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TRANSFER OPERATORS ACTING ON ZYGMUND FUNCTIONS

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Abstract. We obtain a formula for the essential spectral radius ρ_{ess} of transfer-type operators associated with families of $C^{1+\delta}$ diffeomorphisms of the line and Zygmund, or Hölder, weights acting on Banach spaces of Zygmund (respectively Hölder) functions. In the uniformly contracting case the essential spectral radius is strictly smaller than the spectral radius when the weights are positive.

1. Introduction

During the last decade, a generalised theory of Fredholm determinants has been obtained using tools from statistical mechanics, often in a dynamical setting. Typically, one considers

- a transformation f, with finitely or countably many inverse branches, of a metric space M to itself,
- a weight $g : M \to \mathbb{C}$;

and one defines the associated *transfer operator*

$$\mathcal{L}\varphi(z) = \sum_{f(w)=z} g(w)\varphi(w)$$

acting on a Banach space of functions $\varphi : M \to \mathbb{C}$. Transfer operators are useful in the study of "interesting" invariant measures for f. They sometimes arise in a surprising fashion: It has been proved that the period-doubling renormalization spectrum is exactly the spectrum of a suitably defined transfer operator (see e.g. Jiang-Morita-Sullivan [6]). Transfer operators are usually bounded but non-compact; however, it has been possible in many cases to compute an upper bound, or even an exact value for the *essential spectral radius* ρ_{ess} of $\mathcal{L}$. This is the first step towards a generalised Fredholm theory. The second step is to introduce a *generalised Fredholm determinant,* which is often closely connected to weighted *dynamical zeta functions* (see Section 5). One then shows under suitable assumptions that the determinant is an analytic function in a subset of the complex plane, or that the zeta function is meromorphic in some domain, where its zeroes (respectively poles) describe exactly the spectrum of $\mathcal{L}$ outside of a disc of radius $r \geq \rho_{\text{ess}}$.

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This program has been successfully carried out in an Axiom A framework with various degrees of smoothness (Hölder, analytic, differentiable: see Parry-Pollicott [12]; and Rugh [18] for more recent developments), for families of contractions on finite dimensional manifolds and $C^{k+\alpha}$ smoothness, $0 \le k \le \omega$, $0 \le \alpha \le 1$ (Ruelle [15, 16], Fried [4]). In dimension one, one may consider test functions of bounded variation (see Ruelle [17] and references therein, Baladi-Ruelle [1]), and under Markov-type assumptions also C^k Banach spaces (Collet-Isola [2]).

One Banach space which had not yet been investigated in this context is the space $Z(I)$ of Zygmund functions on an interval or circle I (see Section 2 for definitions). The space $Z(I)$, which has been much used in dynamical systems in recent years, notably in Sullivan's analysis of renormalisation (Sullivan [19]), is interesting not only because $\Lambda^1 \subsetneq Z \subsetneq \Lambda^\alpha$ for all $0 < \alpha < 1$, where Λ^α denotes the space of α-Hölder functions ($\Lambda^1 = \mathrm{Lip(I)}$) but also because it arises in the study of quasiconformal mappings and Teichmüller theory, as we explain now.

Let I denote the circle $\mathbb{R}/\mathbb{Z}$, and choose three points $p_1 < p_2 < p_3$ in I. A homeomorphism h of I fixing p_i for $i = 1$, 2, and 3 is called quasisymmetric if

$$||h||_{qs} = \sup_{x\in I; x+t, x-t \in I} \frac{|h(x+t) - h(x)|}{|h(x) - h(x-t)|} < \infty \,.$$

Let T be the set of all orientation preserving quasisymmetric homeomorphisms of I which fix p_i for $i = 1$, 2, 3, endowed with the distance $d(h_1, h_2) = \log ||h_1 \circ h_2^{-1}||_{qs}$. The set T with distance d is a model for universal Teichmüller space (see Lehto [9]). For a fixed quasisymmetric homeomorphism h_0 in T, the right composition $R_{h_0}(h) = h \circ h_0$ acting on T is a continuous map, and sends a neighborhood of the identity to a neighborhood of h_0. This makes T into a homogeneous space. It is also known that T is a complex manifold (see Gardiner [5], Lehto [9]). Thus T has a tangent space at the identity, which is also the tangent space at any point h_0. This tangent space is a Banach space of continuous vector fields $\phi(x)d/dx$ defined on I, and, when factored by the two-dimensional subspace of affine functions, can be identified with $Z(I)$, the Zygmund function space (Reimann [14]). Therefore a transfer operator $\mathcal{L}$ acting on $Z(I)$ can be viewed as acting on the tangent space of universal Teichmüller space. It is hoped that the knowledge of the spectral properties of such operators may be applied to the study of Teichmüller theory. An especially interesting case is when $\mathcal{L}$ is the tangent map $\mathcal{DR}$ to some nonlinear operator $\mathcal{R}$ acting on universal Teichmüller space.

In this paper, we carry out the first step towards a generalised Fredholm determinant theory on Zygmund spaces: We obtain an exact formula (Theorem 1) for the essential spectral radius of transfer operators $\mathcal{L}$ acting on $Z(I)$, or Λ^α for $0 < \alpha \le 1$ (the Λ^α case was treated by Lanford [8]), and under additional assumptions a strict inequality between the essential spectral and the spectral radii. Section 2 contains definitions and results on the essential spectral radius. To obtain the essential spectral radius, we prove an upper bound in Section 3, and a lower bound in Section 4 (our method to get the lower bound differs from the one used by Pollicott [13] and Collet-Isola [2], but is similar to the one applied by Keller [7]). Section 5 contains results on the spectral radius and two conjectures on the second part of the program mentioned above.

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2. Definitions and statement of results

Throughout, I denotes a compact interval (minor modifications yield results for $I = \mathbb{R}/\mathbb{Z}$), and $C > 0$ a generic constant (in particular we admit identities such as $C = 2C$).

Zygmund functions. The Zygmund space Z on I (Zygmund [20]) is the complex vector space of continuous (or, equivalently, locally bounded) functions $\varphi : I \to \mathbb{C}$ such that

$$Z(\varphi) = \sup_{\substack{x \in I \\ t>0: x \pm t \in I}} |Z(\varphi, x, t)| < \infty,$$

where $Z(\varphi, x, t) = (\varphi(x+t) + \varphi(x-t) - 2\varphi(x))/t$. The vector space Z becomes a Banach space when endowed with the norm $\|\varphi\|_Z = \max(\sup_I |\varphi|, Z(\varphi))$.

For $0 < \alpha \le 1$, let Λ^α denote the space of α-Hölder functions, i.e. functions $\varphi : I \to \mathbb{C}$ satisfying

$$|\varphi|_\alpha = \sup_{x \neq y \in I} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\alpha} < \infty\,.$$

In particular, Λ^1 is the space of Lipschitz functions. Each Λ^α is a Banach space for the norm $\|\varphi\|_\alpha = \max(\sup_I |\varphi|, |\varphi|_\alpha)$; $Z \subsetneq \Lambda^\alpha$ for $0 < \alpha < 1$; and $\Lambda^1 \subsetneq Z$. (For a proof of the second assertion, see e.g. de Melo-van Strien [10, p. 293]; for an example showing that $\Lambda^1 \neq Z$, see the remark following the proof of Lemma 1.) We shall also consider the Banach space $\mathcal{B}$ of bounded functions on I endowed with the supremum norm.

