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SHADOW FORMS OF BRASSELET-GORESKY-MacPHERSON

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ABSTRACT. Brasselet, Goresky and MacPherson constructed an explicit morphism, providing a De Rham isomorphism between the intersection homology of a singular variety X and the cohomology of some complex of differential forms, called "shadow forms" and generalizing Whitney forms, on the smooth part of X. The coefficients of shadow forms are integrals of Dirichlet type. We find an explicit formula for them; from that follows an alternative proof of Brasselet, Goresky and MacPherson's theorem. Next, we give a *duality formula* and a *product formula* for shadow forms and construct the correct algebra structure, for which shadow forms yield a morphism.

INTRODUCTION

With the Whitney forms, we have an explicit proof of the De Rham isomorphism for a compact oriented variety M (see *Differential Forms in Algebraic Topology*, by Bott and Tu). Given a triangulation $\mathscr{T}$ of M, the idea of Whitney is to associate, for each simplex S of $\mathscr{T}$, a differential form $W(S)$ which is nothing but the volume form of S. He thus constructs explicitly an isomorphism between the homology of simplicial chains of $\mathscr{T}$ and the De Rham cohomology of M. There are also singular versions for the De Rham theorem (i.e., M need not be a smooth variety). One of them asserts that, for a singular space X with a "good" metric which is riemannian on the smooth part X_0 of X, an isomorphism exists between the *intersection homology* group [GM] of X and the cohomogy of a complex of differential forms on X_0 which are p-integrable in the sense of Hodge (Goresky-MacPherson, Cheeger, Nagase,...).

Recently, Brasselet, Goresky and MacPherson [BGM] have proposed for the singular case an similar approach to that of Whitney : if X is an oriented compact pseudovariety with a triangulation $\mathscr{T}$ (exactly, X must be a polyhedron of some $\mathbf{R}^N$), they associate to each simplex σ of a barycentric subdivision $\mathscr{T}'$ of $\mathscr{T}$, a differential form $\omega(\sigma)$ on X_0, they call the "shadow form" of σ. Finally, what they obtain is exactly the singular De Rham theorem but their method is new and rather explicit since it avoids introducing the *perverse sheaves theory*.

However, the coefficients of shadow forms given in [BGM, Theorem 4.1] are Dirichlet integrals. Our first purpose is to provide effective formulæ for

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the shadow forms (Theorem 1.1), in order to simplify the proof of [BGM] (Proposition 1.2, Theorem 1.3). The second part of this paper is devoted to a question of [BGM, §10] about the multiplicative structure of shadow forms. The results we obtain are the following :

(i) On simplicial chains level, a *duality* $\sigma \longmapsto \hat{\sigma}$ compatible with shadow forms (Theorem 2.6) exists.

(ii) We construct an associative product for simplicial chains, denoted " $*$ ", for which the shadow forms verify $\omega(\sigma * \tau) = \omega(\sigma) \wedge \omega(\tau)$ (Theorem 2.8).

The method we develop uses the Chen theory of *iterated integrals* [Ch].

Our conclusion is the following (Theorem 2.11, Proposition 2.13): *for suitable algebras, the shadow forms yield a graded differential algebra morphism.* So, thanks to shadow forms, we hope to state in the future a *multiplicative* singular De Rham theorem between, on one hand, the intersection product and, on the other hand, a kind of *Hölder product* (coming from the Hölder $\mathbf{L}^p$-inequalities). There are also other reasons to look after shadow forms: in his thesis [Be], the author has related them with polylogarithms, which are useful analytic functions in different fields (in K-theory for example, to state the Zagier's Conjecture). From this follows some new functional equations for polylogarithms, coming directly from the geometry of shadow forms.

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1. Shadow forms

1.1. Shadow forms and intersection homology. Throughout this section, $\triangle \subset \mathbf{R}^{n+1}$ denotes the standard n-simplex naturally oriented and its $n+1$ vertices are denoted by $a^i = (0, \dots, 0, 1, 0, \dots, 0)$, for $0 \leq i \leq n$, where 1 appears at the $(i+1)$-th coordinate. Any point $x \in \triangle$ admits an unique decomposition $x = \sum_{i=0}^n x_i a^i$, where the *barycentric coordinates* x_i of x satisfy

$$0 \leq x_i \leq 1 \quad \text{and} \quad \sum_{i=0}^n x_i = 1.$$

1.1.1. *Encoding simplices.* To the standard simplex $\triangle$, we associate the *barycentric abstract complex* $\triangle'$ simplices of which, called *abstract simplices of* $\triangle'$, are the sequences of strictly increasing faces of $\triangle$. For $k \in [0, n]$, a k-simplex σ will be written $F_0 < \dots < F_k$ or, more simply, $\sigma = (F_0, \dots, F_k)$. The last face F_k of σ will be denoted by F_σ.

Given an abstract k-simplex $\sigma = (F_0, \dots, F_k)$, one has a corresponding sequence of disjoint nonempty subsets of $[0, n]$, $S = S(\sigma) = (S_0, \dots, S_k)$ in the following way:

– the elements of S_0 are the labels of the vertices of F_0,

and, for $i = 1, \dots, k$:

– S_i is the set of the labels of vertices of F_i which do not lie in F_{i-1}.

Conversely, such a sequence $S = (S_0, \dots, S_k)$ of disjoint nonempty sets of $[0, n]$, provides an unique abstract k-simplex $\sigma = (F_0, \dots, F_k)$, such that the labels of the vertices of the face F_i are the elements of $S_0 \cup \cdots \cup S_i$. The map $\sigma \mapsto S(\sigma)$ is hence a bijection; we will denote by $S(\sigma)$ the *coded simplex* associated to the abstract simplex σ, or *the encoding* $S(\sigma)$ of σ. To a coded k-simplex $S = (S_0, \dots, S_k)$, we associate a sequence of integers $s = (s_0, \dots, s_k)$, defined by $s_i = \mathrm{card}(S_i)$.

Denote by $\mathring{\Delta}$ the interior of Δ. For each point p of $\mathring{\Delta}$, we can define the *affine barycentric subdivision of* Δ *with barycenter* p. Denote it by $\Delta'(p)$; it is a simplicial complex, canonically isomorphic to Δ'. We shall denote $\sigma(p)$, the image of $\sigma \in \Delta'$ by this isomorphism and we shall call $\sigma(p)$, the *geometric simplex* asociated to σ.

Definition. We call a *central simplex* any abstract simplex $\sigma \in \Delta'$ verifying one of the equivalent following properties:

(i) $F_\sigma = \Delta$;

(ii) S_σ is a partition of $[0, n]$ (thus, if $S_\sigma = (S_0, \dots, S_k)$ then $\sum_i s_i = n+1$);

(iii) Any point p of $\mathring{\Delta}$ is a vertex of $\sigma(p)$.

The 0-simplex (Δ) is central. The set of central simplices, denoted by Δ^c, spans a real vector space $\mathscr{C}_*(\Delta^c)$; it is a graded complex with differential morphism ∂_Δ, defined on each central simplex σ by: $\partial_\Delta \sigma = (\partial\sigma) \cap \Delta^c$. We identify $\mathscr{C}_*(\Delta^c)$ with the quotient $\mathscr{C}_*(\Delta')/\mathscr{C}_*(\partial\Delta)$, where $\partial\Delta$ denotes the simplicial sub-complex whose simplices are those of the proper faces of Δ.

