ISOMORPHISMS OF ADJOINT CHEVALLEY GROUPS
OVER INTEGRAL DOMAINS

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Abstract. It is shown that every automorphism of an adjoint Chevalley group
over an integral domain containing the rational number field is uniquely ex-
pressible as the product of a ring automorphism, a graph automorphism and an
inner automorphism while every isomorphism between simple adjoint Cheval-
ley groups can be expressed uniquely as the product of a ring isomorphism, a
graph isomorphism and an inner automorphism. The isomorphisms between
the elementary subgroups are also found having analogous expressions.

1. Introduction and main theorems

Let $G$ and $G'$ be simple Chevalley-Demazure group schemes of adjoint type.
Suppose $R$ and $R'$ are commutative integral domains containing the rational number
field $\mathbb{Q}$. The main purpose of this paper is to determine the isomorphisms between
Chevalley groups $G(R)$ and $G'(R')$, as well as the isomorphisms between their
elementary subgroups $E(R)$ and $E'(R')$. When $G$ is semisimple, the automorphisms
of $G(R)$ and $E(R)$ are also determined in this paper. When $R$ is a field, the
automorphisms of simple adjoint Chevalley groups over $R$ have been determined
by Steinberg [9] and Humphreys [8]. Our main results are as follows.

Theorem 1.1. Let $R$ and $R'$ be commutative integral domains containing $\mathbb{Q}$. Sup-
pose $G$ and $G'$ are simple adjoint Chevalley-Demazure group schemes whose ranks
are greater than 1. Then

(i) every isomorphism between $E(R)$ and $E'(R')$ can be extended uniquely to an
isomorphism between $G(R)$ and $G'(R')$;

(ii) if $\alpha$ is an isomorphism from $G(R)$ to $G'(R')$, the restriction of $\alpha$ to $E(R)$ is
an isomorphism from $E(R)$ to $E'(R')$.

Suppose $G$ and $G'$ are adjoint Chevalley-Demazure group schemes. Let $\Phi$ (resp.
$\Phi'$) be a root system of $G$ (resp. $G'$) and let $\Delta$ (resp. $\Delta'$) be a fundamental root
system of $\Phi$ (resp. $\Phi'$). We refer to [7] for the properties of Chevalley-Demazure
group schemes. In particular, for each root $a \in \Phi$ and for each commutative ring
$R$ with a unit, there is a canonical monomorphism (cf. [7, XXII])

$$u_{a,R} : R^+ \rightarrow G(R).$$
We rewrite this monomorphism simply as \( u_a \) for whatever commutative ring with a unit. We denote by \( U_a(R) \) the subgroup consisting of the elements \( u_a(r) \) for all \( r \in R \) and \( a \in \Phi \). The elementary subgroup \( E(R) \) of \( G(R) \) is generated by \( U_a(R) \) for all \( a \in \Delta \) or \(-\Delta\), where \(-\Delta\) is the set of the negative fundamental roots of \( \Phi \).

Since both \( G \) and \( G' \) are of adjoint type, it follows from Demazure’s fundamental theorem (see [7, XXIII, §5.1]) that every isomorphism of root systems between \( \Phi \) and \( \Phi' \) implies an isomorphism between \( G \) and \( G' \). Hence, if \( \gamma : \Phi \to \Phi' \) is an isomorphism of root systems such that \( \gamma(\Delta) = \Delta' \), it gives rise canonically to an isomorphism \( \gamma : G(R) \to G'(R) \) satisfying
\[
\gamma(u_a(r)) = u_{\gamma(a)}(r) \quad \text{for} \quad a \in \Delta \text{ or } -\Delta, \quad r \in R.
\]
We call this \( \gamma \) a graph isomorphism related to \( \gamma \). It is obvious that
\[
\gamma(E(R)) = E'(R).
\]
This identity allows us to define a graph isomorphism from \( E(R) \) to \( E'(R) \) related to \( \gamma \) to be an isomorphism \( \gamma : E(R) \to E'(R) \) which satisfies
\[
\gamma(u_a(r)) = u_{\gamma(a)}(r) \quad \text{for} \quad a \in \Delta \text{ or } -\Delta, \quad r \in R.
\]

Suppose \( G \) is a simple Chevalley-Demazure group scheme. Let \( R \) and \( R' \) be commutative rings with units. Since \( G \) is a covariant group functor on the category of commutative rings with units, every isomorphism \( \varphi : R \to R' \) gives rise to an isomorphism \( \varphi : G(R) \to G(R') \) in a canonical way, which is called the ring isomorphism related to \( \varphi \). In particular, we have
\[
\varphi(u_a(r)) = u_a(\varphi(r)) \quad \text{for} \quad a \in \Delta \text{ or } -\Delta, \quad r \in R.
\]
Thus
\[
\varphi(E(R)) = E'(R).
\]
We shall also call an isomorphism \( \varphi : E(R) \to E'(R') \) to be a ring isomorphism related to \( \varphi \) if it satisfies
\[
\varphi(u_a(r)) = u_a(\varphi(r)) \quad \text{for} \quad a \in \Delta \text{ or } -\Delta, \quad r \in R.
\]

**Theorem 1.2.** Let \( R, R', G \) and \( G' \) be as in Theorem 1.1. If \( \alpha \) is an isomorphism from \( G(R) \) to \( G'(R') \), then there exist an element \( g \in G'(R') \), an isomorphism of root system \( \gamma : \Phi \to \Phi' \) with \( \gamma(\Delta) = \Delta' \) and an isomorphism of rings \( \varphi : R \to R' \) such that
\[
\alpha = \text{Int } g \cdot \gamma \cdot \varphi.
\]
Moreover, \( g, \gamma \) and \( \alpha \) are uniquely determined by \( \alpha \).

**Remark 1.3.** The isomorphisms between \( E(R) \) and \( E'(R') \) have similar expressions where \( \hat{\gamma} \) and \( \hat{\varphi} \) are replaced by \( \gamma \) and \( \varphi \) respectively (see Theorem 3.9).

**Theorem 1.4.** Let \( R \) be a commutative integral domain which contains \( \mathbb{Q} \) and let \( G \) be an adjoint Chevalley-Demazure group scheme which has no simple component of type \( A_1 \). Then
(i) every automorphism of \( E(R) \) can be extended uniquely to an automorphism of \( G(R) \);
(ii) the restriction of each automorphism of \( G(R) \) to \( E(R) \) is an automorphism of \( E(R) \).

In particular, \( \text{Aut } G(R) \cong \text{Aut } E(R) \).
Let \( G \) be an adjoint Chevalley-Demazure group scheme and let \( \{ G_i \}_{i=1}^n \) be its simple components. Since \( G = \prod_{i=1}^n G_i \) (cf. \( \text{T, XXIV,§5.5} \)), \( G \) is a covariant group functor on the category of which the objects are of form \( \prod_{i=1}^n R \) for some commutative ring \( R \) with a unit. Hence, if \( \varphi_i : R \to R \) is an automorphism for \( 1 \leq i \leq n \), then the automorphism \( \prod_{i=1}^n \varphi_i \in \text{Aut} \ \prod_{i=1}^n R \) gives rise canonically to an automorphism of \( G(R) \), which is easily seen to be \( \prod_{i=1}^n \tilde{\varphi}_i \), and is called the ring automorphism of \( G(R) \) related to \( \prod_{i=1}^n \varphi_i \). The automorphism \( \prod_{i=1}^n \tilde{\varphi}_i \in \text{Aut} E(R) \) is also called the ring automorphism of \( E(R) \) related to \( \prod_{i=1}^n \varphi_i \).

**Theorem 1.5.** Let \( R \) and \( G \) be as in Theorem 1.4, then every automorphism \( \alpha \) of \( G(R) \) has an expression

\[
\alpha = \text{Int} \ g \cdot \tilde{\gamma} \cdot \prod_{i=1}^n \tilde{\varphi}_i \tag{1.5.1}
\]

where \( g \in G(R) \), \( \gamma : \Phi \to \Phi \) is an automorphism of root system which keeps the fundamental root system \( \Delta \) invariant and \( \varphi_i \in \text{Aut} R \) for all \( 1 \leq i \leq n \). Moreover, \( g, \gamma \) and \( \varphi_i (1 \leq i \leq n) \) are uniquely determined by \( \alpha \).

**Remark 1.6.** The automorphisms of \( E(R) \) have similar expressions where \( \tilde{\gamma} \) and \( \prod_{i=1}^n \tilde{\varphi}_i \) are replaced by \( \gamma \) and \( \prod_{i=1}^n \varphi_i \) respectively (see Theorem 4.2).

