

GEOMETRIZING INFINITE DIMENSIONAL LOCALLY COMPACT GROUPS

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ABSTRACT. We study groups having invariant metrics of curvature bounded below in the sense of Alexandrov. Such groups are a generalization of Lie groups with invariant Riemannian metrics, but form a much larger class. We prove that every locally compact, arcwise connected, first countable group has such a metric. These groups may not be (even infinite dimensional) manifolds. We show a number of relationships between the algebraic and geometric structures of groups equipped with such metrics. Many results do not require local compactness.

1. INTRODUCTION AND MAIN RESULTS

The traditional notion of infinite dimensional manifold excludes, among other things, all locally compact spaces. This exclusion means that Hilbert's Fifth Problem (Sophus Lie's theory of transformation groups without differentiability assumptions) must, for infinite dimensional groups, be interpreted differently from the way it was for finite dimensional groups. Instead of finding suitable differentiable coordinates for a group, we have to take the problem in a more general sense, for example by seeking invariant geometries associated with groups, which can be used to understand their algebraic structure. In this paper we approach the problem by extending the notion of invariant Riemannian metric on a Lie group to a reasonable class of infinite dimensional groups. Roughly speaking, the "weakly convex" metric we define has curvature bounded below and local convexity on finite dimensional "slices." We believe that groups with invariant weakly convex metrics are a suitable generalization of Lie groups with invariant Riemannian metrics for several reasons. First, every first countable, arcwise connected, locally compact group has such a metric (Theorem 1.1). Second, there is some useful interplay between the geometric and algebraic structures on such groups. In the process of proving Theorem 1.1 we use purely metric arguments to obtain a number of results in this direction, usually assuming neither local compactness nor weak convexity. For example, we show that for bi-invariant metrics, any lower curvature bound implies non-negative curvature, geodesic completeness always holds, and positive cut radius implies that geodesics and 1-parameter subgroups coincide (Section 4). We also obtain various results extending the orthogonal splitting of the tangent space induced from the

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quotient of a Lie group by a closed subgroup (Section 2). We believe the potential for such results is far from exhausted.

A third reason we believe our metrics are appropriate is that a finite dimensional group equipped with a weakly convex metric is isometric to a Lie group with invariant Riemannian metric. This follows from either [4] or [19], [20]. It is also true that a weakly convex metric on any finite dimensional topological space forces it to be a smooth manifold (see below).

All spaces in this paper are Hausdorff and connected; unless otherwise stated, metric spaces are assumed complete, and rectifiable curves are parameterized proportional to arclength. “Dimension” refers to covering dimension. For first countable groups, local connectedness, local arcwise connectedness, and arcwise connectedness are equivalent (cf. [22]).

In the next four paragraphs we will state some basic definitions, classical results, and some recent results about metric spaces of curvature bounded below, an area initiated by A. D. Alexandrov in the 1940’s. Good references for the basics of Alexandrov’s theory are [24], [2], [17], [7]; basic theory for spaces which are not locally compact, including an extension of the Hopf-Rinow Theorem, may be found in [21]; good references for metric Riemannian geometry are [6] and [11].

A metric space (X, d) is called an *inner metric space* (also *length* or *intrinsic metric space*) if for any two points $x, y \in X$, $d(x, y)$ is the infimum of the lengths of curves joining x and y . The length $L(c)$ of a curve c is measured in the usual way, by summing distances over finer and finer partitions of c . In locally compact spaces, a theorem of Cohn-Vossen ([8]) states that there is always a *minimal* curve joining each pair of points, whose length realizes the distance between them. A *geodesic* γ is a locally minimizing curve; i.e. every point on γ lies in the interior of a segment σ of γ which is minimal between its endpoints. For any number k , an inner metric space X has curvature $\geq k$ if every quadruple in X isometrically embeds in S_l^2 for some $l \geq k$. Here S_l^2 denotes the simply connected, 2 dimensional space form of curvature l : a sphere for $l > 0$, the plane for $l = 0$, and a hyperbolic space for $l < 0$. This definition is due to Wald and Berestovskii ([3]), and is equivalent to Alexandrov’s in the locally compact case. Curvature bounded below is a generalization of the notion of sectional curvature bounded below in a Riemannian manifold M (cf. [6]). Here we consider M as a metric space with the metric induced by measuring lengths of curves using the Riemannian metric. Piecewise differentiable curves may be equivalently measured using either the Riemannian metric or this induced metric (distance); therefore the induced metric is inner. The Riemannian and metric notions of geodesic are also equivalent in this setting.

For the rest of this discussion we will suppose that X is an inner metric space of curvature bounded below. The angle $\alpha(\gamma_1, \gamma_2)$ between any two geodesics γ_1, γ_2 starting at a common point p in X is measured using the Euclidean cosine law: if $\alpha_t = \cos^{-1}(1 - \frac{d(\gamma_1(t), \gamma_2(t))^2}{2t^2})$ then the curvature bound implies $\lim_{t \rightarrow 0} \alpha_t$ exists, and we define this limit to be $\alpha(\gamma_1, \gamma_2)$. An important property of angles in X is the usual Euclidean property that the sum of complementary angles is π .

We will use Alexandrov’s triangle comparisons in this paper: any triangle T in X whose sides are formed by minimal curves has the property that the three angles of T are not smaller than the corresponding angles in a triangle T' of the same side lengths in S_k^2 . Roughly speaking, triangles are always “fatter” in X than in S_k^2 ; equivalently, geodesics starting at a common point diverge more slowly in X than

in S_k^2 . An immediate consequence of these comparisons is that two geodesics which coincide on an open interval are identical (up to reparameterization) wherever both are defined. Given any point $p \in X$, a *geodesic direction* at p is an equivalence class of geodesics, where two geodesics are equivalent if they coincide on an open interval. Equivalently, a geodesic direction is a unit (arclength parameterized) geodesic of maximal domain of definition. X is *geodesically complete* if the maximal domain of definition of every geodesic is \mathbb{R} . For any $p \in X$, S_p denotes the *space of geodesic directions*, which is given the angle metric; *the space of directions*, \overline{S}_p , is the metric completion of S_p . The tangent cone and completed tangent cones, which are the metric cones on S_p and \overline{S}_p , respectively, are denoted by T_p and \overline{T}_p (cf. [17]). If X is geodesically complete, \overline{T}_p is a Hilbert space ([17]), the cone on its unit sphere \overline{S}_p . Recall that the Hopf-Rinow Theorem states that a complete Riemannian manifold is always geodesically complete, and every direction is a geodesic direction; in this case $T_p = \overline{T}_p$ can be identified with the usual tangent space of the underlying smooth manifold; the angle (and distance) are also induced by the Riemannian metric.

The *exponential map* $\exp_p : T_p \rightarrow X$ carries radial lines from the cone point (0) to the corresponding geodesics in X . In a Riemannian manifold this notion of \exp_p coincides with the usual one. Recall that for a Lie group with invariant Riemannian metric, there is also an algebraic exponential map which carries radial lines onto 1-parameter subgroups. For infinite dimensional groups (cf. [14]) the difference between these maps is much greater (even their domains are different); this situation is investigated further in a paper presently in preparation by the author, V.N. Berestovskii, and C. Stallmann.

The *cut radius map* (cf. [17]) $c_p : \overline{S}_p \rightarrow \mathbb{R}$ assigns to each direction the maximal length for which the corresponding geodesic (if there is one) is minimal; if there is no corresponding geodesic, the cut radius is 0. Note that S_p is by definition the subset of \overline{S}_p on which c_p is positive. The cut radius map is semi-continuous but (if the space is not Riemannian) not necessarily continuous (cf. [17]). Two Euclidean disks glued together along their boundaries is a simple example ([1]) of a geodesically complete space of curvature ≥ 0 which is not Riemannian.

Any Banach space has an induced inner metric, but the metric generally does not have curvature bounded below (unless the norm comes from an inner product, in which case it is immediate from the definition that the curvature is non-negative). In general, curvature bounded below is a very strong condition, which often has powerful geometric and topological consequences.

Definition 1.1. A real valued (possibly not continuous) function f on a subset Y of a topological vector space \mathcal{V} is called *weakly positive* if for every finite dimensional linear subspace \mathcal{W} of \mathcal{V} , $f|_{Y \cap \mathcal{W}}$ has a positive lower bound.

Definition 1.2. A metric on a space X is said to be *weakly convex* if it is a geodesically complete inner metric of curvature $\geq k$ (for some k) and for each $p \in X$, there is a vector subspace W_p of T_p such that

- 1) $c_p|_{S_p \cap W_p}$ is weakly positive, and
- 2) for every $\gamma \in \overline{S}_p$ there exist $\gamma_i \in W_p \cap S_p$ such that $\alpha(\gamma_i, \gamma) \rightarrow 0$ and $c_p(\gamma_i) \rightarrow c_p(\gamma)$.

Condition 2) is a kind of strong density of W_p , and implies that W_p is dense in \overline{T}_p . This condition also implies that what should be the “normal coordinates” for

this metric have dense image in X . If X is finite dimensional then we immediately have $W_p = \overline{T}_p$, and c_p is positive for all p . This implies that X is a smooth manifold ([18]). If X is infinite dimensional and locally compact, we conjecture that W_p is *always* a (noncomplete) proper subspace of \overline{T}_p , since c_p must be positive on a locally convex subset of W_p which is not open. However, W_p is geometrically as “complete” as \overline{T}_p in the sense that the set of points joined to p by a unique geodesic in W_p is dense in X .

Let G be a topological group. A metric on G is called *invariant* if it is invariant under left translation, and *bi-invariant* if it is also invariant under right translation.

Theorem 1.1. *Every arcwise connected, first countable, locally compact group G admits an invariant, weakly convex inner metric d of curvature $\geq k$. In addition,*

1. *Every direction in \overline{S}_e is tangent to a unique unit 1-parameter subgroup, and every non-trivial rectifiable 1-parameter subgroup is tangent to a direction.*
2. *If d is bi-invariant, then $k = 0$, every geodesic through e is a 1-parameter subgroup, and every 1-parameter subgroup tangent to a geodesic direction is a geodesic.*
3. *If G is compact we can take d to be bi-invariant.*

Note that arcwise connectedness and first countability are necessary conditions for the existence of any inner metric on G . Infinite dimensional locally compact spaces with weakly invariant metric cannot have positive cut radius, since then they would be locally homeomorphic (via \exp_e) to an infinite dimensional Hilbert space (and these are never locally compact). In particular, these spaces cannot have an upper curvature bound in the usual sense.

For the meaning of “tangent” see Definition 2.3. There are non-rectifiable 1-parameter subgroups in every such metric on an infinite dimensional locally compact group (see Corollary 5.3 and subsequent comment). Note that even in the compact case, rectifiable 1-parameter subgroups need not be geodesics, because in infinite dimensions the cut radius is not positive, so there are always directions which are not geodesic directions.

