

## SIMULTANEOUS RATIONAL APPROXIMATION TO BINOMIAL FUNCTIONS

MICHAEL A. BENNETT

ABSTRACT. We apply Padé approximation techniques to deduce lower bounds for simultaneous rational approximation to one or more algebraic numbers. In particular, we strengthen work of Osgood, Fel'dman and Rickert, proving, for example, that

$$\max \left\{ \left| \sqrt{2} - p_1/q \right|, \left| \sqrt{3} - p_2/q \right| \right\} > q^{-1.79155}$$

for  $q > q_0$  (where the latter is an effective constant). Some of the Diophantine consequences of such bounds will be discussed, specifically in the direction of solving simultaneous Pell's equations and norm form equations.

### 0. INTRODUCTION

In 1964, Baker [1, 2] utilized the method of Padé approximation to hypergeometric functions to obtain explicit improvements upon Liouville's theorem on rational approximation to algebraic numbers. By way of example, he showed that

$$(0.1) \quad \left| \sqrt[3]{2} - \frac{p}{q} \right| > 10^{-6} q^{-2.955}$$

for all positive integers  $p$  and  $q$  and used such bounds to solve related Diophantine equations. Chudnovsky [6] subsequently refined Baker's results, primarily through a detailed analysis of the arithmetical properties of certain Padé approximants. Analogous to (0.1), he proved that

$$(0.2) \quad \left| \sqrt[3]{2} - \frac{p}{q} \right| > q^{-2.42971}$$

for all integers  $p$  and  $q$  with  $q$  greater than some effectively computable constant  $q_0$ . By working out the implicit constants in (0.2), Easton [8] deduced

$$\left| \sqrt[3]{2} - \frac{p}{q} \right| > 6.6 \times 10^{-6} q^{-2.795}$$

for positive integers  $p$  and  $q$  (as well as related bounds for other cubic irrationalities).

Similar results exist for simultaneous approximation to several algebraic numbers. In particular, Baker [3] derived bounds of the form

$$(0.3) \quad \max_{1 \leq u \leq m} \left\{ \left| \theta_u - \frac{p_u}{q} \right| \right\} > q^{-\lambda}$$

---

Received by the editors June 30, 1994 and, in revised form, January 31, 1995.

1991 *Mathematics Subject Classification*. Primary 11J68, 11J82; Secondary 11D57.

*Key words and phrases*. Simultaneous approximation to algebraic numbers, irrationality and linear independence measures, Padé approximants, Pell-type equations, norm form equations.

Research supported by an NSERC Postdoctoral Fellowship.

for certain algebraic numbers  $\theta_1, \theta_2, \dots, \theta_m$  with  $1, \theta_1, \theta_2, \dots, \theta_m$  linearly independent over the rationals,  $\lambda = \lambda(\theta_1, \dots, \theta_m)$  an explicit real number and  $p_1, \dots, p_m, q$  positive integers with  $q$  greater than an effective  $q_0(\lambda, \theta_1, \dots, \theta_m)$ . To be precise, he considered

$$(0.4) \quad (\theta_1, \theta_2, \dots, \theta_m) = (r^{\nu_1}, r^{\nu_2}, \dots, r^{\nu_m})$$

with  $r, \nu_1, \nu_2, \dots, \nu_m$  rational, via approximation to the system of binomial functions

$$1, (1+x)^{\nu_1}, \dots, (1+x)^{\nu_m}.$$

Chudnovsky [6], generalizing his approach from the case of a single approximation, sharpened these inequalities. Along somewhat different lines, Osgood [13], Fel'dman [9] and Rickert [15] obtained results like (0.3) with

$$(0.5) \quad (\theta_1, \theta_2, \dots, \theta_m) = (r_1^\nu, r_2^\nu, \dots, r_m^\nu)$$

for  $r_1, r_2, \dots, r_m$  and  $\nu$  rational. These also utilized Padé approximation, this time to the functions

$$1, (1+a_1x)^\nu, \dots, (1+a_mx)^\nu$$

where  $a_1, \dots, a_m$  are distinct integers. Through use of an elegant contour integral representation for the desired Padé approximants, Rickert proved the inequality

$$(0.6) \quad \max \left\{ \left| \sqrt{2} - \frac{p_1}{q} \right|, \left| \sqrt{3} - \frac{p_2}{q} \right| \right\} > 10^{-7} q^{-1.913}$$

for  $p_1, p_2$  and  $q$  integral.

In this paper, we will strengthen the work of Osgood, Fel'dman and Rickert on simultaneous approximation to algebraic numbers satisfying (0.5), in analogy to Chudnovsky's results for those with (0.4). This is primarily accomplished through careful estimation of both "analytic" and "arithmetic" asymptotics (in the same sense as Chudnovsky [6]) of Padé approximants to binomial functions. A particularly striking result along these lines (with rather different approximating forms) is due to Hata [11] who proved that

$$\left| \pi - \frac{p}{q} \right| \geq q^{-8.0161}$$

for sufficiently large positive integers  $p$  and  $q$ .

In the special case  $m = 1$ , we obtain Chudnovsky's Theorem 5.3 of [6] on approximation to a single algebraic number (see §7). For larger values of  $m$ , we can prove, for example, that

$$\max \left\{ \left| \sqrt{2} - \frac{p_1}{q} \right|, \left| \sqrt{3} - \frac{p_2}{q} \right| \right\} > q^{-1.79155}$$

for  $p_1$  and  $p_2$  integral and  $q \geq q_0$  effectively computable (compare to (0.6)). Similarly, we have

$$\max \left\{ \left| \sqrt{3} - \frac{p_1}{q} \right|, \left| \sqrt{5} - \frac{p_2}{q} \right| \right\} > q^{-1.82227}$$

for  $q \geq q_1$  effective. Optimally, one would like to derive (0.3) for any  $\lambda > 1 + 1/m$ . Theorems of Roth [16] ( $m = 1$ ) and Schmidt [17] ( $m > 1$ ) assert that such bounds exist for any independent algebraic  $\theta_1, \dots, \theta_m$ , but are ineffective in that they do not permit the explicit calculation of  $q_0$ . For specific classes of algebraic numbers, however, we will be able to obtain effective bounds with  $\lambda$  arbitrarily close to  $1 + 1/m$ . These correspond to the situations described by previous authors where

the rationals  $r$  or  $r_1, r_2, \dots, r_m$  in (0.4) or (0.5), respectively, are suitably close to 1. We will also prove a theorem on linear forms, of the type

$$(0.7) \quad |x_0 + x_1\theta_1 + \dots + x_m\theta_m| > X^{-\lambda_1}$$

for  $x_0, \dots, x_m$  integers,  $\theta_1, \dots, \theta_m$  as in (0.5) and  $X = \max_{0 \leq i \leq m} |x_i|$  satisfying  $X \geq X_0(\lambda_1, \theta_1, \dots, \theta_m)$ . Standard transference arguments (see e.g. Cassels [5]) ensure that (0.3) implies (0.7) with exponent

$$\lambda_1 = \frac{m(\lambda - 1)}{m(-\lambda + 1) + \lambda}$$

provided  $\lambda < 1 + 1/(m - 1)$ . Our result, however, is somewhat stronger. These results have direct applications to Diophantine equations which we will address in §8 and §9. For example, they permit solution of the norm form equation

$$N_{K/\mathbb{Q}} \left( x + y \sqrt[m]{M^4 - 1} + z \sqrt[m]{M^4 + 1} \right) = u$$

(where  $K = \mathbb{Q}(\sqrt[m]{M^4 - 1}, \sqrt[m]{M^4 + 1})$ ,  $x, y$  and  $z$  are integers and  $u$  is constant) for  $M \geq 6$ .