2. Preliminaries

Let \( H \) be a group. If \( M \) and \( P \) are subgroups of \( H \), we denote by \( \mathcal{C}_P(M) \) and \( \mathcal{N}_P(M) \) the centralizer and the normalizer of \( M \) in \( P \) respectively. The centre of \( H \) is denoted by \( C(H) \). A subgroup of \( H \) generated by subsets \( M_1, M_2, \ldots \) is written as \( \langle M_1, M_2, \ldots \rangle \) and \( [M_1, M_2] \) stands for the subgroup of \( H \) generated by the elements of the form \( xyx^{-1}y^{-1} \) for all \( x \in M_1, y \in M_2 \). If \( H \) is an algebraic group, we denote by \( L(H) \) the Lie algebra of \( H \). Suppose \( M \) is an abstract subgroup of \( H \), we denote by \( \overline{M} \) the Zariski closure of \( M \) in \( H \) and by \( M^0 \) the connected component of \( \overline{M} \) which contains the identity element of \( H \). Throughout this paper we fix a universal domain \( K \) of \( \mathbb{Q} \) and \( R \) (resp. \( R' \)) stands for a subring of \( K \) which contains \( \mathbb{Q} \).

In this section we give some preliminary properties of algebraic groups and Chevalley groups over a ring which are needed in the development of our discussion.

**Lemma 2.1.** The Zariski closure of every infinite abstract simple subgroup of an algebraic group is connected.

**Proof.** Suppose \( H \) is an infinite abstract simple subgroup of an algebraic group. Let \( \iota \) be the natural embedding of \( H \) into its Zariski closure \( \overline{H} \) and let \( \pi \) be the natural homomorphism from \( \overline{H} \) to its quotient group \( \overline{H}/\overline{H}^\circ \). Consider a composition of homomorphisms

\[
H @> \iota >> \overline{H} @> \pi >> \overline{H}/\overline{H}^\circ.
\]

Since \( \overline{H}/\overline{H}^\circ \) is a finite group, \( |H/\ker \pi| < \infty \). This yields, since \( H \) is infinite and simple,

\[
H = \ker \pi \iota = H \cap \overline{H}^\circ \subseteq \overline{H}.
\]

Taking Zariski closures of the above groups, we obtain immediately the connectedness of \( \overline{H} \). \( \square \)
Let $G$ be an adjoint Chevalley-Demazure group scheme with its root system $\Phi$ and fundamental root system $\Delta$. We denote by $U(R)$ (resp. $U^-(R)$) the subgroup of $G(R)$ generated by $U_a(R)$ for all $a \in \Phi^+$ (resp. $-a \in \Phi^+$), where $\Phi^+$ is the subset of positive roots of $\Phi$. Let $B$ (resp. $B^-$) be the normalizer of $U(K)$ (resp. $U^-(K)$) in $G(K)$, which is a Borel subgroup of $G(K)$ and let $T$ be the maximal torus of $G(K)$ which is contained in both $B$ and $B^-$. If $G'$ is also an adjoint Chevalley-Demazure group scheme, we denote analogically by $\Phi'$, $\Delta'$, $U'_a(R)$ ($a \in \Phi'$), $U'(R)$ (resp. $U'^-(R)$), $T'$ and $B'$ for related root systems and subgroups. Suppose $\gamma : \Phi \to \Phi'$ is an isomorphism of root systems such that $\gamma(\Delta) = \Delta'$, then $\gamma$ gives rise to a homomorphism of algebraic groups $\gamma : G(K) \to G'(K)$ defined by

$$\gamma(u_a(k)) = u_{\gamma(a)}(k) \text{ for } a \in \Delta \text{ or } -\Delta, \quad k \in K,$$

which is called the isogeny related to $\gamma$ (cf. [6, exp. 18]).

**Lemma 2.2.** Suppose $\varepsilon$ is an isogeny from $G(K)$ to $G'(K)$, then there exist an element $g \in G'(K)$ and an isomorphism of root systems $\gamma : \Phi \to \Phi'$ with $\gamma(\Delta) = \Delta'$ such that

$$\varepsilon = \Int g \cdot \gamma$$

where $\gamma$ is the isogeny related to $\gamma$.

**Proof.** Since $\varepsilon(B)$ is a Borel subgroup of $G'(K)$, there exists an element $g_1 \in G'(K)$ such that $\Int g_1 \varepsilon(B) = B'$. Moreover, $\Int g_1 \varepsilon(T)$ is a maximal torus of $B'$ since $T$ is contained in $B$. Hence there exists an element $g_2 \in B'$ such that

$$\Int (g_2 g_1) \varepsilon(T) = T'.$$

This, together with the fact that $\text{char} K = 0$ and

$$\Int (g_2 g_1) \varepsilon(B) = B',$$

gives rise to an isomorphism of root systems $\gamma : \Phi \to \Phi'$ with $\gamma(\Delta) = \Delta'$ such that (cf. [6, exp. 18])

$$\Int (g_2 g_1) \varepsilon(u_a(k)) = u_{\gamma(a)}(q_a k) \text{ for } a \in \Delta \text{ or } -\Delta, \quad k \in K,$$

where $q_a \in K^*$. Since the fundamental roots in $\Delta'$ are linearly independent, there exists an element $t \in T'$ such that

$$\gamma(a)(t) = q_a^{-1} \text{ for } a \in \Delta \text{ or } -\Delta.$$

Let $g = (t g_2 g_1)^{-1}$, we then have for each root $a \in \Delta$ or $-\Delta$

$$\Int g \varepsilon(u_a(k)) = u_{\gamma(a)}(k) \text{ for } k \in K,$$

from which follows (2.2.1) immediately. \qed

**Lemma 2.3.**

(i) $\overline{U}(R) = U(K)$; $\overline{U}^-(R) = U^-(K)$;

(ii) $T \cap E(R) = T$;

(iii) $B \cap E(R) = B$.

**Proof.** (i) Suppose $a$ is a positive root of $\Phi$. Since $U_a(R)$ is an infinite group while $\overline{U}_a(R)/U_a(R)$ is a finite group, $\overline{U}_a(R)$ must be infinite. In other words, $\dim \overline{U}_a(R) \geq 1$. On the other hand, since

$$\overline{U}_a(R) \subseteq \overline{U}_a(R) \subseteq U_a(K),$$

(2.3.1)
we have
\[ \dim \overline{U_a(R)} \leq \dim U_a(K) = 1. \]
Hence \( \dim \overline{U_a(R)} = 1 \) and (2.3.1) implies that
(2.3.2) \( \overline{U_a(R)} = U_a(K) \).
Therefore
\[ U(R) = \overline{U_a(R)} = \langle U_a(R) \mid \forall a \in \Phi^+ \rangle = U(K) \]
By taking negative roots instead of positive roots and by following a similar argument as above, we obtain also the Zariski density of \( U^-(R) \) in \( U^-(K) \).
(ii) Let \( \{a_1, a_2, \ldots, a_n\} \) be the fundamental roots of \( \Phi \) and write \( T_i \) for the one dimensional torus \( T \cap \langle U_{a_i}(K), U_{-a_i}(K) \rangle \) for all \( 1 \leq i \leq n \). Then
(2.3.3) \[ T = \prod_{i=1}^{n} T_i. \]
Let \( T_i(R) \) be the \( R \)-rational points of \( T_i \) for \( 1 \leq i \leq n \), then we have
\[ T \cap E(R) \supseteq \prod_{i=1}^{n} T_i(R). \]
Note that \( T_i(R) \) is Zariski dense in \( T_i \) by [1, Ch.V, Cor.18.3] since \( R \) contains rational field \( \mathbb{Q} \). Hence we obtain from (2.3.3) that
\[ T \supseteq T \cap E(R) \supseteq \prod_{i=1}^{n} T_i(R) = T. \]
This means that \( T \cap E(R) \) is Zariski dense in \( T \).
(iii) We have
\[ B \supseteq B \cap E(R) \supseteq (T \cap E(R)) \cdot U(R). \]
This yields
\[ B \supseteq B \cap E(R) \supseteq (T \cap E(R)) \cdot U(R) = T \cdot U = B. \]
Hence \( B \cap E(R) \) is Zariski dense in \( B \).

Recall that the semisimple complex Lie algebra \( L(G(\mathbb{C})) \) has a \( \mathbb{Z} \)-form \( \mathfrak{g} \) with a Chevalley basis related to the root system \( \Phi \) (cf. [10]). We denote by \( \mathfrak{g}_K \) the \( R \)-Lie algebra \( \mathfrak{g} \otimes_{\mathbb{Z}} R \) and let \( \text{ad} : \mathfrak{g}_K \to M_n(K) \) be the adjoint representation of \( \mathfrak{g}_K \), where \( n \) is the dimension of the Lie algebra \( \mathfrak{g}_K \) over \( K \) and \( M_n(K) \) is the algebra of \( n \times n \) matrices over \( K \).

**Lemma 2.4.** Suppose \( z \) is an element of \( \mathfrak{g}_K \) such that \( \text{ad}(z) \in M_n(R) \), then \( z \) lies in \( \mathfrak{g}_R \).