Note that the norms $\|\varphi\|_{Z,\alpha} = \max(\sup_I |\varphi|, Z(\varphi), |\varphi|_\alpha)$ for $0 < \alpha < 1$ on Z are all equivalent with the norm $\|\cdot\|_Z$. (Indeed, for each $0 \le \alpha < 1$ the space Z is a Banach space for the norm $\|\cdot\|_{Z,\alpha}$; the open mapping theorem may then be applied to the identity maps $(Z, \|\cdot\|_{Z,\alpha}) \to (Z, \|\cdot\|_Z)$.) In other words, for each $0 \le \alpha < 1$, there is a constant $K = K(\alpha)$ such that

$$|\varphi|_\alpha \le K(\alpha)\,(\sup|\varphi| + Z(\varphi))\,, \quad \forall \varphi \in Z\,.$$

The following key lemma may be proved by direct computation:

Zygmund derivation of a product. *For all φ, ψ in $Z(I)$, $x \in I$, and $t > 0$,*

$$\begin{aligned} Z(\varphi\psi, x, t) &= \varphi(x) Z(\psi, x, t) + \psi(x) Z(\varphi, x, t) \\ &\quad + t \cdot \Delta_+(\varphi, x, t)\Delta_+(\psi, x, t) + t \cdot \Delta_-(\varphi, x, t)\Delta_-(\psi, x, t)\,, \end{aligned} \tag{2.1}$$

where $\Delta_+(\upsilon, x, t) = (\upsilon(x+t) - \upsilon(x))/t$ and $\Delta_-(\upsilon, x, t) = (\upsilon(x) - \upsilon(x-t))/t$.

The following result is also useful (the constant $1/2$ is not optimal):

Skewed Zygmund bound. *For all $\varphi \in Z$, $x, y \in I$, $0 < t < 1$,*

$$\big|\big((1-t)\varphi(x) + t\varphi(y)\big) - \varphi((1-t)x + ty)\big| \le \frac{1}{2} Z(\varphi)|x-y|.$$

Proof of the skewed Zygmund bound. Fix x and y. There is nothing to prove if $Z(\varphi) = 0$. Otherwise, by subtracting off an affine function, making an affine change of variables, and multiplying by a constant, we can reduce to the case $x = 0$, $y = 1$, $\varphi(0) = \varphi(1) = 0$, $Z(\varphi) = 4$. We then have

$$\left|\frac{1}{2}\varphi(u) + \frac{1}{2}\varphi(v) - \varphi(\frac{1}{2}u + \frac{1}{2}v)\right| \le |u - v|,$$

and we want to prove that $|\varphi(t)| \le 2$ for $0 \le t \le 1$. By continuity, it is enough to prove the desired bound for t a dyadic rational. We will construct recursively an increasing sequence of bounds γ_n such that $|\varphi(t)| \le \gamma_n$ for t of the form $\frac{j}{2^n}$.

We start with $\gamma_1 = 1$. For the induction step, it is evidently enough to consider

$$t = \frac{2j+1}{2^{n+1}} = \frac{1}{2}\frac{j}{2^n} + \frac{1}{2}\frac{j+1}{2^n}.$$

By the induction hypothesis, $\varphi(\frac{j}{2^n}) \le \gamma_n$ and $\varphi(\frac{j+1}{2^n}) \le \gamma_n$; by the Zygmund condition

$$\left|\varphi(t) - \big(\frac{1}{2}\varphi(\frac{j}{2^n}) + \frac{1}{2}\varphi(\frac{j+1}{2^n})\big)\right| \le \frac{1}{2^n}.$$

Hence, the bound holds inductively if we set $\gamma_{n+1} = \gamma_n + \frac{1}{2^n}$, and, since $\lim_{n\to\infty} \gamma_n = 2$, the assertion follows. □

The transfer operator. The basic data entering into the definition of the transfer operator are a dynamical system and a weight. Let $\mathcal{I}$ be a finite or countable set and $0 \le \delta < 1$. The *dynamical system* here is a family of $C^{1+\delta}$ diffeomorphisms, $f_i : I \to J_i$, for $i \in \mathcal{I}$, where the intervals $J_i \subset I$ have disjoint interiors. We assume further that $\sup_i \|f_i'\|_\delta < \infty$, in particular $\lambda := 1/\sup_{i,x} |f_i'(x)| > 0$.

The *weight* is a family of functions $g_i : I \to \mathbb{C}$, $i \in \mathcal{I}$. Such a family g_i is called *summably bounded* if $\sup^{\Sigma}|g| = \sum_i \sup|g_i| < \infty$.

A summably bounded family is called *summably* Λ^α if $|g|_\alpha^\Sigma = \sum_i |g_i|_\alpha < \infty$ for some $0 < \alpha \le 1$; it is called *summably Zygmund* if $Z(g)^\Sigma = \sum_i Z(g_i) < \infty$.

Define formally the transfer operator $\mathcal{L}$ associated with the families f_i and g_i, and acting on functions $\varphi : I \to \mathbb{C}$, by

$$\mathcal{L}\varphi(x) = \sum_{i\in\mathcal{I}} g_i(x)\varphi(f_i(x)) \,. \tag{2.2}$$

A typical example is when the f_i are the finitely many *inverse branches* of a piecewise expanding, piecewise surjective interval map f, or the finitely many inverse branches of a one-dimensional hyperbolic repeller, and $g_i = |f_i'|$.

The following lemma is a "warm-up":

Lemma 1. *The linear operator $\mathcal{L}$ is bounded when acting on $\mathcal{B}$* (*respectively Λ^α, for any $0 < \alpha \le 1$*) *if the family g_i is summably bounded* (*respectively summably Λ^α*) *and $\delta \ge 0$; the operator $\mathcal{L}$ is bounded when acting on Z if the family is summably Zygmund and $\delta > 0$.*

Proof of Lemma 1. It follows immediately from the definitions that

$$\sup_I |\mathcal{L}\varphi| \le \sup_I |\varphi| \sum_{i\in\mathcal{I}} \sup_I |g_i| \le \sup{}^{\Sigma}|g| \sup_I |\varphi| \,.$$

To bound the α-Hölder seminorm, we use $|x-y| \geq \lambda |f_i x - f_i y|$ for all i and get

$$
\begin{aligned}
|\mathcal{L}\varphi|_\alpha &= \sup_{x,y\in I} \frac{|\sum_i g_i(x)\varphi(f_i x) - g_i(y)\varphi(f_i y)|}{|x-y|^\alpha} \\
&\leq \sup_{x,y\in I} \frac{\sum_i |g_i(x)(\varphi(f_i x) - \varphi(f_i y))| + |\varphi(f_i y)(g_i(x) - g_i(y))|}{|x-y|^\alpha} \\
&\leq \sup{}^\Sigma |g| \frac{|\varphi|_\alpha}{\lambda^\alpha} + |g|_\alpha^\Sigma \sup_I |\varphi| \, .
\end{aligned} \tag{2.3}
$$

For the Zygmund bound, we first note that for each $x \in I$ and $t > 0$ with $x \pm t \in I$, the Zygmund derivation formula yields for any $0 < \alpha < 1$:

$$
\begin{aligned}
\left|Z(\mathcal{L}\varphi, x, t)\right| &= \left|\sum_{i\in\mathcal{I}} Z(g_i \cdot (\varphi \circ f_i), x, t)\right| \\
&\leq \sup{}^\Sigma |g| \sup_i \left|Z(\varphi \circ f_i, x, t)\right| + Z^\Sigma(g) \sup|\varphi| + \frac{2}{\lambda^\alpha}|g|_{1-\alpha}^\Sigma |\varphi|_\alpha \, .
\end{aligned} \tag{2.4}
$$

Defining $0 < |t_i| \leq t/\lambda$ for each $i \in \mathcal{I}$ by $f_i(x+t) = f_i(x) + t_i$, we observe that, since $\delta > 0$, there is a constant $C > 0$ such that for all i, and all $x \in I$, $t > 0$ with $x \pm t \in I$,

$$
\begin{aligned}
|f_i(x-t) - (f_i(x) - t_i)| &= |(f_i(x+t) - f_i(x)) - (f_i(x) - f_i(x-t))| \\
&= |f_i'(x+u)t - f_i'(x-v)t| \leq |f_i'|_\delta 2^\delta t^{1+\delta} \leq C t^{1+\delta} \, ,
\end{aligned} \tag{2.5}
$$

where we used $0 \leq u + v \leq 2t$ and $\sup_i |f_i'|_\delta < \infty$. For each $i \in \mathcal{I}$, we decompose

$$
Z(\varphi \circ f_i, x, t) = \frac{t_i}{t} Z(\varphi, f_i(x), t_i) - \frac{\varphi(f_i(x) - t_i) - \varphi(f_i(x-t))}{t} = \mathrm{I}_i + \mathrm{II}_i \, . \tag{2.6}
$$