1. A PAIR OF THEOREMS ON RATIONAL APPROXIMATION

Henceforth, we will suppose that  $a_0, a_1, \dots, a_m$  are distinct integers ( $m \geq 1$ ) with one of them equal to zero, satisfying

$$a_0 < a_1 < \dots < a_m.$$

Let us also assume that  $N$  is a positive integer with

$$N > \max_{0 \leq u \leq m} |a_u|$$

and that  $s$  and  $n$  are integers with  $1 \leq s < n$  and  $(s, n) = 1$ . Define

$$(1.1) \quad c_1 = \text{lcm} \left\{ \prod_{\substack{l=0 \\ l \neq v}}^m |a_l - a_v| : 0 \leq v \leq m \right\},$$

$$(1.2) \quad c_2 = \text{lcm} \{ |a_l - a_v| : 0 \leq v < l \leq m \},$$

$$(1.3) \quad c_3 = \prod_{p|n} p^{\max\{\text{ord}_p(n/c_2) + \frac{1}{p-1}, 0\}}$$

and

$$c_4 = c_1 \cdot c_2 \cdot c_3.$$

If, following Rickert [15], we set  $A(z) = \prod_{u=0}^m (z - a_u)$ , then the polynomial

$$(1.4) \quad A(z) - (z + N)A'(z)$$

(where we write  $A'(z)$  for  $dA(z)/dz$ ) is readily seen to have  $m + 1$  real zeros, one of them, say  $z_0$ , satisfying  $z_0 < -N$  and the remaining  $m$ , say  $z_1, z_2, \dots, z_m$ , lying between successive values of the  $a_i$ 's. Without loss of generality, we suppose that  $a_{u-1} < z_u < a_u$  ( $1 \leq u \leq m$ ) and define

$$c_5 = |A'(z_0)|,$$

$$c_6(v) = \begin{cases} |A'(z_1)| & \text{if } v = 0, \\ \min\{|A'(z_v)|, |A'(z_{v+1})|\} & \text{if } 1 \leq v < m, \\ |A'(z_v)| & \text{if } v = m, \end{cases}$$

$$c_7 = \min_{1 \leq u \leq m} |A'(z_u)|$$

and

$$c_8 = \exp \left( -\gamma - \frac{1}{\phi(n)} \sum_{\substack{1 \leq r < n \\ (r,n)=1}} \psi \left( \max \left\{ \frac{nm-r}{nm}, \frac{r}{n} \right\} \right) \right)$$

where  $\gamma$  is Euler’s constant,  $\phi(n)$  is Euler’s totient function and  $\psi(z) = d \log \Gamma(z)/dz$  is the digamma function. We may conclude

**Theorem 1.1.** *Let  $a_0, a_1, \dots, a_m$  be distinct integers with  $a_r = 0$  for some  $r$  and  $N > \max_{0 \leq u \leq m} |a_u|$  an integer. If, further,  $s$  and  $n$  are relatively prime with  $1 \leq s < n, \epsilon > 0$  and  $c_7 \cdot c_8 < c_4 < c_5 \cdot c_8$ , then*

$$\max_{\substack{0 \leq u \leq m \\ a_u \neq 0}} \left\{ \left| \left( 1 + \frac{a_u}{N} \right)^{s/n} - \frac{p_u}{q} \right| \right\} > q^{-\lambda-\epsilon}$$

for all integers  $p_0, \dots, p_m$  and with  $q \geq q_0(\epsilon, s, n, a_0, \dots, a_m, N)$ , where  $\lambda = 1 + \frac{\log(c_4/c_7 \cdot c_8)}{\log(c_5 \cdot c_8/c_4)}$  and  $q_0$  is effectively computable.

As mentioned previously, in §7 we will show that, in the case  $m = 1$ , the above theorem implies Chudnovsky’s result [6, Theorem 5.3] (see also Heimonen, et al. [12]).

For linear forms, we will prove

**Theorem 1.2.** *If  $a_0, \dots, a_m, N, s$  and  $n$  are integers satisfying the hypotheses of the previous theorem,  $x_0, \dots, x_m$  integers,  $X = \max_{0 \leq u \leq m} |x_u|, \epsilon > 0$ , and*

$$\prod_{v=1}^m c_6(v) < (c_4/c_8)^m < c_5 \cdot \min_{1 \leq l \leq m} \prod_{\substack{v=1 \\ v \neq l}}^m c_6(v),$$

then

$$\left| \sum_{u=0}^m x_u \cdot \left( 1 + \frac{a_u}{N} \right)^{s/n} \right| > X^{-\lambda_1-\epsilon}$$

for all  $X \geq X_0(\epsilon, s, n, a_0, \dots, a_m, N)$ , where

$$\lambda_1 = \frac{m \log(c_4/c_8) - \sum_{1 \leq v \leq m} \log(c_6(v))}{m \log(c_8/c_4) + \log(c_5) + \min_{1 \leq l \leq m} \sum_{\substack{1 \leq v \leq m \\ v \neq l}} \log(c_6(v))}$$

and  $X_0$  is effectively computable.

Examples and applications of this result will be briefly described in §8 and §9.

2. THE NATURE OF THE APPROXIMATING FORMS

To construct our approximants to the system of binomial functions (0.3), we consider the contour integral

$$(2.1) \quad I_u(x) = \frac{1}{2\pi i} \int_{\gamma} \frac{(1+zx)^k(1+zx)^\nu}{(z-a_u)(A(z))^k} dz \quad (0 \leq u \leq m).$$