**Proof.** Let \( \{e_1, e_2, \ldots, e_n\} \) be a Chevalley basis of \( L(G(\mathbb{C})) \) related to \( \Phi \). Then \( \text{ad}(e_i) = M_{a_i}(\mathbb{Z}) \) for all \( 1 \leq i \leq n \). Suppose \( z \) has an expression \( \sum_{i=1}^{n} e_i \otimes k_i \), where \( k_i \in K \) for all \( 1 \leq i \leq n \). Then
(2.4.1) \[ \text{ad}(z) = \sum_{i=1}^{n} \text{ad}(e_i \otimes k_i). \]
On the other hand, we way assume \( \text{ad}(z) = (z_{pq}) \in M_n(R) \), where \( z_{pq} \in R \) for all \( 1 \leq p \leq n, 1 \leq q \leq n \) and suppose \( \text{ad}(e_i \otimes 1) = (e_i^{(i)}_{pq}) \in M_n(\mathbb{Z}) \), where \( e_i^{(i)} \in \mathbb{Z} \) for
all $1 \leq i \leq n, 1 \leq p \leq n$ and $1 \leq q \leq n$. Then the equation (2.4.1) implies the following $n^2$ equations:

\[
\begin{align*}
z_{11} &= k_1 e_{11}^{(1)} + k_2 e_{11}^{(2)} + \cdots + k_n e_{11}^{(n)} \\
z_{12} &= k_1 e_{12}^{(1)} + k_2 e_{12}^{(2)} + \cdots + k_n e_{12}^{(n)} \\
& \vdots \\
z_{nn} &= k_1 e_{nn}^{(1)} + k_2 e_{nn}^{(2)} + \cdots + k_n e_{nn}^{(n)}.
\end{align*}
\]

Since $ad(e_1 \otimes 1), ad(e_2 \otimes 1), \ldots, ad(e_n \otimes 1)$ are linearly independent, there are $n$ linearly independent equations in the above system. Therefore the unique solution for $k_1, k_2, \ldots, k_n$ in the above equations is given by Cramer's rule as the quotient of the determinant of a matrix in $M_n(R)$ factored by the determinant of a matrix in $M_n(Z)$. Consequently, $k_i$ lies in $R$ for all $1 \leq i \leq n$, which implies that $z$ belongs to $g_R$.

\[\Box\]

**Lemma 2.5.** Let $g$ be an element of $G(K)$, then $g$ lies in $G(R)$ if $Int g(u_a(1))$ belongs to $G(R)$ for all $a \in \Phi$.

**Proof.** Let $\{e_a, h_b | \forall a \in \Phi, b \in \Delta\}$ be a Chevalley basis of the semisimple Lie algebra $L(G(\mathbb{C}))$, where $[e_a, e_{-b}] = h_b$ for $b \in \Delta$. Considering $G(K)$ as a subgroup of $GL_n(g_K)$ through the adjoint representation of $G(K)$ where $n = \dim g_K$, we obtain that $u_a(1) = \exp ad(e_a \otimes 1)$ for all $a \in \Phi$ (cf. [10]) and

\[\text{(2.5.1)} \quad Int g(u_a(1)) = \exp ad(g(e_a \otimes 1))\]

where $\exp$ is the canonical exponential map which sends the nilpotent elements of $M_n(R)$ to the unipotent elements of $GL_n(K)$. Recall that the logarithm map $log$ sends the unipotent subset of $M_n(R)$ to the nilpotent subset of $M_n(R)$ and the composite $log \circ \exp$ is the identity map on the nilpotent subset (cf. [3, Ch.II.6.1]). We have by (2.5.1)

\[\log(Int g(u_a(1))) = ad(g(e_a \otimes 1)) \in M_n(R) \text{ for } a \in \Phi.\]

Hence $g(e_a \otimes 1)$ belongs to $g_R$ for all $a \in \Phi$ by Lemma 2.4. Moreover, we have

\[g(h_a \otimes 1) = [g(e_a \otimes 1), g(e_{-a} \otimes 1)] \in g_R \text{ for } a \in \Delta.\]

Hence $g \in GL_n(g_R) \cap G(K) = G(R)$. \[\Box\]

**Remark.** Lemma 2.4 and Lemma 2.5 have been shown in [4] for the case when $R$ is a Laurent polynomial ring over the complex number field.

Let $a$ be a root in $\Phi$, we denote by $g_a$ the root subspace of $g_K$ related to $a$ and by $u$ the subalgebra generated by $g_a$ for all $a \in \Phi^+$. If $b$ is a subalgebra of $g_K$, we denote by $c_u(b)$ the centralizer of $b$ in $u$.

**Lemma 2.6.** Let $a$ be a positive root and $I = \{c \in \Phi^+ | a + b \in \Phi^+ \Rightarrow c + b \in \Phi^+, \forall b \in \Phi^+\}$.

\[\text{(2.6.1)} \quad C_{U(K)}(C_{U(K)}(U_a(K))) = \prod_{c \in I} U_c(K).\]

**Proof.** It is easily seen that the Lie algebra $L(U(K))$ of $U(K)$ is $u$, hence we have

\[L(C_{U(K)}(U_a(K))) = C_u(L(U_a(K))) = C_u(g_a).\]
Note that $C_{U(K)}(U_n(K))$ is connected since it is a $T$-stable closed subgroup of $U(K)$ (cf. [1, Ch.IV.14.4]). Then we have
\[
L(C_{U(K)}(C_{U(K)}(U_n(K)))) = C_u(L(C_{U(K)}(U_n(K))))
\]
\[
= C_u(C_u(g_a)) = C_u(\sum_{b \in \Phi^+} b_q) = \bigcap_{b \in \Phi^+} C_u(b_q)
\]
\[
= \bigcap_{b \in \Phi^+} \sum_{c \in I} b_q = \sum_{c \in I} b_q = L(U_n(K) \forall c \in I)).
\]

Moreover, it is easily seen that
\[
(U_n(K) \forall c \in I)) = \prod_{c \in I} U_n(K).
\]

Note that, since $C_{U(K)}(C_{U(K)}(U_n(K)))$ is a $T$-stable closed subgroup of $U(K)$, it is also connected. Thus (2.6.1) follows from the above identities since the correspondence between the connected subgroups of $U(K)$ and the Lie subalgebras of $u$ is bijective.

**Lemma 2.7.** Let $a$ be a positive root, then
\[(2.7.1)\]
\[
\overline{C_{U(R)}(U_n(Q))} = C_{U(K)}(U_n(K)).
\]

**Proof.** It is obvious that $U_b(R) \subseteq C_{U(R)}(U_n(Q))$ for all $b \in \Phi^+$ with $a + b \notin \Phi$. Note that $U_n(Q)$ is a Zariski dense subgroup of $U_n(K)$ by (2.3.2). We then have
\[(2.7.2)\]
\[
\langle U_b(R) \rangle \subseteq \langle U_n(Q) \rangle \subseteq C_{U(R)}(U_n(K)) \subseteq C_{U(K)}(U_n(K)).
\]

Moreover, since $U_b(R)$ is Zariski dense in $U_b(K)$ for all $b \in \Phi^+$ by (2.3.2), we have
\[
\langle U_b(R) \rangle \subseteq \langle U_n(K) \rangle \subseteq \langle U_n(Q) \rangle \subseteq C_{U(K)}(U_n(K)).
\]

Therefore, taking Zariski closures of the subgroups in (2.7.2), we obtain immediately (2.7.1).

**Proposition 2.8.** Every normal subgroup of $G(R)$ that contains $E(Q)$ must contain the elementary subgroup $E(R)$.

**Proof.** For each root $a \in \Phi$ and each element $q \in \mathbb{Q}^*$, let
\[
h_a(q) = u_a(q)u_{-a}(-q^{-1})u_a(q)u_{-a}(1)u_a(-1)u_{-a}(1) \in T \cap E(Q).
\]
Then (cf. [?])
\[
h_a(q)u_a(r)h_a(q)^{-1} = u_a(q^2r) \text{ for } q \in \mathbb{Q}^*, r \in R.
\]

Suppose $H$ is a normal subgroup of $G(R)$ which contains $E(Q)$ and let $q \neq \pm 1$, then for all $r \in R$ and $a \in \Phi$ we have
\[
u_a(r) = h_a(q)u_a((q^2 - 1)^{-1}r)h_a(q)^{-1}u_a((q^2 - 1)^{-1}r)^{-1} \in H.
\]
This implies that $H$ contains $E(R)$.

