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DISTINGUISHED REPRESENTATIONS AND QUADRATIC BASE CHANGE FOR $GL(3)$

HERVE JACQUET AND YANGBO YE

ABSTRACT. Let E/F be a quadratic extension of number fields. Suppose that every real place of F splits in E and let H be the unitary group in 3 variables. Suppose that Π is an automorphic cuspidal representation of $GL(3, E_{\mathbb{A}})$. We prove that there is a form ϕ in the space of Π such that the integral of ϕ over $H(F)\backslash H(F_{\mathbb{A}})$ is non zero. Our proof is based on earlier results and the notion, discussed in this paper, of Shalika germs for certain Kloosterman integrals.

1. INTRODUCTION

Let E/F be a quadratic extension of number fields. We denote by $z \mapsto \overline{z}$ the Galois conjugation in E. We denote by F^+ the group of elements of $F^\times$ which are norms and define similarly F_v^+ for any place v of F inert in E. We let $F_{\mathbb{A}}^+$ be the group of elements $z = (z_v)$ in $F_{\mathbb{A}}^\times$ such that $z_v \in F_v^+$ for v inert. Of course $F^+ = F_{\mathbb{A}}^+ \cap F^\times$. We let E^1 be the group of elements of norm 1 in $E^\times$. We let S be the variety of invertible Hermitian matrices:

$$S(F) = \{s \in GL(n, E) | s^* = s\} \tag{1}$$

where we have set

$$s^* = {}^t\overline{s}.$$

Finally, we let $S^+(F)$ be the set of $s \in S(F)$ such that $\det s \in F^+$ and define similarly S_v^+ and $S^+(F_{\mathbb{A}})$. Let Π be an automorphic cuspidal representation of $GL(n, E_{\mathbb{A}})$. For $\sigma \in S(F)$ let H_σ be the corresponding unitary group:

$$H_\sigma = \{h | h^*\sigma h = \sigma\}. \tag{2}$$

Following [HLR], we say that Π is H_σ-distinguished if there exists a form $\phi \in \Pi$ such that

$$\int_{H_\sigma(F)\backslash H_\sigma(F_{\mathbb{A}})} \phi(h)\, dg \neq 0. \tag{3}$$

Let Z be the center of $GL(n)$ so that $Z \simeq GL(1)$. Implicit in this definition is the requirement that the central character ω of Π be trivial on the set of elements of norm 1, that is, be distinguished with respect to E^1, the unitary group in 1 variable. According to an argument due to [HLR], if Π is distinguished with respect to a unitary group H_σ then Π is invariant under Galois conjugation and thus is the base change of a cuspidal representation π of $GL(n, F_{\mathbb{A}})$ ([AC]). It is natural to

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conjecture the converse is true: any base change representation is distinguished with respect to some unitary group. According to [HLR], this is the case if $n = 2$. We consider more particularly the case where n is odd and all the Archimedean places of F split in E. At a (finite) place v of F inert in E the only invariant in $S(F_v)$ is the class of the determinant modulo F_v^+. It follows that the only invariant in $S(F)$ is the class of the determinant modulo F^+. Thus two Hermitian matrices are equivalent if and only if their determinants are equivalent modulo the norms. As a consequence, if σ_1 and σ_2 are given in $S(F)$, there is $\zeta \in F^\times$ such that $\zeta\sigma_1$ and σ_2 are equivalent. In particular, two given matrices have isomorphic unitary groups so that we can speak of the unitary group without ambiguity.

We prove the converse for $n = 3$ under some restriction on the quadratic extension:

Theorem 1.1. *Assume that any Archimedean place of F splits in E. Let Π be a cuspidal automorphic representation of $GL(3, E_{\mathbb{A}})$ which is a base change. Then Π is distinguished for the unitary group.*

We may also view the above result in a more geometric light. Let Φ be a smooth function of compact support on $S(F_{\mathbb{A}})$. Let χ be an idèle class character of F. Define a function K_Φ on $G(E_{\mathbb{A}})$ by

$$K_\Phi(g) = \int_{F_{\mathbb{A}}^+/F^+} \sum_{\xi \in S(F)} \Phi(g^*\xi z g)\chi(z) d^\times z. \tag{4}$$

If Φ is supported on $S^+(F_{\mathbb{A}})$ then the sum can be taken over $S^+(F)$ since $S^+(F) = S(F) \cap S^+(F_{\mathbb{A}})$. Clearly

$$K_\Phi(\gamma z g) = K_\Phi(g)\chi(z\overline{z})^{-1}$$

for every $\gamma \in G(E)$ and $z \in E_{\mathbb{A}}^\times$. An automorphic representation Π with central character $z \mapsto \chi(z\overline{z})^{-1}$ is distinguished if and only if

$$\int_{G(E)Z(E_{\mathbb{A}})\backslash G(E_{\mathbb{A}})} K_\Phi(g)\overline{\phi}(g)\, dg \neq 0,$$

for at least one function Φ and one ϕ in the space of Π. Thus the problem of determining which cuspidal representations are distinguished is tantamount to finding the "projection" of the space spanned by the functions K_Φ on the space of cusp forms. In other words, our result amounts to saying that the forms which are quadratic base change are the "automorphic spectrum" of the symmetric variety S.

In general, if θ is any involutive automorphism of a reductive group G, one can define similarly the notion of an automorphic representation distinguished for the group of fixed points H of θ. As is the case here, one expects that distinguished representations have a simple characterization, in terms of the "principle of functoriality" (suitably extended to metaplectic covering groups): see [JR3]. Other examples are discussed in [J4], [M2], [FJ], [JR2], [yF1]. Of course the general theory remains to be developed. It will have many applications to special values of L-functions, cohomology and the principle of functoriality. The case at hand is a prototype case where distinguished representations have a specially simple characterization.

To prove the theorem, we use a form of the trace formula: see (18) and (23) below. This form of the trace formula has been used in other cases than the case at hand: see [J3], [J4], [M1], [M2] and also the flawed article [yF2]. More classical contributions to the same area are [I] and [Z].

The notion of distinguished representation is specially interesting when the pair (G, H) is a Gelfand pair: this means that for any place v and any irreducible representation π_v of G_v the dimension of the space of linear forms on π_v which are H_v invariant is at most 1. Indeed the period integral

$$\mu(\phi) = \int_{H(F)\backslash H(F_{\mathbb{A}})} \phi(h)\, dh$$

is a linear form which is $H(F_{\mathbb{A}})$ invariant and one expects it is a product of local invariant forms μ_v times a number which is the special value of an L function: see [G]. In the case at hand, for $n > 2$, we have an example of the opposite situation: the pair is not a Gelfand pair. Nonetheless, the trace formula that we are using suggests that the linear form μ (defined by the global integral) is an infinite tensor product of local ones. However, our result is not really sufficient to establish this assertion, because of the restrictions on the functions we are using in the trace formula.

To describe the trace formula in question we let Φ' be a smooth function of compact support on $GL(n, F_{\mathbb{A}})$. We shall assume that Φ' is the product of local functions Φ'_v and that for any v inert in E the function Φ'_v is supported on the group

$$G_v^+ = \{g | \det g \in F_v^+\}.$$

We also define $G^+(F), G^+(F_{\mathbb{A}})$ similarly, as well as $Z^+(F), Z_v^+$ and $Z^+(F_{\mathbb{A}})$. Thus $Z^+(F_{\mathbb{A}}) \simeq F_{\mathbb{A}}^+$. Let χ be an idèle class character of F. We will set

$$K_{\Phi'}(g_1, g_2) = \int_{F^+\backslash F_A^+} \sum_{\xi\in GL(n,F)} \Phi'(g_1^{-1}\xi z g_2)\chi(z) d^\times z. \tag{5}$$

We let ψ be a nontrivial additive character of $F_{\mathbb{A}}/F$. We let N be the group of upper triangular matrices with unit diagonal. We define an algebraic additive character of N, i.e. an algebraic morphism of algebraic groups from N to F, by:

$$\theta_0(n) = \sum_i n_{i,i+1}$$

and then set $\theta(n) = \psi(\theta_0(n))$. We compute the integral:

$$\int K_{\Phi'}({}^t n_1, n_2)\theta(n_1^{-1} n_2)\, dn_1\, dn_2$$

where $n_i \in N(F) \setminus N(F_{\mathbb{A}})$.

To that end (compare with [F], [dG], [g.S]), we introduce a notion of orbital integral. We let A be the group of diagonal matrices, $B = AN$ the group of upper triangular matrices in $G = GL(n)$, $W = W(A, G)$ the Weyl group of A identified with the group of permutation matrices. We first go to a local situation and let F be a local field, ψ a nontrivial additive character of F. We define θ as above. Then an element $g \in G(F)$ is said to be relevant if the character $(n_1, n_2) \mapsto \theta_0(n_1 n_2)$ is trivial on the stabilizer N^g of g in the group $N \times N$. The stabilizer is defined by the equation ${}^t n_1 g n_2 = g$. We will denote by $\mathcal{C}(G(F))$ the space of smooth functions of compact support on $G(F)$. If g is relevant and $\Phi' \in \mathcal{C}(G(F))$, we define the orbital integral $I(g, \Phi')$ by:

$$I(g, \Phi') = \int_{N^g(F)\backslash N(F)\times N(F)} \Phi'({}^t n_1 g n_2)\theta(n_1 n_2)\, dn_1\, dn_2. \tag{6}$$

Recall that the elements of the form wa with $w \in W, a \in A(F)$ form a set of representatives for the orbits of $N(F) \times N(F)$ on $G(F)$. We now describe a set of representatives for the relevant orbits. We say that a Levi-subgroup of M is standard if it contains A and a parabolic subgroup is standard if it contains B. We say that an element $w \in W$ is relevant if $w^2 = 1$ and for every simple root α (i.e. simple with respect to B) such that $w(\alpha)$ is negative, there is another simple root β with $w(\alpha) = -\beta$. We denote by $R(G)$ the set of relevant elements in W. An element w of W is relevant if and only if there is a standard Levi-subgroup M of G such that w is the longest element of $W \cap M$. We then denote by A_w or A_M the center of M. We also write $M = M_w$ and $w = w_M$. If $g = wa$ with $a \in A(F)$ is relevant then $w \in R(G)$. Assuming $w \in R(G)$ then wa is relevant if and only if $a \in A_w(F)$. Suppose that $w \in R(G)$. Then let $P_w = M_w U_w$ be the parabolic subgroup which is standard and has Levi factor M_w. Set also $V_w = N \cap M_w$. Then $N^{wa} = N^w$ for every $a \in A_w$. Furthermore, N^w is the set of pairs (n_1, n_2) with $n_i \in V_w$ and $n_2 = w^t n_1^{-1} w$. It follows that any point of the orbit of wa under $N(F) \times N(F)$ can be written uniquely in the form:

$$\xi(u_1, u_2, v) = {}^t u_1 wav u_2$$

with $u_i \in U_w(F)$ and $v \in V_w(F)$. Since the orbits of $N(F) \times N(F)$ are closed, the map ξ is an isomorphism of $U_w(F) \times U_w(F) \times V_w(F)$ onto the orbit of wa. Recall we have fixed a nontrivial additive character ψ; we let dx be the self-dual Haar measure on F. If α is a root let X_α be the corresponding root vector in the Lie algebra of N (one entry is 1, the other entries are 0). If U is a subgroup of N generated by roots (i.e. whose Lie algebra is spanned by vectors X_α) we set $du = \otimes dx_\alpha$ if $u = 1 + \sum_\alpha x_\alpha X_\alpha$. We take for invariant measure on the orbit the product measure $du_1\, dv\, du_2$. Thus:

$$I(wa, \Phi') = \int_{U_w(F)\times U_w(F)\times V_w(F)} \Phi'({}^t u_1 wav u_2)\theta(u_1 u_2)\theta(v)\, du_1\, dv\, du_2. \tag{7}$$