Here,  $k$  is some fixed positive integer,  $\nu$  a nonintegral positive rational,  $\gamma$  a closed, counter-clockwise contour enclosing the poles of the integrand and  $x$  a real satisfying

$$(2.2) \quad |x|^{-1} > \max_{0 \leq u \leq m} |a_u|.$$

By application of Cauchy’s residue theorem, we can write

$$(2.3) \quad I_u(x) = \sum_{v=0}^m P_{uv}(x)(1+a_vx)^\nu \quad (0 \leq u \leq m)$$

where the  $P_{uv}(x)$  are polynomials with rational coefficients and degree at most  $k$  in  $x$ . Explicitly, from Rickert [15, Lemma 3.3], we have

$$(2.4) \quad P_{uv}(x) = \sum \binom{k+\nu}{h_v} (1+a_vx)^{k-h_v} x^{h_v} \prod_{\substack{0 \leq l \leq m \\ l \neq v}} \binom{-k_{ul}}{h_l} (a_v - a_l)^{-k_{ul}-h_l}$$

where  $\sum$  denotes summation over all nonnegative integers  $h_0, \dots, h_m$  with sum  $k_{uv} - 1$ , for  $k_{ab} = k + \delta_{ab}$  and  $\delta_{ab}$  the Kronecker delta. To guarantee the “independence” of the approximants, we require that  $\det_{0 \leq u, v \leq m} (P_{uv}(x))$  does not vanish for nonzero  $x$ , a consequence of Rickert’s Lemma 3.4. To be precise, one may write

$$\det_{0 \leq u, v \leq m} (P_{uv}(x)) = \left( \prod_{v=-1}^{m-1} \binom{\nu - vk}{k} \right) \cdot \left( \prod_{l=0}^m \prod_{\substack{s=0 \\ s \neq l}}^m (a_s - a_l)^{-k} \right) x^{(m+1)k}.$$

In the sections that follow, we will find asymptotics for  $|P_{uv}(1/N)|$  and  $|I_u(1/N)|$  and then study the arithmetic properties of the coefficients of  $P_{uv}(x)$ .

3. CONTOUR INTEGRAL ESTIMATES

To begin, we note that the value  $P_{uv}(x)(1+a_vx)^\nu$  ( $0 \leq u \leq m$ ) is obtained from the integral (2.1), only with the contour  $\gamma$  changed so as to enclose the integer  $a_v$  and no other  $a_l$ ’s (for  $l \neq v$ ). Setting  $x = 1/N$ , one sees that (2.2) is satisfied and it follows that the integrand of (2.1) is analytic in a suitable deleted neighbourhood of  $a_v$ . Following Hata [11], we may apply the saddle-point method as described in Dieudonné [7, chapter IX] to estimate the principal part of  $P_{uv} \left( \frac{1}{N} \right) (1+a_v/N)^\nu$  for large values of  $k$ . Explicitly, we set

$$F(z) = \log \left( 1 + \frac{z}{N} \right) - \log (|A(z)|)$$

and

$$G(z) = \left( 1 + \frac{z}{N} \right)^\nu (z - a_u)^{-1}$$

so that

$$(3.1) \quad P_{uv} \left( \frac{1}{N} \right) (1 + a_v/N)^\nu = \int_\gamma G(z) e^{kF(z)} dz.$$

The saddles of the surface  $|F(z)|$  are given by the zeros of the derivative of  $F(z)$  which, since  $x = 1/N$ , are the zeros of the polynomial (1.4) (say  $z_0, z_1, \dots, z_m$  as in §1). Since  $G(z)e^{kF(z)}$  vanishes as  $z$  tends to  $-N$  or  $\infty$  (avoiding the real branch cut from  $-N$  to  $-\infty$ ), the saddle-point method yields

**Lemma 3.1.** *As  $k \rightarrow \infty$ , the principal part of is given by*

$$P_{uv} \left( \frac{1}{N} \right) \left( 1 + \frac{a_v}{N} \right)^\nu \sim \sum_{v \leq t \leq v+1} e^{kF(z_t)} G(z_t) \sqrt{\frac{-2\pi}{kF''(z_t)}},$$

where the summation is restricted to  $t \in [1, m]$ . In particular, since the roots of satisfy

$$e^{F(z_l)} = |NA'(z_l)|^{-1} \quad (0 \leq l \leq m)$$

we may conclude that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \left| P_{uv} \left( \frac{1}{N} \right) \right| = -\log(c_6(v) \cdot N) \leq -\log(c_7 \cdot N)$$

for all  $0 \leq u, v \leq m$ .

To find asymptotics for  $|I_u(1/N)|$  requires a more delicate analysis. Since the integrand of (2.1) has a branch point at  $z = -N$ , we cannot simply apply the saddle-point method for the saddle  $z_0$  without justification (recall that  $z_0 < -N$  is real). If, however, we make the change of variables  $1 + \frac{z}{N} \rightarrow -w$ , then we may write

$$(3.2) \quad I_u \left( \frac{1}{N} \right) = \frac{(-1)^{mk} e^{\pi i \nu}}{2\pi i N^{(m+1)k}} \int_{\gamma'} \frac{w^k w^\nu dw}{(w + 1 + \frac{a_u}{N})(B(w))^k}$$

where  $B(w) = \prod_{l=0}^m \left( w + 1 + \frac{a_l}{N} \right)$  and  $\gamma' = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$  is a contour containing the poles of the integrand of (3.2) while avoiding a branch cut along the nonnegative real axis (see Figure 1).

Since

$$\left| \int_{\gamma_l} \frac{w^k w^\nu dw}{(w + 1 + \frac{a_u}{N})(B(w))^k} \right| \leq \int_0^{2\pi} \left| \frac{w^k w^\nu}{(w + 1 + \frac{a_u}{N})(B(w))^k} \right| d\theta, \quad l = 2 \text{ or } 4$$

(where  $w = Re^{i\theta}$  or  $re^{i\theta}$  respectively), we have that the contribution to (3.2) associated with the arcs  $\gamma_2$  and  $\gamma_4$  becomes negligible as  $r \rightarrow 0$  and  $R \rightarrow \infty$ . Therefore, from

$$\frac{w^k w^\nu}{(w + 1 + \frac{a_u}{N})(B(w))^k} = \begin{cases} \frac{x^k x^\nu}{(x + 1 + \frac{a_u}{N})(B(x))^k} \text{ on } \gamma_1, \\ \frac{e^{2\pi i \nu} x^k x^\nu}{(x + 1 + \frac{a_u}{N})(B(x))^k} \text{ on } \gamma_3, \end{cases}$$

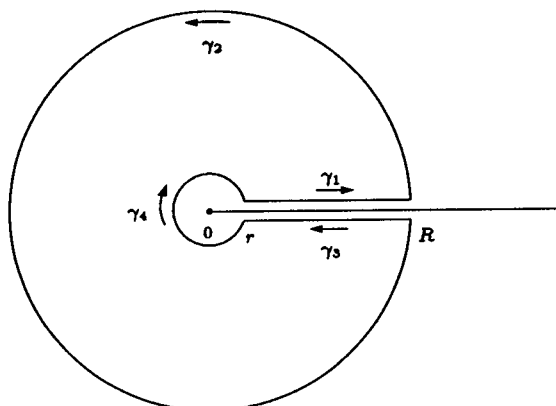


FIGURE 1

we may conclude, letting  $r \rightarrow 0$  and  $R \rightarrow \infty$ , that

$$I_u \left( \frac{1}{N} \right) = \frac{(-1)^{mk} e^{\pi i \nu} (1 - e^{2\pi i \nu})}{2\pi i N^{(m+1)k}} \int_0^\infty \frac{x^k x^\nu dx}{(x + 1 + \frac{a_u}{N}) (B(x))^k}.$$