**Proposition 2.9.** If $\alpha$ is an automorphism of $G(R)$ which fixes each element of $E(R)$, then $\alpha$ is the identity map on $G(R)$.
Proof. Since $E(R)$ is a normal subgroup of $G(R)$ by \[11\], we have for all $g \in G(R)$
\[ gxg^{-1} = \alpha(gxg^{-1}) = \alpha(g)x\alpha(g)^{-1} \text{ for } x \in E(R). \]

This yields
\[ \alpha(g)^{-1}g = x(\alpha(g)^{-1}) x \text{ for } x \in E(R), \ g \in G(R), \]

which means that, since $E(R)$ is Zariski dense in $G(K)$ (cf. \[2\]),
\[ \alpha(g)^{-1} g \in C_{G(R)}(E(R)) = C(G(K)). \]

Note that $C(G(K))$ is trivial since $G$ is of adjoint type. Then we obtain
\[ \alpha(g) = g \text{ for all } g \in G(R). \]

\[ \square \]

3. Isomorphisms of Simple Chevalley Groups

In this section we assume that $G$ (resp. $G'$) is a simple adjoint Chevalley-Demazure group scheme with its root system $\Phi$ (resp. $\Phi'$) and fundamental root system $\Delta$ (resp. $\Delta'$) whose rank is greater than 1. Let $R$ and $R'$ stand for subrings of $K$ containing $\mathbb{Q}$. The elementary subgroup of $G'(R')$ is denoted by $E'(R')$.

**Lemma 3.1.** If there exists a nontrivial homomorphism from $E(\mathbb{Q})$ to $G'(K)$, then
\[ \dim G(K) = \dim G'(K). \]

**Proof.** See \[5, Cor.2.4\].  \[ \square \]

**Lemma 3.2.** Let $H$ be a connected algebraic group, then

(i) $\dim G(K) \leq \dim H$ if there exists a nontrivial homomorphism from $E(\mathbb{Q})$ to $H$;

(ii) the image of a nontrivial homomorphism from $E(\mathbb{Q})$ to $H$ is Zariski dense in $H$ if $\dim G(K)$ is equal to $\dim H$.

**Proof.** (i) Let $\alpha : E(\mathbb{Q}) \to H$ be a nontrivial homomorphism. Since $E(\mathbb{Q})$ is a simple group, $\alpha(E(\mathbb{Q}))$ is a connected and non-solvable subgroup of $H$ by Lemma 2.1. Therefore, if $\mathfrak{R}$ is the solvable radical of $\alpha(E(\mathbb{Q}))$, the quotient group $\alpha(E(\mathbb{Q}))/\mathfrak{R}$ is a semisimple algebraic group of positive dimension. Let $\{H_i\}_{i=1}^m$ be the family of the simple components of $\alpha(E(\mathbb{Q}))/\mathfrak{R}$ and let $H_i^{ad}$ be an adjoint simple algebraic group of the same type as $H_i$ for all $1 \leq i \leq m$. Then there exists an isogeny $\varepsilon : \overline{\alpha(E(\mathbb{Q}))}/\mathfrak{R} \to \prod_{i=1}^m H_i^{ad}$. Let $\pi$ be the natural homomorphism from $\overline{\alpha(E(\mathbb{Q}))}$ to $\overline{\alpha(E(\mathbb{Q}))}/\mathfrak{R}$ and let $p_j$ be the canonical projection of $\prod_{i=1}^m H_i^{ad}$ to the $j$-th factor $H_i^{ad}$ for $1 \leq j \leq m$. Note that, since $p_j$ (1 $\leq j \leq m$), $\varepsilon$ and $\pi$ are isomorphisms which preserve the Zariski density, so does their composite $p_j \circ \varepsilon \circ \pi$. In particular we have for all $1 \leq j \leq m$
\[ p_j \circ \varepsilon \circ \pi = p_j \circ \varepsilon \circ \pi(\overline{\alpha(E(\mathbb{Q}))}) = H_i^{ad}, \]

which means that the composite $p_j \circ \varepsilon \circ \pi$ is a homomorphism from $E(\mathbb{Q})$ to $H_i^{ad}$ with a Zariski dense image. It follows from Lemma 3.1 that for all $1 \leq j \leq m$
\[ \dim G(K) = \dim H_j^{ad} = \dim H_j. \]

Hence
\[ (3.2.1) \quad \dim G(K) \leq \overline{\alpha(E(\mathbb{Q}))}/\mathfrak{R} \leq \overline{\alpha(E(\mathbb{Q}))} \leq \dim H. \]
(ii) Suppose $G(K)$ and $H$ have the same dimension, then it follows from (3.2.1) that for a nontrivial homomorphism $\alpha : E(\mathbb{Q}) \to H$ we have

$$\dim \alpha(E(\mathbb{Q})) = \dim H.$$ 

Since $H$ is connected, this implies by Lemma 2.1 that $\alpha(E(\mathbb{Q})) = H$ as required. □

**Corollary 3.3.** If $E(R)$ and $E'(R')$ are isomorphic to each other, then

$$\dim G(K) = \dim G'(K).$$

**Proof.** This comes directly from Lemma 3.2(i). □

**Proposition 3.4.** Suppose $\alpha$ is an isomorphism from $E(R)$ (resp. $G(R)$) to $E'(R')$ (resp. $G'(R')$), then there exist an element $g \in G'(R')$ and an isomorphism of root systems $\gamma : \Phi \to \Phi'$ with $\gamma(\Delta) = \Delta'$ such that

$$(3.4.1) \quad \text{Int } \alpha(E(\mathbb{Q})) = \epsilon(\gamma_\Delta(q)) \text{ for } a \in \Delta \text{ or } \Delta, q \in \mathbb{Q}.$$ 

In particular

$$(3.4.2) \quad \text{Int } \alpha(E(\mathbb{Q})) = E'(\mathbb{Q}).$$

**Proof.** Since $E(R)$ (resp. $G(R)$) and $E'(R')$ (resp. $G'(R')$) are isomorphic to each other, $G(K)$ and $G'(K)$ have the same dimension by Corollary 3.3. Hence the restriction of $\alpha$ to $E(\mathbb{Q})$, which is a nontrivial homomorphism from $E(\mathbb{Q})$ to $G'(K)$, has a Zariski dense image by Lemma 3.2. It follows from the Borel-Tits theorem [2, Th.A] that there exist a homomorphism of fields $\varphi : \mathbb{Q} \to K$ and an isogeny $\varepsilon$ from $\varphi G(K)$, the group obtained from the base change through $\varphi$, to $G'(K)$ such that

$$\alpha(x) = \varepsilon \varphi^a(x) \text{ for } x \in E(\mathbb{Q})$$

where $\varphi^a$ is the canonical homomorphism from $G(K)$ to $\varphi G(K)$ induced by $\varphi$ (see [2] for the notation). Note that there is no other possibility for $\varphi$ but of the natural embedding, which implies that $\varphi^a$ is the identity map. This yields

$$(3.4.3) \quad \alpha(x) = \varepsilon(x) \text{ for } x \in E(\mathbb{Q}).$$

It follows from Lemma 2.2 that there exist an isomorphism of root systems $\gamma : \Phi \to \Phi'$ with $\gamma(\Delta) = \Delta'$ and an element $g \in G'$ such that

$$(3.4.4) \quad \varepsilon = \text{Int } g^{-1} \cdot \gamma$$

where $\gamma$ is the isogeny from $G(K)$ to $G'(K)$ related to $\gamma$. Hence the identity (3.4.1) comes from the definition of $\gamma$ and the fact that

$$\text{Int } \alpha(E(\mathbb{Q})) = \text{Int } \alpha(E(\mathbb{Q})) = \gamma(\alpha(E(\mathbb{Q}))) \text{ for } a \in \Delta \text{ or } \Delta, q \in \mathbb{Q}.$$ 

We claim that $g$ lies in $G'(R')$. This is because, for each root $a' \in \Phi'$, we have by (3.4.3) and (3.4.4)

$$\text{Int } g^{-1}(u_{a'}(1)) = \text{Int } g^{-1}(\gamma^{-1}(u_{a'}(1))) = \varepsilon(\gamma^{-1}(u_{a'}(1)))$$
$$= \alpha(\gamma^{-1}(u_{a'}(1))) = \alpha(u_{\gamma^{-1}(a')}(1)) \in G'(R'),$$

which implies by Lemma 2.5 that $g^{-1}$, hence also $g$, lies in $G'(R')$. □
Lemma 3.5. Suppose $\gamma : \Phi \to \Phi'$ is an isomorphism of root systems with $\gamma(\Delta) = \Delta'$. If $\alpha : E(R) \to E'(R')$ is an isomorphism such that
\[ \alpha(u_\alpha(q)) = u_{\gamma(\alpha)}(q) \text{ for } a \in \Delta \text{ or } -\Delta, \quad q \in \mathbb{Q}, \]
then $\alpha(U(R)) = U'(R')$.

