gordon and Breach, New York-London-Paris, 1969.

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