Since we will only be treating spaces having a lower curvature bound, we will often refer to a metric of the type described in Theorem 1.1 (invariant or otherwise) as simply a “weakly convex” metric. We denote the identity element of a group G by e . When G has an invariant metric, we will often use implicitly the fact that G is homogeneous; i.e. has a transitive group of isometries, namely the left translations $L_a(x) = ax$. This allows us to prove local metric conditions only near e . When we specify a geodesic γ through e , it will be understood that γ is unit and $\gamma(0) = e$.

A number of the properties of the metric given by Theorem 1.1 are true for more general metrics. In the next theorems we collect some of these results, which are proved in this paper:

Theorem 1.2. *Let G be a group with bi-invariant inner metric of curvature $\geq k$. Then*

1. *G has curvature ≥ 0 ,*
2. *every geodesic through e is a 1-parameter subgroup, and*
3. *G is geodesically complete.*

Theorem 1.3. *If G is a group with left invariant inner metric of curvature $\geq k$, then every non-trivial rectifiable 1-parameter subgroup of G is tangent to a direction in \overline{S}_e .*

In fact a stronger statement than Theorem 1.3, is true, see Corollary 5.1.

We conclude this section with a simple motivating example, which, although very special, is really not too different from what one finds in general. We only sketch our construction here; details carried out in greater generality may be found later in the paper.

Example 1.1. Let $T^\infty = S^1 \times S^1 \times \dots$ with the product topology. Then T^∞ is a compact, abelian group. T^∞ is not a manifold; arbitrarily small neighborhoods are homeomorphic to products of T^∞ itself with Euclidean open sets of arbitrarily large dimension. We metrize T^∞ by taking the usual metric on each copy of S^1 but letting the diameter of the i^{th} copy be 2^{-i} . We then take the product metric with more and more factors. Each finite product is naturally embedded in its successor; passing to the limit we obtain a bi-invariant inner metric on T^∞ . Although it is not a manifold, T^∞ , metrized this way, “ought” to be a flat Riemannian manifold. Any triangle of strictly minimal curves (uniquely joining their endpoints) embeds isometrically in the plane, i.e. the “sectional curvature” is 0. However, there is no neighborhood in which all minimal curves are unique between their endpoints. Thus, although there is a lower curvature bound of 0 in Alexandrov’s sense, there is no upper curvature bound. On the other hand, the lower curvature bound gives us a “tangent space” which is a separable Hilbert space, and an exponential map defined on this Hilbert space. From the standpoint of Riemannian geometry, this exponential map is about as reasonable as one could expect in this generality—namely, it is injective (and surjective onto a dense subset) when restricted to a relatively compact, locally convex (but non-open) subset of a non-complete infinite dimensional inner product space. This exponential map differs from the algebraic one ([14]) and in particular does not “cover” non-rectifiable 1-parameter subgroups.

2. QUOTIENTS

Definition 2.1. Let (X, d) be a complete metric space and G be a group of isometries of X whose orbits are all closed. The orbit space X/G is given the *induced metric* $d(Gx_1, Gx_2) = \inf\{d(y_1, y_2) : y_1 \in Gx_1, y_2 \in Gx_2\} = \inf\{d(x_1, y_2) : y_2 \in Gx_2\}$. The *projection map* $p : X \rightarrow X/G$ is defined by $p(x) = Gx$.

In this paragraph we recall some of the basic results of Berestovskii ([4]) on orbit spaces of inner metric spaces. If X is an inner metric space then X/G is also an inner metric space. Roughly speaking, an “almost” midpoint for orbits Gx_1 and Gx_2 is the orbit of a point x_3 which is almost a midpoint between points almost realizing the distance between Gx_1 and Gx_2 . Such “almost” midpoints can be used to construct “almost” minimal curves between Gx_1 and Gx_2 (cf. also [24],[10],[17]). If x_3 is a midpoint between x_1 and x_2 then Gx_3 is a midpoint between Gx_1 and Gx_2 .

Definition 2.2. A metric space X is *finitely compact* if closed bounded subsets of X are compact.

If X is an inner metric space, finite compactness and local compactness is equivalent ([8]). If the orbits of G are finitely compact in the induced metric, then the distance between orbits is always realized as the distance between two points. In this case, a curve γ is minimal from Gx_1 to Gx_2 if and only if $\gamma = p(\Gamma)$, where Γ is minimal from $y_1 \in Gx_1$ to $y_2 \in Gx_2$ such that $d(y_1, y_2) = d(Gx_1, Gx_2)$. We will call Γ a *lift* of γ ; $p|_\Gamma$ is an isometry. By translating with an isometry we can assume

the lift starts at any point in Gx_1 . If X has curvature $\geq k$ then it is immediate from the curvature definition that X/G also has curvature $\geq k$. In this case, given minimal curves γ_1, γ_2 in X/G starting at a single point and lifts Γ_1, Γ_2 starting at a single point, it is immediate from the definition of the angle and the fact that p is distance decreasing, that $\alpha(\gamma_1, \gamma_2) \leq \alpha(\Gamma_1, \Gamma_2)$. We note here that it is easy to verify that the quotient metric coincides with the induced Riemannian metric in case p is a Riemannian submersion ([16]). We say that a geodesic is extendable past a point if it can be extended as a geodesic beyond the point.

From now on we will assume that X is an inner metric space and G is a group of isometries of X with finitely compact orbits. The latter assumption is automatic if X is locally compact and the orbits of G are closed.

Lemma 2.1. *Suppose X has curvature $\geq k$ and γ_1, γ_2 are minimal in X/G starting at a point $p(x)$. If γ_1 is extendable past $p(x)$, then for any lifts Γ_1, Γ_2 of γ_1, γ_2 , respectively, at x , $\alpha(\gamma_1, \gamma_2) = \alpha(\Gamma_1, \Gamma_2)$.*

Proof. Let γ_3 be minimal starting at $p(x)$ such that $\alpha(\gamma_1, \gamma_3) = \pi$ and let Γ_3 be a lift of γ_3 at x . Then $\alpha(\Gamma_1, \Gamma_3) \geq \alpha(\gamma_1, \gamma_3) = \pi$, so $\alpha(\Gamma_1, \Gamma_3) = \pi$. Then $\alpha(\gamma_1, \gamma_2) \leq \alpha(\Gamma_1, \Gamma_2) = \pi - \alpha(\Gamma_2, \Gamma_3) \leq \pi - \alpha(\gamma_2, \gamma_3) = \alpha(\gamma_1, \gamma_2)$. \square

Corollary 2.2. *If γ is minimal in X/G and extendable past $p(x)$, then γ has a unique lift at x .*

Definition 2.3. Let X be a space of curvature $\geq k$, P be a connected subset of X , and $p \in P$. We say that $\gamma \in \overline{S}_p$ is *tangent* to P if there exist $p_i \in P$ with $p_i \rightarrow p$ and γ_i minimal from p to p_i such that $\lim_{i \rightarrow \infty} \alpha(\gamma, \gamma_i) = 0$. We say that a curve $c : [-1, 1] \rightarrow X$ such that $c(0) = p$ is *tangent* to $\gamma \in \overline{S}_p$ if $\lim_{t \rightarrow 0} \alpha(\gamma, \gamma_t) = 0$, where γ_t is any minimal curve from p to $c(t)$.

We let H_x denote the metric completion of the subset of \overline{T}_x whose elements are scalar multiples of lifts of minimal curves in X/G . If $p(x)$ is not a geodesic terminal (i.e. every geodesic can be extended through it), H_x is isometric, via the linear, lift-induced map, to the Hilbert space $\overline{T}_{p(x)}$. Let V_x denote all elements of \overline{T}_x orthogonal to H_x (i.e., if $\gamma \in \overline{T}_x$ then for any $\beta \in H_x$, $\alpha(\gamma, \beta) = \pi/2$). We let $S(H_x) = \overline{S}_x \cap H_x$ and $S(V_x) = \overline{S}_x \cap V_x$. Note that the geodesic directions are dense in $\overline{S}_{p(x)}$, and so also in $S(H_x)$.

Lemma 2.3. *If X/G is geodesically complete then if γ is tangent to Gx at x , $\gamma \in S(V_x)$.*

Proof. Suppose we have $g_i \rightarrow e$ and minimal curves γ_i from x to $g_i x$ such that $\lim \alpha(\gamma_i, \gamma) = 0$. It suffices to prove that for any $\beta \in S(H_x)$, $\lim \alpha(\gamma_i, \beta) = \pi/2$. Since the geodesic directions are dense in $S(H_x)$ we can assume that β is in fact a minimal geodesic. Since G/X is geodesically complete we can find another geodesic β' such that $\alpha(\beta, \beta') = \pi$; we also assume β and β' are minimal. If p is the endpoint of β then $d(p, x) = d(p, Gx)$. Now if $\liminf \alpha(\gamma_i, \beta) = \pi/2 - \epsilon < \pi/2$ then the curvature bound implies that $d(p, g_i x) < d(p, x)$ for some large i , a contradiction; hence $\liminf \alpha(\gamma_i, \beta) \geq \pi/2$. Applying the same argument to β' we obtain $\limsup \alpha(\gamma_i, \beta) \leq \pi/2$ from knowing $\alpha(\beta', \gamma_i) + \alpha(\gamma_i, \beta) = \pi$. \square

Recall that a space of curvature $\geq k$ is called *almost Riemannian* ([17]) if it is geodesically complete and finite dimensional. One of the important properties of an

almost Riemannian space is that the space of directions at each point is compact. For the next proposition, recall that the *strong excess* of a triple $(b; a, c)$ in a metric space is $\sigma(b; a, c) = \frac{d(a,b)+d(b,c)-d(a,c)}{\min\{d(a,b),d(b,c)\}}$, which, for fixed k , is small if and only if the corresponding representative angle $\alpha_k(b; a, c)$ in S_k is close to π .