This is readily evaluated for large  $k$  via Laplace’s method (see e.g. Dieudonné [7, Chapter IV, §2]). Since the function  $x/B(x)$  has only one critical point on the positive real axis, say  $x_0$ , and vanishes as  $x \rightarrow 0^+$  or  $x \rightarrow \infty$ , we conclude that

$$(3.3) \quad \lim_{k \rightarrow \infty} \frac{1}{k} \log \left| I_u \left( \frac{1}{N} \right) \right| = \log \left| \frac{x_0}{N^{m+1} B(x_0)} \right|.$$

Changing variables back to our original  $z$ , we find that

$$\left| \frac{x_0}{N^{m+1} B(x_0)} \right| = \left| \frac{1 + z_0/N}{A(z_0)} \right| = \left| \frac{1}{N A'(z_0)} \right|$$

for  $z_0$  as in §1. Combining this with (3.3) yields

**Lemma 3.2.**  $\lim_{k \rightarrow \infty} \frac{1}{k} \log \left| I_u \left( \frac{1}{N} \right) \right| = -\log(c_5 \cdot N) \quad (0 \leq u \leq m).$

#### 4. COEFFICIENTS OF THE APPROXIMATING POLYNOMIALS

We wish to determine certain arithmetic properties of the coefficients of the polynomials  $P_{uv}(x)$  defined in §2. Let us write

$$P_v = \prod_{\substack{0 \leq l \leq m \\ l \neq v}} \binom{k + h_l - 1}{h_l} (a_l - a_v)^{-k - h_l}.$$

From (2.4), if  $u = v$ , then

$$(4.1) \quad P_{uv}(x) = (-1)^{mk} \sum \binom{k + \nu}{h_v} (1 + a_v x)^{k - h_v} x^{h_v} P_v$$

where  $\sum$  implies the sum over nonnegative  $h_0, \dots, h_m$  satisfying  $\sum_{l=0}^m h_l = k$ . If, however,  $u \neq v$ , we have

(4.2)

$$P_{uv}(x) = (-1)^{mk} \sum \binom{k + \nu}{h_v} (1 + a_v x)^{k-h_v} x^{h_v} \left( \frac{k + h_u}{k(a_u - a_v)} \right) P_v$$

where in this latter case, the summation is over nonnegative  $h_0, \dots, h_m$  with  $\sum_{l=0}^m h_l = k - 1$ . From here on we will fix  $\nu = s/n$ . The following elementary lemma concerning primes dividing binomial coefficients will be the chief tool in determining our “arithmetic” asymptotics. It enables us to identify certain classes of prime numbers which are guaranteed to divide the numerators of the coefficients of the polynomials  $P_{uv}(x)$  defined in (2.4). Similar results have been utilized by Chudnovsky [6], Hata [11] and Heimonen, et al. [12], amongst others, in the pursuit of irrationality and linear independence measures. Suppressing dependence on  $m, n, s$  and  $k$ , let

(4.3)

$$S(r) = \{p \text{ prime} : p > \sqrt{nk + s}, (p, nk) = 1 \text{ (and, if } m = 1, (p, nk - s - n) = 1) \text{ and } \left\{ \frac{k-1}{p} \right\} > \max \left( \frac{nm-r}{nm}, \frac{r}{n} \right) \text{ for } r \text{ with } 1 \leq r < n \text{ and } pr \equiv s \pmod{n} \}$$

where we adopt the notation  $\{x\} = x - [x]$  for the fractional part of a real number  $x$ .

**Lemma 4.1.** *If  $p \in S(r)$ , then*

$$\text{ord}_p \left( \binom{k + s/n}{h_0} \binom{k + h_1 - 1}{h_1} \dots \binom{k + h_m - 1}{h_m} \right) \geq 1$$

for all nonnegative integers  $h_0, \dots, h_m$  with  $\sum_{l=0}^m h_l = k$  or  $k - 1$ .

*Proof.* Suppose that  $p \in S(r)$  does not divide the product

$$\binom{k + h_1 - 1}{h_1} \dots \binom{k + h_m - 1}{h_m}.$$

Then it follows that

$$\left\{ \frac{h_l}{p} \right\} < 1 - \left\{ \frac{k-1}{p} \right\} \quad (1 \leq l \leq m)$$

and thus, since  $p \nmid k$ , that

$$(4.4) \quad \left\{ \frac{h_l}{p} \right\} \leq 1 - \left\{ \frac{k}{p} \right\} \quad (1 \leq l \leq m).$$

From (4.3), we may therefore write

$$(4.5) \quad \sum_{l=1}^m \left\{ \frac{h_l}{p} \right\} \leq m \left( 1 - \left\{ \frac{k}{p} \right\} \right) < m \left( \frac{r}{nm} - \frac{1}{p} \right) = \frac{r}{n} - \frac{m}{p}.$$



Hence  $\sum_{l=0}^m h_l = k$  or  $k - 1$  in conjunction with

$$\left\{ \frac{k-1}{p} \right\} > \frac{r}{n} > \sum_{l=1}^m \left\{ \frac{h_l}{p} \right\}$$

yields

$$(4.6) \quad \left\{ \frac{k-h_0}{p} \right\} \leq \sum_{l=1}^m \left\{ \frac{h_l}{p} \right\} + \frac{1}{p}.$$

We wish to show that  $\text{ord}_p \binom{k+s/n}{h_0} \geq 1$ . Since  $p \nmid n$ , we have

$$\text{ord}_p \binom{k+s/n}{h_0} = \text{ord}_p \left( \frac{(nk+s)(n(k-1)+s) \cdots (n(k-h_0)+n+s)}{h_0!} \right)$$

and so by a result of Chudnovsky [6, Lemma 4.5] (recalling that  $p > \sqrt{nk+s}$ )

$$(4.7) \quad \text{ord}_p \binom{k+s/n}{h_0} = \left\{ \frac{k-\theta-h_0}{p} \right\} + \left\{ \frac{h_0}{p} \right\} - \left\{ \frac{k-\theta}{p} \right\}$$

where  $\theta = \frac{pr-s}{n}$ . We thus have that  $\text{ord}_p \binom{k+s/n}{h_0} \geq 1$  exactly when  $\left\{ \frac{h_0}{p} \right\} > \left\{ \frac{k-\theta}{p} \right\}$  or, equivalently, when

$$(4.8) \quad \left\{ \frac{k-h_0}{p} \right\} < \frac{r}{n} - \frac{s}{pn}.$$

Now, combining (4.5) and (4.6) gives

$$(4.9) \quad \left\{ \frac{k-h_0}{p} \right\} \leq \frac{r}{n} - \frac{(m-1)}{p}$$

which for  $m \geq 2$  implies  $\left\{ \frac{k-h_0}{p} \right\} \leq \frac{r}{n} - \frac{1}{p} < \frac{r}{n} - \frac{s}{pn}$ , as desired (since  $1 \leq s < n$ ).