Clearly,

$$
\sup_{i\in\mathcal{I}} |\mathrm{I}_i| \leq \frac{1}{\lambda} Z(\varphi) \, . \tag{2.7}
$$

Now, using (2.5), we get for all i with $\mathrm{II}_i \neq 0$:

$$
|\mathrm{II}_i| \leq C \frac{|\varphi(f_i(x) - t_i) - \varphi(f_i(x-t))|}{|f_i(x) - t_i - f_i(x-t)|^{1/(1+\delta)}} \leq C\, |\varphi|_{1/(1+\delta)} \, . \tag{2.8}
$$

To finish, put (2.4) and (2.6)–(2.8) together, observing that for any $(1+\delta)^{-1} \leq \alpha < 1$ there is a constant $K(\alpha)$ with $|\varphi|_{1/(1+\delta)} \leq |\varphi|_\alpha \leq K(\alpha)\|\varphi\|_Z$, and $|g|_\alpha^\Sigma \leq K(\alpha) Z^\Sigma(g)$. $\square$

Remark. We would like to point out that the transfer operator $\mathcal{L}$ acting on Z may be unbounded if $\delta = 0$ (even for constant weights). Indeed, it is well known that there exist Zygmund functions φ and C^1 diffeomorphisms f such that $\varphi \circ f$ is not Zygmund. For example, let $I = [-\epsilon, \epsilon]$ be a small neighbourhood of 0, let $\varphi(x) = x \log|x|$ on I, and let $f : I \to f(I) \subset I$ be a C^1 diffeomorphism with $f(0) = 0$, $f'(x) = 1$ for $x \leq 0$ and $f'(x) = 1 - 1/\sqrt{|\log(x)|}$ for $x > 0$ (in particular, there is a constant $C > 0$ with $C < f'(x) \le 1$ on I). To check that $\varphi \circ f$ is not Zygmund, we first show, by straightforward computation, that

$$Z(\varphi \circ f, 0, t) = \left(\frac{f(t)}{t} - 1\right) \log(t) + \frac{f(t)}{t} \log \frac{f(t)}{t}\,, \quad \text{for } t > 0\,.$$

The second term on the right goes to zero as $t \to 0^+$; the first, on the other hand, is unbounded since

$$\left(\frac{f(t)}{t} - 1\right)\sqrt{|\log t|} = -\frac{\sqrt{|\log t|}}{t}\int_0^t \frac{ds}{\sqrt{|\log s|}} \to -1 \text{ when } t \to 0^+.$$

The essential spectral radius of the transfer operator. For each $n \ge 1$ and $i_\ell \in \mathcal{I}$, $1 \le \ell \le n$, introduce the maps $f^{(n)}_{\vec\imath} = f_{i_n} \circ \cdots \circ f_{i_1}$, and the weights $g^{(n)}_{\vec\imath}(x) = g_{i_n}(f_{i_{n-1}} \cdots f_{i_1}(x)) \cdots g_{i_2}(f_{i_1}(x)) \cdot g_{i_1}(x)$. Note that for all $n \ge 1$

$$\mathcal{L}^n \varphi(x) = \sum_{\vec\imath \in \mathcal{I}^n} g^{(n)}_{\vec\imath}(x) \varphi(f^{(n)}_{\vec\imath} x)\,.$$

Our main result is:

Theorem 1.

1. *Assume that the family g_i is summably Zygmund and that $\delta > 0$. The essential spectral radius $\rho_{\text{ess}}(\mathcal{L})$ of the operator $\mathcal{L}$ acting on Z is equal to*

$$\rho_{\text{ess}}(\mathcal{L}) = \lim_{n\to\infty} \left[\sup_{x\in I} \sum_{\vec\imath \in \mathcal{I}^n} |g^{(n)}_{\vec\imath}(x)|\, |{f^{(n)}_{\vec\imath}}'(x)|\right]^{1/n}$$

   (*in particular, the limit on the right exists*).

2. *If the family g_i is summably Λ^α for some $0 < \alpha \le 1$, the essential spectral radius $\rho_{\text{ess}}(\mathcal{L})$ of the operator $\mathcal{L}$ acting on Λ^α is equal to*

$$\rho_{\text{ess}}(\mathcal{L}) = \lim_{n\to\infty} \left[\sup_{x\in I} \sum_{\vec\imath \in \mathcal{I}^n} |g^{(n)}_{\vec\imath}(x)|\, |{f^{(n)}_{\vec\imath}}'(x)|^\alpha\right]^{1/n}\,.$$

The proof of Theorem 1 is based on the following result of Nussbaum [11], which holds for any bounded linear operator $\mathcal{L}$ on a Banach space:

$$\rho_{\text{ess}}(\mathcal{L}) = \lim_{n\to\infty} \left(\inf\{\|\mathcal{L}^n - \mathcal{K}\| \mid \mathcal{K} \text{ compact }\}\right)^{1/n}\,.$$

Indeed, using the above equality and the expression of $\mathcal{L}^n$ as a sum over $\mathcal{I}^n$, the theorem will be an immediate consequence of the two following lemmas:

Lemma 2 (Upper bound). *There is a universal constant $C > 0$ so that, for any family f_i with $\delta > 0$ and summably Zygmund g_i,*

$$\inf\{\|\mathcal{L} - \mathcal{K}\|_Z \mid \mathcal{K} : Z \to Z \text{ compact }\} \le C \cdot \sup_{x\in I} \sum_{i\in\mathcal{I}} |g_i(x)|\, |f_i'(x)|\,;$$

and for any family f_i with $\delta \ge 0$ and summably Λ^α weights g_i

$$\inf\{\|\mathcal{L} - \mathcal{K}\|_\alpha \mid \mathcal{K} : \Lambda^\alpha \to \Lambda^\alpha \text{ compact }\} \le C \cdot \sup_{x\in I} \sum_{i\in\mathcal{I}} |g_i(x)|\, |f_i'(x)|^\alpha\,.$$

Lemma 3 (Lower bound). *For any family f_i with $\delta > 0$ and summably Zygmund g_i,*

$$\inf\{\|\mathcal{L} - \mathcal{K}\|_Z \mid \mathcal{K} : Z \to Z \quad \textit{compact}\ \} \geq \sup_{x\in I} \sum_{i\in\mathcal{I}} |g_i(x)|\,|f_i'(x)|\,.$$

For any family f_i with $\delta \geq 0$ and summably Λ^α weights g_i,

$$\inf\{\|\mathcal{L} - \mathcal{K}\|_\alpha \mid \mathcal{K} : \Lambda^\alpha \to \Lambda^\alpha \ \textit{compact}\ \} \geq \sup_{x\in I} \sum_{i\in\mathcal{I}} |g_i(x)|\,|f_i'(x)|^\alpha\,.$$