1.1.2. *Shadow forms associated to central simplices.* We associate to each face $F \le \Delta$ whose vertices are $a^{i_0}, \dots, a^{i_p}$, the coded 0-simplex $(S_F) = \{i_0, \dots, i_p\}$. Whitney associates to this simplex the differential form:

$$W(F) = W(S_F) = W(x_{i_0}, \dots, x_{i_p}) = p! \sum_{j=0}^{p} (-1)^j x_{i_j} dx_{i_0} \wedge \dots \wedge \widehat{dx_{i_j}} \wedge \dots \wedge dx_{i_p} .$$

Definition. The *antipode* of the coded k-simplex $S = (S_0, \dots, S_k)$ is the coded k-simplex $\hat{S} = (S_k, S_{k-1}, \dots, S_1, S_0)$. Similarly, one constructs the *antipode* $\hat{\sigma}$ for any abstract k-simplex $\sigma = (F_0, \dots, F_k)$. It is the abstract k-simplex $(G_0, \dots, G_k)$, where G_i is the face of Δ whose vertices are those of F_k which do not lie in F_{k-i-1} ($F_{-1} = \phi$).

For every abstract simplex σ, one has the following properties:

(i) $S_{\hat{\sigma}} = \hat{S}_\sigma$ and $F_{\hat{\sigma}} = F_\sigma$;

(ii) $\sigma \longmapsto \hat{\sigma}$ is an involution of Δ' leaving Δ^c fixed;

(iii) $\sigma \cap \hat{\sigma} = S_{F_\sigma}$. If σ is central, the simplex $\sigma \cap \hat{\sigma}$ is reduced to the point (Δ);

(iv) *geometric duality:* if p and q are points in the interior of the face F_σ, then

$$q \in \sigma(p) \Longleftrightarrow p \in \hat{\sigma}(q) .$$

Definition. Let σ be a central simplex, we define its *incidence variety*:

$$D_\sigma = \{(p, x) \in \mathring{\Delta} \times \Delta : x \in \sigma(p)\} = \bigsqcup_{p \in \mathring{\Delta}} \hat{\sigma}(p) \times \{p\} .$$

It is a variety of dimension $n+k$ with corners and canonically oriented. Let us denote π_1 and π_2 the canonical projections of D_σ on the factors $\mathring{\Delta}$ and Δ

respectively. The map $\pi_2 : D_\sigma \longrightarrow \Delta$ is a locally trivial bundle with compact fibers $\hat{\sigma}$. Write

$$\pi_1^* : \Omega^*(\mathring{\Delta}) \longrightarrow \Omega^*(D_\sigma) ,$$

the classical pullback and write

$$\pi_{2*} : \Omega^*(D_\sigma) \longrightarrow \Omega^{*-k}(\mathring{\Delta}) ,$$

the "integration along the fibers of π_2".

Definition. The *shadow form* $\omega(\sigma)$ is the C^∞ $(n-k)$-differential form on $\mathring{\Delta}$, the image of the Whitney form $W(\Delta)$ by the morphism $\pi_{2*} \circ \pi_1^*$. Extending the definition of shadow forms to the chains, one may define the *shadow morphism* $\omega : \mathscr{C}_*(\Delta^c) \to \Omega^{n-*}(\mathring{\Delta})$.

1.1.3. *Properties.* (Refer to [BGM] for proofs.)

(i) *Morphism.* The shadow morphism ω commutes with differentials: $\omega(\partial_\Delta \sigma) = d\omega(\sigma)$; it is a morphism of graded differential complexes.

(ii) *The shadow formula.* Let σ be a central k-simplex and c a piecewise smooth $(n-k)$-chain with support in $\mathring{\Delta}$. The *shadow of the chain c with respect to the abstract simplex σ* is the n-chain $O_\sigma(c)$ defined by its support

$$|O_\sigma(c)| = \{p \in \mathring{\Delta} : \ \sigma(p) \cap |c| \neq \phi\}.$$

The coefficients of the chain $O_\sigma(c)$ are those of c transported by the projection; they are constant on each fiber of $|O_\sigma(c)|$. Denoting the algebraic volume by "Vol", one has the *shadow formula*

$$\int_c \omega(\sigma) = \mathrm{Vol}(O_\sigma(c)) .$$

In this formula, the "geometric shadows" $\hat{\sigma}(p)$ are directly involved, in fact one has

$$|O_\sigma(c)| = \bigsqcup_{p \in c} \hat{\sigma}(p).$$

(iii) *Equations of shadow forms.* Letting σ be a central k-simplex and $S = (S_0, \dots, S_k)$ its encoding, one has

$$\omega(\sigma) = (-1)^{\sum_i is_i} \frac{\Gamma(n+1)}{\Gamma(s_0)\dots\Gamma(s_k)} \frac{W(S_0)}{v_0^{s_0}} \wedge \cdots \wedge \frac{W(S_k)}{v_k^{s_k}} \phi_k(v_0, \dots, v_k ; s_0, \dots, s_k),$$

where $v_i = \sum_{j \in S_i} x_j$ (thus $\sum_i v_i = 1$) and where

$$\phi_k(v_0, \dots, v_k ; s_0, \dots, s_k) \\ = \int_{\mathscr{R}(v)} u_0^{s_0-1} \dots u_{k-1}^{s_{k-1}-1} (1 - u_0 - \cdots - u_{k-1})^{s_k-1} du_0 \dots du_{k-1},$$

with, for $v = (v_0, \dots, v_k)$,

$$\mathscr{R}(v) = \left\{ (u_0, \dots, u_{k-1}) \in \mathbf{R}^k \mid 0 \leq \frac{u_0}{v_0} \leq \dots \leq \frac{u_{k-1}}{v_{k-1}} \leq \frac{1 - u_0 - \cdots - u_{k-1}}{v_k} \right\}.$$

1.1.4. *A singular De Rham theorem.* (Refer to [GM] for intersection homology theory.)

Definition. For σ a central k-simplex and $S = (S_0, \dots, S_k)$ its encoding, put $\lambda_0 = 0$ and, for $1 \le i \le k$, $\lambda_i = s_{k-i+1} + \cdots + s_k$. The *profile* of σ is a function $p_\sigma : [0, n] \longrightarrow \mathbf{R}$ defined, for $\lambda \in [0, n]$, by

$$p_\sigma(\lambda) = \begin{cases} p_\sigma(\lambda_i) & \text{if } \lambda_i \le \lambda \le \lambda_i + 1; \\ p_\sigma(\lambda_i) + \lambda - \lambda_i - 1 & \text{if } \lambda_i + 1 \le \lambda \le \lambda_{i+1}. \end{cases}$$

Whenever $s_k = 1$, the profile of σ is a *perversity* in the sense of Goresky-MacPherson; it is the smallest perversity for which the simplex σ is *allowed.*

Let $q \ge 1$ and let X be a triangulated pseudomanifold of dimension n (embedded in a euclidean space), [BGM] defines the complex of "shadow forms of type $\mathbf{L}^q$":

$$\mathscr{L}_q^*(X) = \{\eta \in \operatorname{Im}(\omega) \cap \Omega^*(X_{reg}, \mathbf{L}_q(X)) : d\eta \in \Omega^{*+1}(X_{reg}, \mathbf{L}_q(X))\}$$

where $\mathbf{L}_q(X)$ denotes the space of real C^∞ functions f over X_{reg} such that f^q is integrable and where ω is the morphism defined in [BGM] by "glueing" the shadow morphisms associated to the maximal simplices of the triangulation of X. We define also the perversity $\bar{p}(q) = \max\{\bar{p} \mid \forall \lambda \in]0, n],\ \bar{p}(\lambda) < \lambda/q\}$. The homology of the intersection complex for the perversity $\bar{p}(q)$ is denoted by $IH_*^{\bar{p}(q)}(X)$.