It remains to consider the case  $m = 1$ . If we have strict inequality in (4.4), then

$$\left\{ \frac{h_1}{p} \right\} \leq 1 - \left\{ \frac{k}{p} \right\} - \frac{1}{p} < \frac{r}{n} - \frac{2}{p}$$

whence

$$\left\{ \frac{k-h_0}{p} \right\} \leq \left\{ \frac{h_1}{p} \right\} + \frac{1}{p} < \frac{r}{n} - \frac{1}{p}$$

which again implies (4.8). Similarly, we attain the inequality (4.8) if  $h_0 + h_1 = k$ .

If  $\left\{ \frac{h_0}{p} \right\} < \left\{ \frac{k-\theta}{p} \right\}$ , then

$$\left\{ \frac{k-h_0}{p} \right\} \geq \frac{r}{n} - \frac{s}{pn} + \frac{1}{p} > \frac{r}{n}$$

contradicting (4.9). To complete the proof of the lemma, we have only to show that the simultaneous equalities

$$\left\{ \frac{h_1}{p} \right\} = 1 - \left\{ \frac{k}{p} \right\},$$

$$\left\{ \frac{h_0}{p} \right\} = \left\{ \frac{k - \theta}{p} \right\}$$

and

$$h_0 + h_1 = k - 1$$

produce a contradiction. Well, the first of these implies that  $p|k + h_1$ , while the second yields  $p|n(k - h_0) + s$ . But these together with  $h_0 + h_1 = k - 1$  imply that  $p$  divides  $nk - s - n$ , contradicting our initial assumptions.  $\square$

Now if by  $\pi(x, n, s)$  we denote the number of primes  $p \leq x$  in the arithmetic progression  $bn + s$ , then, analogous to the standard prime number theorem, we have

$$\pi(x, n, s) = \frac{x}{\phi(n) \log x} \left( 1 + O\left(\frac{1}{\log x}\right) \right).$$

It follows that if  $\beta > \alpha > 0$ , then

$$(4.10) \quad \lim_{k \rightarrow \infty} \frac{1}{k} \log \prod p = \frac{\beta - \alpha}{\phi(n)}$$

where the product is over all primes  $p$  in the interval  $\alpha k < p < \beta k$  and satisfying  $p \equiv a \pmod{n}$ . The inequality  $\left\{ \frac{k-1}{p} \right\} > \max\left(\frac{nm-r}{nm}, \frac{r}{n}\right)$  defines a collection of open intervals for primes  $p$  in  $S(r)$ , of the form

$$\left( \frac{k}{l+1}, \min\left(\frac{nmk}{(l+1)nm-r}, \frac{nk}{ln+r}\right) + O(\sqrt{k}) \right), \quad l = 0, 1, 2, \dots,$$

where the shape of the error term follows from the assumption that  $p > \sqrt{nk+s}$ . From(4.10), we may therefore write

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \prod_{p \in S(r)} p = \frac{1}{\phi(n)} \sum_{l=0}^{\infty} \left( \min\left(\frac{nm}{(l+1)nm-r}, \frac{n}{ln+r}\right) - \frac{1}{l+1} \right).$$

Since, from Bateman and Erdélyi [4], the function  $\psi(z) = \frac{d \log \Gamma(z)}{dz}$  satisfies

$$\psi(\beta) - \psi(\alpha) = \sum_{l=0}^{\infty} \left( \frac{1}{l+\alpha} - \frac{1}{l+\beta} \right),$$

we may conclude that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \prod_{p \in S(r)} p = \frac{1}{\phi(n)} \left( \psi(1) - \psi\left(\max\left\{\frac{nm-r}{nm}, \frac{r}{n}\right\}\right) \right)$$

whence, recalling the equality  $\psi(1) = -\gamma$ ,

$$(4.11) \quad \lim_{k \rightarrow \infty} \frac{1}{k} \log \prod_{\substack{1 \leq r < n \\ (r,n)=1}} \prod_{p \in S(r)} p = -\gamma - \frac{1}{\phi(n)} \sum_{\substack{1 \leq r < n \\ (r,n)=1}} \psi\left(\max\left\{\frac{nm-r}{nm}, \frac{r}{n}\right\}\right).$$

We thus have

**Lemma 4.2.** *For each  $k$ , there is a rational number  $C_k$  such that*

$$C_k P_{uv} \left( \frac{1}{N} \right) \in \mathbb{Z} \text{ for all } 0 \leq u, v \leq m$$

and

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \log |C_k| \leq \log(c_4 N / c_8)$$

for  $c_4$  and  $c_8$  as in §1.

*Proof.* If we define  $\mu_{n,r} = \prod_{p|n} p^{[r/(p-1)]}$ , then by Chudnovsky [6, Lemma 4.1], we have

$$(4.12) \quad \mu_{n,r} \cdot n^r \binom{k + s/n}{r} \in \mathbb{Z}$$

for  $k \geq r \geq 0$ . Let

$$\alpha = \max_{0 \leq l < v \leq m} \text{ord}_p(a_l - a_v)$$

and

$$\beta_v(h_0, \dots, h_m) = \mu_{n,h_v} \cdot n^{h_v} \prod_{\substack{0 \leq l \leq m \\ l \neq v}} (a_l - a_v)^{h_l}$$

where  $\sum_{0 \leq l \leq m} h_l = k$  or  $k - 1$ .

If  $p$  is prime and  $p$  does not divide  $n$ , then

$$\begin{aligned} \text{ord}_p(\beta_v(h_0, \dots, h_m)) &= \text{ord}_p \prod_{\substack{0 \leq l \leq m \\ l \neq v}} (a_l - a_v)^{h_l} \\ &= \sum_{\substack{0 \leq l \leq m \\ l \neq v}} h_l \text{ord}_p(a_l - a_v) \leq \sum_{0 \leq l \leq m} \alpha h_l \leq \alpha k. \end{aligned}$$

On the other hand, if  $p|n$ , then

$$\begin{aligned} \text{ord}_p(\beta_v(h_0, \dots, h_m)) &\leq \left\lfloor \frac{h_v}{p-1} \right\rfloor + h_v \text{ord}_p n + \sum_{\substack{0 \leq l \leq m \\ l \neq v}} h_l \text{ord}_p(a_l - a_v) \\ &\leq h_v \left( \text{ord}_p n + \frac{1}{p-1} \right) + \sum_{\substack{0 \leq l \leq m \\ l \neq v}} \alpha h_l \\ &\leq \max \left( \text{ord}_p n + \frac{1}{p-1}, \alpha \right) \sum_{0 \leq l \leq m} h_l \\ &\leq \max \left( \text{ord}_p n + \frac{1}{p-1}, \alpha \right) k. \end{aligned}$$

Since  $h_l \leq k$  for  $0 \leq l \leq m$ , (1.1) and (1.2) yield that

$$(4.13) \quad \beta_v(h_0, \dots, h_m)^{-1} \cdot (c_2 \cdot c_3)^k$$

is an integer for all  $0 \leq v \leq m$  and nonnegative integers  $h_0, \dots, h_m$  with sum  $k - 1$ , or  $k$ .