Proposition 2.4. *If X is geodesically complete and X/G is almost Riemannian then γ is tangent to Gx if and only if $\gamma \in S(V_x)$. In other words, the tangent space splits orthogonally into “tangent” and “lifted” components.*

Proof. By the above lemma we need to show that if $\gamma \in S(V_x)$ then γ is tangent to Gx . Choose geodesic directions β_i such that $\alpha(\beta_i, \gamma) \rightarrow 0$ and positive $t_i \rightarrow 0$ such the restriction of β_i to $[-t_i, t_i]$ is strictly minimal. We claim that $d(\beta_i(t_i), Gx)/t_i \rightarrow 0$. Suppose that there exists an $\epsilon > 0$ such that for all sufficiently large i , $[d(\beta_i(t_i), Gx)/t_i] \geq \epsilon$. If $p(\beta_i(t_i)) = p(x)$ for all large i , then $\beta_i(t_i) \in Gx$, a contradiction, so $p(\beta_i(t_i))$ and $p(x)$ are distinct. Let ζ_i be minimal in X/G from $p(x)$ to $p(\beta_i(t_i))$. Choosing a subsequence, if necessary (\bar{S}_p is compact), we can assume $\{\zeta_i\}$ is Cauchy. We will obtain a contradiction by showing that for any unit geodesic η at $p(x)$, $\limsup_{i \rightarrow \infty} \alpha(\eta, \zeta_i) \leq \pi/2$. Let $y_i = \eta(t_i)$, $Y_i = \psi(t_i)$, where ψ is a lift of η at x . Finally, let $d_i = d(y_i, p(\beta_i(t_i)))$ and $D_i = d(Y_i, \beta_i(t_i))$. By Lemma 1.3, [17], since β_i is Cauchy, $\lim_{i \rightarrow \infty} (D_i - \sqrt{2}t_i)/t_i = 0$. Using the distance decreasing property of p , we have

$$\begin{aligned} \limsup_{i \rightarrow \infty} [(d_i - \sqrt{2}t_i)/d(p(x), p(\beta_i(t_i)))] \\ \leq \limsup_{i \rightarrow \infty} [(D_i - \sqrt{2}t_i)/d(\beta_i(t_i), Gx)] \\ \leq \left| \limsup_{i \rightarrow \infty} [(D_i - \sqrt{2}t_i)/d(\beta_i(t), Gx)] \right| \\ \leq \frac{1}{\epsilon} \left| \limsup_{i \rightarrow \infty} [(D_i - \sqrt{2}t_i)/t_i] \right| = 0. \end{aligned}$$

Lemma 1.3, [17] now implies that $\limsup_{i \rightarrow \infty} \alpha(\eta, \zeta_i) \leq \pi/2$.

To complete the proof of the proposition, suppose that $\delta_i := d(\beta_i(t_i), Gx)$ is realized by $d(\beta_i(t_i), g_i x)$ and let $m_i = \min\{t_i = d(\beta_i(t_i), x), d(x, g_i x)\}$. Then the triangle inequality implies that

$$t_i + d(x, g_i x) - d(\beta_i(-t_i), g_i x)/m_i \leq (2t_i + \delta_i - 2t_i + \delta_i)/m_i = 2\delta_i/m_i.$$

Now $m_i \leq t_i \leq m_i + \delta_i$, so $t_i/m_i \rightarrow 1$ and $\lim \delta_i/m_i = \lim(\delta_i/t_i)(t_i/m_i) = 0$. In other words, $\sigma(x; g_i x, \beta_i(-t_i))$ tends to zero, so if τ_i is minimal from x to $g_i x$ then $\alpha(-\beta_i, \tau_i) \rightarrow \pi$ and $\lim \alpha(\gamma, \tau_i) = \lim \alpha(\beta_i, \tau_i) = 0$. \square

Remark 2.1. The above results are valid when replacing “extendable” by “almost extendable” ([21]). For example, in this way it is possible to find “good” orbits at which lifts are angle preserving, by taking orbits whose tangent space in the quotient is Euclidean. However, we will not use these facts in this paper.

3. PRODUCTS

To prove our main theorem we need to consider products of countably infinitely many factors. The key to understanding countable products is to regard elements in any tangent cone of a factor not as unit parameterized geodesics of various lengths

(which is the interpretation suggested by the exponential map), but as constant-multiple reparameterizations of the corresponding unit geodesics.

If (X, d_1) and (Y, d_2) are inner metric spaces, the *product* of X and Y is the space $(X \times Y, d_1 \times d_2)$, where $d_1 \times d_2$ is the metric given by $d_1 \times d_2[(x_1, y_1), (x_2, y_2)] = \sqrt{d_1(x_1, x_2)^2 + d_2(y_1, y_2)^2}$. It is not hard to verify that $d_1 \times d_2$ is an inner metric. Minimal curves in $X \times Y$ are precisely curves of the form $(\gamma(t), \alpha(t))$ where γ and α are minimal or constant in X and Y , respectively, and $L(\gamma)^2 + L(\alpha)^2 = L(\gamma, \alpha)^2$. When no confusion will result, we denote the metric on the factors and the product all by d .

Lemma 3.1. *If (γ, β) is a minimal geodesic in $X \times Y$ from (x, y) to (w, z) , with γ, β non-constant, then the triangle made by the curves (γ, β) , (γ, y) , and (w, β) embeds isometrically in the plane. In particular, the angle $\alpha((\gamma, \beta), (\gamma, y))$ between the geodesic and its projection onto $X \times y$ exists and is equal to $\tan^{-1}(\frac{L(\beta)}{L(\gamma)})$.*

Proof.

$$d((x, y), (\gamma(t), y))^2 + d((\gamma(t), y), (\gamma(t), \beta(t)))^2 = d((x, y), (\gamma(t), \beta(t)))^2.$$

Since (γ, β) is proportional to arclength, this equation can be rewritten as

$$(t \cdot L(\gamma))^2 + (t \cdot L(\beta))^2 = (t \cdot L(\gamma, \beta))^2;$$

i.e. representatives of the triple $((x, y); (\gamma(t), y), (\gamma(t), \beta(t)))$ in the plane are vertices of triangles similar to the right triangle having legs of lengths $L(\gamma)$ and $L(\beta)$. In other words, for all t , the representative angles are the same, namely, $\tan^{-1}(\frac{L(\beta)}{L(\gamma)})$. \square

If γ is a unit geodesic in X , we denote by $k\gamma$ the curve defined by $k\gamma(t) = \gamma(kt)$. If β is a unit geodesic in Y , we denote by $k\gamma + m\beta$ the curve $(k\gamma, m\beta)$ in $X \times Y$, which is either a constant map ($k = m = 0$) or a geodesic; by Lemma 3.1, $k\gamma + m\beta$ is a unit geodesic if and only if $k^2 + m^2 = 1$. When we use this notation it will always be assumed that γ and β are unit parameterized and $m, k \geq 0$. Every unit geodesic (c_1, c_2) in $X \times Y$ can be written in this form (if one factor, say c_1 , is constant, we pick a geodesic γ_1 in X arbitrarily and let the coefficient be 0). The expression of a unit geodesic in the form $k\gamma + m\beta$ is unique if one ignores terms with 0 as coefficient. In what follows, $k_i = 0$, we define $c_{p_i}(\gamma)/k_i$ to be ∞ . (Recall that c_p denotes the cut radius at p , cf. Section 1).

Lemma 3.2. *For any $(p_1, p_2) \in X_1 \times X_2$ and minimal $\gamma = k_1\gamma_1 + k_2\gamma_2$, with $k_1^2 + k_2^2 = 1$, starting at $p = (p_1, p_2)$, $c_p(\gamma) = \min\{c_{p_i}(\gamma_i)/k_i\}$.*

Proof. Let $c = \min\{c_{p_i}(\gamma_i)/k_i\}$. Then each curve $k_i\gamma_i$ is minimal on $[0, c]$, so γ is also minimal on $[0, c]$, and $c_p(\gamma) \geq c$. Conversely, γ is minimal on $[0, c_p(\gamma)]$ so each $k_i\gamma_i$ is minimal on this interval. But then $c_{p_i}(\gamma_i) \geq k_i c_p(\gamma)$ for both i and $c \geq c_p(\gamma)$. \square

If X and Y are Riemannian manifolds, $d_1 \times d_2$ coincides with the distance induced by the usual product Riemannian metric on $X \times Y$. From this last fact it follows that if X and Y are inner metric spaces of curvature $\geq k$ and $\geq k'$, respectively, then any four points in $X \times Y$ can be isometrically embedded in $S_k \times S_{k'}$. By the definition of bounded curvature this means that $X \times Y$ has curvature $\geq \min\{0, k, k'\}$ (cf. also [7]). In a similar way, we see that angles in $X \times Y$ are measured in the same way as angles in a product of Riemannian manifolds. In other words:

Lemma 3.3. *Suppose X and Y have curvature $\geq k$ and $\gamma = k_1\gamma_1 + k_2\gamma_2$ and $\beta = m_1\beta_1 + m_2\beta_2$ are unit minimal starting at a common point in $X \times Y$. Then $\cos \alpha(\gamma, \beta) = k_1m_1 \cos \alpha(\gamma_1, \beta_1) + k_2m_2 \cos \alpha(\gamma_2, \beta_2)$. In particular, $\alpha(\gamma, \beta) \geq \alpha(\gamma_i, \beta_i)$ for all i such that $k_i, m_i \neq 0$.*

Proposition 3.4. *For any $\epsilon > 0$ and k , there exists a $\delta > 0$ depending only k such that if X and Y have curvature $\geq k$, and $\gamma = k_1\gamma_1 + k_2\gamma_2$ and $\beta = m_1\beta_1 + m_2\beta_2$ are unit minimal in $X \times Y$ such that $\alpha(\gamma, \beta) < \delta$ then $|k_i - m_i| < \epsilon$.*

Proof. Let $k' = \min\{k, 0\}$. Fix $\delta > 0$ small enough that if Γ_1 and Γ_2 are unit minimal in $S_{k'}$ such that if $\alpha(\Gamma_1, \Gamma_2) < \delta$ then for any $t < 1$, $d(\Gamma_1(t), \Gamma_2(t))/t < \epsilon/2$. Assume first that $k_i, m_i > 0$. Then for any positive $t < \min\{c_{p_i}(\gamma_i)/m_i, c_{p_i}(\gamma_i)/k_i\}$, if $\alpha(\gamma, \beta) < \delta$ then by Lemma 3.3, $\alpha(\gamma_i, \beta_i) < \delta$ and

$$\begin{aligned} |k_i - m_i| &= \left(\frac{1}{t}\right) |k_it - m_it| = \frac{1}{t}d(\gamma_i(k_it), \gamma_i(m_it)) \\ &\leq \frac{1}{t}(d(\gamma_i(k_it), \beta_i(m_it)) + d(\beta_i(m_it), \gamma_i(m_it))) \\ &< \frac{1}{t}d(\gamma(t), \beta(t)) + \epsilon/2 < \epsilon. \end{aligned}$$

If, say, $m_i > 0$ and $k_i = 0$ then $m_i = \frac{d(\beta_i(m_it), p_i)}{t} = \frac{d(\beta_i(m_it), \gamma_i(k_it))}{t} \leq \frac{d(\beta(t), \gamma(t))}{t} < \epsilon/2$. If, say, $m_1 = k_1 = 0$ then γ and β lie in a single Y slice, and the proof is immediate from our choice of δ . The other cases are essentially the same. \square

Since \overline{S}_p is by definition the space of equivalence classes of Cauchy sequences in S_p , we immediately have the following corollaries of Lemma 3.3 and Proposition 3.4. Note that we identify $k_1\gamma_1 + k_2\gamma_2$ with $k'_1\gamma'_1 + k_2\gamma_2$ if $k_1 = k'_1 = 0$, and let $-k\gamma(t) = \gamma(-kt)$ whenever this makes sense (i.e. γ extends past $\gamma(0)$). To avoid technicalities later, if $k_i = 0$, we will always assume that γ_i is a geodesic direction (we can, of course, always choose it this way).