The essential spectral radius of restrictions of linear operators. If the family g_i is summably Zygmund and $\delta > 0$, it follows from Theorem 1 that the essential spectral radius of $\mathcal{L}$ acting on $Z(I)$ is the limit of its essential spectral radii on Λ^α as $\alpha \to 1$. Moreover, if the family g_i is summably Lipschitz, $\mathcal{L}$ has the same essential spectral radius when acting on Λ^1 or Z. Although this is hardly surprising, we believe that part 1 of Theorem 1 cannot be easily deduced from part 2, i.e., that the Zygmund result cannot be deduced immediately from the Λ^α, $0 < \alpha \leq 1$, results: The essential spectrum of a bounded operator contains its residual spectrum, which can be very badly behaved under restriction (see e.g. Dowson [3]). In this respect, we recall the very well known example of the shift operator acting on the Hilbert space $\ell^2 = \{(x_k)_{k\in\mathbb{Z}} \mid x_k \in \mathbb{C}\,, \sum_k |x_k|^2 < \infty\}$ by $(T(\vec{x}))_j = x_{j-1}$, whose spectrum is the unit circle, but which has the property that the spectrum of its restriction to the closed invariant space of sequences $\{(x_k)_{k\in\mathbb{Z}} \in \ell^2 \mid x_k = 0\,, k \leq 0\}$ fills the whole unit *disc.*

It can happen that the essential spectral radius *decreases* when one lets $\mathcal{L}$ act on the bigger spaces Λ^α for $\alpha < 1$ instead of Z. A simple example can be constructed as follows: We take $I = [0, 1]$ and the index set $\mathcal{I}$ to have one member 1. We then take for f_1 an analytic diffeomorphism $I \to I$ satisfying $f_1'' < 0$, and having exactly two fixed points 0 and 1, with $f_1'(0) > 1$ and $f_1'(1) < 1$. If g_1 is analytic and satisfies $g_1(0) = 1$ and $0 \leq g_1(x) \leq 1$ for all $x \in I$, then Theorem 1 yields that the essential spectral radius of $\mathcal{L}$ acting on Z or Λ^1 is $f_1'(0) > 1$, but shrinks to $f_1'(0)^\alpha$ when $\mathcal{L}$ acts on Λ^α for $0 < \alpha < 1$. If $\sup |f_i'| \leq 1$ for all i, this shrinking phenomenon is of course not possible.

3. The upper bound

To prove the upper bound we consider an explicit sequence of compact projections. Assuming that $I = [0, 1]$ to fix ideas, define for integers $n \geq 1$

$$\tau_j^{(n)} = \frac{j}{n}, \qquad j = 0, \ldots, n, \tag{3.1}$$

and let $P^{(n)}$ be the compact operator of piecewise affine interpolation at the $\tau_j^{(n)}$. (I.e., $P^{(n)}\varphi$ is the unique function which is affine on each interval $[\tau_{j-1}^{(n)}, \tau_j^{(n)}]$ and which agrees with φ at the points $\tau_j^{(n)}$.) We write $Q^{(n)} = 1 - P^{(n)}$, where 1 denotes the identity operator. For simplicity, we often drop the superscript (n). We will use the compact operators $\mathcal{K} = \mathcal{K}^{(n)} = \mathcal{L} - Q^{(n)}\mathcal{L}Q^{(n)}$.

For each fixed $n \geq 1$, it will be convenient to use the auxiliary seminorms

$$
\begin{aligned}
|\varphi|_\alpha^{(n)} &= \sup_{0\le j\le n-1} \sup_{\tau_j\le x<y\le\tau_{j+1}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\alpha}\,, \text{ for } \varphi\in\Lambda^\alpha(I)\,, 0<\alpha\le 1\,,\\
Z^{(n)}(\varphi) &= \sup_{0\le j\le n-2} \sup_{\substack{x\in I\\ t>0: x\pm t\in[\tau_j,\tau_{j+2}]}} |Z(\varphi,x,t)|\,, \text{ for } \varphi\in Z(I)\,.
\end{aligned}
\tag{3.2}
$$

Obviously, $|\varphi|_\alpha^{(n)} \le |\varphi|_\alpha$ and $Z^{(n)}(\varphi) \le Z(\varphi)$. We summarize properties of the operators $Q^{(n)}$ and the seminorms $|\cdot|_\alpha^{(n)}$ and $Z^{(n)}(\cdot)$:

Sublemma 4. *For any $n \geq 1$ and $0 < \alpha \le 1$:*
1. $\sup|Q^{(n)}\varphi| \le 2\sup|\varphi|$ *and* $\sup|Q^{(n)}\varphi| \le (2n)^{-\alpha}|Q^{(n)}\varphi|_\alpha^{(n)}$*, for each* $\varphi\in\Lambda^\alpha$.
2. $|Q^{(n)}\varphi|_\alpha^{(n)} \le 2|\varphi|_\alpha^{(n)}$ *and* $|Q^{(n)}\varphi|_\alpha \le 2|Q^{(n)}\varphi|_\alpha^{(n)}$*, for each* $\varphi\in\Lambda^\alpha$.
3. $Z(Q^{(n)}\varphi) \le 4Z^{(n)}(\varphi)$*, for each* $\varphi\in Z$.

Proof of Sublemma 4. Clearly, $\sup|P\varphi| \le \sup|\varphi|$, which yields the first bound by the definition of Q. The other claim is immediate too, since $Q\varphi$ vanishes at the τ_j and any point x is within distance at most $1/(2n)$ of some τ_j.

To prove the first bound for the α-Hölder seminorm it suffices to control P. Consider a pair of points $x < y$ belonging to the same interval $[\tau_{j-1},\tau_j]$. Then $(P\varphi(y)-P\varphi(x))/(y-x) = (\varphi(\tau_j)-\varphi(\tau_{j-1}))/(\tau_j-\tau_{j-1})$. Therefore

$$
\frac{|P\varphi(y)-P\varphi(x)|}{|y-x|^\alpha} = \frac{|\varphi(\tau_j)-\varphi(\tau_{j-1})|}{|\tau_j-\tau_{j-1}|^\alpha}\cdot\left(\frac{|y-x|}{|\tau_j-\tau_{j-1}|}\right)^{1-\alpha} \le |\varphi|_\alpha^{(n)}\,.
\tag{3.3}
$$

To prove the second bound, write $\psi = Q\varphi$ and consider $x < y$. If there is some j with $\tau_{j-1}\le x<y\le\tau_j$, then we have by definition $|\psi(y)-\psi(x)| \le |\psi|_\alpha^{(n)}|y-x|^\alpha$. Otherwise, there are j and k such that $\tau_{j-1}\le x<\tau_j\le\tau_{k-1}<y\le\tau_k$. Then, since $\psi(\tau_j)=0=\psi(\tau_{k-1})$, we have

$$
\begin{aligned}
|\psi(y)-\psi(x)| &\le |\psi(y)-\psi(\tau_{k-1})| + |\psi(\tau_j)-\psi(x)|\\
&\le |\psi|_\alpha^{(n)}\big(|y-\tau_{k-1}|^\alpha + |\tau_j-x|^\alpha\big)\\
&\le 2|\psi|_\alpha^{(n)}|y-x|^\alpha\,.
\end{aligned}
$$

To prove the claim on the Zygmund seminorm, we first show that

$$
Z^{(n)}(P^{(n)}\varphi) \le Z^{(n)}(\varphi)\,.
\tag{3.4}
$$

Since both $P^{(n)}$ and $Z^{(n)}$ can be built a pair of successive intervals at a time, it is enough to consider the case $n=2$, in which case we can write simply Z rather than $Z^{(n)}$. By an affine change of variable, we can assume that the working interval is $[-1,1]$, and, by subtracting a linear function from φ, then multiplying by an overall constant, we can assume that $\varphi(-1)=\varphi(1)=1$ and $\varphi(0)=0$ or $\varphi(0)=1$, i.e., $P^{(n)}\varphi(x)=|x|$ or $P^{(n)}\varphi(x)\equiv 0$. It suffices to consider the case $\varphi(0)=0$. Then, on the one hand,

$$
Z(\varphi) \ge |\varphi(1)+\varphi(-1)-2\varphi(0)| = 2,
$$

and, on the other hand, for $t > 0$,

$$0 \leq \frac{|x+t| - |x|}{t} - \frac{|x| - |x-t|}{t} \leq 1 - (-1) = 2,$$

so that $Z(|\,.\,|) = 2$, proving (3.4).