From (4.1), (4.2), (4.12), (4.13) and Lemma 4.1, we therefore have that

$$(c_4 \cdot N)^k \cdot c_2 \cdot k \left( \prod_{\substack{1 \leq r \leq n \\ (r,n)=1}} \prod_{p \in S(r)} p \right)^{-1} P_{uv} \left( \frac{1}{N} \right)$$

is also an integer for  $0 \leq u, v \leq m$  (using the fact that  $P_{uv}(x)$  has degree at most  $k$ ). The lemma then follows from equality (4.11).  $\square$

### 5. PROOF OF THEOREM 1.1

To apply the results of the previous sections, we first state a lemma that connects our approximating polynomials with bounds of the form (0.1). Let  $\theta_0, \theta_1, \dots, \theta_m$  be distinct real numbers with  $\theta_r = 1$  for some  $r$ . Then

**Lemma 5.1.** *Suppose there are positive real numbers  $P$  and  $Q$  such that, for  $\epsilon > 0$  and each positive integer  $k$  greater than an effective constant  $k_0$ , we can find integers  $P_{uvk}$  ( $0 \leq u, v \leq m$ ) with nonzero determinant,*

$$\frac{1}{k} \log |P_{uvk}| \leq P + \epsilon \quad (0 \leq u, v \leq m)$$

and

$$\frac{1}{k} \log \left| \sum_{v=0}^m P_{uvk} \theta_v \right| \leq -Q + \epsilon \quad (0 \leq u \leq m).$$

Then (0.1) holds for any  $\lambda > 1 + \frac{P}{Q}$  and  $q \geq q_0(\theta_0, \dots, \theta_m, \lambda)$  effectively computable.

*Proof.* This follows directly from Lemma 2.1 in [14].  $\square$

To complete the proof of Theorem 1.1, we write

$$\theta_l = \left( 1 + \frac{a_l}{N} \right)^{s/n} \quad (0 \leq l \leq m)$$

and

$$P_{uvk} = C_k P_{uv} \left( \frac{1}{N} \right) \quad (0 \leq u, v \leq m)$$

where  $C_k$  is as in Lemma 4.2. Then from Lemmas 3.1 and 4.2, we may take

$$P = \log \left( \frac{c_4}{c_7 \cdot c_8} \right)$$

and

$$Q = \log \left( \frac{c_5 \cdot c_8}{c_4} \right)$$

in Lemma 5.1. If  $c_7 \cdot c_8 < c_4 < c_5 \cdot c_8$ , then both of these quantities are positive and, recalling that  $\det_{0 \leq u, v \leq m} (P_{uv}(1/N))$  is nonzero, we can utilize this last lemma to obtain the desired result. Since we can make Lemma 5.1 effective, the same is true for Theorem 1.1.  $\square$

6. PROOF OF THEOREM 1.2

Let  $x_0, \dots, x_m, a_0, \dots, a_m, s, n, N$  and  $X$  be as in §1,  $C_k$  and  $P_{uvk}$  as in §4 and §5 and write

$$L = \sum_{u=0}^m x_u \cdot \left(1 + \frac{a_u}{N}\right)^{s/n}.$$

Then the fact that  $\det_{0 \leq u, v \leq m} (P_{uv}(1/N)) \neq 0$  implies that we can find  $m$  of the  $C_k I_u(1/N)$  ( $0 \leq u \leq m$ ), say  $C_k I_1(1/N), \dots, C_k I_m(1/N)$  with

(6.1) 
$$L, C_k I_1\left(\frac{1}{N}\right), \dots, C_k I_m\left(\frac{1}{N}\right)$$

independent forms in the numbers  $(1 + a_0/N)^{s/n}, \dots, (1 + a_m/N)^{s/n}$ . Following Fel'dman [9], we consider the determinant

(6.2) 
$$\Delta_k = \begin{vmatrix} x_0 & x_1 & \dots & x_m \\ P_{10k} & P_{11k} & \dots & P_{1mk} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ P_{m0k} & P_{m1k} & \dots & P_{mmk} \end{vmatrix} = L \cdot \Delta_{k,0} + \sum_{l=1}^m C_k I_l\left(\frac{1}{N}\right) \Delta_{k,l}$$

where  $\Delta_{k,l}$  ( $0 \leq l \leq m$ ) are the cofactors of the elements of the first column of  $\Delta_k$ . Since the  $P_{uvk}$ 's and  $x_v$ 's are integers, the independence of the forms in (6.1) ensures that  $|\Delta_k| \geq 1$  and thus (6.2) implies that

(6.3) 
$$|L| \cdot |\Delta_{k,0}| + \sum_{l=1}^m |C_k I_l(1/N)| \cdot |\Delta_{k,l}| \geq 1.$$

Define, for  $\sigma \in S_m$  (i.e.  $\sigma$  a permutation on  $\{1, 2, \dots, m\}$ ) and  $v = 1, 2, \dots, m$ ,

$$P(\sigma) = P_{\sigma(1)1k} P_{\sigma(2)2k} \dots P_{\sigma(m)mk}$$

and

$$P_v(\sigma) = P_{\sigma(1)1k} \dots P_{\sigma(v-1)(v-1)k} P_{\sigma(v+1)(v+1)k} \dots P_{\sigma(m)mk}.$$

Since

$$\Delta_{k,0} = \begin{vmatrix} P_{11k} & \dots & P_{1mk} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ P_{m1k} & \dots & P_{mmk} \end{vmatrix}$$

we have that

$$|\Delta_{k,0}| \leq m! \cdot \max \{|P(\sigma)| : \sigma \in S_m\}$$

and thus Lemmas 3.1 and 4.2 yield

(6.4) 
$$\limsup_{k \rightarrow \infty} \frac{1}{k} \log |\Delta_{k,0}| \leq m \log \left(\frac{c_4}{c_8}\right) - \sum_{v=1}^m \log(c_6(v)).$$

Similarly, if  $1 \leq l \leq m$ , then  $\Delta_{k,l}$  satisfies

$$|\Delta_{k,l}| \leq (m-1)! \cdot \sum_{v=1}^m |x_v| \cdot \max\{|P_v(\sigma)| : \sigma \in S_m\}$$

and Lemmas 3.2 and 4.2 imply (for  $1 \leq v, l \leq m$ ) that

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \frac{1}{k} \log \left( C_k I_l \left( \frac{1}{N} \right) \max\{|P_v(\sigma)| : \sigma \in S_m\} \right) \\ (6.5) \quad & \leq m \log \left( \frac{c_4}{c_8} \right) - \log(c_5) - \min_{\substack{1 \leq l \leq m \\ v \neq l}} \sum_{v=1}^m \log(c_6(v)). \end{aligned}$$