Theorem 1 ([BGM], Theorem 9.2). *The cohomology of $\mathscr{L}_q^*(X)$ is isomorphic to $IH_{n-*}^{\bar{p}(q)}(X)$.*

In [BGM], the authors introduce the shadow forms in order to construct a morphism of differential complexes, inducing the isomorphism of the theorem. The following result is the key to their proof ([BGM], Theorem 5.2).

Theorem 2. *For every central simplex σ, one has $\omega(\sigma) \in \mathscr{L}_q^*(\Delta) \iff p_\sigma < \bar{p}(q)$.*

This theorem relates the position of σ with respect to the faces of Δ (controlled by the profile p_σ) and the integrability order of $\omega(\sigma)$.

1.2. Explicit formulæ for shadow forms. Our purpose is to give an explicit formula for the coefficient $\phi_k(v; s)$ occurring in the *equation of shadow forms* §1.1.3 (iii).

Let σ be a central k-simplex and let $S_\sigma = (S_0, \dots, S_k)$ be its encoding. Using the notations of 1.1 and denoting succesively $\#S = \sum_{0 \le i \le k} i s_i$,

$$\widetilde{\phi}(v; s) = \frac{n!}{{v_0}^{s_0} \dots {v_k}^{s_k}} \phi_k(v_0, \dots, v_k; s_0, \dots, s_k),$$

$$\frac{W(S)}{\Gamma(s)} = \frac{W(S_0)}{\Gamma(s_0)} \wedge \cdots \wedge \frac{W(S_k)}{\Gamma(s_k)},$$

then, the shadow form associated to σ can be written

$$\omega(\sigma) = (-1)^{\#S} \frac{W(S)}{\Gamma(s)} \widetilde{\phi}(v; s).$$

Theorem 1.1. *For every $x \in \mathring{\Delta}$, one has*

$$\widetilde{\phi}_k(v_0, \dots, v_k; s_0, \dots, s_k) = \Big(-\frac{\partial}{\partial v_0}\Big)^{s_0 - 1} \dots \Big(-\frac{\partial}{\partial v_k}\Big)^{s_k - 1} \frac{1}{v_k(v_k + v_{k-1}) \dots (v_k + \cdots + v_0)}.$$

We shall write briefly

$$\widetilde{\phi}(v\,;\,s)=\left(-\frac{\partial}{\partial v}\right)^{s-1}\frac{1}{v_k(v_k+v_{k-1})\dots(v_k+\cdots+v_0)}\,.$$

Proof. The proof is an immediate consequence of the following lemmas.

Lemma 1 (A. Zelevinsky [Ze]).

$$\widetilde{\phi}_k(v_0\,,\,\dots,\,v_k\,;\,s_0\,,\,\dots,\,s_k)=\int_{0\le u_0\le\cdots\le u_k}u_0^{s_0-1}\dots u_k^{s_k-1}e^{-u_0v_0-\cdots-u_kv_k}\mathrm{d}u_0\dots\mathrm{d}u_k\,.$$

We shall abreviate this as

$$\widetilde{\phi}(v\,;\,s)=\int u^{s-1}e^{-\langle v\,,\,u\rangle}\mathrm{d}u\,.$$

Proof of Lemma 1. Since the simplex σ is central, we have

$$n!=\Gamma(s_0+\cdots+s_k)=\int_0^\infty t^{s_0-1}\dots t^{s_k-1}e^{-t}t^k\mathrm{d}t\,.$$

Put $u_k=1-u_0-\cdots-u_{k-1}$. The Fubini formula gives

$$\widetilde{\phi}_k(v_0\,,\,\dots,\,v_k\,;\,s_0\,,\,\dots,\,s_k)=\int_{\mathscr{R}(v)}\int_0^\infty\left(t\frac{u_0}{v_0}\right)^{s_0-1}\cdots\left(t\frac{u_k}{v_k}\right)^{s_k-1}\frac{e^{-t}t^k}{v_0\dots v_k}\mathrm{d}t\mathrm{d}u\,.$$

For the change of variables $t_i=tu_iv_i^{-1}$, the jacobian determinant of the transformation $(u_0\,,\,\dots,\,u_{k-1}\,,\,t)\mapsto(t_0\,,\,\dots,\,t_k)$ is $t^k(v_0\dots v_k)^{-1}$. The proof of the lemma is completed if one notices that $t=\sum_i t_iv_i$ and that the image of $\mathscr{R}(v)\times\mathbf{R}_+$ by the transformation is the set $\{(t_0\,,\,\dots\,,\,t_k):0\le t_0\le\cdots\le t_k\}$. □

Lemma 2. *Let* $t_0\,,\,\dots\,,\,t_k$ *be nonnegative integers. One has the formula*

$$\left(-\frac{\partial}{\partial v_0}\right)^{t_0}\cdots\left(-\frac{\partial}{\partial v_k}\right)^{t_k}\widetilde{\phi}_k(v_0\,,\,\dots\,,\,v_k\,;\,s_0\,,\,\dots\,,\,s_k)$$
$$=\widetilde{\phi}_k(v_0\,,\,\dots\,,\,v_k\,;\,s_0+t_0\,,\,\dots\,,\,s_k+t_k).$$

Namely, using our notation, we have

$$\left(-\frac{\partial}{\partial v}\right)^{t}\widetilde{\phi}(v\,;\,s)=\widetilde{\phi}(v\,;\,s+t)\,.$$

Proof of Lemma 2. It suffices to verify the following formula:

$$\forall i\in[0\,,\,k],\ \frac{\partial}{\partial v_i}\widetilde{\phi}_k(v_0\,,\,\dots\,,\,v_k\,;\,s_0\,,\,\dots\,,\,s_k)$$
$$=-\phi_k(v_0\,,\,\dots\,,\,v_k\,;\,s_0\,,\,\dots\,,\,s_i+1\,,\,\dots\,,\,s_k).$$

Lemma 2 is a consequence of Lemma 1 by commutation of symbols $\frac{\partial}{\partial v_i}$ and $\int$:

$$\frac{\partial}{\partial v_i}\widetilde{\phi}_k(v_0\,,\,\dots\,,\,v_k\,;\,s_0\,,\,\dots\,,\,s_k)=\int u^{s-1}(-u_i)e^{-\langle v\,,\,u\rangle}\mathrm{d}u$$
$$=-\int u_0^{s_0-1}\dots u_i^{s_i}\dots u_k^{s_k-1}e^{-\langle v\,,\,u\rangle}\mathrm{d}u \qquad \square$$
$$=-\widetilde{\phi}_k(v_0\,,\,\dots\,,\,v_k\,;\,s_0\,,\,\dots\,,\,s_i+1\,,\,\dots\,,\,s_k)\,.\quad\square$$

Remark. Whenever $s_i = 1$ for every i, we write $\tilde{\phi}_k(v\,;1)$ for $\tilde{\phi}_k(v\,;1,\dots,1)$, thus

$$\forall s = (s_0,\dots,s_k), \quad \tilde{\phi}_k(v\,;s) = \left(-\frac{\partial}{\partial v}\right)^{s-1}\tilde{\phi}_k(v\,;1).$$

Lemma 3.

$$\tilde{\phi}_k(v\,;1) = \frac{1}{v_k(v_k+v_{k-1})\cdots(v_k+\cdots+v_0)}.$$

Proof of Lemma 3. If $s_0 = \cdots = s_k = 1$, the function $\tilde{\phi}_k$ becomes

$$\begin{aligned}\tilde{\phi}_k(v\,;1) &= \int_{0\le u_0\le\cdots\le u_{k-1}} e^{-u_0v_0-\dots-u_{k-1}v_{k-1}}\mathrm{d}u_0\dots\mathrm{d}u_{k-1}\int_{u_{k-1}}^{\infty} e^{-u_kv_k}\mathrm{d}u_k\\ &= \frac{1}{v_k}\int_{0\le u_0\le\cdots\le u_{k-1}} e^{-u_0v_0-\cdots-u_{k-1}(v_{k-1}+v_k)}\mathrm{d}u_0\dots\mathrm{d}u_{k-1}\\ &= \frac{1}{v_k}\tilde{\phi}_{k-1}(v_0,\dots,v_{k-1}+v_k\,;1)\,.\end{aligned}$$

The lemma follows by induction on k. □

1.3. Proof of [BGM's] theorem. As a first application of the Theorem 1.1, we give a short proof of the principal theorem of [BGM].