We now show that $Z(Q^{(n)}\varphi) \leq 4Z^{(n)}(\varphi)$. Recall that $Z(Q^{(n)}\varphi)$ is defined as the supremum of $|Z(Q^{(n)}\varphi, x, t)|$ over an appropriate set of pairs x, t; $Z^{(n)}(Q^{(n)}\varphi)$ as the supremum of the same quantity over the set of pairs such that $x \pm t$ lie in the union of some pair of successive subintervals. By (3.4), this latter supremum can be majorized by $Z^{(n)}(P^{(n)}\varphi) + Z^{(n)}(\varphi) \leq 2Z^{(n)}(\varphi)$, so the asserted bound holds when $x \pm t$ lie in the union of a pair of successive intervals.

If, on the other hand, $x \pm t$ do not lie in the union of two successive subintervals, then $|t|$ must be $> 1/2n$. By the skewed Zygmund bound and the fact that $Q^{(n)}(\varphi)$ vanishes at the division points,

$$|Q^{(n)}(\varphi)(s)| \leq \frac{1}{2} Z^{(n)}(\varphi) \frac{1}{n} \quad \text{for all relevant } s,$$

so we can estimate

$$|Z(Q^{(n)}\varphi, x, t)| \leq 4 \cdot \frac{1}{2} Z^{(n)}(\varphi) \frac{1}{n} \cdot \frac{1}{|t|} \leq 4Z^{(n)}(\varphi),$$

using $|t| \geq 1/(2n)$. Thus, the asserted bound also holds when $x \pm t$ do not lie in the union of two successive subintervals. $\square$

For each fixed $n \geq 1$ and each $0 < \alpha \leq 1$, we define

$$\beta_\alpha^{(n)} = \sup_{0 \leq j \leq n-1} \sup_{x,y \in [\tau_j^{(n)}, \tau_{j+1}^{(n)}]} \sum_{i \in \mathcal{I}} |g_i(x)|\, |f_i'(y)|^\alpha .$$

For $0 < \alpha \leq 1$, and large enough n, $\beta_\alpha^{(n)}$ is arbitrarily close to

$$\sup_{x \in I} \sum_{i \in \mathcal{I}} |g_i(x)|\, |f_i'(x)|^\alpha .$$

The next sublemma shows the usefulness of the seminorms $|\cdot|_\alpha^{(n)}$, $Z^{(n)}(\cdot)$:

Sublemma 5. *If g_i is summably Λ^α for $0 < \alpha \leq 1$, then for each $n \geq 1$ and $\varphi \in \Lambda^\alpha$*

$$|\mathcal{L}\varphi|_\alpha^{(n)} \leq |g|_\alpha^\Sigma \sup|\varphi| + \beta_\alpha^{(n)} |\varphi|_\alpha .$$

If g_i is summably Zygmund and $\delta > 0$, there are constants $K > 0$ and $\epsilon > 0$, depending only on the families f_i and g_i, such that for any $n \geq 1$, and $\varphi \in Z$,

$$Z^{(2n)}(\mathcal{L}\varphi) \leq K \sup|\varphi| + \big(\beta_1^{(n)} + \frac{K}{n^\epsilon}\big) Z(\varphi) .$$

Proof of Sublemma 5. We first prove the bound on the Λ^α seminorm by refining (2.3). Let $\varphi \in \Lambda^\alpha$ and $\tau_{j-1} \le x < y \le \tau_j$. Then there are points $z_i \in [x, y]$ with

$$\begin{aligned}
|\mathcal{L}\varphi(y) - \mathcal{L}\varphi(x)| &\le \sum_{i\in\mathcal{I}} \Big(|g_i(y) - g_i(x)|\,|\varphi(f_i(y))| + |g_i(x)|\,|\varphi(f_i(y)) - \varphi(f_i(x))| \Big) \\
&\le |g|_\alpha^\Sigma \sup|\varphi|\,|x-y|^\alpha + \sum_{i\in\mathcal{I}} |g_i(x)|\,|\varphi|_\alpha |f_i(y) - f_i(x)|^\alpha \\
&= \Big(|g|_\alpha^\Sigma \sup|\varphi| + \sum_{i\in\mathcal{I}} |g_i(x)|\,|f_i'(z_i)|^\alpha |\varphi|_\alpha \Big) |x-y|^\alpha \\
&\le \Big(|g|_\alpha^\Sigma \sup|\varphi| + \beta_\alpha^{(n)} |\varphi|_\alpha \Big) |x-y|^\alpha ,
\end{aligned}$$

as claimed.

To prove the Zygmund bound, we fix $0 < \alpha < 1$ and consider x, $x \pm t$ in some $[\tau_j^{(2n)}, \tau_{j+2}^{(2n)}]$. We first rewrite (2.4) more carefully:

(3.5)
$$\big|Z(\mathcal{L}\varphi, x, t)\big| \le \sum_{i\in\mathcal{I}} |g_i(x)|\,|Z(\varphi\circ f_i, x, t)| + Z^\Sigma(g)\sup|\varphi| + \frac{2}{\lambda^\alpha}\left(\frac{1}{n}\right)^\epsilon |g|^\Sigma_{1-\alpha+\epsilon}|\varphi|_\alpha ,$$

where $0 < \epsilon < \delta$ is such that $1 - \alpha + \epsilon < 1$, and we used $t < 1/n$. To bound the first term in the right-hand side of (3.5), we may use the decomposition (2.6) of $Z(\varphi\circ f_i, x, t)$ into $\mathrm{I}_i + \mathrm{II}_i$. Then, by definition of the t_i, there are points $z_i \in [x, x+t]$ so that

$$\mathrm{I}_i = f_i'(z_i) Z(\varphi, f_i(x), t_i)\,. \tag{3.6}$$

Using again $t < 1/n$, we may rewrite (2.8) as

$$|\mathrm{II}_i| \le C\left(\frac{1}{n}\right)^\epsilon |\varphi|_{\frac{1+\epsilon}{1+\delta}}\,. \tag{3.7}$$

Setting $\alpha = (1+\epsilon)/(1+\delta) < 1$, the bounds (3.5)–(3.7) yield a constant $C > 0$, depending only on the f_i, with

$$\begin{aligned}
Z^{(n)}(\mathcal{L}\varphi) &\le Z^\Sigma(g)\sup|\varphi| + \beta_1^{(n)} Z(\varphi) \\
&\quad + \left(\frac{1}{n}\right)^\epsilon \Big(\frac{2}{\lambda^{\frac{1+\epsilon}{1+\delta}}} |g|^\Sigma_{\delta\cdot\frac{1+\epsilon}{1+\delta}} + C\sup{}^\Sigma|g|\Big)|\varphi|_{\frac{1+\epsilon}{1+\delta}}\,,
\end{aligned}$$

To finish the proof, we proceed as in Lemma 1 to bound the Λ^α seminorms. $\square$

Proof of Lemma 2. It suffices to show that there is a universal constant $C > 0$ so that for each $n \ge 1$

$$\limsup_{n\to\infty} \|Q^{(n)}\mathcal{L}Q^{(n)}\|_\alpha \le C \sup_{x\in I} \sum_{i\in\mathcal{I}} |g_i(x)|\,|f_i'(x)|^\alpha\,,$$

when the g_i are summably Λ^α and $\delta \ge 0$, and

$$\limsup_{n\to\infty} \|Q^{(2n)}\mathcal{L}Q^{(2n)}\|_Z \le C \sup_{x\in I} \sum_{i\in\mathcal{I}} |g_i(x)|\,|f_i'(x)|\,,$$

when the g_i are summably Zygmund and $\delta > 0$.