Since the orbit is closed, for $f \in \mathcal{C}(G(F))$, the integrand on the right has compact support. Thus the integral converges and defines a smooth function on $A_w(F)$ (see section 2). We let $\Delta_i(g)$ be the minor formed with the first i rows and i columns of a matrix g. Thus $\Delta_i({}^t n_1 a_1 g a_2 n_2) = \Delta_i(a_1)\Delta_i(g)\Delta_i(a_2)$. On the support of a function $\Phi' \in \mathcal{C}(G(F))$ the functions $\Delta_i(g)$ remain in a compact support of F, while the function $\Delta_n(g)$ remains in a compact support of $F^\times$. Thus an orbital integral $I(wa, \Phi')$ has support in a set defined by inequalities of the form:

$$|\Delta_i(a)| \le C_i \quad \text{if } \Delta_i(w) \ne 0, \qquad A \le |\det(a)| \le B.$$

Explicitly, for $n = 3$ the relevant elements can be defined as follows. For $M = A$, every a in $A(F)$ is relevant. Let $P_1 = M_1 N_1$ be the parabolic subgroup of type $(2, 1)$ and A_1 the center of M_1, that is, the group of matrices of the form $a = \mathrm{diag}(a_1, a_2, a_3)$ with $a_1 = a_2$. The longest element of $W \cap M_1$ is

$$w_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

Then for $a \in A_1(F)$ the element $w_1 a$ is relevant. Similarly, let $P_2 = M_2 N_2$ be the parabolic subgroup of type $(1,2)$ and A_2 the center of M_2, that is, the group of elements of the form $a = \mathrm{diag}(a_1, a_2, a_3)$ with $a_2 = a_3$. The longest element in $W \cap M_2$ is

$$w_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{9}$$

Then any $w_2 a$ with a in A_2 is relevant. Finally set

$$w_G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{10}$$

Then for $a \in Z(F) = A_{w_G}(F)$ the product $w_G a$ is relevant. For $GL(3)$ the orbital integrals are the functions $I(a, \Phi), I(w_1 a, \Phi), I(w_2 a, \Phi), I(w_G a, \Phi)$ on the groups of F points of $A, A_1, A_2, Z = A_{w_G}$ respectively.

Coming back to the global situation we define similarly the orbital integrals $I(wa, \Phi')$ for relevant elements wa with $w \in R(G), a \in A_w(F_{\mathbb{A}})$. If V is an algebraic subgroup of N we normalize the Haar measure on $V(F_{\mathbb{A}})$ by $\mathrm{vol}(V(F)\backslash V(F_{\mathbb{A}})) = 1$. The integrals are infinite products of convergent integrals, almost all of which are equal to 1 because the integrand is 1 on its support which has measure 1 (see Proposition 2.1). Thus they are convergent. A simple formal manipulation gives then

$$\int K_{\Phi'}({}^t n_1, n_2)\theta(n_1^{-1} n_2)\, dn_1\, dn_2 = \int_{F_{\mathbb{A}}^+/F^+} \sum_w \sum_\alpha I(w\alpha z, \Phi')\chi(z) d^\times z. \tag{11}$$

Here the sum is for $w \in R(G)$ and $\alpha \in A_w(F)$. Indeed, one replace $K_{\Phi'}$ by its expression as a sum and collects the terms belonging to one orbit of $N(F) \times N(F)$. An irrelevant orbit contributes a zero integral. The integral on the left is over a compact set and thus converges absolutely (even when Φ' and θ are replaced by their absolute values). As before there are $A > 0, B > 0$ such that $I(wa, \Phi') \neq 0$ implies $A \leq |\Delta_n(wa)| \leq B$; there is also a compact subset ω of $F_{\mathbb{A}}$ such that $I(wa, \Phi') \neq 0$ implies $\Delta_i(wa) \in \omega$ for $1 \leq i < n$. As a result $I(w\alpha z, \Phi') \neq 0$ implies that z is in a compact set of $F_{\mathbb{A}}^+/F^+$ and α in a finite set. Thus the expression on the right is well defined.

Now let Φ be a smooth function of compact support on $S(F_{\mathbb{A}})$. Recall K_Φ defined in (4). We shall assume that $\Phi = \prod \Phi_v$ where Φ_v is supported on S_v^+ for v inert. Thus Φ is supported on $S^+(F_{\mathbb{A}})$. We consider the integral

$$\int K_\Phi(n)\theta(\overline{n}n)\, dn$$

over $N(E)\backslash N(E_{\mathbb{A}})$. Note that the product of $\overline{n}n$ and a suitable element of the derived group of $N(E_{\mathbb{A}})$ is in $N(F_{\mathbb{A}})$ so that the expression $\theta(\overline{n}n)$ is well defined. Alternatively, we define an algebraic additive character θ_1 of N (regarded as a group over E) to F by

$$\theta_1(n) = \sum_i (n_{i,i+1} + \overline{n}_{i,i+1}).$$

Then $\theta(\overline{n}n) = \psi(\theta_1(n))$.

To compute this integral, we define a notion of orbital integral for the action of $N(E_{\mathbb{A}})$ on $S(F_{\mathbb{A}})$. To that end we again go to a local situation where F is a local field and E a quadratic extension of F. The group $G(E)$ operates on $S(F)$ by $s \overset{g}{\mapsto} g^*sg$. We say that an element $s \in S(F)$ is relevant if the character θ_1 is trivial on the stabilizer $N(E)^s$ of s in $N(E)$. The corresponding orbital integral is defined by

$$J(s, \Phi) = \int_{N(E)^s \backslash N(E)} \Phi(n^*sn)\theta(\overline{n}n)\, dn.$$

Mutatis mutandis, the previous discussion applies to the present situation where $G(F)$ is replaced by $S(F)$ and the group $N(F) \times N(F)$ by the group $N(E)$ acting on $S(F)$. Recall that the intersection of an orbit of $N(E) \times N(E)$ on $GL(n, E)$ with $S(F)$ is a single orbit of $N(E)$ acting on $S(F)$ ([t.S]). Otherwise said, each orbit of $N(E)$ on $S(F)$ has a unique representative of the form wa with $w \in W$ and $a \in A(E)$ satisfying $waw = \overline{a}$. In particular, each relevant orbit has a unique representative of the form wa with $w \in R(G)$ and $a \in A_w(F)$ (see [JR1] or [Y4]). Consider elements $w \in R(G)$ and $a \in A_w(F)$. Introduce as before $P_w = M_wU_w$ and $V_w = M_w \cap N$. Then any element of the orbit of wa can be written in the form

$$u^*v^*wavu$$

with $v \in V_w(E), u \in U_w(E)$. The stabilizer of w (or wa) is the set of $v \in V_w(E)$ such that $v^*wv = w$. On the other hand, it is immediate that $v_1 = wv^*wv$ is an element of $V_w(E)$ which satisfies the equation

$$wv_1^*w = v_1.$$

This equation determines an F-subgroup V_w^1 of $V_w(E)$. Thus $v_1 \in V_w^1(F)$. Thus in fact any point of the orbit of wa can be written uniquely in the form $u^*wav_1u, u \in U_w(E), v_1 \in V_w^1(F)$. This gives a diffeomorphism of $U_w(E) \times V_w^1(F)$ onto the orbit of wa in $S(F)$. We have already defined a measure dv on $V_w(F)$. The additive character $z \mapsto \psi(z + \overline{z})$ of E gives rise to a self-dual Haar measure on E and thus, as before, to a measure on $U_w(E)$. We can write an element v_1 of $V_w^1(F)$ in the form

$$v_1 = \sum(z_\alpha X_\alpha + \overline{z}_\alpha X_{-w_G\alpha} + \sum_{\beta=-w} x_\beta X_\beta.$$

Here the second sum is over all positive roots β in V_w such that $\beta = -w$ and the first sum is over all remaining positive roots in V_w; also $z_\alpha \in E, x_\beta \in F$. We set $dv_1 = \otimes\, dz_\alpha \otimes\, dx_\beta$. We take for invariant measure on the orbit of wa the product measure $du\, dv_1$. Then

$$I(wa, \Phi) = \int_{U_w(E) \times V_w^1(F)} \Phi(u^*wav_1u)\theta(u\overline{u})\psi(\theta_1(v_1))\, du\, dv_1. \tag{12}$$

As before the orbital integrals are absolutely convergent and define smooth functions on $A_w(F)$. For $GL(3)$ the orbital integrals are the functions $J(a, \Phi)$, $J(w_1a, \Phi)$, $J(w_2a, \Phi)$, $J(w_Ga, \Phi)$ on the groups of F-points of $A, A_1, A_2, Z = A_{w_G}$ respectively.

We can apply the same discussion to the situation where the local quadratic extension is replaced by the semisimple F-algebra $E = F \oplus F$. Then

$$S(E) = \{(g, {}^tg) | g \in G(F)\}$$

and we can consider the relevant orbits of $N(E) = N(F) \times N(F)$ on $S(E)$. The situation is then equivalent to the situation discussed earlier for $N(F) \times N(F)$ acting on $G(F)$.