Let $\sigma = (F_0,\dots,F_k)$ be a central k-simplex. With the previous notations, the face F_i is the intersection (in $\mathbf{R}^{n+1}$) of the hyperplane $\{x : \sum_i v_i(x) = 1\}$ with the standard simplex $\triangle$. We have $\dim(F_i) = -1 + \sum_{j=0}^{i} s_j$.

Proposition 1.2. *The poles of* $\omega(\sigma)$ *lie on the faces* F_i $(i<k)$ *of* $\triangle$. *For* $i<k$, *the order of the pole of* $\omega(\sigma)$ *on* F_i *is equal to* $\sum_{j=i+1}^{k}(s_j-1)$.

Proof. By definition, the poles of $\omega(\sigma)$ are those of the function f such that

$$\omega(\sigma)\wedge *\omega(\sigma) = f^2 dx_0\wedge\cdots\wedge dx_n,$$

where "$*$" denotes the *Hodge star* operator. We have first

$$\begin{aligned}W(S)\wedge *W(S) &= \Gamma(s)^2 \sum_{i_0\in S_0,\dots,i_k\in S_k} x_{i_0}^2\dots x_{i_k}^2\; dx_0\wedge\cdots\wedge dx_n\\ &= \Gamma(s)^2\prod_{i=0}^{k}\sum_{j\in S_i} x_j^2\; dx_0\wedge\cdots\wedge dx_n,\end{aligned}$$

thus

$$f^2(x) = \tilde{\phi}^2(v\,;s)\prod_{i=0}^{k}\sum_{j\in S_i} x_j^2.$$

For the computation of the orders of the poles of $\tilde{\phi}(v\,;s)$, we use a "generalized Leibniz formula", applied to the function $\tilde{\phi}(v\,;s)$ as written in Theorem 1.1. We get

$$\tilde{\phi}(v\,;s) = \sum_i \frac{\Gamma(s)}{i!}\left(-\frac{\partial}{\partial v_k}\right)^{i_{kk}}\frac{1}{v_k}\cdots\left(-\frac{\partial}{\partial v_0}\right)^{i_{00}+\cdots+i_{0k}}\frac{1}{v_0+\cdots+v_k},$$

where the "triangle of integers" $i = (i_{00}\,; i_{01}, i_{11}\,; i_{02}, i_{12}, i_{22}\,; \ldots; i_{0k}, i_{1k}, \ldots, i_{kk})$ verifies:

$$\begin{aligned} i_{00} &= s_0 - 1\,,\\ i_{01} + i_{11} &= s_1 - 1\,,\\ &\;\vdots\\ i_{0k} + i_{1k} + \cdots + i_{kk} &= s_k - 1. \end{aligned}$$

Hence, we have

$$\widetilde{\phi}(v\,;s) = \frac{\widetilde{N}(v\,;s)}{\widetilde{D}(v\,;s)}\,,$$

where $\widetilde{D}(v\,;s) = \prod_{0\le i\le k}(v_k+\cdots+v_i)^{s_k+\cdots+s_i-(k-i)}$ and $\widetilde{N}(v\,;s)$ is a polynomial whose restrictions to the hyperplanes $P_i = \{x : v_k + \cdots + v_i = 0\}$ are not identically zero. The poles of $\widetilde{\phi}(v\,;s)$ are the hyperplanes P_i with orders $s_k + \cdots + s_i - (k-i)$. One verifies that $\triangle \cap P_0 = \varnothing$ and that, for all $i < k$, $\triangle \cap P_{i+1} = F_i$. Moreover, the order of the pole F_i of $\omega(\sigma)$ is the order of P_{i+1} minus one (the contribution of $W(S)\wedge *W(S)$). Therefore, the order of F_i is equal to

$$(s_k + \cdots + s_{i+1} - (k-(i+1))) - 1 = \sum_{j=i+1}^{k}(s_j - 1)\,. \quad \square$$

We are now able to give an alternative proof of the following result:

Theorem 1.3 ([BGM], Theorem 5.2). *Let* $q \ge 1$. *One has*: $\omega(\sigma) \in \mathscr{L}^q(\triangle) \iff p_\sigma < \bar{p}(q)$.

Proof. Let $\mathbf{L}^1_{loc}(\mathbf{R}^n)$ denote the space of locally integrable functions over $\mathbf{R}^n$ for the Lebesgue measure. The *generalized Riemann criterion of local integrability* implies that, for every homogeneous polynomial function P defined on $\mathbf{R}^n$, one has

$$\frac{1}{P} \in \mathbf{L}^1_{loc}(\mathbf{R}^n) \iff \text{degree}(P) < \text{codim}(P^{-1}\{0\}) \quad (\text{codim}(\varnothing) = +\infty)\,.$$

The general case follows easily: for $q \ge 1$, one has

$$\frac{1}{P} \in \mathbf{L}^q_{loc}(\mathbf{R}^n) \iff q.\text{degree}(P) < \text{codim}(P^{-1}\{0\})\,,$$

where $\mathbf{L}^q_{loc}(\mathbf{R}^n)$ is the space of functions f such that $f^q \in \mathbf{L}^1_{loc}(\mathbf{R}^n)$.

Let us apply this criterion to the shadow forms $\omega(\sigma)$. The Proposition 1.2 gives the orders of the poles of $\omega(\sigma)$. Therefore

$$\begin{aligned} \omega(\sigma) \in \mathscr{L}^q(\triangle) &\iff \forall i < k, \quad q\sum_{j=i+1}^{k}(s_j - 1) < n - \Big(\sum_{j=0}^{i} s_j - 1\Big)\\ &\iff \forall i > 0, \quad q\sum_{j=i}^{k}(s_j - 1) < \sum_{j=i}^{k} s_j\\ &\iff p_\sigma < \bar{p}(q)\,. \end{aligned}$$

The last equivalence follows from the fact that only the vertices of the graph of p_σ occur in the previous inequality. □

Let σ be a central abstract k-simplex of $\triangle$ and $S = (S_0, \dots, S_k)$ denote its encoding. From Theorem 1.3, we deduce the following corollary.

Corollary. *The following properties hold:*
(i) $\omega(\sigma) \in \mathscr{L}^1(\triangle)$;
(ii) $\omega(\sigma) \in \mathscr{L}^\infty(\triangle) \iff \forall i > 0,\ s_i = 1 \iff \omega(\sigma) \in \mathscr{L}^n(\triangle)$.

2. Generalized shadow forms

Theorem 1.1 shows that the shadow form of a central simplex σ is rational and that its poles are located on the sequence of faces "defining σ". This remark motivates the construction of complexes more appropriate to the study of shadow forms, especially of their duality (Theorem 2.6) and of their product (Theorem 2.8, Theorem 2.11).

2.1. Duality formula. The aim of this section is to extend the definition of shadow forms to all simplices of $\triangle'$, even those which are situated in proper faces of $\triangle$. For this reason we will modify the usual differential map on the simplicial chains, by "cutting" it. We define a new complex called "complex of coded chains", on which will be defined a "generalized shadow morphism". Then, we obtain a duality formula for shadow forms.