Corollary 3.5. *For any $p_1 \in X$ and $p_2 \in Y$, $\overline{S}_{(p_1, p_2)} = \{k_1\gamma_1 + k_2\gamma_2 : k_1^2 + k_2^2 = 1, k_i \geq 0, \text{ and } \gamma_i \in \overline{S}_{p_i}\}$, and $c_{(p_1, p_2)}(\gamma) = \min\{c_{p_i}(\gamma_i)/k_i\}$.*

Corollary 3.6. *If X and Y are geodesically complete then for all $p = (p_1, p_2) \in X \times Y$, $\overline{T}_p = \overline{T}_{p_1} \times \overline{T}_{p_2}$ (as Hilbert spaces).*

The above constructions are easily extended to any finite number of factors by simple inductive arguments; the details are left to the reader. We only verify the following:

Lemma 3.7. *If each X_i has curvature $\geq k$ and $\gamma = \sum_{i=1}^k k_i\gamma_i$ is a unit geodesic then $c_p(\gamma) = \min\{c_{p_i}(\gamma_i)/k_i\}$.*

Proof. The case $k = 2$ is Lemma 3.2; we proceed by induction. Suppose the lemma holds for $k - 1 > 1$. If any $k_i = 0$ we can eliminate a factor and apply the induction hypothesis, so we suppose that $k_i > 0$ for all i . By re-indexing the product, if necessary, we can suppose that

$$c_{p_k}(\gamma_k)/k_k \leq c_{p_{k-1}}(\gamma_{k-1})/k_{k-1} \leq \dots \leq c_{p_1}(\gamma_1)/k_1.$$

Write $\sum_{i=1}^k k_i \gamma_i = k_k \gamma_k + \sum_{i=1}^{k-1} k_i \gamma_i$. By the induction hypothesis,

$$c_{(p_1, \dots, p_{k-1})} \left(\sum_{i=1}^{k-1} k_i \gamma_i \right) = \frac{c_{p_k}(\gamma_{k-1}) \left(\sum_{i=1}^{k-1} k_i^2 \right)^{1/2}}{k_{k-1}}.$$

Now

$$c_{p_k}(\gamma_k)/k_k \leq c_{p_{k-1}}(\gamma_{k-1})/k_{k-1} = \frac{c_{(p_1, \dots, p_{k-1})} \left(\sum_{i=1}^{k-1} k_i \gamma_i \right)}{\left(\sum_{i=1}^{k-1} k_i^2 \right)^{1/2}}$$

and the proof is complete by Lemma 3.2. \square

Now suppose $\{X_i, d_i\}$ is a countable collection of inner metric spaces such that $\{diam(X_i)\}$ is square summable. We define a metric on $X = \prod_{i=1}^{\infty} X_i$ in the following way. Choose a fixed element (x_1, x_2, \dots) of X . For each j , let X^j denote $\prod_{i=1}^j X_i$. Then X^j isometrically embeds in X^{j+1} via the mapping $(y_1, \dots, y_j) \mapsto (y_1, \dots, y_j, x_{j+1})$, and with this embedding, $d_H(X^j, X^{j+1}) \leq diam(X_{j+1})$. Thus the sequence $\{X^j\}$ is Cauchy in the Gromov-Hausdorff metric ([10]), and the unique limit of this sequence is identified with X . It is easy to see that this definition is equivalent to setting

$$\begin{aligned} & d((y_1, y_2, \dots), (z_1, z_2, \dots)) \\ &= \lim_{j \rightarrow \infty} d_1 \times \dots \times d_j((y_1, \dots, y_j), (z_1, \dots, z_j)) = \sqrt{\sum d_i(y_i, z_i)^2}. \end{aligned}$$

The topology induced by this metric is equivalent to the usual product topology, and so X is a compact inner metric space if each X_i is compact. Also, it is immediate from the definition that for any j , X is isometric to $\prod_{i=1}^j X_i \times \prod_{i=j}^{\infty} X_i$. Each finite product of the factors X_i is embedded naturally in X by fixing all the remaining coordinates. We recall the following definition (cf. [21]):

Definition 3.1. A subset Y of an inner metric space X is called *metrically embedded* if the induced metric and the induced inner metric coincide on Y .

When Y is metrically embedded in X , every minimal curve between points $x, y \in Y$ is also a minimal curve in X . Consequently, if X has curvature $\geq k$ then so does Y , and the angle between minimal curves γ and β in Y (if it exists) is the same as the angle in X . In other words, for any point $p \in Y$, the inclusion of $\overline{S}_p Y$ in $\overline{S}_p X$ is an isometry, and $\overline{S}_p Y$ is a metrically embedded subset of $\overline{S}_p X$ if both are inner metric spaces.

It is immediate from the definition of the product metric that each X_i is metrically embedded in X . If each X_i has curvature $\geq k$ then X has curvature $\geq \min\{0, k\}$, since we have expressed it above as a Gromov-Hausdorff limit. If $(c_1, c_2, \dots) : [0, 1] \rightarrow X$ is minimal then each c_i is minimal in X_i . In fact, we need only show that if we set $c_i(0) = x_i$, $c_i(1/2) = y_i$, and $c_i(1) = z_i$ then y_i is a midpoint between x_i and z_i . However, the Minkowski and triangle inequalities imply that (setting $x = (x_1, \dots)$, etc.) $d(x, z) = d(x, y) + d(y, z) = \sqrt{\sum d(x_i, y_i)^2} +$

$\sqrt{\sum d(y_i, z_i)^2} \geq \sqrt{\sum (d(x_i, y_i) + d(y_i, z_i))^2} \geq \sqrt{\sum d(x_i, z_i)^2} = d(x, z)$. If y_i were not a midpoint for some i then the second (triangle) inequality would be a strict inequality, a contradiction.

Proposition 3.8. *Let c_i be a unit parameterized curve in X_i . Then for $k_i \geq 0$, a curve $c(t) = (c_1(k_1t), c_2(k_2t), \dots)$ in $X = \prod_{i=1}^{\infty} X_i$ is rectifiable if and only if $\sum_{i=1}^{\infty} k_i^2 < \infty$ (and c is unit parameterized if and only if $\sum_{i=1}^{\infty} k_i^2 = 1$).*

Proof. It suffices to assume c is defined on a fixed closed interval $[a, b]$. For any $s < t$,

$$\begin{aligned} d(c(s), c(t))^2 &= \sum_{i=1}^{\infty} d(c_i(k_i s), c_i(k_i t))^2 \\ &\leq \sum_{i=1}^{\infty} |k_i t - k_i s|^2 = \sum_{i=1}^{\infty} k_i^2 |t - s|^2. \end{aligned}$$

If $\sum_{i=1}^{\infty} k_i^2 = K^2 < \infty$ then we have that $d(c(s), c(t)) \leq K(t - s)$; i.e. c is rectifiable and $L(c|_{[a,b]}) \leq K(b - a)$. Now for any $\epsilon > 0$ we can find M such that $\sum_{i=M}^{\infty} k_i^2 (b - a)^2 < \epsilon$. Choose a partition $a = s_1 < \dots < s_k = b$ such that for all $i < M$, $k_i^2 (b - a)^2 - \sum_{j=1}^{k-1} d(c_i(k_i s_j), c_i(k_i s_{j+1}))^2 < \epsilon/M$. Then

$$\begin{aligned} L(c)^2 &\geq \left(\sum_{j=1}^{k-1} d(c(s_j), c(s_{j+1})) \right)^2 \geq \sum_{j=1}^{k-1} \sum_{i=1}^{\infty} d(c_i(s_j), c_i(s_{j+1}))^2 \\ &\geq \sum_{j=1}^{k-1} \sum_{i=1}^M d(c_i(k_i s_j), c_i(k_i s_{j+1}))^2 > \sum_{i=1}^M k_i^2 (b - a)^2 - \epsilon > K^2 (b - a)^2 - 2\epsilon. \end{aligned}$$

In other words, $L(c) = K(b - a)$, and c is unit parameterized if and only if $K = 1$.

Now suppose $\sum_{i=1}^{\infty} k_i^2 = \infty$. For any N fix M such that $\sum_{i=1}^M k_i^2 > N$ and choose a partition $a = s_1 < \dots < s_k = b$ such that for all $i < M$, $k_i^2 (b - a)^2 - \sum_{j=1}^{k-1} d(c_i(k_i s_j), c_i(k_i s_{j+1}))^2 < 1/M$. Then

$$\begin{aligned} \left(\sum_{j=1}^{k-1} d(c(k_i s_j), c(k_i s_{j+1})) \right)^2 &= \sum_{j=1}^{k-1} \sum_{i=1}^{\infty} d(c_i(k_i s_j), c_i(k_i s_{j+1}))^2 \\ &> \sum_{j=1}^{k-1} \sum_{i=1}^M d(c_i(k_i s_j), c_i(k_i s_{j+1}))^2 \\ &> \sum_{i=1}^M (k_i^2 (b - a)^2 - 1/M) > N(b - a)^2 - 1. \quad \square \end{aligned}$$

Corollary 3.9. *Every unit parameterized geodesic in $X = \prod_{i=1}^{\infty} X_i$ can be written uniquely (except for γ_i when $k_i = 0$) in the form*

$$\sum_{i=1}^{\infty} k_i \gamma_i(t) = (\gamma_1(k_1 t), \gamma_2(k_2 t), \dots)$$

where each γ_i is a unit geodesic in X_i , $k_i \geq 0$, and $\sum_{i=1}^{\infty} k_i^2 = 1$.

As in the case of finitely many factors, when we use the notation $\sum_{i=1}^{\infty} k_i \gamma_i$, it will always be assumed that $k_i \geq 0$ and γ_i is unit. From now on we assume that $X = \prod_{i=1}^{\infty} X_i$ and that each X_i has curvature $\geq k$. In general, continuity of the angle fails with Gromov-Hausdorff limits: the angle of the limit of curves can be strictly smaller than the limit of the angles. In the present case, however, this does not occur. For the purposes of the next proposition, we take the angle between any geodesic and a constant map to be 0.