Proof. It follows from the definition of $\gamma$ that $\gamma(B) = B'$ and
\[ (3.5.1) \quad \alpha(g) = \gamma(g) \text{ for } g \in E(Q). \]
Therefore
\[ \alpha(B \cap E(Q)) = \gamma(B \cap E(Q)) = B' \cap E(Q). \]
Hence
\[ \alpha(B \cap E(R)) \supseteq \alpha(B \cap E(Q)) = B' \cap E(Q). \]
By taking the Zariski closures of the above groups, we obtain from Lemma 2.3(iii)
\[ \alpha(B \cap E(R)) \supseteq B' \cap E(Q) = B'. \]
However, since $\overline{\alpha(B \cap E(R))}$ is a solvable group, we have
\[ \overline{\alpha(B \cap E(R))} = B'. \]
In particular, we obtain that
\[ (3.5.2) \quad \alpha(U(R)) \subseteq \alpha(B \cap E(R)) \subseteq B' \cap E'(R'). \]
Let $a$ be a positive root in $\Phi$, we can choose an element $t \in T \cap E(Q)$ such that $a(t) \neq 1$ since $T \cap E(Q)$ is Zariski dense in $T$ by Lemma 2.3(ii). Note that $a(t)$ lies in $Q$, we have
\[ (3.5.3) \quad u_a(r) = tu_a((a(t) - 1)^{-1}r)t^{-1}u_a((a(t) - 1)^{-1}r)^{-1} \text{ for } r \in R. \]
This implies that
\[ u_a(R) \subseteq [T \cap E(Q), U(R)] \text{ for } a \in \Phi^+. \]
Thus $U(R)$ is contained in $[T \cap E(Q), U(R)]$. Hence we obtain by (3.5.1) and (3.5.2)
\[
\alpha(U(R)) \subseteq [\alpha(T \cap E(Q)), \alpha(U(R))] = [\gamma(T \cap E(Q)), \alpha(U(R))]
\]
\[
= [T' \cap E'(Q), \alpha(U(R))] \subseteq [T', B'] \cap E'(R')
\]
\[
= U' \cap E'(R') = U'(R').
\]
Replacing $\alpha$ by $\alpha^{-1}$ and following a similar argument as above, we obtain on the other hand that $\alpha(U(R)) \supseteq U'(R')$. Hence $\alpha(U(R))$ is equal to $U'(R')$ as required.

Lemma 3.6. Let $\alpha$ and $\gamma$ be as in Lemma 3.5, then
\[ (3.6.1) \quad \alpha(U_a(R)) = U_{\gamma(\alpha)}(R') \text{ for } a \in \Phi. \]

Proof. We first show (3.6.1) for the case where $a$ is a positive root. Using Lemma 2.6 and Lemma 2.7, we have
\[
\mathcal{C}_{U(R)}(\mathcal{C}_{U(R)}(U_a(Q))) = U(R) \cap \mathcal{C}_{U(K)}(\mathcal{C}_{U(R)}(U_a(Q)))
\]
\[
= U(R) \cap \mathcal{C}_{U(K)}(\mathcal{C}_{U(K)}(U_a(K))) = U(R) \cap \prod_{e \in I} U_a(K),
\]
where $I$ is as in Lemma 2.6. Moreover, since $\gamma$ is an isomorphism of root systems, we also have by Lemma 2.6 that

$$C_{U'(R')}(C_{U'(R')}(U_{\gamma(a)}(Q))) = U'(R') \cap \prod_{c \in I} U_{\gamma(c)}(K).$$

Hence, applying Lemma 3.5, we obtain that

$$\alpha(U_a(R)) \subseteq \alpha(C_{U'(R')}(C_{U'(R')}(U_a(Q))))$$

$$= C_{U'(R')}(C_{U'(R')}(\alpha(U_a(Q)))) = U'(R') \cap \prod_{c \in I} U_{\gamma(c)}(K).$$

(3.6.2)

Suppose $I = \{c_1, c_2, \ldots, c_m\}$ where $c_1 = a$. If $m = 1$, then

$$\alpha(U_a(R)) \subseteq U'(R') \cap U_{\gamma(a)}(K) = U_{\gamma(a)}(R'),$$

from which follows (3.6.1) since $\alpha$ is an isomorphism. Suppose $m \geq 2$. Then $(\ker c_m) \cap \ker a$ is an open subset of $(\ker c_m) \circ$. Note that, since $(\ker c_m) \circ$ splits over $Q$ (cf. [1, Ch.III, Cor.8.7]), $(\ker c_m) \circ \cap E(Q)$ is Zariski dense in $(\ker c_m) \circ$ by [2, Cor.6.8]. Therefore

$$\{(\ker c_m) \circ - \ker a\} \cap E(Q) = \{(\ker c_m) \circ \cap E(Q)\} \cap \{(\ker c_m) \circ - \ker a\} \neq \emptyset.$$  

Let $t \in \{(\ker c_m) \circ - \ker a\} \cap E(Q)$. Then the coincidence of the restrictions of $\alpha$ and $\tilde{\gamma}$ to $E(Q)$ implies that $\alpha(t)$ lies in $T'_{\gamma}$ since $\tilde{\gamma}(T) = T'$ (see §2 for the notation). Moreover, for any root $\beta \in \Phi$, $t$ lies in $\ker \gamma(\beta)$ if and only if $\alpha(t)$ lies in $\ker \gamma(\beta)$ because

$$u_{\gamma(\beta)}(b(t)) = \alpha(u_{\beta}(b(t))) = \alpha(t u_{\beta}(1) t^{-1})$$

$$= \alpha(t) u_{\gamma(\beta)}(1) \alpha(t)^{-1} = u_{\gamma(\beta)}(\gamma(\beta)(\alpha(t))).$$

Therefore, $\alpha(t)$ lies in $(\ker c_m) - \ker a \cap E'(Q)$. This yields that

$$[\alpha(t) \prod_{i=1}^{m} U_{\gamma(c_i)}(K)] \subseteq \prod_{i=1}^{m-1} U_{\gamma(c_i)}(K).$$

Note that $U_a(R) = \{t, U_a(R)\}$ by (3.5.3). We then have by (3.6.2)

$$\alpha(U_a(R)) = [\alpha(t), \alpha(U_a(R))]$$

$$\subseteq [\alpha(t), U'(R') \cap \prod_{i=1}^{m} U_{\gamma(c_i)}(K)] \subseteq U'(R') \cap \prod_{i=1}^{m-1} U_{\gamma(c_i)}(K).$$

This results in (3.6.1) if $m = 2$. When $m \geq 3$, (3.6.1) follows from the repetitions of analogous arguments as above.

We show now that (3.6.1) holds also for all $-a$, where $a \in \Phi^+$. Let $w_a = u_{a}(1) u_{-a}(1) u_{a}(1)$ for $a \in \Phi^+$. Then we have $\text{Int} \ w_a(U_a(R)) = U_{-a}(R)$. Note that for all $a \in \Phi^+$

$$\alpha(w_a) = \tilde{\gamma}(w_a) = w_{\gamma(a)}.$$

This yields

$$\alpha(U_{-a}(R)) = \text{Int} \ w_{\gamma(a)}(\alpha(U_a(R))) = \text{Int} \ w_{\gamma(a)}(U_{\gamma(a)}(R')) = U_{-\gamma(a)}(R').$$

Hence (3.6.1) holds for all $a \in \Phi$. 

$\square$
Let \( \alpha \) and \( \gamma \) be as in Lemma 3.5. Thanks to Lemma 3.6, we can assign a map \( \varphi_a : R \to R' \) to each root \( a \in \Phi \) satisfying
\[
\alpha(u_a(r)) = u_{\gamma(a)}(\varphi_a(r)) \text{ for } r \in \Phi.
\]
It is easily seen that \( \varphi_a \) is an isomorphism between the additive groups \( R^+ \) and \( R'^+ \).

**Lemma 3.7.** For each root \( a \) in \( \Phi \), \( \varphi_a \) is an isomorphism of rings and
\[
\varphi_a = \varphi_b \text{ for } b \in \Phi.
\]

**Proof.** We first consider the case where \( a \) is a fundamental root. Since \( G \) is not of type \( A_1 \), there exists a positive root \( b \) such that \( a + b \in \Phi \). We have by the commutator formula [7, Exp.XXII,§5]
\[
u_a(r)u_b(s)u_a(r)^{-1}u_b(s)^{-1} = u_{a+b}(n_{a,b}rs) \prod_{h(c) > h(a+b)} u_c(r_c) \text{ for } r, s \in R,
\]
where \( n_{a,b} \) is an integer determined uniquely by \( a \) and \( b \) while \( r_c \in R \), and \( h \) is the height function of \( \Phi \). Applying \( \alpha \) on both sides, we obtain that
\[
u_{\gamma(a)}(\varphi_a(r))u_{\gamma(b)}(\varphi_b(s))u_{\gamma(a)}(\varphi_a(r))^{-1}u_{\gamma(b)}(\varphi_b(s))^{-1} = u_{\gamma(a+b)}(n_{a,b}\varphi_a(rs))u
\]
where \( u \) is a product of elements of the form \( u_{\gamma(c)}(p) \) for some positive root \( c \) such that \( h(c) > h(a+b) \) and for some \( p \in R' \). On the other hand, it follows from the commutator formula that
\[
u_{\gamma(a)}(\varphi_a(r))u_{\gamma(b)}(\varphi_b(s))u_{\gamma(a)}(\varphi_a(r))^{-1}u_{\gamma(b)}(\varphi_b(s))^{-1} = u_{\gamma(a+b)}(n_{a,b}\varphi_a(r)\varphi_b(s))u_1
\]
where \( u_1 \) is also a product of elements of the form \( u_{\gamma(c)}(p) \) for some \( c \in \Phi^+ \) with \( h(c) > h(a+b) \) and \( p \in R' \). Note that, if \( h' : \Phi' \to \mathbb{Z} \) is the height function of \( \Phi' \), then \( h'(\gamma(c)) > h'(\gamma(a+b)) \) for all the factors \( u_{\gamma(c)}(p) \) of \( u \) (resp. \( u_1 \)). Thus, comparing these two identities, we have
\[
\varphi_{a+b}(rs) = \varphi_a(r)\varphi_b(s) \text{ for } r, s \in R.
\]
Taking \( r \) and \( s \) to be 1 alternately, we obtain that \( \varphi_{a+b} = \varphi_a = \varphi_b \). Note that for each fundamental root \( c \in \Delta \) there exists a sequence of fundamental roots
\[
a = a_1, a_2, \ldots, a_m = c
\]
such that \( a_i + a_{i+1} \in \Phi \) for all \( 1 \leq i \leq m - 1 \). Hence we have, by following similar arguments as above, that
\[
\varphi_a = \varphi_{a_2} = \cdots = \varphi_{a_m} = \varphi_c.
\]
Thus we may simply write \( \varphi \) in stead of \( \varphi_a \) for all \( a \in \Delta \). It follows from (3.7.2) that \( \varphi(rs) = \varphi(r)\varphi(s) \) for all \( a \in \Delta \), which means that \( \varphi \) is a homomorphism of rings and therefore is an isomorphism of rings.