Coming back to the global situation, we define global orbital integrals for relevant elements wa and we have

$$\int K_\Phi(n)\theta(\overline{n}n)\,dn = \int_{F_{\mathbb{A}}^\times/F^\times} \sum_w \sum_\alpha J(w\alpha z, \Phi)\chi(z)d^\times z. \tag{13}$$

We can also describe the function K_Φ as follows. To begin with, every $\sigma \in S^+(F)$ is equivalent to the identity matrix so that we can write

$$K_\Phi(g) = \int \sum_{\gamma \in H(F)\backslash G(E)} \Phi(g^*\gamma^* z\gamma g)\chi(z)d^\times z.$$

Here H is the unitary group attached to the identity matrix. There exists a smooth function of compact support f on $GL(n, E_{\mathbb{A}})$ such that

$$\Phi(g^*g) = \int_{H(F_{\mathbb{A}})} f(hg)\,dh.$$

We use the exact sequence

$$1 \to E_{\mathbb{A}}^1 \to E_{\mathbb{A}}^\times \to F_{\mathbb{A}}^+ \to 1$$

to define a measure on $E_{\mathbb{A}}^1$. Then

$$\mathrm{vol}(E_{\mathbb{A}}^1/E^1(F)) = 1$$

and

$$\int_{F_{\mathbb{A}}^+/F^+} \sum_{\xi \in S(F)} \Phi(g^*\xi zg)\chi(z)d^\times z = \int_{E_{\mathbb{A}}^\times/E^\times} \sum_{\xi \in S(F)} \Phi(g^*\xi z\overline{z}g)\chi(z\overline{z})d^\times z.$$

Set as usual

$$K_f(x,y) = \int_{E^\times\backslash E_{\mathbb{A}}^\times} \sum_{\gamma \in GL(n,E)} f(x^{-1}\gamma zy)\chi(z\overline{z})d^\times z.$$

Then

$$K_\Phi(g) = \int_{H(F)\backslash H(F_{\mathbb{A}})} K_f(h,g)\,dh,$$

and

$$\int_{N(E)\backslash N(E_{\mathbb{A}})} K_\Phi(n)\theta(\overline{n}n)\,dn = \int_{N(E)\backslash N(E_{\mathbb{A}})} \left(\int_{H(F)\backslash H(F_{\mathbb{A}})} K_f(h,n)\,dh \right) \theta(\overline{n}n)\,dn.$$

In our trace formula, $n = 3$ and the functions Φ and Φ' are related by the following global matching orbital integral conditions, where $\eta = \eta_{E/F}$ is the quadratic idèle class character of F attached to E:

$$I(a, \Phi') = \eta(a_2)J(a, \Phi), \tag{14}$$

$$I(w_1a, \Phi') = \eta(a_2)J(w_1a, \Phi), \tag{15}$$

$$I(w_2a, \Phi') = \eta(a_2)J(w_2a, \Phi), \tag{16}$$

$$I(w_Ga, \Phi') = J(w_Ga, \Phi) = 0. \tag{17}$$

It follows that, for all $w \in R(G), \alpha \in A_w(F), z \in F_{\mathbb{A}}^+$:

$$I(w\alpha z, \Phi') = J(w\alpha z, \Phi).$$

Taking in account the equalities (11) and (13) we get

$$\int K_{\Phi'}({}^t n_1, n_2)\theta(n_1^{-1} n_2)\, dn_1\, dn_2 = \int K_\Phi(n)\theta(\overline{n}n)\, dn. \tag{18}$$

In a more precise way, we assume that $\Phi = \prod \Phi_v, f = \prod f_w$ and $\Phi' = \prod \Phi'_v$. The Haar measure dh on $H(F_{\mathbb{A}})$ is written as a product of local Haar measures in the usual way. We let S be a finite set of places so chosen that a place v not in S is finite and split, or (finite) inert and unramified in E; in addition, the residual characteristic of v is odd and the conductor of ψ_v is the ring of integers $\mathcal{O}_v$ of F_v. If $v \notin S$ is inert and w is the place of E above v, we set $K_w = GL(3, \mathcal{O}_w)$ and assume that $\mathrm{vol}(H_v \cap K_w) = 1$. We will also assume that S does contain at least one finite place of F inert in E.

Recall the function Φ'_v is assumed to be supported on G_v^+ for all places v inert in E. For $v \notin S$, the function Φ'_v is bi-invariant under $K'_v = GL(3, \mathcal{O}_v)$ and equal to the characteristic function of K'_v for almost all $v \notin S$. If v is in S and inert in E then we assume that $I(w_G a, \Phi'_v) = 0$.

Similarly, let v be a place of F inert in E. Let us denote by w the unique place of E above v. We define Φ_v by $\Phi_v(s) = 0$ if $v \notin S_v^+$ and

$$\Phi_v(s) = \int_{H_v} f_w(h_v g)\, dh_v$$

if $s \in S_v^+$ and $s = g^* g$. Thus Φ_v is indeed supported on S_v^+. If $v \in S$ we will assume that $J(w_G a, \Phi_v) = 0$. If $v \notin S$ we will take f_w to be bi-invariant under $K_w = GL(3, \mathcal{O}_w)$ and will assume that Φ'_v is the image of f_w under the base change homomorphism of Hecke algebras. For almost all $v \notin S$ and inert we assume that Φ'_v is the characteristic function of K'_v and f_w the characteristic function of K_w. This implies that for almost all v inert and not in S, Φ_v is the characteristic function Φ_v^0 of $K_w \cap S_v$, provided we assume, as we do, that the measure of $H_v \cap K_w$ is 1. Indeed, if $\Phi_v(s) \neq 0$ then $s = g^* g$ with $g \in H_v K_w$. Thus in fact $s = g^* g$ with $g \in K_w$. Moreover $\Phi_v(s) = 1$ then because $\mathrm{vol}(H_v \cap K_w) = 1$. Conversely, if s is in $S_v \cap K_w$ then $s = g^* g$ with $g \in K_w$ by the theory of elementary divisors for Hermitian matrices and then $\Phi_v(s) = 1$. Note that in general for v inert and not in S we have

$$\Phi_v(s) = \int \Phi_v^0(g_w^* s g_w) f_w(g_w)\, dg_w.$$

For v inert in S or not in S, we will assume that the following local matching orbital integral conditions are satisfied:

$$I(a, \Phi'_v) = \eta_v(a_2) J(a, \Phi_v), \tag{19}$$

$$I(w_1 a, \Phi'_v) = \eta_v(-a_2) J(w_1 a, \Phi_v) c(E_w/F_v, \psi_v), \tag{20}$$

$$I(w_2 a, \Phi'_v) = \eta_v(a_2) J(w_2 a, \Phi_v) c(E_w/F_v, \psi_v). \tag{21}$$

The constant $c(E_w/F_v, \psi_v)$ will be defined in Proposition 3.1. The essential point of this article is to show that for v in S and inert and a given Φ'_v with $I(w_G a, \Phi'_G) = 0$ there is a function Φ_v with $J(w_G a, \Phi_v) = 0$ satisfying these conditions (Proposition 3.2). This is the only missing element in the proof of the trace formula at hand. For v inert not in S, the constant $c(E_w/F_v, \psi_v)$ is 1 and the "fundamental lemma" ([JY2] and [Y2]) asserts that these conditions are indeed satisfied by our chosen functions. The product of the constants $c(E_w/F_v, \psi_v)$ for v inert is 1.

Finally, consider a place v of F (finite or infinite) which splits into two places v_1 and v_2. After choosing one of the two places, we may identify S_v with the set of pairs $(g, {}^tg)$ with $g \in G_v$ and H_v with the group of pairs $(h, {}^th^{-1})$ with $h \in G_v$. The group $G_{v_1} \times G_{v_2} \simeq G_v \times G_v$ operates on S_v and we have again a notion of orbital integrals. To insure the analogue of the matching conditions (19) to (21) (with $c = 1$) we take

$$\Phi'_v(g) = \Phi_v(g, {}^t g) = \int f_{v_1}(hg) f_{v_2}({}^th^{-1})\, dh. \tag{22}$$

For all places $v \notin S$ which split we take Φ'_v to be a bi-K'_v invariant function and f_{v_i} to be a bi-K_{v_i} invariant function. Since such a function is invariant under transposition, the function Φ'_v is simply the convolution of f_{v_1} and f_{v_2}. For almost all such v we take f_{v_i} to be the characteristic function of K_{v_i}. Finally at each Archimedean place w of E (resp. v of F), we denote by K_w (resp. K'_v) the standard maximal compact subgroup. If v is an Archimedean place of F then (by assumption) it splits in E; we assume that each function f_{v_i} is itself a convolution of two K_{v_i} finite functions. It follows that Φ'_v is a K'_v finite function which is a quadruple convolution product of smooth K'_v finite functions of compact support. With these choices, the global matching orbital integral conditions (14) to (17) are satisfied.

We define in the usual way the cuspidal components $K^{\mathrm{cusp}}_{\Phi'}$ and K^{cusp}_f of $K_{\Phi'}$ and K_f and define the cuspidal component of K_Φ by

$$K^{\mathrm{cusp}}_\Phi(g) = \int K^{\mathrm{cusp}}_f(h, g)\, dh.$$

Then according to [J6] the difference:

$$\int K_\Phi(n)\theta(\overline{n}n)\, dn - \int K^{\mathrm{cusp}}_\Phi(n)\theta(\overline{n}n)\, dn$$

can be represented by an **absolutely** convergent expression taken over the continuous spectrum (see [J6] for details). The same assertion is trivially true for the difference

$$\int K_{\Phi'}({}^tn_1, n_2)\theta(n_1^{-1}n_2)\, dn_1\, dn_2 - \int K^{\mathrm{cusp}}_{\Phi'}({}^tn_1, n_2)\theta(n_1^{-1}n_2)\, dn_1\, dn_2.$$

Equating the continuous parts and the discrete parts, we conclude from a standard argument that:

$$\int K^{\mathrm{cusp}}_\Phi(n)\theta(\overline{n}n)\, dn = \int K^{\mathrm{cusp}}_{\Phi'}({}^tn_1, n_2)\theta(n_1^{-1}n_2)\, dn_1\, dn_2. \tag{23}$$

This is the trace formula we had in mind. It will imply the theorem.

Finally we remark that for an arbitrary extension E/F we would have to consider all unitary groups. If H_σ is a set of representatives for the finitely many isomorphism classes (n is odd), then for each σ we would have to introduce a function f_σ.

The material is arranged as follows. In section 2 we study the local orbital integrals $I(wa, \Phi')$ and in section 3 the local orbital integrals $J(wa, \Phi)$ together with their matching with the integrals $I(wa, \Phi')$. The theorem is then quickly proved in section 4. In section 2, it is more convenient to discuss the qualitative asymptotic behavior of our integrals in the context of $GL(n)$. We thus define the *Shalika germs* for our integrals and prove the existence of the germs in the context of $GL(n)$. As a matter of fact, one of the aims of this paper is to introduce the correct notion of Shalika germs for our integrals. However, for the purpose at hand, we only need very partial results on $GL(2)$ and $GL(3)$. A discussion with S. Rallis and J. Bernstein was very helpful in formulating the definition of the germs.