Let us denote the right-hand side of (6.4) by  $D_1$  and the right-hand side of (6.5) by  $D_2$ . Given  $\epsilon_1 > 0$ , from (6.4) we can find a  $k_0 = k_0(\epsilon_1)$  for which

$$(6.6) \quad |\Delta_{k,0}| \leq \frac{1}{2} \cdot e^{(D_2 + \epsilon_1)k}$$

for  $k \geq k_0$ . Inequality (6.5) implies that for  $X \geq X_0(\epsilon_1)$ , we may find a  $k = k(X)$  with both

$$(6.7) \quad m! \cdot \sum_{l=1}^m |C_k I_l \left( \frac{1}{N} \right)| \cdot \max\{|P_v(\sigma)| : \sigma \in S_m, 1 \leq v \leq m\} \leq \frac{1}{2X}$$

and

$$(6.8) \quad X \leq e^{-(D_1 - \epsilon_1)k}.$$

Now (6.3) and (6.7) yield  $|L| \cdot |\Delta_{k,0}| \geq 1/2$  whence

$$|L| \geq \frac{1}{2|\Delta_{k,0}|} \geq e^{-(D_2 + \epsilon_1)k}$$

via (6.6). Applying (6.8), then, gives

$$|L| \geq X^{\frac{D_2 + \epsilon_1}{D_1 - \epsilon_1}} \geq X^{D_2/D_1 + \epsilon}$$

for suitably small  $\epsilon_1$  relative to  $\epsilon$  and  $X \geq X_0(\epsilon)$ . This completes the proof of Theorem 1.2.  $\square$

## 7. THE CASE OF A SINGLE BINOMIAL FUNCTION

In the following analysis, we specialize Theorem 1.1 by fixing  $m = 1$ , and writing  $a_1 = a$  (the ordering of the  $a_u$ 's, as given in §1, is unimportant here). Further, take  $n \geq 3$  and define

$$c_9 = \prod_{p|n} p^{\min\{\text{ord}_p a, \text{ord}_p n + \frac{1}{p-1}\}}$$

and

$$c_{10} = \exp \left( \frac{1}{\phi(n)} \sum_{\substack{1 \leq r < n/2 \\ (r,n)=1}} \left( \psi \left( \frac{n-r}{n} \right) - \psi \left( \frac{r}{n} \right) \right) \right).$$

We then obtain the following result of Chudnovsky [6, Theorem 5.3] as a corollary to Theorem 1.1:

**Theorem 7.1.** *If  $a, N, s$  and  $n$  are integers with  $N > |a|, 1 \leq s < n, (s, n) = 1, \epsilon > 0$  and  $(\sqrt{N+a} - \sqrt{N})^2 \cdot c_{10} < c_9$ , then*

$$(7.1) \quad \left| \left(1 + \frac{a}{N}\right)^{s/n} - \frac{p}{q} \right| > q^{-\lambda-\epsilon}$$

for  $p$  and  $q$  integers with  $q \geq q_0(\epsilon, a, N, s, n)$  and

$$\lambda = 1 - \frac{\log \left\{ (\sqrt{N+a} + \sqrt{N})^2 \cdot c_{10}/c_9 \right\}}{\log \left\{ (\sqrt{N+a} - \sqrt{N})^2 \cdot c_{10}/c_9 \right\}}.$$

*Proof.* From (1.1) and (1.2), we have that  $c_1 = c_2 = a$ . Since  $A(z) = z(z - a)$ , the zeros of polynomial (1.4) are given by  $z_0 = -N - \sqrt{N^2 + aN}$  and  $z_1 = -N + \sqrt{N^2 + aN}$ . It follows that

$$c_5 = 2N + a + 2\sqrt{N^2 + aN} = (\sqrt{N+a} + \sqrt{N})^2$$

and

$$c_7 = \left| 2N + a - 2\sqrt{N^2 + aN} \right| = (\sqrt{N+a} - \sqrt{N})^2$$

and so we may apply Theorem 1.1 to deduce a bound of the form (7.1) with

$$\begin{aligned} \lambda &= 1 + \frac{\log(c_4/c_7 \cdot c_8)}{\log(c_5 \cdot c_8/c_4)} \\ &= 1 + \frac{\log \left( a^2 \cdot c_3 / (\sqrt{N+a} - \sqrt{N})^2 \cdot c_8 \right)}{\log \left( (\sqrt{N+a} + \sqrt{N})^2 \cdot c_8 / a^2 \cdot c_3 \right)} \\ &= 1 - \frac{\log \left( (\sqrt{N+a} + \sqrt{N})^2 \cdot c_3 / c_8 \right)}{\log \left( (\sqrt{N+a} - \sqrt{N})^2 \cdot c_3 / c_8 \right)} \end{aligned}$$

provided

$$(\sqrt{N+a} - \sqrt{N})^2 \cdot c_3 < c_8 < (\sqrt{N+a} + \sqrt{N})^2 \cdot c_3.$$

It remains to show that  $c_3/c_8 = c_{10}/c_9$  or, equivalently, that

$$(7.2) \quad \log(c_8 \cdot c_{10}) = \log(c_3 \cdot c_9).$$

Recalling the various definitions of our constants, the left-hand side of equation (7.2) becomes

$$(7.3) \quad \begin{aligned} &\frac{2}{\phi(n)} \sum_{\substack{1 \leq r < n/2 \\ (r,n)=1}} \left( \psi(1) - \psi\left(\frac{n-r}{n}\right) \right) + \frac{1}{\phi(n)} \sum_{\substack{1 \leq r < n/2 \\ (r,n)=1}} \left( \psi\left(\frac{n-r}{n}\right) - \psi\left(\frac{r}{n}\right) \right) \\ &= \frac{1}{\phi(n)} \sum_{\substack{1 \leq r < n/2 \\ (r,n)=1}} \left( 2\psi(1) - \psi\left(\frac{r}{n}\right) - \psi\left(\frac{n-r}{n}\right) \right). \end{aligned}$$

In order to express  $\psi(z)$  with rational arguments as a finite combination of elementary functions, we appeal to a result of Gauss (see e.g. [4]):

$$\begin{aligned} \psi\left(\frac{r}{n}\right) &= -\gamma - \log n - \frac{1}{2}\pi \cot\left(\frac{\pi r}{n}\right) \\ &\quad + \sum'_{1 \leq k \leq n/2} \cos(2\pi kr/n) \log(2 - 2 \cos(2\pi k/n)). \end{aligned}$$

Here the prime over the summation indicates that only half the value associated with  $k = n/2$  is applied to the sum. The finite sum in (7.3) thus may be written as (again recalling that  $\psi(1) = -\gamma$ )

$$(7.4) \quad \log n - \frac{1}{\phi(n)} \sum_{\substack{1 \leq r < n \\ (r,n)=1}} \sum'_{1 \leq k \leq n/2} \cos(2\pi kr/n) \log(2 - 2 \cos(2\pi k/n)).$$