We show now that
\[
\varphi_a = \varphi \text{ for } a \in \Phi^+.
\]
We use induction on the height of the roots. Suppose \( a \) is not a fundamental root and \( \varphi_a = \varphi \) for all \( c \in \Phi^+ \) such that \( h(c) < h(a) \). Since \( a \) can be written as the sum of two positive roots, say \( b \) and \( c \), with \( h(b) < h(a) \) and \( h(c) < h(a) \), we have

\[
\gamma \in \Phi^+ \quad \text{such that} \quad h(\gamma) > h(a).
\]

Then \( \varphi_a = \varphi \). This completes our proof. \( \square \)

**Corollary 3.8.** Let \( \alpha \) and \( \gamma \) be as in Lemma 3.5, then there exists an isomorphism of rings \( \varphi : R \to R' \) such that \( \alpha = \hat{\varphi} \).

**Proof.** This is a consequence of Lemma 3.6, Lemma 3.7 and the definition of \( \hat{\varphi} \). \( \square \)

**Theorem 3.9.** If \( \alpha : E(R) \to E'(R') \) is an isomorphism, then there exist an element \( g \in G'(R') \), an isomorphism of root systems \( \gamma : \Phi \to \Phi' \) with \( \gamma(\Delta) = \Delta' \) and an isomorphism of rings \( \varphi : R \to R' \) such that

\[
\alpha = \text{Int}_g \cdot \hat{\gamma} \cdot \hat{\varphi}.
\]

Moreover, \( g, \gamma \) and \( \varphi \) are uniquely determined by \( \alpha \).

**Proof.** It follows from Proposition 3.4 that there exist an element \( g \in G'(R') \) and an isomorphism of root systems \( \gamma : \Phi \to \Phi' \) such that

\[
\text{Int} g^{-1} \alpha(a(q)) = u_{\gamma(a)}(q) = \hat{\gamma}(u_a(q)) \quad \text{for} \quad a \in \Delta \text{ or } -\Delta, \quad q \in Q.
\]

Since \( E'(R') \) is a normal subgroup of \( G'(R') \) (cf. [11]), \( \text{Int} g^{-1} \alpha \) is an isomorphism from \( E(R) \) to \( E'(R') \). Hence \( \hat{\gamma}^{-1} : \text{Int} g^{-1} \cdot \alpha \) is also an isomorphism from \( E(R) \) to \( E'(R') \). Therefore by Corollary 3.8 there exists an isomorphism of rings \( \varphi : R \to R' \) such that

\[
\hat{\gamma}^{-1} \cdot \text{Int} g^{-1} \cdot \alpha = \hat{\varphi}.
\]
from which follows (3.9.1). Suppose there exist an element $g_1 \in G'(R')$, an isomorphism of root systems $\gamma_1 : \Phi \to \Phi'$ with $\gamma_1(\Delta) = \Delta'$ and an isomorphism of rings $\varphi_1 : R \to R'$ such that

$$\alpha = \text{Int} \ g \cdot \hat{\gamma} \cdot \hat{\varphi} = \text{Int} \ g_1 \cdot \hat{\gamma}_1 \cdot \hat{\varphi}_1,$$

then

(3.9.2) $$\text{Int} \ g_1^{-1}g = \hat{\gamma}_1 \cdot \hat{\varphi}_1 \cdot \hat{\varphi}_1^{-1} \cdot \hat{\gamma}_1^{-1}.$$ Let $U'(R')$ (resp. $U'^-(R')$) be the subgroup of $G'(R')$ generated by $u_a(r)$ for all $a \in \Phi'$ (resp. $-a \in \Phi'$) and $r \in R'$. Since

$$\hat{\gamma}(U(R')) = \hat{\gamma}_1(U(R')) = U'(R'),$$

and

$$\hat{\varphi}(U(R)) = \hat{\varphi}_1(U(R)) = U(R'),$$

we have by (3.9.2)

$$\text{Int} \ g_1^{-1}g(U'(R')) = U'(R').$$

Similarly we also have

$$\text{Int} \ g_1^{-1}g(U'^-(R')) = U'^-(R').$$

Therefore, if we denote by $B'^-$ the opposite Borel subgroup of $B'$, then

$$g_1^{-1}g \in \mathcal{N}_{G'(R')}(U'(R')) \cap \mathcal{N}_{G'(R')}(U'^-(R'))$$

$$\subseteq \mathcal{N}_{G'(R')}(U'(R')) \cap \mathcal{N}_{G'(R')}(U'^-(R')) = \mathcal{N}_G(R')(U'(K)) \cap \mathcal{N}_G(R')(U'^-(K))$$

$$= G'(R') \cap B' \cap B'^- = G'(R') \cap T'.$$

This yields that, for each fundamental root $a \in \Delta'$,

$$\text{Int} \ g_1^{-1}g(u_a(1)) = u_a(a(g_1^{-1}g)).$$

On the other hand, we have

$$\hat{\gamma}_1 \hat{\varphi}_1 \hat{\varphi}_1^{-1}(u_a(1)) = u_a(\gamma_1^{-1}(a)) \text{ for } a \in \Delta'.$$

Comparing these two identities, we obtain that $\gamma_1 = \gamma$ and $a(g_1^{-1}g) = 1$ for all $a \in \Delta'$, which means that

$$g_1^{-1}g \in \bigcap_{a \in \Delta'} \ker a = \mathcal{O}(G'(K)).$$

This implies immediately that $g_1 = g$ and that, by (3.9.2), $\varphi_1 = \varphi$. Hence the expression (3.9.1) of $\alpha$ is unique. \qed

**Proof of Theorem 1.1.** (i) Suppose $\alpha : E(R) \to E'(R')$ is an isomorphism. It follows from Theorem 3.9 that $\alpha$ has an expression of the form $\text{Int} \ g \cdot \hat{\gamma} \cdot \hat{\varphi}$ where $g \in G'(R')$, $\gamma : \Phi \to \Phi'$ is an isomorphism of root systems with $\gamma(\Delta) = \Delta'$ and $\varphi : R \to R'$ is an isomorphism of rings. It is evident from the definitions that $\hat{\gamma}$ can be extended to the graph isomorphism $\hat{\gamma}$ from $G(R')$ to $G'(R')$ and that $\hat{\varphi}$ can be extended to the ring isomorphism $\hat{\varphi}$ from $G(R)$ to $G(R')$. Hence $\alpha$ can be extended to an isomorphism $\tilde{\alpha}$ from $G(R)$ to $G'(R')$ in an obvious way. If $\tilde{\alpha} : G(R) \to G'(R')$ is an isomorphism which is also an extension of $\alpha$, then $\tilde{\alpha} \cdot \tilde{\alpha}^{-1}$ is an automorphism,
of $G(R)$ which fixes each element of $E(R)$ and, therefore, $\tilde{\alpha} = \bar{\alpha}$ by Proposition 2.9. Thus the extension of $\alpha$ to an isomorphism between $G(R)$ and $G'(R')$ is unique.