2.1.1. *Complex of coded chains.* For every face F of $\triangle = \triangle^n$, let us put $v_F = \sum_{i \in S_F} x_i$ and define the complex $\Omega^*(F)$ of "differential forms with poles on F":

$$\Omega^*(F) = \Lambda^*(\mathbf{R}^{n+1}) \otimes \mathbf{R}[F],$$

where $\mathbf{R}[F]$ is a sub-algebra of "Laurent polynomials" of $\mathbf{R}(x_0, \dots, x_n)$,

$$\mathbf{R}[F] = \mathbf{R}\left[\left\{\frac{1}{v_G}\right\}_{G<F}\right].$$

Remark that, since the coefficients are rational, given $F \overset{j}{\hookrightarrow} G$ two faces of $\triangle$, there is a canonical *covariant inclusion*: $j_* : \Omega^*(F) \hookrightarrow \Omega^*(G)$.

Each face F of $\triangle$ is canonically isomorphic to the standard simplex:

$$(x_0, \dots, x_n) \in F \overset{i_F}{\longmapsto} (x_j)_j \in \triangle^{\dim F},$$

where j runs over all labels of ordered vertices a^j in F. Through this isomorphism, we can define central simplices in F and construct the complex $(\mathscr{C}_*(\dot F), \dot\partial_F)$ of the *F-central chains*: $\dot F$ denotes the set of coded simplices S which are representations of simplices σ of F' which are central relatively to F, *i.e.* having the barycenter of F as a vertex. One defines $\dot\partial_F S$ by $\dot\partial_F(\sigma) = \partial\sigma \cap \dot F$. Geometrically speaking, if σ is a simplex lying in the face F then $\dot\partial_F(\sigma)$ is the part of $\partial\sigma$ which does not lie in the boundary of F. It is clear that $\dot\partial_F \circ \dot\partial_F = 0$.

Given a coded simplex S, there exists an unique face F_S of $\triangle$ such that S is central relatively to F_S. We have the natural decompositions:

$$\triangle' = \bigsqcup_{F \le \triangle} \dot F \quad \text{and} \quad \mathscr{C}_*(\triangle') = \bigoplus_{F \le \triangle} \mathscr{C}_*(\dot F).$$

Definition. The complex of coded chains is defined by

$$\dot{\mathscr{C}}_*(\Delta) = (\mathscr{C}_*(\Delta'), \dot{\partial}) \quad \text{where} \quad \dot{\partial} = \bigoplus_{F<\Delta} \dot{\partial}_F .$$

The value of $\dot{\partial}$ on a coded simplex $S = (S_0, \dots, S_k)$ is given by the formula

$$\dot{\partial}(S_0, \dots, S_k) = \sum_{i=0}^{k-1} (-1)^{n_S - i} (S_0, \dots, S_i \cup S_{i+1}, \dots, S_k) ,$$

where n_S denotes the dimension of F_S. Roughly speaking, the differential map $\dot{\partial}$ differs from the usual one ∂, only by the absence of the "last" simplex $(S_0, \dots, S_{k-1})$.

Definition. The *shadow morphism* $\omega : \mathscr{C}_*(\Delta') \longrightarrow \Omega^*(\Delta)$ is defined on each factor $\mathscr{C}_*(\dot{F}) \longrightarrow \Omega^{*-\dim(F)}(\Delta)$, $F < \Delta$, by $\omega = j_* \circ \omega_F \circ i_{F*}$, where ω_F is the classical shadow morphism of [BGM] defined over the standard simplex F, j_* is the covariant inclusion and i_{F*} is the isomorphism deduced from i_F.

For a coded simplex S (hence F_S-central), the corresponding shadow form $\omega(S)$ has degree $n_S - \dim(S)$. The following proposition is a corollary of Theorem 1.1.

Proposition 2.1. *Using the previous notations, one has*

$$\omega(S) = (-1)^{\#S} \frac{W(S)}{\Gamma(s)} \widetilde{\phi}(v\,; s).$$

2.1.2. *Functional equations of* $\widetilde{\phi}$. Let us fix a coded k-simplex $S = (S_0, \dots, S_k)$. The application $\widetilde{\phi}(v\,; s)$ is homogeneous of degree $-\sum_{i=0}^{k} s_i$, hence it is a solution of the Euler equation:

$$\sum_{i=0}^{k} \Big(s_i + v_i \frac{\partial}{\partial v_i}\Big) \widetilde{\phi}(v\,; s) = 0 .$$

A refinement of the Euler formula (Proposition 2.2) will provide the relation between the shadow form of a simplex and those of its faces (see Theorem 2.3).

A coded k-simplex S admits $k+1$ faces:

$$S^{(i)} = \begin{cases} (S_0, \dots, S_i \sqcup S_{i+1}, \dots, S_k) & \text{for } 0 \le i < k, \\ (S_0, \dots, S_{k-1}) & \text{for } i = k. \end{cases}$$

Let $i \in [0, k]$ be fixed. Rewrite the i-th face of S, $S^{(i)} = (S_0^{(i)}, \dots, S_{k-1}^{(i)})$ and associate to it the sequence of integers $s^{(i)} = (s_0^{(i)}, \dots, s_{k-1}^{(i)})$ defined by $s_j^{(i)} = \mathrm{card}(S_j^{(i)})$. In the same way, denote $v^{(i)} = (v_0^{(i)}, \dots, v_{k-1}^{(i)})$ where, for every j, $v_j^{(i)} = \sum_{h \in S_j^{(i)}} x_h$. As usual, define also $\#S^{(i)} = \sum_j j s_j^{(i)}$.

Notation. For every $i \in [0, k]$, we will write $\widetilde{\phi}^{(i)}(v\,; s)$ instead of $\widetilde{\phi}(v^{(i)}, s^{(i)})$. For convenience, we also define $\widetilde{\phi}^{(-1)}(v\,; s) = \widetilde{\phi}^{(k+1)}(v\,; s) = 0$.

Proposition 2.2 (Functional equations for $\widetilde{\phi}$). *For* $i \in [0, k]$, *one has*

$$\Big(s_i + v_i \frac{\partial}{\partial v_i}\Big) \widetilde{\phi} = \widetilde{\phi}^{(i)} - \widetilde{\phi}^{(i-1)} .$$

Proof. Consider the particular case $s_0 = \cdots = s_k = 1$. We have (Theorem 1.1, Lemma 3):

$$\begin{aligned}\widetilde{\phi}(v\,;\,1) =& \frac{1}{v_k \cdots (v_k + \cdots + v_{i+1})(v_k + \cdots + v_i) \cdots (v_k + \cdots + v_0)} \\ =& \frac{1}{v_i}\Big[\widetilde{\phi}(v_0, \ldots, v_{i-1} + v_i, v_{i+1}, \ldots, v_k\,;\,1) \\ & \qquad - \widetilde{\phi}(v_0, \ldots, v_{i-1}, v_i + v_{i+1}, \ldots, v_k\,;\,1)\Big].\end{aligned}$$

Therefore, applying the differential operator $(-\frac{\partial}{\partial v})^{s-1}$ (and using the second lemma of the proof of Theorem 1.1), we obtain

$$\left(-\frac{\partial}{\partial v}\right)^{s-1} v_i\widetilde{\phi}(v\,;\,1) = \widetilde{\phi}^{(i)}(v\,;\,s) - \widetilde{\phi}^{(i-1)}(v\,;\,s)\,.$$

On the other hand, it is easy to check the following formula:

$$\left(-\frac{\partial}{\partial v}\right)^{s-1}\left(-\frac{\partial}{\partial v_i}\right)v_i = \left(-s_i - v_i\frac{\partial}{\partial v_i}\right)\left(-\frac{\partial}{\partial v}\right)^{s-1}.$$

The proof ends by applying the left operator to $\widetilde{\phi}(v\,;\,1)$ (and using again the second lemma). □

The following result gives the connection between the shadow form associated to a simplex and the ones associated to its faces.