Proposition 3.10. *Suppose $\gamma = (\gamma_1, \dots)$ and $\beta = (\beta_1, \dots)$ are geodesics starting at a common point in X . Then $\alpha(\gamma, \beta)$ is the limit of the increasing sequence $\alpha((\gamma_1, \dots, \gamma_k), (\beta_1, \dots, \beta_k))$.*

Proof. The sequence is increasing by Lemma 3.3. Since we are only interested in the angle, we can assume that both γ and β are minimal and have the same length $L > 0$. For any i , split X as the product $\prod_{k=1}^i X_k \times \prod_{k=i+1}^{\infty} X_k$. Observe that $L(\gamma_k), L(\beta_k) < d_k := \text{diam}(X_k)$. Set $\gamma^i = (\gamma_1, \dots, \gamma_i), \beta^i = (\beta_1, \dots, \beta_i)$ and identify γ^i, β^i with the geodesics $\gamma^i \times (e, e, \dots)$ and $\beta^i \times (e, e, \dots)$ in the slice $\prod_{k=1}^i X_k$. According to Lemma 3.1 and the triangle inequality, this means that $\alpha(\gamma^i, \gamma)$ and $\alpha(\beta^i, \beta)$ are both smaller than $\tan^{-1}(\frac{\sum_{k=i}^{\infty} d_k^2}{L_i})$, where $L_i = \min\{L(\gamma^i), L(\beta^i)\}$ and tends to L as i becomes large. In other words, both $\alpha(\gamma^i, \gamma)$ and $\alpha(\beta^i, \beta)$ tend to 0 as i becomes large. \square

We wish to extend Corollary 3.9 to any *direction* in X . Since directions may not be uniquely associated with curves, our statement is necessarily about limits. What we are essentially doing, however, is identifying \bar{S}_p with $\{\sum_{i=1}^{\infty} k_i \gamma_i : \sum_{i=1}^{\infty} k_i^2 = 1, \gamma_i \in \bar{S}_{p_i}\}$.

Proposition 3.11. *Let $p \in X$ and $\gamma \in \bar{S}_p$. Then there are unique $k_i \geq 0$ with $\sum_{i=1}^{\infty} k_i^2 = 1$ and unique (for $k_i \neq 0$) directions $\gamma_i \in \bar{S}_{p_i}$ so that $c_p(\gamma) = \inf\{c_{p_i}(\gamma_i)/k_i\}$, and for any geodesic directions $\beta_j = \sum_{i=1}^{\infty} k_{ij} \beta_{ij} \in S_p$ such that $\alpha(\beta_j, \gamma) \rightarrow 0, \lim_{j \rightarrow \infty} k_{ij} = k_i$ and $\lim_{j \rightarrow \infty} \beta_{ij} = \gamma_i$ when $k_i \neq 0$. Conversely, suppose we are given directions γ_i and $k_i \geq 0$ such that $\sum_{i=1}^{\infty} k_i^2 = 1$. Then for any geodesic*

directions β_{ij} and $k_{kj} \geq 0$ such that $\sum_{i=1}^{\infty} k_{ij}^2 = 1$ and $\alpha(\beta_{ij}, \gamma_i) \rightarrow 0$ for all i such that $k_i \neq 0$, the geodesics $\beta^k := \sum_{i=1}^k k_{ik} \beta_{ik}$ form a Cauchy sequence.

Proof. Every unit geodesic γ in X is uniquely represented in the form $\gamma = \sum_{i=1}^{\infty} k_i \gamma_i$, with non-negative coefficients square summable to 1. By the semicontinuity of the cut radius map, $c_p(\gamma) \geq \inf\{c_{p_i}(\gamma_i)/k_i\}$. On the other hand, writing

$$\gamma = \sum_{i=1}^k \frac{k_i}{\left(\sum_{i=1}^k k_i^2\right)^{1/2}} \gamma_i + \sum_{i=k+1}^{\infty} \frac{k_i}{\left(\sum_{i=k+1}^{\infty} k_i^2\right)^{1/2}} \gamma_i$$

we see that

$$c_p(\gamma) \leq \inf_{i \leq k} \left\{ \left(\sum_{i=1}^k k_i^2\right)^{1/2} c_{p_i}(\gamma_i)/k_i \right\} \rightarrow \inf\{c_{p_i}(\gamma_i)/k_i\},$$

so $c_p(\gamma) = \inf\{c_{p_i}(\gamma_i)/k_i\}$.

If γ is not a geodesic direction (i.e. $c_p(\gamma) = 0$), choose $\beta_j \in S_p$ such that $\beta_j \rightarrow \gamma$, where $\beta_j = \sum_{i=1}^{\infty} k_{ij} \beta_{ij}$. By Lemma 3.3 and Proposition 3.4, for fixed k the sequence $\beta_j^k = \sum_{i=1}^k k_{ij} \beta_{ij}$ is Cauchy with limit $\gamma^k = \sum_{i=1}^k k_i \gamma_i$, where the coefficients $k_i = \lim_{j \rightarrow \infty} k_{ij}$ satisfy $\sum_{i=1}^{\infty} k_i^2 = 1$. Now for j sufficiently large, $\alpha(\gamma^k, \beta_j^k)$ is small. Also, if $s_{jk} := \sum_{i=k}^{\infty} k_{ij}$, then for j, k sufficiently large, s_{jk} is small. In other words, for j, k sufficiently large, $\alpha(\beta_j, \beta_j^k)$ is small; that is, $\gamma^k \rightarrow \gamma$ in \bar{S}_p , and we have identified γ with $\sum_{i=1}^{\infty} k_i \gamma_i$. It now follows that $\inf\{c_{p_i}(\gamma_i)/k_i\} = 0$ by the semicontinuity of the cut radius. Thus we have associated to each direction a sum $\sum_{i=1}^{\infty} k_i \gamma_i$ and verified the cut radius formula. Our choice of β_j was arbitrary and we proved convergence without taking a subsequence. From this it follows that the sum $\sum_{i=1}^{\infty} k_i \gamma_i$ is unique.

The converse statement is an immediate consequence of Lemma 3.3. □

If Y is a geodesically complete space of curvature bounded below then the inner product induced on the tangent space is completely determined by angles and lengths of curves; hence we have:

Corollary 3.12. *If each X_i is geodesically complete, then for any $p \in X$, \bar{T}_p is identified with the Hilbert space of formal sums $\sum_{i=1}^{\infty} k_i \gamma_i$, where each γ_i is unit minimal in X_i , $\sum_{i=1}^{\infty} k_i^2 < \infty$, and the inner product is given by*

$$\begin{aligned} \left\langle \sum_{i=1}^{\infty} k_i \gamma_i, \sum_{i=1}^{\infty} m_i \beta_i \right\rangle &= \lim_{k \rightarrow \infty} \left\langle \sum_{i=1}^k k_i \gamma_i, \sum_{i=1}^k m_i \beta_i \right\rangle \\ &= \lim_{k \rightarrow \infty} \sum_{i=1}^k k_i m_i \langle \gamma_i, \beta_i \rangle. \end{aligned}$$

Note that X need not be geodesically complete in general, although this requires a certain amount of asymmetry from the cut radius at each p_i .

Proposition 3.13. *If each X_i has positive cut radius, then X is weakly convex and for each direction $\gamma = \sum_{i=1}^{\infty} k_i \gamma_i$, the corresponding unit curve $c = (k_1 \gamma_1, k_2 \gamma_2, \dots)$ is tangent to γ .*

Proof. For any $p \in X$, let W_k be the subspace of \bar{T}_p consisting of all geodesics $\sum_{i=1}^k k_i \gamma_i$. Since X_i has positive cut radius, the fact that $W_p = \bigcup_{k=1}^{\infty} W_k$ satisfies the requirements of Definition 1.2 follows easily from Proposition 3.11. Since each X_i has positive cut radius, each γ_i is a bona fide geodesic, so c is in fact a curve. We need to show that c is tangent to γ . Suppose that $\beta_t = \sum_{i=1}^{\infty} m_{ti} \beta_{ti}$ is minimal from $c(0)$ to $c(t)$ parameterized so that $\beta_t(t) = c(t)$. Then each $\beta_t^k = \sum_{i=1}^k m_{ti} \beta_{ti}$ is minimal from $\gamma^k(0)$ to $\gamma^k(t)$, where $\gamma^k = \sum_{i=1}^k k_i \gamma_i$. As soon as $t < \min_{i \leq k} \{c_{p_i}(\gamma_i)/k_i\}$, β_t^k must coincide with γ^k (even up to parameterization, since $\beta_t^k(t) = \gamma^k(t)$), i.e. $m_{ti} = k_i$. Now $\sum_{i=1}^k k_i^2 = \sum_{i=1}^k m_{ti}^2 \leq \sum_{i=1}^{\infty} m_{ti}^2 = d(c(t), c(0))^2/t^2 \leq \sum_{i=1}^{\infty} k_i^2 = 1$. In other words, by Lemma 3.1, for k fixed and small t , $\alpha(\beta_t^k, \beta_t)$ is uniformly small. For any $\epsilon > 0$ there exists a k such that $\alpha(\gamma^k, \gamma) < \epsilon$, so there exists a $t > 0$ such that $\alpha(\beta_t^k, \gamma) < \epsilon$ and hence $\alpha(\beta_t, \gamma) < \epsilon$. \square

4. BI-INVARIANT INNER METRICS

The Gleason-Iwasawa-Yamabe theory of locally compact groups (cf. [9]) implies that a connected locally compact group splits locally as a product between a Lie group and a compact group. This theorem essentially reduces our results on locally compact groups to results on compact groups. For compact groups it is important to consider metrics which are bi-invariant; such metrics are also very interesting in their own right.

Lemma 4.1. *A metric on a group G is bi-invariant if and only if it is invariant and the inverse map $a \rightarrow a^{-1}$ is an isometry.*

Proof. If G has a bi-invariant metric then $d(a, b) = d(e, a^{-1}b) = d(b^{-1}, a^{-1})$. Conversely, if the metric is invariant and the inverse is an isometry, $d(ab, cb) = d(b^{-1}a^{-1}, b^{-1}c^{-1}) = d(a^{-1}, c^{-1}) = d(a, c)$. \square

Proposition 4.2. *If G is a group with a bi-invariant inner metric of curvature bounded below then every geodesic γ through e is a homomorphism (i.e. the image of the curve is a 1-parameter subgroup).*

Proof. Assume γ is unit parameterized. We will show first that $\gamma(-t) = \gamma(t)^{-1}$. Since the inverse mapping is an isometry, the curve $\gamma(t)^{-1}$ is a geodesic. Consider the geodesic $\beta_s(t) = \gamma(s)^{-1}\gamma(t)$ for fixed s , which has the property that $\beta_s(s) = e$ and $\beta_s(0) = \gamma(s)^{-1}$. As $s \rightarrow 0$, β_s uniformly converges to γ ; since γ is extended through e , this means that $\alpha(\gamma, \beta_s) \rightarrow 0$. But then the representative angle $\alpha_k(e; \gamma(-s), \gamma(s)^{-1})$, which is smaller than $\alpha(\gamma, \beta_s)$, also tends to 0.