Applying Sublemma 4, we get for each $\varphi \in \Lambda^\alpha$, $n \geq 1$:

$$\begin{aligned}\sup |Q^{(n)} \mathcal{L} Q^{(n)} \varphi| &\leq C \sup |\mathcal{L} Q^{(n)} \varphi| \leq C \sum_{i \in \mathcal{I}} \sup |g_i| \, \sup |Q^{(n)} \varphi| \\ &\leq C \sup{}^{\Sigma} |g| \, \frac{|\varphi|_\alpha}{(2n)^\alpha} \, .\end{aligned}$$

Applying again Sublemma 4, and also Sublemma 5, we get for any $\varphi \in \Lambda^\alpha$, $n \geq 1$:

$$\begin{aligned}|Q^{(n)} \mathcal{L} Q^{(n)} \varphi|_\alpha &\leq C |\mathcal{L} Q^{(n)} \varphi|_\alpha^{(n)} \leq C \cdot \big(|g|_\alpha^\Sigma \sup |Q^{(n)} \varphi| + \beta_\alpha^{(n)} |Q^{(n)} \varphi|_\alpha \big) \\ &\leq C \cdot \big(|g|_\alpha^\Sigma \frac{|\varphi|_\alpha}{(2n)^\alpha} + \beta_\alpha^{(n)} |\varphi|_\alpha \big) \, .\end{aligned}$$

Finally, with Sublemmas 4 and 5, we obtain for each $\varphi \in Z(I)$, $0 < \alpha < 1$, and $n \geq 1$:

$$\begin{aligned}Z(Q^{(2n)} \mathcal{L} Q^{(2n)} \varphi) &\leq C Z^{(2n)} (\mathcal{L} Q^{(2n)} \varphi) \\ &\leq C \big(K \sup |Q^{(2n)} \varphi| + (\beta_1^{(n)} + \frac{K}{n^\epsilon}) Z(Q^{(2n)} \varphi) \big) \\ &\leq C \cdot \big(\frac{K}{n^\alpha} |\varphi|_\alpha + (\beta_1^{(n)} + \frac{K}{n^\epsilon}) Z(\varphi) \big) \, ,\end{aligned}$$

where $C > 0$ is universal and $K > 0$, $\epsilon > 0$ depend on the f_i and g_i (but not on n). $\square$

4. The lower bound

The idea for the argument yielding the lower bound on the Banach spaces Λ^α ($0 < \alpha \leq 1$) is originally due to A. Davies (Lanford [8]). The Zygmund case can be treated similarly, as will be shown now.

Proof of Lemma 3. To prove the Zygmund claim, we introduce the continuous function

$$\beta_1(x) = \sum_{i \in \mathcal{I}} |g_i(x)| \, |f_i'(x)| \, .$$

Writing $\bar{\beta}_1 = \sup_{x \in I} \beta_1(x)$, the first assertion of Lemma 3 is that the infimum of $\|\mathcal{L} - \mathcal{K}\|_Z$ for $\mathcal{K}$ compact is not less than $\bar{\beta}_1$. Fix $\epsilon > 0$ small. We may assume that $\mathcal{I}$ is finite, since otherwise replacement of $\mathcal{I}$ by a large finite subset of $\mathcal{I}$ in the definition of $\beta_1(x)$ yields a supremum arbitrarily close to $\bar{\beta}_1$. The strategy is now to construct an infinite-dimensional subspace $\chi_\epsilon \subset Z(I)$ (with, in fact, $\chi_\epsilon \subset \Lambda^1$) such that $\|\mathcal{L}\varphi\|_Z \geq (\bar{\beta}_1 - \epsilon) \|\varphi\|_Z$ for each $\varphi \in \chi_\epsilon$.

Then, if $\mathcal{K}$ is a compact operator on $Z(I)$, there is a function $\varphi \in \chi_\epsilon$ with $\|\varphi\|_Z = 1$ and such that $\|\mathcal{K}\varphi\|_Z \leq \epsilon$, and hence such that $\|(\mathcal{L} - \mathcal{K})\varphi\|_Z \geq (\bar{\beta}_1 - 2\epsilon)$. Therefore the norm of $\mathcal{L} - \mathcal{K}$ cannot be less than $\bar{\beta}_1 - 3\epsilon$.

The construction of these subspaces goes as follows: We take a point x_∞ where $\beta_1(x_\infty) = \bar{\beta}_1$ and choose—with some care—a sequence $x_1, x_2, \ldots$ of distinct points in I converging to x_∞. We then construct a sequence of functions $\psi_1, \psi_2, \ldots$ in $\Lambda^1(I)$ such that

(P1) $\|a_1 \psi_1 + \cdots + a_N \psi_N\|_Z = \max_j \{|a_j|\}$ for any $N \geq 1$ and complex numbers $a_1, \ldots, a_N$—in particular $\|\psi_j\|_Z = 1$ for every j;

(P2) $\limsup_{t\to 0} |Z(\mathcal{L}\psi_j, x_j, t)| = \beta_1(x_j) \to \bar\beta_1$ as $j \to \infty$;
(P3) $\mathcal{L}\psi_j$ vanishes on a neighbourhood of x_ℓ for all $j \neq \ell$.

From (P2) and (P3) we get $\|\mathcal{L}(a_1\psi_1 + \cdots + a_N\psi_N)\|_Z \geq \max_{1\leq j\leq N}\{\beta_1(x_j)|a_j|\}$, and hence, using (P1), we get for any φ in the linear span of ψ_k, $\psi_{k+1}, \cdots$

$$\|\mathcal{L}\varphi\|_Z \geq \inf_{j\geq k}\{\beta_1(x_j)\}\|\varphi\|_Z\,.$$

Thus, we can take χ_ϵ to be the closed linear span of the ψ_j's with $j \geq k$ for any sufficiently large $k = k(\epsilon)$. The problem is therefore reduced to constructing (x_j), (ψ_j) so that (P1), (P2) and (P3) hold.

We first specify how to choose the x_j's. For x_∞ as defined above, we choose inductively a sequence of x_j's converging to, but distinct from, x_∞, assuming furthermore that the $f_i(x_j)$, for $i \in \mathcal{I}$ and $j \geq 1$, are distinct from each other and from the $f_i(x_\infty)$. Suppose $x_1, \ldots, x_k$ have been chosen so that the $f_i(x_j)$ for $1 \leq j \leq k$ are distinct from each other and from the $f_i(x_\infty)$. We then choose a point x'_{k+1} near enough to x_∞ so that each $f_i(x'_{k+1})$ is nearer to its one or two neighbours in the set of $\{f_\ell(x_\infty)\}$ than *any* previous $f_\ell(x_j)$ but still not in this set. (We use here that no f_i can be locally constant.) Then, by moving x'_{k+1} a little, and using the fact that no two f_is coincide on any non-trivial interval, we find x_{k+1} so that the $f_i(x_{k+1})$ are distinct from each other, but so that the preceding "inequalities" still hold. Constructed in this way, the $f_i(x_j)$ for $i \in \mathcal{I}$ and $j \geq 1$ are all different and no $f_i(x_j)$ is an accumulation point of the others.

Now, let $\phi \in \Lambda^1(]-1,1[)$ be of Zygmund norm one, with compact support, and such that

$$\phi(t) = |t|/2 \quad \text{for small } t. \tag{4.1}$$

For any γ between 0 and 1, the rescaled function $\gamma\phi(t/\gamma)$ has the same properties, and by taking γ small we can make its support and supremum norm as small as we like. We simply construct ψ_j as a sum of functions $\psi_{i,j}$, for $i \in \mathcal{I}$, each of which is a rescaled ϕ translated to $f_i(x_j)$ (up to a complex phase), i.e., has the form

$$\psi_{i,j}(x) = \omega_{i,j} \cdot \gamma_{i,j} \cdot \phi((x - f_i(x_j))/\gamma_{i,j}),$$

where $|\omega_{i,j}| = 1$ will be chosen later, and $\gamma_{i,j} > 0$ is such that the support of $\psi_{i,j}$ is a subset of the interior of $f_i(I)$, and may be reduced further in Sublemma 6 below.