(ii) It follows from Proposition 3.4 that there exists an element $g \in G'(R')$ such that

$$\alpha(E(Q)) = \text{Int} \, g(E'(Q)).$$

Thus $H$ is a normal subgroup of $G(R)$ which contains $E(Q)$ if and only if $\alpha(H)$ is a normal subgroup of $G'(R')$ containing $E'(Q)$. This implies that $\alpha$ induces a bijection between the set of normal subgroups of $G(R)$ containing $E(Q)$, which is denoted by $N$, and the set of normal subgroups of $G'(R')$ containing $E'(Q)$, which is denoted by $N'$. Note that by Proposition 2.8

$$E(R) = \bigcap_{H \in N} H.$$

Hence we have

$$\alpha(E(R)) = \bigcap_{H \in N} \alpha(H) = \bigcap_{H' \in N'} H' = E'(R').$$

Proof of Theorem 1.2. It follows from Theorem 1.1(ii) and Theorem 3.9 that the restriction $\alpha|_{E(R)}$ of $\alpha$ to $E(R)$ is an isomorphism between $E(R)$ and $E'(R')$ which has an expression of the form $\text{Int} \, g \cdot \hat{\gamma} \cdot \hat{\varphi}$ where $g \in G'(R')$, $\hat{\gamma}$ is a graph isomorphism from $E(R')$ to $E'(R')$ related to an isomorphism of root systems $\gamma : \Phi \to \Phi'$ with $\gamma(\Delta) = \Delta'$ and $\hat{\varphi}$ is a ring isomorphism from $E(R)$ to $E'(R')$ related to an isomorphism of rings $\varphi : R \to R'$. Thus $\alpha|_{E(R)}$ can be extended to an isomorphism from $G(R)$ to $G'(R')$ by extending $\hat{\gamma}$ (resp. $\hat{\varphi}$) to $\hat{\gamma}$ (resp. $\hat{\varphi}$). This extension of $\alpha|_{E(R)}$ has the form $\text{Int} \, g \cdot \hat{\gamma} \cdot \hat{\varphi}$ and is equal to $\alpha$ by Theorem 1.1(i). The uniqueness of the elements $g, \gamma$ and $\varphi$ comes directly from Theorem 3.9.

4. Automorphisms of $G(R)$ and $E(R)$

In this section, we assume that $G$ is an adjoint Chevalley-Demazure group scheme that has no simple component of type $A_1$. Let $\{G_i\}_{i=1}^n$ be the simple components of $G$ and $\Phi_i$ (resp. $\Delta_i$) be the root (resp. fundamental root) system of $G_i$ for all $1 \leq i \leq n$. Denote by $E_i(R)$ the elementary subgroup of $G_i(R)$ for all $1 \leq i \leq n$.

Proposition 4.1. Suppose $H$ is either $E(R)$ or $G(R)$ and $\alpha$ is an automorphism of $H$. Then there exists a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that

(i) $\alpha(E_i(R)) = E_{\sigma(i)}(R)$ for $1 \leq i \leq n$;
(ii) $\alpha(G_i(R)) = G_{\sigma(i)}(R)$ if $\alpha \in \text{Aut} \, G(R)$.

Proof. We show first that $\alpha(E(Q))$ is a Zariski dense subset of $G(K)$. If $n = 1$, we obtain by Lemma 2.1 and Lemma 3.2(i) that $\dim \, G \leq \dim \, \alpha(E(Q))$. Since $\alpha(E(Q))$ is a subgroup of $G(K)$, this implies that

$$\dim \, G = \dim \, \alpha(E(Q)).$$

Thus we have immediately the Zariski density of $\alpha(E(Q))$ in $G(K)$ since $\alpha(E(Q))$ is connected by Lemma 2.1. Suppose $n > 1$, then

$$E_i(Q) \subseteq C_{E(R)}(E_j(Q)) \text{ for } 1 \leq i \neq j \leq n.$$ (4.1.1)
Lemma 2.1. Let $\mathcal{R}(4.1.7)$ and $G$ be the solvable radical of $\alpha(E_i(\mathbb{Q}))$ and let $Y_i$ be the quotient group of $\mathcal{R} \cdot \alpha(E_i(\mathbb{Q}))$ modulo $\mathcal{R}$ for all $1 \leq i \leq n$, then
\[
\alpha(E(\mathbb{Q})) = \alpha(E_1(\mathbb{Q})) \cdot \alpha(E_2(\mathbb{Q})) \cdots \alpha(E_n(\mathbb{Q})).
\]

Hence $\alpha(E(\mathbb{Q}))$ is connected since each $\alpha(E_i(\mathbb{Q}))$ is connected for $1 \leq i \leq n$ by Lemma 2.1. Let $\mathcal{R}$ be the solvable radical of $\alpha(E(\mathbb{Q}))$ and let $Y_i$ be the quotient group of $\mathcal{R} \cdot \alpha(E_i(\mathbb{Q}))$ modulo $\mathcal{R}$ for all $1 \leq i \leq n$, then
\[
\frac{\alpha(E(\mathbb{Q}))}{\mathcal{R}} = Y_1 \cdot Y_2 \cdots Y_n.
\]

It is obvious that $Y_i$ is a semisimple normal subgroup of $\frac{\alpha(E(\mathbb{Q}))}{\mathcal{R}}$ for all $1 \leq i \leq n$. Moreover $[Y_i, Y_j]$ is trivial for all $1 \leq i \neq j \leq n$ since
\[
[\alpha(E_i(\mathbb{Q})), \alpha(E_j(\mathbb{Q}))] = \{1\}.
\]

This implies that $|Y_i \cap Y_j| < \infty$ for all $1 \leq i \neq j \leq n$ since $\frac{\alpha(E(\mathbb{Q}))}{\mathcal{R}}$ is semisimple. Thus (4.1.4) yields
\[
\dim \frac{\alpha(E(\mathbb{Q}))}{\mathcal{R}} = \sum_{i=1}^{n} \dim Y_i.
\]

Let $\mathcal{R}_i$ be the solvable radical of $\frac{\alpha(E_i(\mathbb{Q}))}{\mathcal{R}}$ for $1 \leq i \leq n$. Note that
\[
\mathcal{R}_i = \mathcal{R} \cap \alpha(E_i(\mathbb{Q})); \quad Y_i \cong \frac{\alpha(E_i(\mathbb{Q}))}{\mathcal{R}_i}.
\]

We obtain from (4.1.5) that
\[
\dim \frac{\alpha(E(\mathbb{Q}))}{\mathcal{R}} = \sum_{i=1}^{n} \dim \frac{\alpha(E_i(\mathbb{Q}))}{\mathcal{R}_i}.
\]

Let $\pi_i (1 \leq i \leq n)$ be the natural homomorphism from $\frac{\alpha(E_i(\mathbb{Q}))}{\mathcal{R}}$ to its quotient group $\frac{\alpha(E_i(\mathbb{Q}))}{\mathcal{R}_i}$. Note that the restriction of $\pi_i \cdot \alpha$ to $E_i(\mathbb{Q})$ is nontrivial. We obtain from Lemma 3.2(i)
\[
\dim G_i \leq \dim \frac{\alpha(E_i(\mathbb{Q}))}{\mathcal{R}_i} \quad \text{for } 1 \leq i \leq n.
\]

Thus we have from (4.1.6) that
\[
\dim G \leq \sum_{i=1}^{n} \dim \frac{\alpha(E_i(\mathbb{Q}))}{\mathcal{R}_i} \leq \dim \frac{\alpha(E(\mathbb{Q}))}{\mathcal{R}} \leq \dim G.
\]

This forces
\[
\alpha(E(\mathbb{Q})) = G(K).
\]

We show now that for each $i \in \{1, 2, \ldots, n\}$, $\alpha(E_i(\mathbb{Q}))$ is a simple component of $G(K)$. From the above identity and (4.1.3) we have
\[
G(K) = \frac{\alpha(E_1(\mathbb{Q})) \cdot \alpha(E_2(\mathbb{Q})) \cdots \alpha(E_n(\mathbb{Q}))}{\mathcal{R}}.
\]

Then (4.1.2) implies that $\alpha(E_i(\mathbb{Q}))$ is a normal subgroup of $G(K)$ for all $1 \leq i \leq n$ and
\[
\alpha(E_i(\mathbb{Q})) \cap \alpha(E_j(\mathbb{Q})) \subseteq C(G(K)) \quad \text{for } 1 \leq i \neq j \leq n.
\]
Note that $\alpha(E_i(\mathbb{Q}))$ is of positive dimension for all $1 \leq i \leq n$. Hence each $\alpha(E_i(\mathbb{Q}))$ contains at least one simple component of $G(K)$ and, meanwhile, is the direct product of all those simple components which are contained in $\alpha(E_i(\mathbb{Q}))$. Moreover, (4.1.8) implies that each simple component $G_k(K)$ ($1 \leq k \leq n$) lies in at most one $\alpha(E_i(\mathbb{Q}))$ for some $1 \leq i \leq n$. Since $G(K)$ has exact $n$ different simple components, each $\alpha(E_i(\mathbb{Q}))$ is in fact a simple component of $G(K)$ for all $1 \leq i \leq n$. In other words, there exists a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that for all $1 \leq i \leq n$

(4.1.8) \[ \alpha(E_i(\mathbb{Q})) = G_{\sigma(i)}(K). \]

Now we come to show (ii). Note that for all $1 \leq i \neq j \leq n$, $[G_i(R), E_j(\mathbb{Q})]$ is trivial and we have

$$ G_i(R) \subseteq C_{G(K)}(\prod_{j \neq i} E_j(\mathbb{Q})) \text{ for } 1 \leq i \leq n. $$

Hence

$$ \alpha(G_i(R)) \subseteq C_{G(K)}(\prod_{j \neq i} \alpha(E_j(\mathbb{Q}))) = C_{G(K)}(\prod_{j \neq i} \alpha(E_j(\mathbb{Q}))) = C_{G(K)}(\prod_{j \neq i} G_{\sigma(j)}(K)) = G_{\sigma(i)}(K). $$

Consequently

$$ \alpha(G_i(R)) \subseteq G(R) \cap G_{\sigma(i)}(K) = G_{\sigma(i)}(R) \text{ for } 1 \leq i \leq n. $$

By taking $\alpha^{-1}$ instead of $\alpha$ and by following a similar argument as above, we obtain on the other hand that $\alpha(G_{\sigma(i)}(R)) \subseteq G_i(R)$ for all $1 \leq i \leq n$. Hence $\alpha(G_i(R)) = G_{\sigma(i)}(R)$ for all $1 \leq i \leq n$.