Then by definition, $\alpha(\gamma^{-1}, -\gamma) = 0$, and so the two geodesics coincide. Now let s be small enough that $\zeta_s = \gamma|_{[-s,s]}$ is the unique minimal curve between its endpoints. Then the isometry $L_{\gamma(s)}$ takes ζ_s to a unit minimal curve of length $2s$ starting at e and passing through $\gamma(s)$; therefore $\gamma(s)\zeta_s$ coincides with the segment of γ from e to $\gamma(2s)$. Since unit geodesics are uniquely determined by any segment (with interior), for any small enough s , γ is invariant under $L_{\gamma(s)}$, and $\gamma(s)\gamma(t) = \gamma(t+s)$. Let $S > 0$ be the supremum of all s such that $\gamma(s)\gamma(t) = \gamma(t+s)$ for all t , and suppose $S < \infty$. Then $\gamma(3S/2) = \gamma(3S/4)\gamma(3s/4)$, and for any t , $\gamma(3S/2)\gamma(t) = \gamma(3S/4)\gamma(3S/4)\gamma(t) = \gamma(3S/4)\gamma(t + 3S/4) = \gamma(t + 3S/2)$, a contradiction. \square

Remark 4.1. From now on, if γ is a geodesic through e , we will denote by $-\gamma$ the geodesic defined by $-\gamma(t) = \gamma(-t)$. For example, it is always true that $\alpha(\gamma, -\beta) = \pi - \alpha(\gamma, \beta)$.

Corollary 4.3. *A group with bi-invariant inner metric of curvature bounded below is geodesically complete.*

Proof. We need to show that an arbitrary minimal curve γ starting at e can be extended as a geodesic through e . For any $\gamma(t)$ in the interior of γ , $\beta(s) = \gamma(t)^{-1}\gamma(s)$ is a minimal curve through e ; hence $\beta(s)$ extends to a 1-parameter subgroup. The left invariance of the metric implies that β restricted to a small interval about 0 is a minimal curve, call it $\sigma(s)$. But then $\psi(s) = \gamma(t)\psi(s)$ is the desired extension of γ . \square

Proposition 4.4. *If G has a bi-invariant inner metric of curvature bounded below, then G has curvature ≥ 0 .*

Proof. Let γ and β be unit minimal curves starting at e of length L, M , respectively. Then for any s, t , $d(\gamma(2s), \beta(2t)) = d(\gamma(s)^2, \beta(t)^2) \leq d(\gamma(s)^2, \gamma(s)\beta(t)) + d(\gamma(s)\beta(t), \beta(t)^2) = 2d(\gamma(s), \beta(t))$. In other words, G has curvature ≥ 0 in a “weak” (cf. [24]) sense. Suppose $\alpha(\gamma, \beta) = A$, fix $\epsilon > 0$ and choose n large enough that $|d(\gamma(L/2^n), \beta(M/2^n))2^n - (L^2 + M^2 - 2LM \cos A)| < \epsilon$. Then from what we proved above, $|d(\gamma(L), \beta(M)) - (L^2 + M^2 - 2LM \cos A)| < \epsilon$. Letting ϵ go to 0 verifies Alexandrov’s comparison condition, and proves the proposition. \square

We say that a space X with inner metric has *positive cut radius* if every point $p \in X$ has a neighborhood in which every point is joined to p by a unique minimal curve. It is well-known that in a Lie group with bi-invariant Riemannian metric, geodesics through e and 1-parameter subgroups coincide (cf. [12]). We have the following extension of this result:

Proposition 4.5. *If G has a bi-invariant inner metric of curvature ≥ 0 and positive cut radius then geodesics through e and non-trivial 1-parameter subgroups coincide.*

Proof. Let γ be a 1-parameter subgroup. Suppose that $c_t : [0, t] \rightarrow G$ is minimal from e to $\gamma(t)$. By Proposition 4.2 c extends to a 1-parameter subgroup which we call by the same name. Now $c_{t/2}(t) = (c_{t/2}(t/2))^2 = \gamma(t/2)^2 = \gamma(t) = c_t(t)$. In other words, if $c_{t/2}$ and c_t are distinct then the cut radius of $c_{t/2}$ is less than $d(e, \gamma(t))$, or twice its length. Since the cut radius of G is positive, for small enough t , all the curves $c_{t/n}$ must coincide with a single geodesic γ' . But then γ and γ'

initially coincide on a dense set of points in their domain of definition, and, being both continuous, must coincide everywhere. \square

We remark that, under the assumptions of 4.5, G has a neighborhood U of e in which each point x lies on a unique segment of a 1-parameter subgroup in U from e to x . That is, U is (uniquely) “ruled” by 1-parameter subgroups. It is a classical (cf. [15]) that a locally compact group having such a neighborhood must be a Lie group.

Let G be a group with invariant metric, H a normal subgroup which is finitely compact (Definition 2.2) with respect to the induced metric. If we identify H with the group of isometries of G consisting of the left translates of elements of H , then orbits of this action are precisely the cosets of the quotient group G/H .

Proposition 4.6. *If G is locally compact and has bi-invariant inner metric of curvature ≥ 0 then G is weakly convex.*

Proof. Since G has an inner metric, G is arcwise connected, and hence connected; this implies that G/G_0 is (trivially) compact. By the Gleason-Iwasawa-Yamabe theory of locally compact groups, G is a projective limit of Lie groups. If G is finite dimensional, G is a Lie group with bi-invariant Riemannian metric, and we are finished. Suppose G is infinite dimensional. Since G is first countable, that means we can find a countable sequence of properly nested (cf. [9]) normal subgroups $G_1 \supset G_2 \supset \cdots$ whose diameters tend to zero, such that $K_i = G/G_i$ is a Lie group. By Lemma 2.1 the tangent space of K_i lifts isomorphically to a subspace H_i of T_e , and c_e has a positive lower bound on $S(H_i) = S_e \cap H_i$ (in fact c_e is no smaller than the cut radius of K_i on $S(H_i)$). Since $G_{i+1} \subset G_i$, Proposition 2.4 implies $T_{i+1}^\perp \subset T_i^\perp$, and therefore H_i is properly contained in H_{i+1} . Let $H_e = \bigcup_{i=1}^{\infty} H_i$. Then H_e satisfies

the requirements of Definition 1.2 except possibly condition 2). Now let $\gamma \in \overline{S}_e$ be arbitrary. Assume first that $c_e(\gamma) = T > 0$. Fix $\epsilon > 0$. For each i , there exists $\gamma_i \in H_i$ from e to the orbit $G_i\gamma(T - \epsilon)$. Since $\text{diam}\{(G_i\gamma(T - \epsilon))\} \rightarrow 0$ and G is geodesically complete, $\alpha(\gamma_i, \gamma) \rightarrow 0$. Furthermore, $\liminf\{c_e(\gamma_i)\} \geq T - \epsilon$. By letting $\epsilon \rightarrow 0$ and choosing a suitable diagonal subsequence we can choose the γ_i such that $\liminf\{c_e(\gamma_i)\} \geq T$. That $\lim c_e(\gamma_i) = T$ now follows from the upper semicontinuity of the cut radius. In particular we have shown that H_e is dense in T_e and therefore in \overline{T}_e . Therefore, even if $c_e(\gamma) = 0$, there exist $\gamma_i \rightarrow \gamma$, and $c_e(\gamma_i) \rightarrow 0$ again by the upper semicontinuity of c_e . \square

5. EXISTENCE OF INVARIANT WEAKLY CONVEX METRICS

Proof of Theorem 1.3. We first claim the following: For any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $s \leq t$ with $0 < s < \delta/4$ and $t < 3s$, $\frac{d_s + d_t - d_{t+s}}{d_s} < \epsilon$, where d_r denotes $d(e, \gamma(r))$. Let γ be unit parameterized, $0 < \delta < 1/2$ be small enough that if P is any partition of $[0, 8]$ of norm $\|P\| < \delta$, then the corresponding sum $\Sigma(P)$ satisfies $8 - \Sigma(P) < \kappa := \frac{\epsilon}{1+\epsilon}$. Then for any $a < 8$, if P' is a partition of $[0, a]$ with $\|P'\| < \delta$, $a - \Sigma(P') < \kappa$. Suppose that $\frac{1}{n} \leq s < \frac{1}{n-1}$ ($n \geq 2$). Then $\frac{2}{n} \leq s + t \leq \frac{4}{n-1}$, and so ns and $n(s+t)$ are both ≤ 8 . Since γ is a 1-parameter subgroup and the metric is invariant, for any real numbers $r_1 < r_2$, $d(\gamma(r_1), \gamma(r_2)) = d(e, \gamma(r_2 - r_1))$. By using the partition of $[0, ns]$ whose points are integral multiples of s , we see that since $s < \delta$, $ns - nd_s < \kappa$, which implies $d_s > s - \frac{\kappa}{n} \geq \frac{1-\kappa}{n}$. Similarly, we obtain

$s + t - d_{s+t} < \frac{\kappa}{n}$. Now $\frac{d_s+d_t-d_{t+s}}{d_s} = \frac{d_s-s+d_t-t+s+t-d_{t+s}}{d_s} \leq \frac{s+t-d_{t+s}}{d_s} < \frac{\kappa}{1-\kappa} = \epsilon$, proving the first claim.

We next claim that for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $s \leq t$ with $0 < s < \delta$ and $t < 2s$, if γ_r denotes a minimal curve from e to $\gamma(r)$ then $\alpha(\gamma_s, \gamma_t) < \epsilon$. In fact, the above claim shows that for small enough s , $\alpha(\gamma_{-s}, \gamma_s)$ and $\alpha(\gamma_{-s}, \gamma_{s+t})$ are both arbitrarily close to π . But then $\alpha(\gamma_s, \gamma_{s+t})$ must be arbitrarily close to 0. In particular, we have shown that a sequence of minimal curves γ_i from e to $\gamma(2^{-i})$ is Cauchy (in angle); hence it converges to some direction we will also call γ .

To complete the proof, consider positive $t_i \rightarrow 0$ and let β_i be minimal curves from e to $\gamma(t_i)$. We need to show that $\beta_i \rightarrow \gamma$. But for each i , there is some j such that $2^{-j} \leq t_i < 2^{-j+1}$, and by the second claim, $\alpha(\beta_i, \gamma_{j+1})$ is arbitrarily small for large j . We remark here that we have used the existence of certain minimal curves to prove the proposition. This is not essential if we use the results of [21]. In the case of present interest, the local compactness provides the needed minimal curves. \square

In the above proof we found $t_i \rightarrow 0$ such that $\lim_{i \rightarrow \infty} \sigma(e; \gamma(t_i), \gamma(-t_i)) = 0$; this implies that the corresponding representative angles must tend to π (see the discussion preceding Proposition 2.4):

Corollary 5.1. *Let G be a group with left invariant inner metric of curvature $\geq k$ and γ be a non-trivial rectifiable 1-parameter subgroup of G . Then there exist positive $t_i \rightarrow 0$, with $\lim_{i \rightarrow \infty} \alpha_k(e; \gamma(t_i), \gamma(-t_i)) = \pi$.*

Note that the product of invariant or bi-invariant metrics is invariant or bi-invariant, respectively.