Now $\mathcal{L}\psi_j(x)$ is non-zero only if some $f_i(x)$ is in the support of some $\psi_{k,j}$. Since we can make the supports of the $\psi_{k,j}$ disjoint by making the $\gamma_{k,j}$ small enough, for any $\ell \neq j$ there is a neighbourhood of x_ℓ on which no $f_i(x)$ is in the support of any $\psi_{k,j}$. Thus, assertion (P3) holds.

We next check that by making the $\gamma_{i,j}$ sufficiently small we can guarantee that (P1) is satisfied. To carry out the verification it is convenient to relabel our objects: We label the pairs (i,j), $i \in \mathcal{I}$, $j \geq 1$, with a positive integer m, and we write $\xi_m = f_i(x_j)$. It suffices to prove the following sublemma:

Sublemma 6. *Let ξ_m, $m \geq 1$, be distinct points in I such that no ξ_m is an accumulation point of the others, and let γ_m be a sequence of positive numbers. For ϕ as defined in (4.1), we set $\phi_m(x) = \omega_m\gamma_m\phi((x-\xi_m)/\gamma_m)$ for arbitrary $|\omega_m| = 1$. If the γ_m's are small enough, then, for any $N \geq 1$, and any $b_1, \ldots, b_N$,*

$$\|b_1\phi_1 + \cdots + b_N\phi_N\|_Z = b_{\max} = \max\{|b_m| \mid 1 \leq m \leq N\}\,. \tag{4.2}$$

Proof of Sublemma 6. We define $\eta = b_1\phi_1 + \cdots + b_N\phi_N$ and set $d_m = \inf\{|\xi_m - \xi_{m'}| \mid m \neq m'\} > 0$.

We claim that it suffices to take γ_m small enough so that

$$\phi_m(x) \text{ vanishes for } |x - \xi_m| \geq d_m/4 \text{ and } \sup|\phi_m| < d_m/8 \tag{4.3}$$

to get (4.2) to hold. Half of (4.2) is immediate: If we take $x = \xi_m$ and $t > 0$ very small, we have (since the supports of the ϕ_m are disjoint)

$$\eta(x+t) + \eta(x-t) - 2\eta(x) = \eta(\xi_m + t) + \eta(\xi_m - t) = \omega_m b_m \cdot t\,,$$

so that $\|\eta\|_Z \geq Z(\eta) \geq |b_m|$ for each m.

To prove the opposite inequality, we first observe that the disjointness of the supports and the fact that $\sup|\phi_m| \leq 1$ imply $\sup|\eta| \leq b_{\max}$. To prove the corresponding estimate for $Z(\eta)$ we consider general $x \in I$ and $t > 0$ with $x \pm t \in I$. We need to show that

$$|\eta(x+t) + \eta(x-t) - 2\eta(x)| \leq t b_{\max}\,.$$

Since $Z(\phi_m) = 1$ for all m, this is immediate unless $\{x, x+t, x-t\}$ intersects the supports of at least two *different* ϕ_m's. Assume thus that x is in the support of ϕ_{m_1}, $x - t$ in the support of ϕ_{m_0}, and $x + t$ in the support of ϕ_{m_2}, where the set $\{m_0, m_1, m_2\}$ is not a singleton. We leave to the reader the easier case where this set has only two elements, and suppose that $m_0 \neq m_1 \neq m_2$. By our assumption (4.3),

$$|x - t - \xi_{m_0}| \leq d_{m_0}/4 \leq |\xi_{m_0} - \xi_{m_1}|/4 \quad \text{and} \quad |x - \xi_{m_1}| \leq d_{m_1}/4 \leq |\xi_{m_1} - \xi_{m_0}|/4\,.$$

Therefore

$$t = |x - t - x| \geq |\xi_{m_0} - \xi_{m_1}|/2\,. \tag{4.4}$$

Similarly, we get $t \geq |\xi_{m_2} - \xi_{m_1}|/2$. On the other hand,

$$|\eta(x)| = |b_{k_1}||\phi_{m_1}(x)| \leq b_{\max}\sup|\phi_{m_1}| \leq b_{\max} d_{m_1}/8 \leq b_{\max}|\xi_{m_0} - \xi_{m_1}|/8\,.$$

Analogously $|\eta(x+t)| \leq b_{\max}|\xi_{m_2} - \xi_{m_1}|/8$, and $|\eta(x-t)| \leq b_{\max}|\xi_{m_0} - \xi_{m_1}|/8$. Finally, recalling (4.4),

$$|\eta(x \pm t) - \eta(x)| \leq |\eta(x \pm t)| + |\eta(x)| \leq \frac{b_{\max}}{4}\max(|\xi_{m_0} - \xi_{m_1}|, |\xi_{m_2} - \xi_{m_1}|) \leq b_{\max}\frac{t}{2}\,.$$

Since

$$|\eta(x+t) + \eta(x-t) - 2\eta(x)| \leq |\eta(x+t) - \eta(x)| + |\eta(x-t) - \eta(x)|\,,$$

this ends the proof of Sublemma 6 and therefore of assertion (P1). $\square$

Going back to the notation with pairs $i \in \mathcal{I}$ and $j \geq 1$, it remains to choose the $\omega_{i,j}$ so that (P2) holds. To do this, we fix j and we decompose as before, using (2.1), (2.6), and the property of the support of $\psi_{i,j}$:

$$\begin{aligned}
&tZ(\mathcal{L}\psi_j, x_j, t) \\
&\quad = \sum_{i\in\mathcal{I}} t_i g_i(x_j) Z(\psi_j, f_i(x_j), t_i) - \sum_{i\in\mathcal{I}} g_i(x_j)(\psi_j(f_i(x_j) - t_i) - \psi_j(f_i(x_j - t)) \\
&\qquad + \sum_{i\in\mathcal{I}} (g_i(x_j + t) - g_i(x_j))(\psi_j(f_i(x_j + t))) \\
&\qquad + \sum_{i\in\mathcal{I}} (g_i(x_j) - g_i(x_j - t))(-\psi_j(f_i(x_j - t))) \\
&\quad = Z_a - Z_b + Z_c + Z_d
\end{aligned}$$

(we used the $t_i = t_{i,j}$ defined by $f_i(x_j + t) = f_i(x_j) + t_i$, and the fact that $\psi_j(f_i(x_j)) = 0$ for all i). Now, since each $\psi_j \in \Lambda^1$ (use e.g. $\#\mathcal{I} < \infty$), we have

$$|\psi_j(f_i(x_j \pm t))| \leq |\psi_j|_1 |f_i(x_j \pm t) - f_i(x_j)| \leq |\psi_j|_1 |f_i'(u)| t,$$

for some u, by construction. Therefore, using $|g|^{\Sigma}_{\alpha} < \infty$, for any $0 < \alpha < 1$, we get $Z_c + Z_d = o(t)$ when $t \to 0$. Since $\sup^{\Sigma} |g| < \infty$, we get, using again $\psi_j \in \Lambda^1$, that $Z_b = o(t)$ by applying (2.5) once more (recall that $\delta > 0$). By definition, $Z(\psi_j, f_i(x_j), u) = \omega_{i,j} + o(u)$ for $u \to 0$, uniformly in $i \in \mathcal{I}$ (using $\#\mathcal{I} < \infty$). Finally, $t_i/t = f_i'(x_j + u) = f_i'(x_j) + O(|u|^{\delta})$ for some $|u| \leq t$. Therefore, if we choose the complex phases $\omega_{i,j}$ properly, we find:

$$tZ(\mathcal{L}\psi_j, x_j, t) = t \sum_{i\in\mathcal{I}} \omega_{i,j} g_i(x_j) f_i'(x_j) + o(t) = t \sum_{i\in\mathcal{I}} |g_i(x_j) f_i'(x_j)| + o(t)\,, \quad t \to 0\,,$$

which gives assertion (P2) above, and thus the first claim of Lemma 3.