2. Shalika germs

We let F be a local field of characteristic 0 and ψ a nontrivial additive character. We often write G for $G(F)$ and $\mathcal{C}(G)$ for the space of smooth functions of compact support on $G(F)$. We use similar notations for other groups or varieties. We denote by $\overline{F}$ an algebraic closure of F.

We first describe the asymptotic behavior of our integrals in the context of $G = GL(n)$. If w, w' are in $R(G)$ we write $w \to w'$ if $A_w \supseteq A_{w'}$. This is equivalent to $M_w \subseteq M_{w'}$ or $w \in M_{w'}$. We write $w \xrightarrow{1} w'$ if $w \to w', w \neq w'$ and there is no $w'' \in R(G)$ such that $w \to w'' \to w$. We can define a graph with $R(G)$ for a set of vertices: the graph is oriented and the edges are the pairs (w, w') with $w \xrightarrow{1} w'$. Note that all oriented paths from a given w to a given w' have the same length which we denote by $d(w, w')$. We write $w \xrightarrow{m} w'$ if $w \to w'$ and $d(w, w') = m$. For each $w \in R(G)$ we have $e \to w \to w_G$. For $0 \leq i \leq n$ we denote by $\pi_i(g)$ the representation of G on the ith exterior power V^i of the standard vector space V of dimension n. Let e_j, $1 \leq j \leq n$, be the canonical basis of V. Let

$$\epsilon_i = e_1 \wedge e_2 \wedge \cdots \wedge e_i, \quad \eta_i = e_{n-i+1} \wedge e_{n-i+2} \wedge \cdots \wedge e_n$$

be the highest vector and the lowest vector respectively in V^i. In particular, the one-dimensional vector space V^n has a basis ϵ_n. If $v \in V^i$ and $v' \in V^{n-i}$ we set

$$v \wedge v' = \langle v, v' \rangle \epsilon_n.$$

Then

$$\Delta_i(g) = \langle \pi_i(g)\epsilon_i, \eta_{n-i} \rangle.$$

We denote by $\Delta(G)$ the set of these functions. Note that $\Delta_0 = 1$ and $\Delta_n(g) = \det g$.

Lemma 2.1. *Suppose $w \in R(G)$ and $\Delta \in \Delta(G)$. Suppose $w \neq w_G$ and $\Delta(w_G w) \neq 0$. Then $\Delta = \Delta_0$ or $\Delta = \Delta_n$. Suppose $\Delta(w) \neq 0$. Then $\Delta(m) \neq 0$ for all $m \in M_w$.*

Proof. For the first assertion suppose that $\Delta = \Delta_i$ with $1 \leq i \leq n-1$ and $\Delta(w_G w) \neq 0$. We have

$$\Delta(w_G w) = \pm \langle \pi_i(w)\epsilon_i, \epsilon_{n-i} \rangle.$$

If $\Delta(w_G w) \neq 0$ then $\pi_i(w)\epsilon_i = \pm\eta_i$. This implies that w has the form

$$w = \begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ A^{-1} & 0 & 0 \end{pmatrix}$$

where $A \in GL(i)$ and $B \in GL(n-i)$. The only standard Levi subgroup which contains w is G. Hence $w = w_G$.

For the second assertion, let us write an element $m \in M = M_w$ as a diagonal matrix of square blocks:

$$m = \operatorname{diag}(g_1, g_2, \ldots, g_r)$$

with $g_i \in G_i \simeq GL(r_i)$. Thus

$$M \simeq G_1 \times G_2 \times \cdots \times G_r.$$

If $\Delta \in \Delta(G)$ then

$$\Delta(m) = \Delta_1'(g_1)\Delta_2'(g_2)\cdots\Delta_r'(g_r), \tag{24}$$

where $\Delta_i' \in \Delta(G_i)$. In addition

$$w = \operatorname{diag}(w_1, w_2, \ldots, w_r)$$

where $w_i = w_{G_i}$. By the previous assertion, each Δ_i' is either the determinant function or the constant function equal to 1 on G_i. The lemma follows. $\square$

Recall that $N \times N$ operates on G:

$$g \overset{n}{\mapsto} {}^t n_1 g n_2$$

if $n = (n_1, n_2)$. The following facts will not be needed but will shed some light on the definition of Shalika germs. The algebra of polynomial functions on $G(\overline{F})$ invariant under $N \times N$ is the polynomial algebra generated by $\Delta(G)$ and Δ_n^{-1}. Similarly, if w is relevant, we consider the closure F_w of the orbit ${}^tNAwN$ (for the Zariski topology) and the algebra of polynomial functions on F_w which are invariant under $N \times N$. Let $P_w = M_w U_w$ be the standard parabolic subgroup of G with Levi-factor M_w. Then ${}^tNwAN = {}^tNwAU_w$. To construct such a polynomial function on F_w it suffices to construct a polynomial function on G invariant under ${}^tN$ on the left and under U_w on the right and restrict it to F_w. The function Δ_w defined by $\Delta_w(g) = \Delta(gw)$ is an example of such a function. The algebra in question is generated by the functions Δ_w with $\Delta \in \Delta(G)$ and the function Δ_n^{-1}. These functions separate the closed orbits in F_w (over $\overline{F}$ for the Zariski topology or over F for the ordinary topology).

If w is relevant we denote by Ω_w the set of $g \in G(F)$ such that $\Delta(w) \neq 0$ implies $\Delta(g) \neq 0$.

Lemma 2.2. *The set Ω_w is open and the map*

$$(u, v, m) \mapsto {}^tumv$$

from $U_w(F) \times U_w(F) \times M_w(F)$ to Ω_w is an isomorphism of analytic varieties over F. If ω_M is a compact subset of $M(F)$ and ω_G a compact subset of $G(F)$ then the relations

$${}^tumv \in \omega_G, \quad m \in \omega_M$$

imply that u and v are in a compact subset of $U_w(F)$. If the orbit of a relevant element $w'a$ intersects Ω_w then $w' \to w$ and the orbit is contained in Ω_w.

Proof. By the previous lemma, if $P = MU$ is the parabolic subgroup attached to w then Ω_w is just the set of g such that $\Delta(g) \neq 0$ if Δ does not vanish on M. We first prove the first assertion of the lemma for a parabolic subgroup $P = MU$ of type (n_1, n_2). Then if $\Delta \in \Delta(G)$ does not vanish on M, we have $\Delta = \Delta_{n_1}$ or $\Delta = \det$ or $\Delta = 1$. Suppose that $\Delta_{n_1}(g) \neq 0$. Write

$$g = \begin{pmatrix} m_1 & v_1 \\ v_2 & m_2 \end{pmatrix}$$

with $m_i \in M(n_i \times n_i, F)$. Since $\det(m_1) = \Delta_{n_1}(g) \neq 0$ we find $m_1 \in GL(n_1, F)$. Thus we can write $v_1 = m_1 U_1$ and $v_2 = {}^tU_2 m_1$. With

$$u_i = \begin{pmatrix} 1 & U_i \\ 0 & 1 \end{pmatrix}$$

we have now $g = {}^t u_1 m u_2, u_i \in U$, and $m \in M$. The first assertion of the lemma follows then from an inductive argument. The second assertion follows from the proof of the first. The third assertion follows from the uniqueness of the Bruhat decomposition. $\square$

We also set

$$Z_w = {}^t N w A N, \quad V_w = N \cap M_w.$$

Then Z_w is a closed subvariety of Ω_w. Indeed, this follows from the previous lemma and the fact that $Z_w \cap M_w = wAV_w$ is closed in M_w.

Proposition 2.1. *If F is any local field (Archimedean or not) the orbital integral $I(wa, \Phi')$ converges and defines a smooth function on $A_w(F)$. Suppose that F is non-Archimedean and that the conductor of ψ is the ring of integers $\mathcal{O}_F$. If f is the characteristic function of $K = GL(n, \mathcal{O}_F)$ and $a \in A_w(F) \cap K$ then $I(wa, f) = 1$.*

By the previous lemma, the map

$$(u_1, u_2, v, a) \mapsto {}^t u_1 wavu_2$$

is a diffeomorphism from $U_w \times U_w \times V_w \times A_w$ onto a closed subvariety of Ω_w. On the other hand, if ω is a compact subset of $A_w(F)$ then the same map is a homeomorphism from $U_w \times U_w \times V_w \times \omega$ onto a closed subset of $GL(n, F)$. The first assertion follows.

We prove the second assertion. We will use two lemmas:

Lemma 2.3. *Suppose $w \in R(G)$ and $g \in M(n, \mathcal{O}_F)$ with $\Delta(g) \in \mathcal{O}_F^\times$ for all Δ with $\Delta(w) \neq 0$. Then we have $g = {}^t u_1 m u_2$ with $u_i \in U_w \cap K$, $m \in M_w \cap K$ and $\Delta(m) \in \mathcal{O}_F^\times$ for all Δ with $\Delta(w) \neq 0$.*

Proof of the lemma. It suffices to prove our assertion when $P_w = MU$ has type (n_1, n_2). We write, as before,

$$g = \begin{pmatrix} m_1 & v_1 \\ v_2 & m_2 \end{pmatrix}$$

with $m_i \in M(n_i \times n_i, F)$. Since m_1 is integral and $\det(m_1) = \Delta_{n_1}(g) \in \mathcal{O}_F^\times$ we find $m_1 \in GL(n_1, \mathcal{O}_F)$. Thus we can write $v_1 = m_1 U_1$ and $v_2 = {}^t U_2 m_1$ where U_i is integral. With

$$u_i = \begin{pmatrix} 1 & U_i \\ 0 & 1 \end{pmatrix},$$

we have now $g = {}^t u_1 m u_2$. Now u_i is in $U \cap K$ so that m is in $K \cap M$ and $\Delta(m) = \Delta(g) \in \mathcal{O}_F^\times$ for all Δ which do not vanish on M. $\square$

Lemma 2.4. *Suppose that g is in K and $g = {}^t u_1 wavu_2$ with $u_i \in U_w, v \in V_w$ and $a \in A_w(F) \cap K$. Then u_i and v are in K as well.*

Proof of the lemma. We can write

$$gw = {}^t u_1 m u_2', \quad m = wvwa, \quad u_2' = wu_2 w.$$

Since $\Delta(m) = \Delta(a)$ we can apply the previous lemma. It follows that u_1, u_2' and m are in K. In turn, this implies that u_2 and v are in K. The lemma follows.

The second assertion of the proposition follows from the two lemmas. $\square$

From now on, we assume F is non-Archimedean.