Proposition 5.2. *If G_i has a bi-invariant inner metric of curvature ≥ 0 and positive cut radius, then a curve β in $G = \prod_{i=1}^{\infty} G_i$ is a 1-parameter subgroup in G if and only if $\beta = \sum_{i=1}^{\infty} k_i \gamma_i$, where each γ_i is a unit geodesic ($\sum_{i=1}^{\infty} k_i^2$ may or may not be finite).*

Proof. Since every geodesic is a 1-parameter subgroup (Lemma 4.2), any curve of the form $\sum_{i=1}^{\infty} k_i \gamma_i$ is a 1-parameter subgroup. Conversely, every 1-parameter subgroup β in G is of the form $(\beta_1, \beta_2, \dots)$, where each β_i is a 1-parameter subgroup in G_i . By Lemma 4.5 each non-trivial β_i is also a geodesic; so $\beta_i = k_i \gamma_i$ for unit γ_i , and $\beta = \sum_{i=1}^{\infty} k_i \gamma_i$. \square

Corollary 5.3. *Let G_i be groups with bi-invariant inner metrics of curvature ≥ 0 and positive cut radius. Then there exist non-rectifiable 1-parameter subgroups in $G := \prod_{i=1}^{\infty} G_i$.*

More generally it follows from the proof of Theorem 1.1 that non-rectifiable 1-parameter subgroups exist in any locally compact infinite dimensional group, with the weakly convex metrics we construct.

Corollary 5.4. *If G_i has a bi-invariant inner metric of curvature ≥ 0 and positive cut radius, then in $G = \prod_{i=1}^{\infty} G_i$ every non-trivial rectifiable 1-parameter subgroup is tangent to a direction, and every direction is tangent to a unique unit 1-parameter subgroup.*

Under the conditions of Corollary 5.4 it makes sense to write $\alpha(\beta_1, \beta_2)$ for two rectifiable 1-parameter subgroups; we let $\alpha(\beta_1, \beta_2)$ be the angle between the corresponding directions.

Lemma 5.5. *Suppose G_i has a bi-invariant inner metric of curvature ≥ 0 and positive cut radius and let $G = \prod_{i=1}^{\infty} G_i$. If β, β_j are unit 1-parameter subgroups such $\alpha(\beta, \beta_j) \rightarrow 0$ then β_j converges uniformly to β on any closed interval.*

Proof. Fix a closed interval $[0, a]$. Let $\beta = \sum_{i=1}^{\infty} k_i \gamma_i$ and $\beta_j = \sum_{i=1}^{\infty} k_{ji} \gamma_{ji}$, fix $\epsilon > 0$, and choose M such that for any $x, y \in G$, $d(x, y)^2 - \sum_{i=1}^M d(x_i, y_i)^2 < \epsilon/2$. By Lemma 3.3, if $\beta^M = \sum_{i=1}^M k_i \gamma_i$ and $\beta_j^M = \sum_{i=1}^M k_{ji} \gamma_{ji}$ then $\alpha(\beta^M, \beta_j^M) \rightarrow 0$. Then for large enough j and $t \in [0, a]$, $\epsilon/2 > d(\beta^M(t), \beta_j^M(t))^2 = \sum_{i=1}^M d(k_i \gamma_i(t), k_{ji} \gamma_{ji}(t))^2 > d(\beta(t), \beta_j(t))^2 - \epsilon/2$. That is, $d(\beta(t), \beta_j(t))^2 < \epsilon$. \square

We finally come to the proof of Theorem 1.1. Because the proof involves some technical details, we first provide a sketch of the argument. The proof is easily reduced to the compact case. For compact groups the strategy is to embed G as a “diagonal” subgroup in the product $P = G/K_1 \times G/K_2 \times \cdots$, where each G/K_i is a compact Lie group with appropriately chosen bi-invariant Riemannian metric. Here $K_1 \supset K_2 \supset \cdots$ is a nested sequence of compact normal subgroups whose intersection is the identity, we embed G as $g \mapsto (gK_1, gK_2, \dots)$, and use the induced inner metric. The main difficulty lies in showing that the induced inner metric is finite. In fact, G can be approximated by the Lie groups $G^n = (gK_1, \dots, gK_n, e, \dots)$, all of which have curvature ≥ 0 . If the inclusion mappings of G^n into P were (metric, not Riemannian) isometries, G^n would converge to G in the Gromov-Hausdorff sense, and we could appeal to known results ([18]). However, the groups G^n are totally geodesic in the factor $G/K_1 \times \cdots \times G/K_n$, but not necessarily convex; the inclusion is locally an isometry on balls of radius λ_n , where λ_n may tend to 0 as n gets large. In other words, the induced inner metric may always be strictly larger than the induced metric. Nonetheless we can use arguments similar to those in [18] once we know that the diameters of the G^n are bounded in the induced inner metric. Control of the metric is accomplished by noting that the arcwise connectedness allows us to assume that the fibers of the maps $G/K_i \rightarrow G/K_{i-1}$ are connected. We can then take these maps to be Riemannian submersions, having fibers of as small diameter as we choose. Thus for every minimal curve γ joining e and p in G/K_{i-1} there is lift $\tilde{\gamma}$ in G/K_i of γ joining e to any point in the fiber over p , and $L(\tilde{\gamma})$ is not much longer than $L(\gamma)$. We then scale the metrics to make the diameters of G/K_i square summable, and in this way we gain control of the metrics of the G^n .

It is useful to see how this procedure fails with a group that is not arcwise connected. The solenoid, Σ , is obtained as the inverse limit of the sequence $S^1 \leftarrow S^1 \leftarrow \dots$, where each connecting map is multiplication by two. The fiber in each case is two points. The only possible Riemannian submersions are local isometries, which forces the diameter of S^1 , and that of the fiber, to double in each step.

Proof of Theorem 1.1. We first assume that G is compact and show that G has a bi-invariant metric of non-negative curvature with the desired properties. Let H_i be a sequence of compact nested normal subgroups of G such that G/H_i is a Lie group and $\bigcap H_i = e$. Let K_i be the identity component of H_i ; K_i is a normal subgroup of G . Then the natural mapping $G/K_i \rightarrow G/H_i$ has totally disconnected kernel, so G/K_i has at most the same (finite) dimension as G/K_i (cf. [13]). Since G is arcwise connected, G/K_i is also arcwise connected, and is therefore a Lie group (cf. [23]). Note that the kernel of the natural map $\pi_{(i+1)i} : G/K_{i+1} \rightarrow G/K_i$ is connected, being the image of the connected group K_i , under the projection $\pi_{i+1} : G \rightarrow G/K_{i+1}$. Now the universal cover of G/K_i is of the form $S_1 \times \dots \times S_{k_i} \times R_1 \times \dots \times R_{n_i}$, where each S_j is a compact simple Lie group and R_j is a copy of the reals (cf. [12]). We choose a bi-invariant Riemannian metric g_1 on G/K_1 of diameter smaller than $1/2$, which lifts to a Riemannian metric \tilde{g}_1 on the universal cover of G/K_1 . We write the universal cover of G/K_2 as $M_1 \times M_2$ where

$$M_1 = (S_1 \times \dots \times S_{k_1} \times R_1 \times \dots \times R_{n_1}),$$

$$M_2 = (S_{k_1+1} \times \dots \times S_{k_2} \times R_{n_1+1} \times \dots \times R_{n_2}).$$

On M_1 have the metric \tilde{g}_1 . On M_2 we choose a bi-invariant metric \tilde{h}_2 , and provide G/K_2 with the metric g_2 induced by $\tilde{g}_2 = \tilde{g}_1 \times k\tilde{h}_2$, where k is a constant to be determined as follows. Since the fibers of the submersion π_{21} are the translates of the compact, connected, totally geodesic kernel of this quotient map, by choosing k small enough we can ensure that the fibers have diameter < 1 with respect to the induced inner (equivalently Riemannian) metric. Note that the projection π_{21} is a Riemannian submersion for any choice of k ; as pointed out in Section 2, the distance from the Riemannian metric on $(G/K_1, g_1)$ is identical to the quotient metric from Definition 2.1. Moreover, for each geodesic γ starting at $a \in (G/K_1, g_1)$ and $b \in \pi_{21}^{-1}(a)$ there is a unique isometric (“horizontal”) lift of γ to a geodesic Γ starting at b . We now consider the product $P_2 = (G/K_1, g_1) \times (G/K_2, \frac{1}{2}g_2)$, with the product metric, and the “diagonal” subspace $G_2 = \{(\pi_{21}(z), z)\} = \{(\pi_1(x), \pi_2(x)) : x \in G\} \subset P_2$ (which is a closed Lie subgroup of the product), with the induced inner metric (equivalently, the induced Riemannian metric). Suppose that $(\pi_1(x), \pi_2(x))$ and $(\pi_1(y), \pi_2(y))$ are in G_2 and let γ_1 be minimal from $\pi_1(x)$ to $\pi_1(y)$ in G/K_1 of length $L = d(\pi_1(x), \pi_1(y))$. The unique horizontal lift Γ_1 of γ_1 at $\pi_2(x)$ is minimal of length $L/2$ (since we have scaled the metric) and ends at a point z in the fiber above $\pi_1(y)$. By our choice of k , we can find a geodesic β_1 in the fiber above $\pi_1(y)$ joining z to $\pi_2(y)$, with $L(\beta) < 1/2$. The concatenated curve $\gamma_2 = \Gamma_1 * \beta_1$ joins $\pi_2(x)$ and $\pi_2(y)$ with length $\frac{L+1}{2}$, and $\pi_{21}(\gamma_2)$ coincides with γ_1 up to reparameterization. That is, $(\pi_{21}(\gamma_2), \gamma_2)$ lies in G_2 , joins $(\pi_1(x), \pi_2(x))$ and $(\pi_1(y), \pi_2(y))$, and has length $< L + \frac{L}{2} + \frac{1}{2} = \frac{3L+1}{2}$. (Here we use the fact that the length of a curve in a product is at most the sum of the lengths of its projections.) The diameter of G_2 with this metric is $< 5/4$.