A simple modification of the construction in the proof yields the second claim of Lemma 3: Instead of $\phi(t) = |t|/2$ for small t, we take $\phi(t) = |t|^{\alpha}$ (assuming that $\|\phi\|_{\alpha} = 1$) and we rescale by $\gamma^{\alpha}(\phi(t/\gamma))$, i.e., we have

$$\phi_m(x) = \omega_m (\gamma_m)^{\alpha} \phi((x - \xi_m)/\gamma_m),$$

where $|\omega_m| = 1$ and the points ξ_m are chosen exactly as above. The scalars γ_m are then chosen similarly as in Sublemma 6, condition (4.3) being naturally replaced by

$$\phi_m(x) \text{ vanishes for } |x - \xi_m| \geq d_m/4 \text{ and } \sup |\phi_m| < (d_m)^{\alpha}/4\,.$$

A slight variation on the above arguments (replacing the Zygmund seminorm by $|\cdot|_{\alpha}$, and using the decomposition (2.3) as a starting point) then yields

(P1$_\alpha$) $\|a_1\psi_1 + \cdots + a_N\psi_N\|_{\alpha} = \max_j\{|a_j|\}$ for any N, $a_1, \ldots, a_N$. In particular, $\|\psi_j\|_{\alpha} = 1$ for every i;

(P2$_\alpha$) $\limsup_{x\to x_j} |\mathcal{L}\psi_j(x) - \mathcal{L}\psi_j(x_j)|/|x_j - x|^{\alpha} = \beta^{\mathcal{I}}_{\alpha}(x_j) \to \bar{\beta}^{\mathcal{I}}_{\alpha}$ as $j \to \infty$ (we set $\beta^{\mathcal{I}}_{\alpha}(x) = \sum_{i\in\mathcal{I}} |g_i(x)|\,|f_i'(x)|^{\alpha}$ and $\bar{\beta}^{\mathcal{I}}_{\alpha} = \sup_{x\in I} \beta^{\mathcal{I}}_{\alpha}(x)$);

(P3$_\alpha$) $\mathcal{L}\psi_j$ vanishes on a neighbourhood of x_ℓ for all $j \neq \ell$. □

5. The spectral radius and two conjectures

In this section, it is convenient to use the notation $\mathcal{L}_g$ instead of $\mathcal{L}$. We have the following result (the statements for Λ^α and $\mathcal{B}$ were obtained previously by Ruelle [15, 16]):

Theorem 2. *If the family g_i is summably bounded, then the spectral radius of $\mathcal{L}_{|g|}$ acting on $\mathcal{B}$ is equal to*

$$e^P := \lim_{n\to\infty} \left(\sup_{x\in I} \sum_{\vec{\imath}\in\mathcal{I}^n} |g_{\vec{\imath}}^{(n)}(x)| \right)^{1/n} ,$$

and the spectral radius of $\mathcal{L}_g$ on $\mathcal{B}$ is bounded above by e^P.

If the g_i are summably Zygmund and $\delta > 0$, respectively Λ^α and $\delta \ge 0$, the spectral radius of $\mathcal{L}_{|g|}$ acting on Z (respectively Λ^α) is equal to $\max(e^P, \rho_{\mathrm{ess}}(\mathcal{L}_{|g|}))$*, and the spectral radius of $\mathcal{L}_g$ acting on Z, respectively Λ^α, is bounded above by* $\max(e^P, \rho_{\mathrm{ess}}(\mathcal{L}_{|g|}))$.

Under the additional assumption that $\lambda > 1$, Theorem 2 together with Theorem 1 yields that $\rho_{\mathrm{ess}}(\mathcal{L}_{|g|})$ is strictly smaller than the spectral radius of $\mathcal{L}_{|g|}$ (except when both vanish) acting on Z (respectively Λ^α).

Proof. Since $e^{n(P-\epsilon)} \le \sup \mathcal{L}_{|g|}^n \psi \le e^{n(P+\epsilon)}$ for $\psi \equiv 1$, $\epsilon > 0$, and $n \ge n(\epsilon)$, the proof of Theorem 2 for the Banach space $\mathcal{B}$ is an immediate consequence of the spectral radius formula together with the easy inequality $\sup |\mathcal{L}_g^n \varphi| \le \sup |\varphi| \sup \mathcal{L}_{|g|}^n \psi$, for all $\varphi \in \mathcal{B}$.

For the other Banach spaces, use the definition of the essential spectral radius.□

Maximal eigenfunctions, zeta functions, and two conjectures. In this subsection, we assume throughout that $\lambda > 1$.

Consider $\mathcal{L}_{|g|}$ acting on Λ^α: When our family f_i consists of the finitely many inverse branches of a (mixing) map $f : I \to I$, it is known that e^P is the only point in the spectrum of modulus e^P, that it is a simple eigenvalue, and that $\mathcal{L}_{|g|}$ admits a positive maximal eigenfunction φ (i.e., $\mathcal{L}_{|g|}\varphi = e^P\varphi$). Finally, P is the topological pressure of $\log |g|$ and f. For all these results, and a theory of equilibrium states, see Ruelle [15], where it was proven that the essential spectral radius of $\mathcal{L}_g$ acting on Λ^α is not bigger than e^P/λ^α, a result which follows from our Theorem 1, part 2. By Theorem 1, part 1, the essential spectral radius of $\mathcal{L}_{|g|}$ acting on Z is smaller than $e^P/\lambda < e^P$. Since each eigenfunction of $\mathcal{L}_{|g|}$ in Z is also an eigenfunction in Λ^α, the eigenvalue e^P is the unique point in the spectrum with modulus e^P, and it is simple with a positive eigenfunction $\varphi \in Z$. The case of countably many branches can be treated similarly.

In Ruelle [15, 16] and Fried [4], zeta functions associated with Λ^α (for $0 < \alpha \le 1$) systems of finitely or countably many branches f_i and weights g_i were studied (in a slightly different setting—in particular the dimension was not limited to one). In our case, the zeta function is defined by

$$\zeta_g(z) = \exp \sum_{n\ge 1} \frac{z^n}{n} \sum_{\substack{\vec{\imath}\in\mathcal{I}^n \\ x\,:\, f_{\vec{\imath}}^{(n)}(x)=x}} \prod_{k=0}^{n-1} g_{i_{k+1}}(f_{i_k}\cdots f_{i_1})(x)\,.$$

The poles ω of $\zeta_g(z)$ in the disc of radius $\lambda^\alpha e^{-P}$ (where the function was shown to be meromorphic) were proved to be in bijection with the eigenvalues $\nu = 1/\omega$ of $\mathcal{L}_g$ acting on Λ^α of modulus $> e^P/\lambda^\alpha$.

For summably Λ^α weights g, we *conjecture* that $\zeta_g(z)$ is meromorphic in the disc of radius $\rho_{\mathrm{ess}}^{-1}(\mathcal{L}_g)$, where $\rho_{\mathrm{ess}}(\mathcal{L}_g)$ is given by part 2 of Theorem 1, and that its poles there are the inverses of the Λ^α-eigenvalues of $\mathcal{L}_g$ of modulus $> \rho_{\mathrm{ess}}(\mathcal{L}_g)$.

Also, if g is summably Zygmund and $\delta > 0$, we *conjecture* that $\zeta_g(z)$ is meromorphic in the disc of radius $\rho_{\mathrm{ess}}^{-1}(\mathcal{L}_g)$ for $\mathcal{L}_g$ acting on $Z(I)$, and that its poles there are in bijection $\omega = 1/\nu$ with the eigenvalues of $\mathcal{L}_g : Z(I) \to Z(I)$ of modulus $> \rho_{\mathrm{ess}}(\mathcal{L}_g)$.

The Λ^α conjectures do not immediately imply the Zygmund one, since Z is a strict subset of $\bigcap_{\alpha<1} \Lambda^\alpha$. The proof of these two conjectures would complete the second part of the program described in the introduction.

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