If $w \to w'$ we denote by $A_w^{w'}$ the set of $b \in A_w(b)$ such that $\Delta(b) = \Delta(w'w)$ for all $\Delta \in \Delta(G)$ such that $\Delta(w'w) \neq 0$. This is a single coset of a subgroup of A_w. More precisely, let $M' = M_{w'}$ and $M = M_w$. Let $A_M^{M'}$ be the algebraic subgroup of A_M defined by the equations $\Delta(a) = 1$ if $\Delta(w'w) \neq 0$. Then $A_w^{w'}$ is a single coset of $A_M^{M'}(F)$. We need simple properties of this construction.

Lemma 2.1 implies that if $w \neq w_G$ then $A_w^{w_G}$ is the set of $b \in A_w$ such that $\det(b) = \det(w_G w)$. On the other hand, it is clear that $A_w^w = \{1\}$ for all w.

Suppose that $M = M_w$ with $w \in R(G)$. Let us write an element $m \in M_w$ as a diagonal matrix of square blocks:

$$m = \operatorname{diag}(g_1, g_2, \ldots, g_r)$$

with $g_i \in G_i \simeq GL(r_i)$ and use the notations of the proof of Lemma 2.1. In particular:

$$w = \operatorname{diag}(w_1, w_2, \ldots, w_r)$$

where $w_i = w_{G_i}$. Similarly, every a in A_w has the form:

$$a = \operatorname{diag}(a_1, a_2, \ldots, a_r)$$

with $a_i \in A_{w_i} \subset G_i$. Thus

$$A_w \simeq \prod A_{w_i}.$$

If $w' \to w$ then $w' \in M$ and

$$w' = \operatorname{diag}(w'_1, w'_2, \ldots, w'_r)$$

where $w'_i \in R(G_i)$ (and $w'_i \to w_i$ in G_i). We have then

$$A_{w'} \simeq \prod A_{w'_i}.$$

If $\Delta(ww') \neq 0$ then, in the notations of (24), $\Delta'_i(w_i w'_i) \neq 0$ for each i. Conversely, if $\Delta'_i \in \Delta(G_i)$ is such that $\Delta'_i(w_i w'_i) \neq 0$ then there is $\Delta \in \Delta(G)$ such that

$$\Delta(m) = \Delta'_1(g_1)\Delta'_2(g_2)\cdots\Delta'_i(g_i),$$

where $\Delta'_j = \det$ on G_j for $j < i$. It follows that

$$A_{w'}^w \simeq \prod A_{w'_i}^{w_i}.$$

Lemma 2.5. *For $w' \to w$, the set $Y_w^{w'}$ of $c \in A_w$ such that $\Delta(c) = 1$ for $\Delta(ww') \neq 0$ is finite. Also $A_{w'}^w A_w^{w''} \subseteq A_{w'}^{w''}$.*

Proof. Writing M_w as a product of linear groups as before, we get that

$$Y_w^{w'} = \prod Y_{w_i}^{w'_i}.$$

If $w_i = w'_i$ then

$$Y_{w_i}^{w'_i} = \{1\}.$$

Thus it suffices to prove the lemma when $w = w_G$ and $w' \neq w_G$. Then $\Delta(w_G w') \neq 0$ implies $\Delta = \det$ or $\Delta = 1$ and $Y_{w_G}^{w'}$ is the set of scalar matrices c with $c^n = 1$. Thus it is finite. The proof of the second assertion is similar. $\square$

We can also describe the group $A^M_{M'}$ where $M' = M_{w'}$ and $M = M_w$. If $M = M'$ then the group is reduced to $\{1\}$. If $M = G$ then

$$A^G_{M'} = \{a \in A_{M'} \,|\, \det a = 1\}.$$

In general, with the previous notations, set $M'_i = M_{w'_i}$ and $M_i = M_{w_i}$. Then

$$A^M_{M'} \simeq \prod A^{M_i}_{M'_i}.$$

In particular, we remark that if Δ does not vanish on M then Δ defines an algebraic character of M and $\Delta(ww') \neq 0$ since ww' is in M. Moreover the group of algebraic characters of M is generated by the restriction to M of those Δ which do not vanish on M. If we let $\widetilde{A}^M_{M'}$ be the group of $a \in A_{M'}$ such that $\chi(a) = 1$ for any character χ of M then $A^M_{M'}$ is clearly contained in $\widetilde{A}^M_{M'}$. In fact, it is easy to check from the above description that $A^M_{M'}$ is just the connected component of the identity in $\widetilde{A}^M_{M'}$ for the Zariski topology. Moreover, the previous lemma amounts to saying that the group $A_{M'}$ is the product, almost direct, of $A^M_{M'}$ and A_M.

A system of Shalika germs will be a family of smooth functions $K^{w'}_w$ defined over the sets $A^{w'}_w$ for $w \to w'$ such that, for any function $f \in \mathcal{C}(G(F))$, there exist functions $\omega_w = \omega^f_w \in \mathcal{C}(A_w(F))$ with:

$$I(wa, f) = \sum_{\{w' : w \to w'\}} \sum_{\{a = bc, b \in A^{w'}_w, c \in A_{w'}\}} K^{w'}_w(b)\omega_{w'}(c).$$

We note that for each w the set A^w_w is reduced to the identity. By convention, we take $K^w_w = 1$. For $w \to w'$, the sum is over all decompositions $a = bc, b \in A^{w'}_w, c \in A_{w'}$. It is finite by the previous lemma. In particular it is empty hence 0 if $a \notin A^{w'}_w A_{w'}$.

For a given function f, the above relations determine the functions ω_w by a triangular system of linear equations. In particular $\omega_{w_G}(a)$ is just the orbital integral $I(w_G a, f)$. When we want to emphasize the dependence of the functions on the system, we will write them as $\omega^{K,f}_w$ or ω^K_w. Given a system of germs K and a family of smooth functions of compact support $\omega_w \in \mathcal{C}(A_w)$, there is $f \in \mathcal{C}(G)$ such that $\omega_w = \omega^{K,f}_w$. This follows from the following observation. Suppose $f \in \mathcal{C}(\Omega_w)$. Then $I(w'a, f) = 0$ unless the orbit of $w'a$ intersects Ω_w, that is, $w' \to w$. In particular, it follows that $\omega^f_{w'} = 0$ unless $w' \to w$. Moreover $\omega^f_w(a) = I(wa, f)$. Now $I(wa, f) = I(wa, f')$ where $f' \in \mathcal{C}(M_w)$, is given by

$$f'(m) = \int_{U_w \times U_w} f({}^t u_1 m u_2)\theta(u_1 u_2)\, du_1\, du_2$$

and

$$I(wa, f') = \int_{V_w} f'(wav)\theta(v)\, dv.$$

Since wA_wV_w is closed in M_w, it follows that ω_w is an arbitrary element of $\mathcal{C}(A_w)$. Our assertion follows now from an easy inductive argument.

Recall that the support of an orbital integral is contained in a set where $|\Delta(a)| \leq C$ for all Δ. We note that in the definition of the germs the relation $b \in A^{w'}_w$ amounts to $\Delta_w(w'c) = \Delta_w(wa)$. Since the functions Δ_w on F_w separate the relevant orbits of wA_w this remark gives some insight in the nature of the germ expansion. Given w' with $w \to w'$, the contribution of w' to the orbital integral $I(wa, f)$ is a function whose support is contained in a set $C_1 \leq |\Delta(a)| \leq C_2$ for all Δ with $\Delta(w'w) \neq 0$.

On the other hand, the behavior at infinity of this contribution, that is, the behavior as $\Delta(a) \to 0$ for those Δ such that $\Delta(w'w) = 0$, is given by the germ $K_w^{w'}$.

The Shalika germs depend on ψ.

It will be convenient to use the following notation: if f and g are functions on $A_w^{w'}$ and $A_{w'}$ respectively then we define a new function $f * g$ on A_w by

$$f * g(a) = \sum_{\{a=bc, b \in A_w^{w'}, c \in A_{w'}\}} f(b)g(c). \tag{25}$$

Then the relations defining the germs read

$$I(w., f) = \sum_{w \to w'} K_w^{w'} * \omega_{w'}.$$

Theorem 2.1. *There exists a system of Shalika germs. If H and K are two systems of Shalika germs, then there are functions $t_w^{w'} \in \mathcal{C}(A_w^{w'})$ such that $t_w^w = 1$ for all w and*

$$K_w^{w'} = \sum_{w \to w_1 \to w'} H_w^{w_1} * t_{w_1}^{w'}.$$

Proof. The existence follows from the following proposition: $\square$

Proposition 2.2. *Suppose $1 \leq m \leq d(e, w_G)$. Then there exist functions $K_w^{w'}$ on $A_w^{w'}$ for $w \to w'$ and $d(w', w_G) < m$ with the following properties: suppose $f \in \mathcal{C}(G(F))$ is given; for any w' with $d(w', w_G) < m$, there exists a function $\omega_{w'} \in \mathcal{C}(A_{w'})$; for any w' with $w' \xrightarrow{m} w_G$, there exists a function $f_{w'} \in \mathcal{C}(\Omega_{w'})$; the following equalities are satisfied:*

$$I(w., f) = \sum_{w' \xrightarrow{m} w_G} I(w., f_{w'}) + \sum_{d(w', w_G) < m, w \to w'} K_w^{w'} * \omega_{w'}.$$

For $m = d(e, w_G)$ this will give the existence of the Shalika germs since $I(wa, f_e) = 0$ unless $w = e$ in which case $I(a, f_e)$ is a smooth function of compact support on $A = A_e$. Note that in the first sum over w' the orbital integral $I(wa, f_{w'})$ vanishes unless the orbit of wa intersects $\Omega_{w'}$, that is, unless $w \to w'$.

We first prove the assertion for $m = 1$. The above relation then reads

$$I(w., f) = \sum_{w' \xrightarrow{1} w_G} I(w., f_{w'}) + K_w^{w_G} * \omega_{w_G}.$$

Let Ω be the complement of Z_{w_G}. Restriction of a function on G to Z_{w_G} gives us an exact sequence

$$0 \to \mathcal{C}(\Omega) \to \mathcal{C}(G) \to \mathcal{C}(Z_{w_G}) \to 0.$$

We denote by $\mathcal{C}(C, \theta)$ the subvector space of $\mathcal{G}(G)$ spanned by the functions of the form

$$f({}^t n_1 g n_2) - \overline{\theta}(n_1 n_2) f(g)$$

and by $\mathcal{C}(G)_\theta$ the quotient space (coinvariants). An element of the dual vector space may be viewed as a distribution on G, relatively invariant under the character $\theta \otimes \theta$ of $N \times N$. We denote by $\mathcal{C}(G)^{*\theta}$ this dual. Examples of such elements are the orbital integrals. We write $f_1 \simeq f_2$ if f_1 and f_2 have the same image in $\mathcal{C}(G)_\theta$. In particular, we have then $I(wa, f_1) = I(wa, f_2)$ for all w. Conversely, by the theorem of density of Bernstein ([B]), if this condition is satisfied then $f_1 \simeq f_2$.