Corollary 2.3 ([BGM], Cororollary 2.2). *The map ω is a morphism of differential modules : $\dot{\mathscr{E}}_*(\Delta) \longrightarrow \Omega^*(\Delta)$.*

Proof. For every i such that $0 \le i \le k$, $v_i^{-s_i}$ is an integrating factor for $W(S_i)$, and

$$\frac{W(S)}{v^s\Gamma(s)} = \frac{W(S_0)}{v_0^{s_0}\Gamma(s_0)} \wedge \cdots \wedge \frac{W(S_k)}{v_k^{s_k}\Gamma(s_k)}$$

is a closed differential form and

$$\begin{aligned}\mathrm{d}\omega(S) =& (-1)^{\#S}\mathrm{d}\big(v^s\widetilde{\phi}(v\,;\,s)\big) \wedge \frac{W(S)}{v^s\Gamma(s)} \\ =& (-1)^{\#S}\sum_{i=0}^{k}\left\{s_i\widetilde{\phi} + v_i\frac{\partial\widetilde{\phi}}{\partial v_i}\right\}\frac{\mathrm{d}v_i}{v_i} \wedge \frac{W(S)}{\Gamma(s)}.\end{aligned}$$

Using Proposition 2.2, the i-th term of the sum becomes:

$$(-1)^{ns+i+\#S^{(i)}+s_i}\left\{\widetilde{\phi}^{(i)} - \widetilde{\phi}^{(i-1)}\right\}\frac{W(S_0)}{\Gamma(s_0)} \wedge \cdots \wedge \mathrm{d}S_i \wedge \cdots \wedge \frac{W(S_k)}{\Gamma(s_k)}\,.$$

Glueing the terms containing $\widetilde{\phi}^{(i)}$ yields:

$$\begin{aligned}(-1)^{ns+i}\omega(S^{(i)}) - (-1)^{\#S^{(i)}+ns+i}\,\widetilde{\phi}^{(i+1)}\frac{W(S_0)}{\Gamma(s_0)} \wedge \cdots \wedge \mathrm{d}S_{i+1} \wedge \cdots \wedge \frac{W(S_k)}{\Gamma(s_k)} \\ - (-1)^{\#S^{(i-1)}+ns+i}\,\widetilde{\phi}^{(i)}\frac{W(S_0)}{\Gamma(s_0)} \wedge \cdots \wedge \mathrm{d}S_i \wedge \cdots \wedge \frac{W(S_k)}{\Gamma(s_k)}\,.\end{aligned}$$

Since the two last terms vanish in the summation, one gets the formula

$$\begin{aligned} d\omega(S_0, \dots, S_k) &= (-1)^{ns}\sum_{i=0}^{k-1}(-1)^i\omega(S_0, \dots, S_i \cup S_{i+1}, \dots, S_k) \\ &= (-1)^{ns}\sum_{i=0}^{k-1}(-1)^i\omega(S^{(i)}) \\ &= \omega\dot{\partial}(S_0, \dots, S_k). \quad \square \end{aligned}$$

2.1.3. *The duality formula.*

Definition. Given a coded k-simplex of $\triangle$, $S = (S_0, \dots, S_k)$, the *antipode* of S is the coded k-chain defined by $\hat{S} = (-1)^k(S_k, S_{k-1}, \dots, S_1, S_0)$. One extends the definition of the antipode by linearity: $\mathscr{C}_*(\triangle) \xrightarrow{\hat{}} \mathscr{C}_*(\triangle)$. We get an involutive isomorphism of the graded module $\mathscr{C}_*(\triangle)$. Furthermore,

Proposition 2.4. *The antipode map anticommutes with the differential map* $\dot{\partial}$:

$$\forall c \in \mathscr{C}_*(\triangle), \quad \dot{\partial}\hat{c} = -\widehat{\dot{\partial}c}\,.$$

A natural question arises here: "Does there exist a relation between the *antipode* of a coded chain and its *shadow form* ? ". The next theorem answers this question affirmatively.

Proposition 2.5. *For every coded* k*-simplex* $S = (S_0, \dots, S_k)$,

$$\sum_{i=0}^{k+1}(-1)^i\widetilde{\phi}_{i-1}(v_0, \dots, v_{i-1}; s_0, \dots, s_{i-1})\, \widetilde{\phi}_{k-i}(v_k, \dots, v_i; s_k, \dots, s_i) = 0.$$

Proof. Here we prove this proposition for $s_0 = \cdots = s_k = 1$. The general case is obtained by using iterated derivations (Theorem 1.1, Lemma 2). From Lemma 3 of the proof of Theorem 1.1, we obtain

$$\widetilde{\phi}_k(v\,; 1) = \sum_{i=0}^{k}\frac{R_i(v)}{v_k + \cdots + v_i}\ ,$$

where

$$\begin{aligned} R_i(v) &= \lim_{v_k \to -v_{k-1}-\cdots-v_i}\frac{v_k + \cdots + v_i}{v_k(v_k+v_{k-1})\dots(v_k+\cdots+v_0)} \\ &= \frac{1}{(-v_{k-1}-\cdots-v_i)(-v_{k-2}-\cdots-v_i)\dots(-v_i)v_{i-1}\dots(v_{i-1}+\cdots+v_0)} \\ &= (-1)^{k-i}\widetilde{\phi}_{k-i-1}(v_{k-1}, \dots, v_i; 1)\widetilde{\phi}_{i-1}(v_0, \dots, v_{i-1}; 1), \end{aligned}$$

Finally, one gets

$$\widetilde{\phi}_k(v\,; 1) = \sum_{i=0}^{k}(-1)^{k-i}\widetilde{\phi}_{k-i}(v_k, \dots, v_i; 1)\widetilde{\phi}_{i-1}(v_0, \dots, v_{i-1}; 1).$$

This formula is exactly the one of the proposition for $s_i = 1$, $0 \le i \le k$. $\square$

An immediate corollary of Proposition 2.5 is the *duality formula* at the level of differential forms; it gives the link between the volume of a simplex and the volume of its antipodal simplex.

Theorem 2.6. *For every k-simplex $S = (S_0, \dots, S_k)$, one has the identity :*

$$\sum_{R \sqcup T = S} \varepsilon_{(R,T)}\, \omega(R) \wedge \omega(\hat{T}) = 0,$$

for suitable signs $\varepsilon_{(R,T)} \in \{-1, +1\}$.

2.2. Product formula. In [BGM], the authors define for any two central simplices, a kind of "transversality", in relation with the vanishing of the product of their shadow forms. We give the geometric interpretation of this product, in terms of "convexity".

2.2.1. *Chen's iterated integrals.* The *duality formula* (cf. Proposition 2.5) is analogous to formulae introduced by Kuo Tsai Chen [Ch] for *iterated integrals.* In the simplest cases, "Chen's data" consist of a path $\alpha : [0, 1] \longrightarrow \mathbf{R}^n$ and of a family of differential 1-forms $\omega_0, \dots, \omega_k$ on $\mathbf{R}^n$. We shall denote by α^t, with $t \in [0, 1]$, the restriction of α to $[0, t]$ and we shall denote by $(i_1, \dots, i_p)$ any *word* of $\{0, \dots, k\}$ with *length* $p > 0$. Chen defines recursively the so-called *iterated integrals of* α: for $p = 1$, $\int_\alpha \mathrm{d}(i_p) = \int_\alpha \omega_{i_p}$ and, for $p > 1$,

$$\int_\alpha \mathrm{d}(i_1, \dots, i_p) = \int_0^1 \left(\int_{\alpha^t} \mathrm{d}(i_1, \dots, i_{p-1}) \right) \alpha^*(\omega_{i_p}).$$

Definition. For two integers p and q such that $0 \le q \le p \le n$, one calls *shuffle of* $[1, q]$ *and of* $[q+1, p]$, any permutation σ of $[1, n]$ such that the restrictions of σ^{-1} to $[1, q]$ and to $[q+1, p]$ are increasing.