Finally we show (i). If $\alpha$ is an automorphism of $G(R)$, then (i) comes as a consequence of the above (ii) and Theorem 1.1(ii). Suppose $\alpha$ is an automorphism of $E(R)$. Note that

$$ E_i(R) \subseteq C_{E(R)}(\prod_{j \neq i} E_j(\mathbb{Q})) \text{ for } 1 \leq i \leq n. $$

We have, by using the identity (4.1.8),

$$ \alpha(E_i(R)) \subseteq C_{E(R)}(\prod_{j \neq i} \alpha(E_j(\mathbb{Q}))) = C_{E(R)}(\prod_{j \neq i} \alpha(E_j(\mathbb{Q}))) = E(R) \cap C_{G(K)}(\prod_{j \neq i} G_{\sigma(j)}(K)) = E(R) \cap G_{\sigma(i)}(K) = E_{\sigma(i)}(R). $$

Since $\alpha$ is an automorphism, we obtain that $\alpha(E_i(R)) = E_{\sigma(i)}(R)$ for all $1 \leq i \leq n$ as required.

**Theorem 4.2.** Suppose $\alpha$ is an automorphism of $E(R)$, then there exist an element $g \in G(R)$, an automorphism of root system $\gamma : \Phi \to \Phi$ which keeps fundamental
root system $\Delta$ invariant and an automorphism $\varphi_i \in Aut R$ for each $1 \leq i \leq n$ such that

\begin{equation}
\alpha = Int g \cdot \hat{\gamma} \prod_{i=1}^{n} \hat{\varphi}_i.
\end{equation}

Moreover, $g$, $\gamma$ and $\varphi_i$ ($1 \leq i \leq n$) are uniquely determined by $\alpha$.

Proof. It is known from Proposition 4.1 that for each $1 \leq i \leq n$, the restriction of $\alpha$ to $E_i(R)$ is an isomorphism from $E_i(R)$ to $E_{\sigma(i)}(R)$ for some permutation $\sigma$ of $\{1, 2, \ldots, n\}$. Hence by theorem 3.9 there exist an element $g_i \in G_{\sigma(i)}(R)$, an isomorphism of root system $\gamma_i : \Phi \rightarrow \Phi_{\sigma(i)}$ with $\gamma_i(\Delta_i) = \Delta_{\sigma(i)}$ and an automorphism $\varphi_i \in Aut R$ such that the restriction of $\alpha$ to $E_i(R)$ has an expression

\begin{equation}
\alpha|_{E_i(R)} = Int g_i \cdot \gamma_i \cdot \varphi_i \quad \text{for} \quad 1 \leq i \leq n.
\end{equation}

Since $\Phi = \bigcup_{i=1}^{n} \Phi_i$, it is easily seen that the isomorphisms $\gamma_1, \gamma_2, \ldots, \gamma_n$, being pieced together, induce an automorphism of root system $\gamma : \Phi \rightarrow \Phi$ defined by

$$
\gamma(a) = \gamma_i(a) \quad \text{for} \quad a \in \Phi_i, \quad 1 \leq i \leq n,
$$

which keeps the fundamental root system $\Delta$ invariant. Moreover, we have by the definition of the graph automorphism that for all $1 \leq i \leq n$

\begin{equation}
\gamma(x) = \gamma_i(x) \quad \text{for} \quad x \in E_i(R).
\end{equation}

Suppose $x$ is an arbitrary element of $E(R)$, we may assume that $x = x_1x_2\ldots x_n$, where $x_i \in E_i(R)$ for all $1 \leq i \leq n$. Then we have by (4.2.2)

\begin{equation}
\alpha(x) = \prod_{i=1}^{n} Int g_i \hat{\gamma}_i \hat{\varphi}_i(x_i).
\end{equation}

Note that, since $G(R)$ is the direct product of $G_i(R)$ for all $1 \leq i \leq n$, we have for each $i \in \{1, 2, \ldots, n\}$

$$
Int g_1 Int g_2 \ldots Int g_n \hat{\gamma}_i \hat{\varphi}_i(x_i) = Int g_i \hat{\gamma}_i \hat{\varphi}_i(x_i).
$$

Let $g = \prod_{i=1}^{n} g_i$, then the identities (4.2.3) and (4.2.4) yield

$$
\alpha(x) = Int g \hat{\gamma}(\prod_{i=1}^{n} \hat{\varphi}_i(x_i)) = Int g \hat{\gamma}(\prod_{i=1}^{n} \hat{\varphi}_i)(x) \quad \text{for} \quad x \in E(R),
$$

from which follows immediately (4.2.1).

Proof of Theorem 1.4. (i) It follows from Theorem 4.2 that every automorphism $\alpha$ of $E(R)$ has an expression of the form $Int g \cdot \hat{\gamma} \cdot \prod_{i=1}^{n} \hat{\varphi}_i$ for some $g \in G(R), \gamma \in Aut \Phi$ with $\gamma(\Delta) = \Delta$ and $\varphi_i \in Aut R$ for $1 \leq i \leq n$. Since $\hat{\gamma}$ and $\prod_{i=1}^{n} \hat{\varphi}_i$ have the extensions $\gamma$ and $\prod_{i=1}^{n} \varphi_i$ in $Aut G(R)$ respectively, $\alpha$ can be extended to an automorphisms of $G(R)$ in an obvious way.

Suppose $\tilde{\alpha}$ and $\hat{\alpha}$ are automorphisms of $G(R)$ and the both are extensions of $\alpha$, then $\tilde{\alpha} \cdot \hat{\alpha}^{-1}$ is an automorphism of $G(R)$ which fixes every element of $E(R)$. Hence $\tilde{\alpha} = \hat{\alpha}$ by Proposition 2.9. Thus the extension of $\alpha$ is unique.

(ii) Suppose $\alpha$ is an automorphism of $G(R)$, then by Proposition 4.1(i) there exists a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ such that $\alpha(E_i(R)) = E_{\sigma(i)}(R)$. Hence

$$
\alpha(E(R)) = \prod_{i=1}^{n} \alpha(E_i(R)) = \prod_{i=1}^{n} E_{\sigma(i)}(R) = E(R).
$$
Proof of Theorem 1.5. Since the restriction of $\alpha$ to $E(R)$ induces an automorphism of $E(R)$ by Theorem 1.4(ii), it follows from Theorem 4.2 that there exist an element $g \in G(R)$, a graph automorphism $\gamma \in \text{Aut} E(R)$ related to an automorphism of root system $\gamma : \Phi \to \Phi$ with $\gamma(\Delta) = \Delta$ and a ring automorphism $\hat{\varphi}_i$ of $E(R)$ related to an automorphism $\varphi_i \in \text{Aut} R$ for each $1 \leq i \leq n$ such that

$$\alpha_{|E(R)} = \text{Int} g \cdot \gamma \cdot \prod_{i=1}^{n} \hat{\varphi}_i.$$

It is easily seen from the definitions that the graph automorphism $\hat{\gamma}$ of $G(R)$ is an extension of $\gamma$ while the ring automorphism $\prod_{i=1}^{n} \hat{\varphi}_i$ is an extension of $\prod_{i=1}^{n} \varphi_i$, hence the automorphism $\text{Int} g \cdot \gamma \cdot \prod_{i=1}^{n} \varphi_i$ is an extension of $\alpha_{|E(R)}$. Since the extension of $\alpha_{|E(R)}$ is unique by Theorem 1.4(i), we obtain immediately the expression $(1.5.1)$. Moreover, we have the uniqueness of $g, \gamma$ and $\varphi_i (1 \leq i \leq n)$ because, by Theorem 4.2 and Theorem 1.4(ii), all of them are uniquely determined by the restriction $\alpha_{|E(R)}$ which is, as a consequence of Theorem 1.4(i), uniquely determined by $\alpha$. This completes our proof.

References


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