It is a fundamental observation of Casselman that the functor of coinvariants is exact. Thus we have an exact sequence

$$0 \to \mathcal{C}(\Omega)_\theta \to \mathcal{C}(G)_\theta \to \mathcal{C}(Z_{w_G})_\theta \to 0.$$

Our first task is to determine the dual space $\mathcal{C}(Z_{w_G})^{*\theta}$. The map

$$\xi : (n_1, n_2, a) \mapsto {}^t n_1 w_G a n_2$$

is surjective and submersive from $N \times N \times A$ to Z_{w_G}. Thus there is a surjective map $\alpha \mapsto f_\alpha$ from $\mathcal{C}(N \times N \times A)$ to $\mathcal{C}(Z_{w_G})$ such that

$$\int_{N\times N\times A} \alpha(n,a) T(\xi(n,a))\, dn\, da = \int_{Z_{w_G}} f_\alpha(z) T(z)\, dz,$$

for $T \in \mathcal{C}(Z_{w_G})$. Here dz is a measure on Z_{w_G} product of an invariant measure on the orbit of wa and the measure da. It follows that for any $T \in \mathcal{C}(Z_{w_G})^{*\theta}$ there is a distribution T^* on A such that

$$\int \alpha(n,a)\theta\otimes\theta(n)\, dn\, dT^*(a) = \int f_\alpha(z)\, dT(z).$$

Suppose $\alpha_1 \in \mathcal{C}(A)$ has support in the complement of A_{w_G}. Choose a smooth map $a \mapsto n_a$ from the support of α_1 to $N^{w_G a}$ with $\theta\otimes\theta(n_a) \neq 1$, a function $\alpha_0 \in \mathcal{C}(N)$ with $\int \theta\otimes\theta(n)\alpha_0(n)\, dn = 1$. Then the function

$$\alpha(n,a) = (\alpha_0(n) - \alpha_0(n_a n))\alpha_1(a)$$

belongs to $\mathcal{C}(N \times N \times A)$. For such a function $f_\alpha = 0$; thus the value of T^* on the function

$$a \mapsto (1 - \overline{\theta}(n_a))\alpha_1(a)$$

vanishes. Thus T^* is supported on A_{w_G}, that is, for $f \in \mathcal{C}(Z_{w_G})$,

$$T(f) = \int_{A_{w_G}} I(w_G a, f)\, dT^*(a).$$

We conclude that a function f on G has a zero image in $\mathcal{C}(Z_{w_G})_\theta$ if and only if all its orbital integrals $I(w_G a, f)$ are 0. This also follows from the theorem of density of Bernstein. The above exact sequence asserts there is then a function $f_0 \in \mathcal{C}(\Omega)$ such that

$$I(wa, f) = I(wa, f_0)$$

for all w and all $a \in A_w$.

We apply this observation as follows: let G_1 be the set of $g \in G$ with $\det g = \det w_G$. We can also define the orbital integrals of a function f_0 on G_1. If $w_G a$ where $a \in A_{w_G}$ is in G_1 then the scalar matrix a verifies $a^n = 1$. In particular, the orbital integral $I(w_G j, f_0)$ is defined as a function on the set of $j \in F$ which are n-roots of 1. We can choose f_0 in such a way that $I(w_G, f_0) = 1$ and $I(w_G j, f_0) = 0$ if $j \neq 1$. Given f define f_1 by

$$f_1(g) = \sum_{g=hz, z\in A_{w_G}, h\in G_1} f_0(h) I(w_G z, f).$$

Note that the sum is empty, hence 0, if $\det g \notin F^{\times n}\det w_G$. This is a smooth function of compact support. Moreover, for $a \in A_{w_G}$:

$$f_1({}^t n_1 w_G a n_2) = \sum_{\{j : j^n = 1\}} f_0({}^t n_1 w_G j^{-1} n_2) I(w_G j a, f).$$

Hence

$$I(w_G a, f_1) = \sum_j I(w_G j^{-1}, f_0) I(w_G j a, f) = I(w_G a, f).$$

Thus there is $f_2 \in \mathcal{C}(\Omega)$ such that

$$f \simeq f_2 + f_1.$$

Next we can find an increasing sequence of open subsets Ω_i, $0 \le i \le r$, invariant under $N \times N$, such that

$$\Omega_0 = \bigcup_{w \xrightarrow{1} w_G} \Omega_w,$$

$\Omega_r = \Omega$ and each difference $Z_i = \Omega_{i+1} - \Omega_i$ has the form ${}^tNwAN$ for a suitable irrelevant w. This follows from the fact that N has finitely many orbits in $B\backslash G$. We have again exact sequences

$$0 \to \mathcal{C}(\Omega_i)_\theta \to \mathcal{C}(\Omega_{i+1})_\theta \to \mathcal{C}(Z_i)_\theta \to 0.$$

An argument similar to the one used before shows that the last term on the right is zero. It follows that

$$\mathcal{C}(\Omega_{i+1})_\theta = \mathcal{C}(\Omega_i)_\theta.$$

Inductively we have then

$$\mathcal{C}\left(\bigcup_{w \xrightarrow{1} w_G} \Omega_w\right)_\theta = \mathcal{C}(\Omega)_\theta.$$

Thus we may assume that f_2 has support in the union of the sets Ω_w with $d(w, w_G) = 1$. Using now a partition of unity we obtain that

$$f \simeq \sum_{\{w : d(w, w_G) = 1\}} f_w + f_1,$$

with $f_w \in \mathcal{C}(\Omega_w)$. Next, we compute the orbital integrals of f. For $w' \neq w_G$ we have

$$I(w'a, f) = \sum_{d(w, w_G) = 1} I(w'a, f_w) + I(w'a, f_1).$$

To compute $I(w'a, f_1)$ we write in all possible ways $g = {}^tn_1 w' a n_2$ in the form $g = g_1 c$ with $g_1 \in G_1$, $c \in A_{w_G}$. We get

$${}^tn_1 w' a n_2 = {}^tn_1 w' b n_2 c$$

with $c \in A_{w_G}$ and $\det b = \det w_G w'$. This amounts to $a = bc$ with $b \in A_{w'}^{w_G}$. We find then

$$I(w'a, f_1) = \sum_{a = bc} I(w'b, f_0) I(w_G c, f).$$

We obtain our assertion for $m = 1$ with

$$K_{w'}^{w_G}(b) = I(w'b, f_0)$$

and $\omega_G(z) = I(w_G z, f)$ for $z \in A_{w_G}$.

To continue, we assume the assertion of the proposition true for m and prove it for $m + 1$. With the notations of the proposition, we consider an element w' such that $w' \xrightarrow{m} w_G$. We recall that the open set $\Omega_{w'}$ is isomorphic to $U_{w'} \times U_{w'} \times M_{w'}$. In particular, there is a function $h_{w'} \in \mathcal{C}(M_{w'})$ such that

$$I(wa, f_{w'}) = I(wa, h_{w'})$$

for all $w \to w'$. Here we extend in an obvious way the notion of orbital integral to the Levi subgroups of $GL(n)$. The assertion of the proposition for $m = 1$ is true for the factors of $M_{w'}$; hence, in an obvious sense, it is true for the group $M_{w'}$ as well. Thus there are functions $K_w^{w'}$ on $A_w^{w'}$ for $w \to w'$ with the following property: for every $w'' \xrightarrow{1} w'$ there is a function ${}_{w'}h_{w''} \in \mathcal{C}(\Omega_{w''} \cap M_{w'})$ such that

$$I(w., h_{w'}) = \sum_{w'' \xrightarrow{1} w'} I(w., {}_{w'}h_{w''}) + K_w^{w'} * \omega_{w'}.$$

Here the function $\omega_{w'} \in \mathcal{C}(A_{w'})$ is given by

$$\omega_{w'}(a) = I(w'a, f_{w'}) = I(w'a, h_{w'}).$$

We can find a function ${}_{w'}f_{w''}$ with support in $\Omega_{w''}$ such that $I(wa, {}_{w'}f_{w''}) = I(wa, {}_{w'}h_{w''})$. We obtain then the assertion of the proposition for $m+1$ by setting:

$$f_{w''} = \sum_{\{w' : w'' \xrightarrow{1} w'\}} {}_{w'}f_{w''}$$

for each w'' with $w'' \xrightarrow{m+1} w_G$. Note that in the above sum $w' \xrightarrow{m} w_G$. The proposition and the existence of the germs are established.

We pass to the proof of the uniqueness. Consider first a system $K_w^{w'}$ of Shalika germs and a system $t_w^{w'}$ of functions in $\mathcal{C}(A_w^{w'})$ (with $t_w^w = 1$). Then the relations

$$K_w^{w'} = \sum_{w \to w_1 \to w'} H_w^{w_1} * t_{w_1}^{w'}$$

form a triangular system of relations which can be solved for the functions $H_w^{w'}$. Moreover:

$$\begin{aligned} I(w., f) &= \sum K_w^{w_1} * \omega_{w'}^{K,f} \\ &= \sum_{w \to w_1 \to w'} H_w^{w_1} * t_{w_1}^{w'} * \omega_{w'}^{K,f}. \end{aligned}$$

It follows that the functions H are also a system of germs. More precisely,

$$\omega_{w_1}^{H,f} = \sum_{w_1 \to w'} t_{w_1}^{w'} * \omega_{w'}^{K,f}.$$

To prove the uniqueness we consider two systems of germs H and K as in the theorem. It is clear that there are functions $t_w^{w'}$ on $A_w^{w'}$ with $t_w^w = 1$ for all w such that

$$K_w^{w'} = \sum_{w \to w_1 \to w'} H_w^{w_1} * t_{w_1}^{w'}$$

for any pair $w \to w'$. Indeed this relation reads

$$K_w^{w'} = t_w^{w'} + H_w^{w'} + \sum_{w \to w_1 \to w', w \neq w_1 \neq w'} H_w^{w_1} * t_{w_1}^{w'}$$

so that the functions t are determined by a triangular system of equations. Similarly we have

$$\omega_w^{H,f} = \sum_{w \to w'} t_w^{w'} * \omega_{w'}^{K,f}.$$

It is clear that the functions t are smooth. What we have to see is that they have compact support. This is clear for $w = w'$ since $t_w^w = 1$. Thus we may assume that for $d(w, w') \le m$ the functions $t_w^{w'}$ have compact support and prove the same assertion for the functions $t_w^{w'}$ with $d(w, w') = m + 1$. Thus we fix a pair $w \overset{m+1}{\to} w'$. To that end, let us consider only functions $f \in \mathcal{C}(\Omega_{w'})$. Then as before $I(w., f) = 0$ unless $w \to w'$ and $\omega_w^{K,f} = \omega_w^{H,f} = 0$ unless $w \to w'$. In particular,

$$\omega_{w'}^{K,f} = \omega_{w'}^{H,f} = I(w'., f)$$

is an arbitrary function $\omega_{w'} \in \mathcal{C}(A_{w'})$. Moreover,

$$\omega_w^H = \sum_{w \to w_1 \to w', w_1 \ne w'} t_w^{w_1} * \omega_{w_1}^K + t_w^{w'} * \omega_{w'}.$$