Let $l < k$ and $s_0, \dots, s_k$ be nonzero integers. One considers, for real variables $v_0, \dots, v_k$ and for words $i = (i_1, \dots, i_p)$ of $[0, k]$, the rational functions

$$\widetilde{\phi}(v_{i_1}, \dots, v_{i_p}; s_{i_1}, \dots, s_{i_p}).$$

Remark that these functions are not necessarily associated to some coded simplex.

Proposition 2.7. *The functions* $\widetilde{\phi}(v; s)$ *satisfy the shuffle relations* [Re]:

$$\begin{aligned}\widetilde{\phi}(v_0, \dots, v_l; s_0, \dots, s_l)&\widetilde{\phi}(v_{l+1}, \dots, v_k; s_{l+1}, \dots, s_k)\\ &= \sum_\sigma \widetilde{\phi}(v_{\sigma(0)}, \dots, v_{\sigma(k)}; s_{\sigma(0)}, \dots, s_{\sigma(k)}),\end{aligned}$$

where σ runs over the shuffles of $[0, l]$ *and of* $[l+1, k]$.

Proof. By Theorem 1.1 (Lemma 2), we have just to prove the following formula:

$$\widetilde{\phi}(v_0, \dots, v_l; 1)\widetilde{\phi}(v_{l+1}, \dots, v_k; 1) = \sum_\sigma \widetilde{\phi}(v_{\sigma(0)}, \dots, v_{\sigma(k)}; 1).$$

But this formula is precisely the illustration given by Chen for his theorem [Ch, p. 176]. Let $v_0, \dots, v_k$ be fixed positive numbers and consider the following data:

—the path $\alpha : [0, 1] \longrightarrow \mathbf{R}^{k+1}$, $\alpha_i(t) = t^{v_i}$, $0 \le i \le n$;
—the differential 1-forms $\omega_i = \mathrm{d}x_i$, $0 \le i \le k$.
Chen finds the following iterated integrals:

$$\int_\alpha \mathrm{d}(i_1, \dots, i_p) = \frac{v_{i_1} \dots v_{i_p}}{v_{i_1}(v_{i_1} + v_{i_2}) \dots (v_{i_1} + \dots + v_{i_p})}.$$

That is to say, with our notation (cf. Lemma 3 of the proof of Theorem 1.1),

$$\int_\alpha \mathrm{d}(i_1, \dots, i_p) = v_{i_1} \dots v_{i_p} \widetilde{\phi}(v_{i_p}, v_{i_{p-1}}, \dots, v_{i_1}; 1).$$

According to Chen [Ch] and Ree [Re], the family (indexed by the words i of the set $[0, k]$)

$$\{v_{i_1} \dots v_{i_p} \widetilde{\phi}(v_{i_p}, v_{i_{p-1}}, \dots, v_{i_1}; 1)\}_i$$

verifies the "shuffle relations". It is then easy to see that this remains also true for the other family $\{\widetilde{\phi}(v_{i_1}, \dots, v_{i_p}; 1)\}_i$. And this proves the proposition. □

2.2.2. *Product of shadow forms.* We give a definition of *affine transversality*, slightly different from the one introduced in [BGM], for example we shall consider all coded simplices of $\triangle$.

Definition. Two coded simplices S and T are said *affinely transversal* iff

$$F_S \cap F_T = \varnothing .$$

Geometrically speaking, two simplices are affinely transversal if they are contained in opposite faces; for example, two simplices with same center could not be. It is also clear that this kind of transversality involves that of [BGM].

Let $S = (S_0, \dots, S_k)$ and $T = (T_0, \dots, T_l)$ be two affinely transversal coded simplex and $R = (R_0, \dots, R_{k+l+1}) = (S_0, \dots, S_k, T_0, \dots, T_l)$: it is a coded $(k + l + 1)$-simplex. We shall write also $r = (r_0, \dots, r_{k+l+1})$, where $r_i = \# R_i$.

Definition. The *join of S and of T* is the coded $(k + l + 1)$-chain

$$S * T = \sum_\sigma \operatorname{sgn}(\sigma)(-1)^{\varepsilon(\sigma, S, T)} (R_{\sigma(0)}, \dots, R_{\sigma(k+l+1)}),$$

where σ runs over the shuffles of $[0, k]$ and of $[k+1, k+l+1]$, $\operatorname{sgn}(\sigma)$ is its signature and where

$$\varepsilon(\sigma, S, T) = (1 + k) \sum_{i > 1+k} r_i + \sum_{i<j,\ \sigma^{-1}(i) > \sigma^{-1}(j)} r_i r_j .$$

For $p \in \mathring{\triangle}$, the geometric realization of the join $S * T$ is the sum (with signs) of the maximal simplices of the smallest convex subset of $\triangle$ containing $S(p)$ and $T(p)$. The following theorem is *the product formula of shadow forms.* It is the main motivation for introducing shadow forms for a noncentral coded simplex and must be compared with the vanishing criterion of the product of shadow forms in [BGM].

Theorem 2.8. *If S and T are affinely transversal, we have the property*

$$\omega(S * T) = \omega(S) \wedge \omega(T).$$

Proof. It is a corollary of Proposition 2.7. We have just to verify that the "signs" $\operatorname{sgn}(\sigma)(-1)^{\varepsilon(\sigma, S, T)}$ previously introduced are the required ones. □

2.3. Morphism of algebras. In this last section, our purpose is to extend the shadow morphism to a complex, canonically endowed with a structure of a graded differential anticommutative algebra, for which the shadow map is a morphism.

2.3.1. *Algebra of singular cochains.* Let k be a natural integer. By definition, a *nonoriented singular k-simplex of* $\triangle$ is a sequence $S = (S_0, \dots, S_k)$ of nonempty words of $\{0, \dots, n\}$. We associate to it, its *weight* $s = (s_0, \dots, s_k)$ where, far all i, s_i is the *length* of the word S_i, also denoted $\#S_i$. The *total weight* is defined by $n_S = \sum_i s_i$.

Denote by $\mathscr{C}_k^{sing}(\triangle)$ the vector space spanned by the singular k-simplices and by $\mathscr{C}_*^{sing}(\triangle)$, their sum:

$$\mathscr{C}_*^{sing}(\triangle) = \bigoplus_{k \in \mathbf{N}} \mathscr{C}_k^{sing}(\triangle).$$

We now define the vector space of *oriented simplices* $\mathscr{C}_{sing}(\triangle)$, as the quotient space (nongraded) of $\mathscr{C}_*^{sing}(\triangle)$ by the "orientations relations":

$$S^\sigma \sim \mathrm{sgn}(\sigma) S \quad \forall k \in \mathbf{N},\ \forall S = (S_0, \dots, S_k),\ \forall \sigma = (\sigma_0, \dots, \sigma_k),$$

where the terms above are the following: σ is a sequence of $k+1$ permutations $\sigma_i : [1, s_i] \to [1, s_i]$, and for $S_i^{\sigma_i} = \sigma_i \circ S_i$, we used the notation: $S^\sigma = (S_0^{\sigma_0}, \dots, S_k^{\sigma_k})$.