We continue this procedure. Choose a Riemannian metric g_3 on G/K_3 so that the mapping π_{32} is a Riemannian submersion having fibers of diameter smaller than 1. Let $G_3 = \{(\pi_1(x), \pi_2(x), \pi_3(x)) : x \in G\} \subset P_3 = (G/K_1, g_1) \times (G/K_2, \frac{1}{2}g_2) \times (G/K_3, \frac{1}{4}g_3)$. If $a = (\pi_1(x), \pi_2(x), \pi_3(x)), b = (\pi_1(y), \pi_2(y), \pi_3(y)) \in G_3$ then we can lift the curve γ_2 constructed above to a piecewise geodesic σ in G/K_3 of length $< \frac{L+1}{4}$, and concatenate with a geodesic lying entirely in the fiber above $\pi_2(y)$, which has length $< \frac{1}{4}$. Thus we can join a and b in G_3 by a piecewise geodesic of length $< L + (\frac{L}{2} + \frac{1}{2}) + (\frac{L}{4} + \frac{1}{4} + \frac{1}{4})$. With similar definitions for G_i , we can join any two points $(\pi_1(x), \dots, \pi_i(x)), (\pi_1(y), \dots, \pi_i(y)) \in G_i$ by a curve γ_i in G_i whose projections γ_i^j onto G/K_j are piecewise geodesics of length $< (2^{-j+1})L + (j-1)2^{-j+1}$, $1 \leq j \leq i$. Then $L(\gamma_i) < 2L + 2$, and the diameter of G_i is smaller than 3. Note also that the diameter of $(G/K_n, g_n)$ is smaller than $\frac{2n-1}{2}$, so each factor $(G/K_n, 2^{-n+1}g_n)$ has diameter smaller than $(2n-1)2^{-n}$.

Let $P = (G/K_1, g_1) \times (G/K_2, \frac{1}{2}g_2) \times \dots$ with the product metric d . We define homomorphisms $\phi_n, \phi : G \rightarrow P$ by $\phi_n(g) = (\pi_1(g), \dots, \pi_n(g), e, e, \dots)$ and $\phi(g) = (\pi_1(g), \dots)$. Note that ϕ is a monomorphism, so we can algebraically identify G with its image in P . We will show that $\phi(G)$ has the desired metric. First, $\phi_n(G)$ lies in the convex subgroup $G/K_1 \times \dots \times G/K_n$, and (with the induced inner metric) is isometric to G_n . That is, $\phi_n(G)$ has curvature ≥ 0 and diameter < 3 . Furthermore, $d(\phi_n(g), \phi(g)) < \text{diam}\{(G/K_{n+1}, 2^{-n+1}g_{n+1}) \times \dots\} \rightarrow 0$, so the Hausdorff distance from $\phi_n(G)$ to $\phi(G)$ tends to 0. We will show that the induced (bi-invariant) inner metric on $\phi(G)$ has curvature ≥ 0 . Let d_n denote the induced inner metric in $\phi_n(G)$ and d' denote the induced inner metric in $\phi(G)$. Note that the lengths of curves in P is the same, whether measured in d , d_n , or d' . For any $x = \phi(a), y = \phi(b) \in \phi(G)$, choose $x_n, y_n \in \phi_n(G)$ so that $d(x_n, x) \rightarrow 0$ and $d(y_n, y) \rightarrow 0$. Let γ_n be minimal (in $(\phi_n(G), d_n)$) from x_n to y_n defined on $[0, 1]$. Since the diameter of $\phi_n(G)$ is < 3 , so is the length of γ_n . We can now apply Ascoli's Theorem to find a uniform limit γ in $\phi(G)$ of a subsequence $\{\gamma_{n_j}\}$ of $\{\gamma_n\}$. Now let Π be a partition of γ . Then we can approximate Π arbitrarily closely by partitions of γ_{n_j} (j large); i.e., the sum the distances given by Π is smaller than $3 - \epsilon$, $\epsilon > 0$ arbitrarily small. In other words, the length $L(\gamma) \leq \liminf L(\gamma_{n_j}) < 3$; we see that the induced inner metric on $\phi(G)$ is finite, and has diameter < 3 . We also have that $d'(x, y) \leq \liminf d_n(x_n, y_n)$ whenever $x_n \rightarrow x$ and $y_n \rightarrow y$. On the other hand, if $\gamma = (\gamma_1(t), \gamma_2(t), \dots) : [0, 1] \rightarrow \phi(G)$ is minimal, then the curve $\gamma^n = (\gamma_1(t), \dots, \gamma_n(t), e, e, \dots)$ is a shorter curve in $\phi_n(G)$, and $\gamma^n(0) \rightarrow \gamma(0)$, and $\gamma^n(1) \rightarrow \gamma(1)$. That is, if $x^n = (\pi_1(a), \dots, \pi_n(a), e, e, \dots)$ and $y^n = (\pi_1(b), \dots, \pi_n(b), e, e, \dots)$, then

$$(5.1) \quad d'(x, y) = \lim d_n(x^n, y^n).$$

It is now an immediate consequence of the definition that, since each $(\phi_n(G), d_n)$ has curvature ≥ 0 , so does $(\phi(G), d')$.

We next need to verify that the topology on $\phi(G)$ from the induced inner metric d' , and the induced (subspace) topology on $\phi(G)$ coincide. We use the same notation as in the preceding paragraph. Since $\phi_n(G)$ is totally geodesic, there exists a constant $\lambda_n > 0$ such that if $w, z \in \phi_n(G)$ and $d(w, z) \leq \lambda_n$, then $d_n(w, z) = d(w, z)$. Given $\epsilon > 0$, note that if n is large enough, then the sum of the diameters of the factors $(G/K_i, 2^{-i}g_i)$, with $i \geq n$ is $< \epsilon/3$; for any $p = (p_1, \dots), q = (q_1, \dots) \in P$, $0 \leq d(p, q) - d_n(p^n, q^n) < \epsilon/3$. We can choose n also large enough that $2^{-n+1} < \epsilon/3$.

Suppose, as above, that $x = \phi(a), y = \phi(b)$. If $d(x, y) < \epsilon/3$ is small enough, then $\lambda_n \geq d(x^n, y^n)$, so $d_n(x^n, y^n) = d(x^n, y^n)$. We have that $0 \leq d(x, y) - d_n(x^n, y^n) \leq \epsilon/3$. Also, $0 \leq d_{n+1}(x^{n+1}, y^{n+1}) - d_n(x^n, y^n) \leq d_n(x^n, y^n)/2 + 2^{-n}$. In fact, we can proceed as before: let γ be minimal in $\phi_n(G)$ from x^n to y^n (in this case, by the way we chose x and y , γ is also minimal in P). We then lift γ horizontally and concatenate with a geodesic in the fiber above y , which has length $< 2^{-n}$. Continuing in this way, we see that, since $d'(x, y) = \lim d_i(x^i, y^i)$, $d'(x, y) - d_n(x^n, y^n) \leq 2\epsilon/3$, and $0 \leq d'(x, y) - d(x, y) < \epsilon$.

We can now identify G with the topological group $\phi(G)$, which is given the the induced inner metric defined above, of curvature ≥ 0 . By Propositions 4.2, 4.4, 4.6, and Theorem 1.3 we need only verify that every direction $\gamma \in \overline{S}_e G$ is tangent to a unique 1-parameter subgroup in G . From the last paragraph we know that for $x, y \in G$, $d(x, y) \leq d'(x, y)$, and that the metrics are equivalent in G . We denote by α the angle metric in $\overline{S}_e P$ and by α' the angle metric in $\overline{S}_e G$. Choose geodesic directions $\gamma_i \in S_e G$ converging to γ . Then each γ_i corresponds to a geodesic in G (also denoted γ_i), which is a rectifiable 1-parameter subgroup. Note that since γ_i is rectifiable in (G, d') , it is rectifiable in (P, d) (of the same length). We write $\gamma_i = \sum k_{ik} \gamma_{ik}$ as in Proposition 5.2. We claim that for any i, j , if $\gamma_i^n = \sum_{k=1}^n k_{ik} \gamma_{ik}$, $\gamma_j^n = \sum_{k=1}^n k_{jk} \gamma_{jk}$, then $\alpha'(\gamma_i, \gamma_j) = \lim_{n \rightarrow \infty} \alpha(\gamma_i^n, \gamma_j^n) = \alpha(\gamma_i, \gamma_j)$. (Here the last equality is Proposition 3.10.) The first equality can be proved using the geodesic completeness in essentially the same way as the proof of Proposition 9 in [18]: From the equation (5.1) and the way angles are defined we know that $\alpha'(\gamma_i, \gamma_j) \leq \lim_{n \rightarrow \infty} \alpha(\gamma_i^n, \gamma_j^n)$. Applying the same argument to complementary directions, we obtain the opposite inequality.

By choosing a diagonal subsequence of the geodesics $\{\gamma_i^n\}$ we can construct a sequence $\{\alpha_i = \sum h_{ik} \alpha_{ik}\}$, Cauchy in $S_e P$, such that $\alpha(\alpha_i, \alpha_j) - \alpha'(\gamma_i, \gamma_j) < 2^{-i}$, $i \leq j$ and $\alpha(\alpha_i, \gamma_i) < 2^{-i}$. By Proposition 3.11 we have that $\lim_{k \rightarrow \infty} h_{ik} = \lim_{k \rightarrow \infty} k_{ik} := m_i$ and $\lim_{i \rightarrow \infty} \alpha_i = \lim_{i \rightarrow \infty} \gamma_i := \beta_i$ (here we may have to change our representations in case $\beta_i = 0$). By Lemma 5.5, the unit 1-parameter subgroup $\beta = \sum m_i \beta_i$ lies in the (closed) subgroup G . We need to show that γ is tangent to β . Let ξ_i be minimal in G from e to $\beta(1/i)$. Theorem 1.3 implies that the sequence $\{\xi_i\}$ is Cauchy in $S_e G$. Since ξ_i is also a rectifiable 1-parameter subgroup, an immediate consequence of Proposition 3.11 is that $\{\xi_i\}$ and $\{\gamma_i\}$ have the same limit in $\overline{S}_e G$, namely γ . Uniqueness follows from the fact that our construction is independent of the choice of the sequence $\{\gamma_i\}$. This completes the proof of Theorem 1.1 in the compact case. We note here that our argument implies that $\overline{S}_e G$ can be considered as a Hilbert subspace of $\overline{S}_e P$ in a natural way.

If G is not compact, the Gleason-Iwasawa-Yamabe theory (as distilled in [9]) implies that there exists a neighborhood U of e in G such that $U = H \times V$, where H is a compact group and V is a local Lie group. Put an invariant Riemannian metric on V , and restrict to a compact, convex neighborhood W of V . If we now put a metric on H of the type constructed above, then we have metrized $H \times W$ as a compact neighborhood of the identity of G . We metrize all of G in the following way. For any curve c in G , we measure the length of c as follows: Divide c into segments short enough that if the endpoint of a segment is left translated to e , then the image of the entire segment lies in $H \times W$. We can measure the length of the segment there, and define the length of c to be the sum of the lengths of the segments. It is easy to see that the length of c is well defined. Letting $d(x, y)$ be the

infimum of lengths of curves measured in this way, it follows that d is an invariant intrinsic metric locally isometric to the metric on $H \times G$. Since the properties we need to verify are all local, the proof is complete. \square

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