Since the functions $t_w^{w_1}$ in the formula have compact support by the induction hypothesis we conclude that the last term has compact support on A_w. Its restriction to $A_w^{w'}$ also has compact support; it can be written as

$$a \mapsto \sum_{c \in Y} t_w^{w'}(ac^{-1}) \omega_{w'}(c)$$

where $Y = Y_{w'}^w$ is the finite subset of $A_{w'}$ introduced earlier. We can choose $\omega_{w'}$ in such a way that $\omega_{w'}(1) = 1$ and $\omega_{w'}(c) = 0$ for $c \in Y$, $c \ne 1$. The conclusion follows. □

The proof of existence gives a way to compute inductively the germs $K_w^{w'}$ in terms of the germs $K_w^{w_G}$. Indeed, if $w \ne w_G$ then $M = M_w$ can be written as a product of linear groups G_i. For w we get

$$w = \mathrm{diag}(w_1, w_2, \ldots, w_r)$$

with $w_i = w_{G_i}$. If $w' \to w$ then

$$w' = \mathrm{diag}(w'_1, w'_2, \ldots, w'_r)$$

with $w'_i \to w_i$ in G_i. Now write a in $A_{w'}^w$ as

$$a = \mathrm{diag}(a_1, a_2, \ldots, a_r)$$

with $a_i \in A_{w'_i}^{w_i}$. We find (for a suitable system of germs)

$$K_{w'}^w(a) = \prod K_{w'_i}^{w_i}(a_i). \tag{26}$$

For $n = 2$ the relations defining the germs read:

$$I(a, f) = \omega(a) + \sum_{a = bc} K_e^w(b) \omega_w(c), \tag{27}$$

$$I(wa, f) = \omega_w(a), \tag{28}$$

where

$$w = w_G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{29}$$

The sum is for $b \in A$ with $\det b = -1$ and $c \in A_{w_G}$. The set A_e^w is the set of matrices of the form

$$\begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix}.$$

The germ K_e^w is supported on a set of matrices of the above form with $|z| \leq C$. It is defined within the addition of a smooth function of compact support.

Proposition 2.3. *The germ $K = K_e^w$ is given, for $|z|$ small enough, by*

$$K \begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix} = \psi\left(\frac{2}{z}\right) \gamma\left(\frac{2}{z}, \psi\right) \left|\frac{1}{2z}\right|_F^{1/2}$$

where γ is the Weil constant.

We recall the definition of the Weil constant:

$$\int \widehat{\Phi}(x) \psi\left(\frac{ax^2}{2}\right) dx = |a|^{-1/2} \gamma(a, \psi) \int \Phi(x) \psi\left(\frac{-x^2}{2a}\right) dx,$$

where $\widehat{\Phi}$ is the Fourier transform of $\Phi \in \mathcal{C}(F)$. In particular, if O is any open neighborhood of 0 in F then

$$\int_O \psi\left(\frac{ax^2}{2}\right) dx = |a|^{-1/2} \gamma(a, \psi)$$

for $|a|$ large enough.

Proof. Let O be an ideal in F which will be taken as small as needed. We may take it so small that $\psi = 1$ on O, $1 \not\equiv -1 \bmod O, 2^{-1}O$ is contained in the maximal ideal of $\mathcal{O}_F$ and the square root function $u \mapsto \sqrt{1+u}$ is defined by the Taylor formula on O; moreover, we may assume that $\sqrt{1+u} \in 1 + 2^{-1}O$ for $u \in O$. Let f be the characteristic function of the set wK_O where K_O is the congruence subgroup determined by O. We have

$$I(w_G, f) = \int_F f \begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} \psi(x)\, dx = \mathrm{Meas}(O).$$

Similarly:

$$I(-w_G, f) = \int_F f \begin{pmatrix} 0 & -1 \\ -1 & -x \end{pmatrix} \psi(x)\, dx = 0.$$

On the other hand,

$$I\left(\begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix}, f\right) = \int f \begin{pmatrix} z & zx_1 \\ zx_2 & -z^{-1} + x_1 x_2 z \end{pmatrix} \psi(x_1 + x_2)\, dx_1\, dx_2.$$

This is 0 unless $z \in O$. We change x_1 to x_1/z and set

$$x_2 = \frac{1 + zu}{zx_1}$$

where $u \in O$. The integral simplifies at once to

$$|z|_F^{-1} \int_{x_1 \equiv 1 \mod 0} \psi\left(\frac{x_1 + x_1^{-1}}{z}\right) dx_1 \int_O du.$$

We can set $x_1 = 1 + v$ with $v \in O$ and then set

$$y = \frac{v}{\sqrt{1+v}}.$$

We find $dx_1 = dv = dy$ if O is small enough. The integral becomes

$$|z|_F^{-1} \operatorname{Meas}(O) \int_O \psi\left(\frac{2+y^2}{z}\right) dy.$$

If z is sufficiently small this becomes

$$\frac{1}{|2z|_F^{1/2}} \psi\left(\frac{2}{z}\right) \gamma\left(\frac{2}{z}, \psi\right) I(w_G, f).$$

Our assertion follows. □

We pass to the group $G = GL(3)$. Then the germs relations are

(30) $$I(., f) = \omega_e + K_e^{w_1} * \omega_{w_1} + K_e^{w_2} * \omega_{w_2} + K_e^{w_G} * \omega_{w_G},$$

(31) $$I(w_1., f) = \omega_{w_1} + K_{w_1}^{w_G} * \omega_{w_G},$$

(32) $$I(w_2., f) = \omega_{w_2} + K_{w_2}^{w_G} * \omega_{w_g},$$

(33) $$I(w_G., f) = \omega_{w_G}.$$

The germs $K_e^{w_1}$ ad $K_e^{w_2}$ are defined within the addition of a smooth function of compact support. The set $A_e^{w_1}$ is the subset of A defined by

$$\Delta_2(b) = -1, \quad \Delta_3(b) = -1,$$

in other words, the set of elements of the form:

$$b = \begin{pmatrix} z & 0 & 0 \\ 0 & -z^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By formula (26) and the uniqueness, we may assume

$$K_e^{w_1}(b) = K \begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix}$$

where K is the $GL(2)$ germ. Similarly, $A_e^{w_2}$ is the set of matrices of the form

$$b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & -z^{-1} \end{pmatrix}.$$

We may assume

$$K_e^{w_2}(b) = K \begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix}.$$

The set $A_{w_1}^{w_G}$ is the set of $a \in A_{w_1}$ with $\det(a) = 1$. The set $A_e^{w_G}$ is the set of $a \in A$ with $\det(a) = -1$.

Finally, suppose that the orbital integrals of f on $w_G A_{w_G}$ vanish; then the above relations simplify to

(34) $$I(., f) = \omega_e + K_e^{w_1} * \omega_{w_1} + K_e^{w_2} * \omega_{w_2},$$

(35) $$I(w_1., f) = \omega_{w_1}, \quad I(w_2., f) = \omega_{w_2}, \quad I(w_G., f) = 0.$$

Moreover, the functions $\omega_e, \omega_{w_1}, \omega_{w_2}$ are arbitrary smooth functions of compact support.

3. Matching

Now we let E be a quadratic extension of our local field F. The group $G(E)$ operates on $S(F)$ by $s \overset{g}{\mapsto} g^* sg$. *Mutatis mutandis*, the discussion of the previous section applies to the present situation where $G(F)$ is replaced by $S(F)$ and the group $N(F)\times N(F)$ by the group $N(E)$ acting on $S(F)$. As before we can define the Shalika germs $L_w^{w'}$ for the integrals $J(wa,\Phi)$ and show they exist and are essentially unique.

We compute the Shalike germ for $GL(2)$. It is defined within the addition of a smooth function of compact support. The defining relations read

$$J(a,\Phi) = \omega(a) + \sum_{a=bc} L_e^w(b)\omega_w(c), \tag{36}$$

$$I(wa,f) = \omega_w(a). \tag{37}$$

Proposition 3.1. *The Shalika germ $L = L_e^w$ is given for $|z|$ small enough by*

$$L\begin{pmatrix} z & 0 \\ 0 & -z^{-1}\end{pmatrix} = \left|\frac{1}{2z}\right|_F^{1/2} \psi\left(\frac{2}{z}\right)\gamma\left(\frac{2\tau}{z},\psi\right)$$

where $E = F(\sqrt{\tau})$. In particular

$$L\begin{pmatrix} z & 0 \\ 0 & -z^{-1}\end{pmatrix} = c(E/F,\psi)\eta_{E/F}(z)K\begin{pmatrix} z & 0 \\ 0 & -z^{-1}\end{pmatrix},$$

where

$$c(E/F,\psi) = \gamma(\tau,\psi)\gamma(1,\psi)^{-1}\eta_{E/F}(2).$$

For future reference, we remark that we may (and will) assume that the relation between L and K holds for all z.

Proof. As before we consider a small enough ideal O in E and the characteristic function Φ of $wK_O \cap S(F)$. We have

$$I(w,\Phi) = \mathrm{Meas}(O\cap F)$$

and $I(-w,\Phi)=0$. Now

$$I\left(\begin{pmatrix} z & 0 \\ 0 & -z^{-1}\end{pmatrix},\Phi\right) = \int_E \Phi\begin{pmatrix} z & yz \\ \overline{y}z & -z^{-1}+y\overline{y}z\end{pmatrix}\psi(y+\overline{y})\,dy.$$

Here dy is the self-dual Haar measure on E. This is 0 unless $z \in O\cap F$. We change y to y/z. The integral becomes

$$|z|_F^{-2}\int \psi\left(\frac{y+\overline{y}}{z}\right)\,dy$$

where $y \in 1+O, y\overline{y}\in 1+zO$. We can then set

$$y = \sqrt{1+zu}(\sqrt{1+x^2\tau}+x\sqrt{\tau})$$

where $u\in O\cap F$ and $x\in O'$, where O' is an open neighborhood of zero in F which does not depend on z. Then

$$dy = |z|_F\,du\,dx|\tau|_F^{1/2}.$$

Here $du = dx$ is the self-dual Haar measure on F. In the formula for y, the first square root belongs to $1 + 2^{-1}z(O \cap F)$. It follows that the integrand does not depend on u (if z is small enough). Thus the integral becomes

$$\mathrm{Meas}(O \cap F)|z|_F^{-1}|\tau|^{1/2} \int_{O'} \psi\left(\frac{2\sqrt{1+x^2\tau}}{z}\right) dx.$$

We can set

$$2\sqrt{1+x^2\tau} = 2 + t^2\tau$$

with $dx = dt$ and $t \in O''$, where O'' is another neighborhood of zero. In fact, we can take

$$t = \frac{x}{\sqrt{\frac{\sqrt{1+x^2\tau}+1}{2}}}.$$

Then the integral takes the form

$$I(w,\Phi)|z|_F^{-1}|\tau|^{1/2}\psi\left(\frac{2}{z}\right)\int_{O''}\psi\left(\frac{t^2\tau}{z}\right) dt.$$

Then for $|z|$ small enough the integral takes the value

$$I(w,\Phi)\left|\frac{1}{2z}\right|^{1/2}\psi\left(\frac{2}{z}\right)\gamma\left(\frac{2\tau}{z},\psi\right)$$

and the first assertion follows.