On the other hand, the "sign" of σ is defined by: $\mathrm{sgn}(\sigma) = \prod_i \mathrm{sgn}(\sigma_i)$.

We shall endow $\mathscr{C}_{sing}(\triangle)$ with the *codimension* grading, defined for all singular simplex S, by $\mathrm{codim}(S) = n_S - \dim(S) - 1$.

For all integer k, denote by $\mathscr{C}_{sing}^k(\triangle)$ the sub-space of $\mathscr{C}_{sing}(\triangle)$ spanned by the oriented singular simplices of codimension k. We have the decomposition

$$\mathscr{C}_{sing}^*(\triangle) = \bigoplus_k \mathscr{C}_{sing}^k(\triangle).$$

Coboundary. Let us define the graded linear map $\mathrm{d} : \mathscr{C}_{sing}^*(\triangle) \longrightarrow \mathscr{C}_{sing}^{*+1}(\triangle)$, by its value on each singular simplex $S = (S_0, \dots, S_k)$,

$$\mathrm{d}(S) = (-1)^{n_S} \sum_{i=0}^{k-1} (-1)^i (S_0, \dots, S_i \sqcup S_{i+1}, \dots, S_k),$$

where "$\sqcup$" is the *concatenation of words.* Let us remark that $\mathrm{codim}(\mathrm{d}S) = \mathrm{codim}(S) + 1$.

Proposition 2.9. *The map* d *is a coboundary map, i.e.* $\mathrm{d}^2 = 0$.

So, the set $\mathscr{C}_{sing}^*(\triangle)$ of singular cochains is a graded differential module. The differential (nongraded) module $\mathscr{C}_*(\triangle)$ of *coded chains* maps naturally into $\mathscr{C}_{sing}^*(\triangle)$. It is true that d is also a *boundary map* for the degree dim. But, in the sequel, it will clearly appear that codim is the degree which gives the "right grading". In the regular central case of [BGM], the gradings associated to dim and codim are the same.

Product of singular cochains. Let $S = (S_0, \dots, S_k)$ and $T = (T_0, \dots, T_l)$ be two singular simplices and put down $R = (S_0, \dots, S_k, T_0, \dots, T_l)$ and $r = (r_0, \dots, r_{k+l+1})$ its weight. The *join* of S and T, which we denote by $S * T$, is the singular cochain

$$S * T = (-1)^{(1+\dim(S))n_T} \sum_\sigma \mathrm{sgn}(\sigma)(-1)^{\epsilon(\sigma, R)} (R_{\sigma(0)}, \dots, R_{\sigma(k+l+1)}),$$

where σ runs the set of all shuffles of $[0, k]$ and of $[k+1, k+l+1]$, $\mathrm{sgn}(\sigma)$ is its sign, and where

$$\epsilon(\sigma, R) = \sum_{i<j\,:\,\sigma^{-1}(i)>\sigma^{-1}(j)} r_i r_j.$$

The operation " $*$ " is then extended to the cochains, by bilinearity:

$$\mathscr{C}^*_{sing}(\Delta) \times \mathscr{C}^*_{sing}(\Delta) \longrightarrow \mathscr{C}^*_{sing}(\Delta).$$

Theorem 2.10. $(\mathscr{C}^*_{sing}(\Delta)$, codim, d$)$ *is a graded differential algebra.*

The sub-module $\mathscr{C}^0_{sing}(\Delta)$ is a commutative sub-algebra of $\mathscr{C}^*_{sing}(\Delta)$; its product is nothing but the *shuffle product* [Re].

2.3.2. *The shadow morphism.* Let $S = (S_0, \dots, S_k)$ be a singular nonoriented simplex. For all i, $0 \le i \le k$, we put $v_i(x) = \sum_{j\in S_i} x_j$ (sum with eventual repetitions).

Definition. The shadow form $\omega(S)$ is the differential form

$$\omega(S) = (-1)^{\sum_i is_i} \frac{W(S_0)}{\Gamma(s_0)} \wedge \cdots \wedge \frac{W(S_k)}{\Gamma(s_k)}\, \tilde{\phi}_k(v_0, \dots, v_k; s_0, \dots, s_k).$$

Then, the shadow morphism ω is extended to a linear morphism on all nonoriented singular cochains. It is easy to see that the map ω is compatible with the orientations relations and it induces a morphism on the oriented cochains: $\omega : \mathscr{C}^*_{sing}(\Delta) \to \Omega^*(\Delta)$.

Theorem 2.11. *The map ω is a graded morphism of differential algebra.*

2.3.3 *Cyclic simplices.* The complex of oriented singular chains is "big": the dimension of each of its graded components is infinite; moreover, it is not bounded. The next definition solves a part of this problem.

Definition. A singular simplex $S = (S_0, \dots, S_k)$ is said to be *cyclic,* if there exist $i_0, \dots, i_p$, distinct integers of $[0, k]$ $(1 \le p \le k)$, such that

$$\forall j : 0 \le j \le p, \quad \mathrm{card}\big((S_{i_{j-1}} \cap S_{i_j}) \cup (S_{i_j} \cap S_{i_{j+1}})\big) \ge 2,$$

where i_{-1} is equal to i_p and i_{p+1} to i_0.

Some remarks. For every cyclic simplex S, it is clear that ∂S is a "cyclic chain". Regular simplices are not cyclic and a zero codimension simplex is not cyclic either. The adjective *cyclic* takes its meaning from Graph Theory [BGM]: denote F_S, the face of Δ, the vertices of which are labelled by the elements of $\bigcup_i S_i$ and represent S_i with any minimal graph G_i with vertices those of F_S labelled by the elements of S_i (so G_i is a tree). One has

$$\bigcup_{i=0}^{k} G_i \text{ contains a cycle} \Longleftrightarrow S \text{ is cyclic.}$$

We shall denote $\mathscr{C}^*_{cycle}(\Delta)$ the sub-space of $\mathscr{C}^*_{sing}(\Delta)$ spanned by the cyclic simplices and we shall speak of *cyclic chains.* With the graph representation, it is now clear that any chain of codimension greater than n is cyclic, *i.e.*

$$\forall p > n, \quad \mathscr{C}^p_{cycle}(\Delta) = \mathscr{C}^p_{sing}(\Delta).$$

Proposition 2.12. *The module $\mathscr{C}^*_{cycle}(\Delta)$ is a differential ideal of $\mathscr{C}^*_{sing}(\Delta)$.*

Let us denote by $\mathscr{C}^*_{reg}(\Delta)$ the quotient of vector spaces $\mathscr{C}^*_{sing}(\Delta)/\mathscr{C}^*_{cycle}(\Delta)$.

Corollary. *The space* $(\mathscr{C}^*_{reg}(\Delta), \dot{\partial})$ *is a bounded graded differential algebra:*

$$\forall p > n, \quad \mathscr{C}^p_{reg}(\Delta) = \{0\}.$$

We also have a characterization of cyclic simplices with the shadow morphism (to compare with [BGM]'s result, Theorem 10.1(1)).

Proposition 2.13. *For any singular simplex* $S \in \mathscr{C}^*_{sing}(\Delta)$, *one has*

$$\omega(S) = 0 \Longleftrightarrow S \text{ is cyclic.}$$

Remark. This proposition is not true in general for the singular chains.

Corollary. *The shadow morphism is well defined on the quotient space and gives us a morphism of graded differential algebras,* $\omega : (\mathscr{C}^*_{reg}(\Delta), \mathbf{d}) \longrightarrow (\Omega^*(\Delta), \mathbf{d})$.

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