For the second assertion, we recall the formulas:

$$\gamma(ab,\psi) = \gamma(a,\psi)\gamma(b,\psi)\gamma(1,\psi)^{-1}(a,b), \quad \gamma(ab^2,\psi) = \gamma(a,\psi)$$

and the fact that $(a,\tau) = \eta_{E/F}(a)$. The assertion follows. $\square$

For the group $GL(3)$ we simply record the minimum information that we need: if $J(w_G a, \Phi) = 0$ then

$$J(a,\Phi) = \omega_e(a) + \sum_{a=bc} L_e^{w_1}(b)\gamma_{w_1}(c) + \sum_{a=bc} L_e^{w_2}(b)\omega_{w_2}(c), \tag{38}$$

$$J(w_1 a,\Phi) = \omega_{w_1}(a), \quad J(w_2 a,\Phi) = \omega_{w_2}(a), \quad J(w_G a,\Phi) = 0. \tag{39}$$

Moreover the functions $\omega_e, \omega_{w_1}, \omega_{w_2}$ are arbitrary smooth functions of compact support. The germs $L_e^{w_i}$ can be computed as before in terms of L. Taking into account the previous proposition we see that the first relation can be written as

$$\begin{aligned} J(a,\Phi) = \omega_e(a) &+ c(E/F,\psi)\sum \eta(-b_2)K_e^{w_1}(b)\omega_{w_1}(c) \\ &+ c(E/F,\psi)\sum \eta(b_2)K_e^{w_2}(b)\omega_{w_2}(c), \end{aligned}$$

or, multiplying by $\eta(a_2)$ and taking into account the relation $a_2 = b_2c_2$:

$$\begin{aligned} \eta(a_2)J(a,\Phi) = \eta(a_2)\omega_e(a) &+ c(E/F,\psi)\sum \eta(-c_2)K_e^{w_1}(b)\omega_{w_1}(c) \\ &+ c(E/F,\psi)\sum \eta(c_2)K_e^{w_2}(b)\omega_{w_2}(c). \end{aligned}$$

Since the functions ω_* are arbitrary the first assertion of the following proposition follows at once:

Proposition 3.2. *If $\Phi' \in \mathcal{C}(G(F))$ has vanishing orbital integrals on $w_G A_{w_G}$ there is $\Phi \in \mathcal{C}(S(F))$ with vanishing orbital integrals on $w_G A_{w_G}$ such that the matching conditions* (19) *to* (21) *are satisfied. Moreover, if Φ' is supported on $G(F)^+$ then we can take Φ supported on $S(F)^+$.*

For the second assertion, one simply multiplies Φ by the characteristic function of $S(F)^+$.

Our last task will be to show that, essentially, we will not lose information by restricting ourselves to the kind of function considered in the Introduction. To that end we consider a unitary generic representation π of $G = GL(3, F)$ (or $G = GL(n, F)$ with n odd). As before let G^+ be the group of $g \in G$ such that $\det g \in F^+$. Set also $A^+ = A \cap G^+$ and $Z^+ = Z \cap G^+$. We let λ be a nonzero linear form on the space of smooth vectors of π such that $\lambda(\pi(n)v) = \theta(n)\lambda(v)$ for all $n \in N(F)$ and all vectors v. We define a distribution Θ on G^+ by

$$\Theta(f) = \sum \lambda(\pi(f)v_i)\overline{\lambda(v_i)},$$

if f is supported on G^+. Here v_i is any orthonormal basis of π (contained in the space of smooth vectors). Clearly Θ transforms on the left and on the right under the character θ of N.

Lemma 3.1. *There is at least one function f which is supported on G^+ and vanishes on A^+N such that $\Theta(f) \neq 0$.*

Proof. Since n is odd, we have $G(F) = G^+Z$ and the restriction of π to G^+ is irreducible. Suppose there is no function f with the required property. Then the support of Θ is contained in the group NA^+. Since N is normal in NA^+ and Z^+ is the stabilizer of the character θ of N the support of Θ is in fact contained in NZ^+. Since Θ transforms under a character ω of Z^+, then in fact

$$\Theta(f) = c \int_{NZ^+} f(nz)\theta(n)\overline{\omega}(z)\, dn\, dz,$$

for a suitable constant c. Now Θ and the integral on the right, call it $M(f)$, are distributions of positive type. Thus $c \geq 0$. Suppose $c > 0$. Since π restricted to G^+ is irreducible, the unitary representation π (restricted to G^+) can be reconstructed from the distribution Θ by considering the positive semidefinite form

$$(f_1, f_2) \mapsto \Theta(f_1 * f_2^*),$$

where we have set $f^*(g) = \overline{f}(g^{-1})$. The corresponding Hilbert space is the space on which π operates, the action of G^+ corresponding to right translations on f_1. The same construction applied to M produces the unitary representation of G^+ induced by the character $\overline{\theta}\omega$ of NZ^+. If $c > 0$ then π must be equivalent to this induced representation which is absurd. Thus $c = 0$; that is $\Theta = 0$. However, the irreducibility of π under G^+ implies there is a f supported on G^+ such that $\Theta(f) \neq 0$. Thus we get a contradiction. □

4. Proof of Theorem 1

We go back to the global situation and the notations of the introduction. If ϕ is a cusp form on $GL(3, F_{\mathbb{A}})$ let us set

$$\lambda(\phi) = \int \phi(n)\overline{\theta}(n)\, dn.$$

Then

$$\int K^{\text{cusp}}_{\Phi'}({}^t n_1, n_2)\theta(n_1^{-1}n_2)\, dn_1\, dn_2 = \sum_\pi \sum_i \lambda(\pi(\Phi'')\phi_i)\overline{\lambda}(\phi_i), \tag{40}$$

where we have set $\Phi''(g) = \Phi'(-w_G g)$. The outer sum is over all unitary cuspidal representations π whose central character is equal to χ on $Z^+(F_{\mathbf{A}})$ and which have a nonzero vector invariant under $K'^S = \prod_{v\notin S} K'_v$. For each such π, the inner sum is over an orthonormal basis of the space of vectors invariant under K'^S. Since Φ' and Φ'' have support in $G^+(F_{\mathbf{A}})$, the representations π and $\pi\otimes\eta$ give the same contribution. For v in S let λ_v be a linear form on the space of π_v transforming under θ_v. Then the above expression can be rewritten as:

$$\sum_\pi \hat{\pi}^S(\Phi'^S)c(\pi, S)\prod_{v\in S}\Theta_{\pi_v}(\Phi''_v),$$

where $\hat{\pi}^S(\Phi'^S)$ is the Hecke eigenvalue associated to the function Φ'^S (we set $\Phi' = (\prod_{v\in S}\Phi'_v)\Phi'^S$). The constant $c(\pi, S)$ depends only on π and the choice of the λ_v. Finally, for $v \in S$:

$$\Theta_{\pi_v}(\Phi''_v) = \sum \lambda_v(\pi_v(\Phi''_v)W_i)\overline{\lambda_v}(W_i);$$

the sum is over an orthonormal basis of the space of vectors invariant under K'_v. Again, each one of these objects gives the same contribution for π and $\pi\otimes\eta$. If v is in S, there is Φ'_v such that $\Theta_{\pi_v}(\Phi''_v) \neq 0$. Indeed, this is clear if v splits and is finite because there is then no constraint on the function Φ''_v. If v is Archimedean, the only constraint is that Φ''_v be a quadruple of K'_v finite functions and our assertion is still true. If v is inert, then it is a finite place and we demand that the orbital integrals of Φ'_v vanish on $w_G A_{w_G}(F_v)$. This is certainly the case if Φ''_v is supported on G^+_v and vanishes on $B(F_v)$. But by Lemma 3.1 we can choose such a function in such a way that $\Theta_{\pi_v}(\Phi''_v) \neq 0$.

Similarly, if ϕ is a cuspidal automorphic representation of $GL(3, E_{\mathbf{A}})$, we set

$$\mu(\phi) = \int \phi(h)\, dh, \quad \Lambda(\phi) = \int_{N(E)\backslash N(E_{\mathbf{A}})} \phi(n)\overline{\theta}(n\overline{n})\, dn.$$

Then

$$\int K^{\text{cusp}}_{\Phi}(n)\theta(n\overline{n})\, dn = \sum_\Pi \sum_i \mu(\Pi(f)\phi_i)\overline{\Lambda}(\phi_i). \tag{41}$$

Let T be the set of places of E above the places of S. The outer sum is over all representations Π having a nonzero vector fixed under $K^T = \prod_{v\notin T} K_v$ and central character $z \mapsto \chi(z\overline{z})$; for each such Π the inner sum is over an orthonormal basis of the space of vectors invariant under K^T. We can also factor out the Hecke eigenvalue to obtain

$$\sum_\Pi \widehat{\Pi}^T(f^T)\sum_i \mu(\Pi(f_T)\phi_i)\overline{\Lambda}(\phi_i).$$

Suppose that Π is the base change of a representation π. Then

$$\widehat{\pi}^S(\Phi'^S) = \widehat{\Pi}^T(f^T)$$

and Π is then the base change of π and $\pi \otimes \eta$ and of no other representation. It follows from the identity of the spectral contributions (see (23)) that

$$\sum_i \mu(\Pi(f_T)\phi_i)\overline{\Lambda}(\phi_i) = 2c(\pi, s) \prod_{v \in S} \Theta_{\pi_v}(\Phi''_v).$$

Choosing the Φ'_v, $v \in S$, in such a way that the right-hand side is $\neq 0$ we see that $\mu \neq 0$ on the space of Π. This concludes the proof of the theorem. $\square$

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Department of Mathematics, Columbia University, New York, New York 10027
E-mail address: `hj@math.columbia.edu`

Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242
E-mail address: `yey@math.uiowa.edu`