Now, if we let  $q(n) = \prod_{p|n} p$ , then

$$(7.5) \quad \sum_{\substack{1 \leq r < n/2 \\ (r,n)=1}} \cos\left(\frac{2\pi kr}{n}\right) = \phi^*\left(\frac{(n,k)}{n}q(n)\right) \cdot \mu\left(\frac{n}{(n,k)}\right) \cdot \frac{n}{q(n)}$$

where  $\phi^*$  is the Euler totient for integer arguments, zero otherwise, and  $\mu$  is the Möbius function. To see this, express the above sum in terms of primitive  $n/(n,k)$ th roots of unity. Also, if  $d > 2$  is a fixed divisor of  $n$ ,

$$\begin{aligned} \prod_{\substack{1 \leq k < n/2 \\ (n,k)=n/d}} \left(2 - 2 \cos\left(\frac{2k\pi}{n}\right)\right) &= \prod_{\substack{1 \leq j < d/2 \\ (j,d)=1}} \left(2 - 2 \cos\left(\frac{2j\pi}{d}\right)\right) \\ &= \prod_{\substack{1 \leq j < d \\ (j,d)=1}} (1 - e^{2\pi ij/d}). \end{aligned}$$

If  $d$  is prime, then this is the product of the roots of the polynomial

$$\frac{(x + 1)^d - 1}{x} = x^{d-1} + dx^{d-2} + \dots + d$$

and hence equal to  $d$ . If, however,  $d$  has at least two distinct prime factors, then it is not difficult to show (see e.g. Washington [19, Prop. 2.8]) that

$$\prod_{\substack{1 \leq j < d \\ (j,d)=1}} (1 - e^{2\pi ij/d}) = 1.$$

Collecting these facts together with (7.5) yields



$$\begin{aligned}
 & \sum_{\substack{1 \leq r < n \\ (r,n)=1}} \sum_{1 \leq k < n/2} \cos\left(\frac{2\pi kr}{n}\right) \log(2 - 2 \cos(2\pi k/n)) \\
 &= \sum_{\substack{2 < d \leq n \\ d|n}} \phi^*(q(n)/d) \cdot \mu(d) \cdot \frac{n}{q(n)} \cdot \log\left(\prod_{\substack{1 \leq k < n/2 \\ (n,k)=n/d}} \left(2 - 2 \cos\left(\frac{2k\pi}{n}\right)\right)\right) \\
 &= - \sum_{\substack{p|n \\ p > 2}} \phi(q(n)/p) \frac{n}{q(n)} \log(p) \\
 &= - \sum_{\substack{p|n \\ p > 2}} \frac{\phi(n)}{p-1} \cdot \log(p)
 \end{aligned}$$

where these last two sums are over  $p$  prime.

This, with (7.3) and (7.4), implies that

$$\log(c_8 \cdot c_{10}) = \log n + \sum_{p|n} \frac{\log p}{p-1} = \log(n\mu_n)$$

where  $\mu_n = \prod_{p|n} p^{1/(p-1)}$  (the contribution from  $p = 2$ , for even  $n$ , is obtained by taking  $k = n/2$  in (7.4)). Now  $c_3 \cdot c_9$  satisfies

$$\begin{aligned}
 c_3 \cdot c_9 &= \prod_{p|n} p^{\max\{\text{ord}_p(n/a) + \frac{1}{p-1}, 0\} + \min\{\text{ord}_p(a), \text{ord}_p(n) + \frac{1}{p-1}\}} \\
 &= \prod_{p|n} p^{\text{ord}_p(n) + \frac{1}{p-1}} = n\mu_n
 \end{aligned}$$

whence equality (7.2) holds, finishing the proof. □

Application of this result with  $n = 3, s = 1, a = 3$  and  $N = 125$  enables one to obtain the bound (0.2) while other choices lead to a variety of examples, including those discussed in [6] and [12].

### 8. SOME APPLICATIONS

As the previous section indicates, it is possible to obtain versions of Theorem 1.1 (and Theorem 1.2, for that matter), for certain  $m$ , which are more explicit than that stated in §1. Though we will not explore these aspects here, we note that Rickert’s Theorem of [15] (compare to Theorem 1.1 with  $n = m = 2$  and  $a_u \in \{-1, 0, 1\}$ ) essentially follows from the fact that the zeros  $z_0, z_1$  and  $z_2$  of polynomial (1.4) are well approximated by  $-3N/2, -1/\sqrt{3}$  and  $1/\sqrt{3}$ , respectively.

In the case  $n = m = 2$ , we obtain nontrivial bounds from Theorem 1.1 whenever  $\lambda < 2$ . For larger values of  $n$  or  $m$ , however, we require not only that  $\lambda < n$ , but also that  $\lambda$  is smaller than that induced by any proper subset of the related numbers  $\theta_i$  in (0.1) ( $1 \leq i \leq m$ ). By way of example, direct application of Theorem 1.1 with  $n = 3, m = 2, N = 8$ , and  $a_i \in \{-2, -1, 0\}$  yields

$$(8.1) \quad \max \left\{ \left| \sqrt[3]{6} - \frac{p_1}{q} \right|, \left| \sqrt[3]{7} - \frac{p_2}{q} \right| \right\} > q^{-2.66985}$$

**Table 1:** Cases of Theorem 1.1 for  $m = 2$  :

$\theta_1, \theta_2$	$N, a_i \neq 0$	$\lambda$	$\theta_1, \theta_2$	$N, a_i \neq 0$	$\lambda$
$\sqrt{2}, \sqrt{3}$	$7^2, -1, 1$	1.79155	$\sqrt{10}, \sqrt{11}$	$3^2, 1, 2$	1.98873
$\sqrt{2}, \sqrt{7}$	$82^2, 3, 4$	1.88908	$\sqrt{10}, \sqrt{22}$	$136^2, -6, 6$	1.76160
$\sqrt{2}, \sqrt{11}$	$10^2, -2, -1$	1.74788	$\sqrt{10}, \sqrt{31}$	$22^2, 6, 12$	1.99858
$\sqrt{2}, \sqrt{17}$	$4^2, 1, 2$	1.89813	$\sqrt{10}, \sqrt{38}$	$6^2, 2, 4$	1.88331
$\sqrt{2}, \sqrt{23}$	$24^2, -1, 2$	1.97500	$\sqrt{14}, \sqrt{15}$	$4^2, -2, -1$	1.91627
$\sqrt{2}, \sqrt{29}$	$140^2, 2, 4$	1.62588	$\sqrt{15}, \sqrt{17}$	$4^2, -1, 1$	1.90671
$\sqrt{2}, \sqrt{51}$	$7^2, 1, 2$	1.79008	$\sqrt{19}, \sqrt{22}$	$61^2, -3, 3$	1.79694
$\sqrt{3}, \sqrt{5}$	$123^2, -6, -4$	1.82227	$\sqrt{21}, \sqrt{23}$	$5^2, -4, -2$	1.96101
$\sqrt{3}, \sqrt{23}$	$5^2, -2, 2$	1.94595	$\sqrt{23}, \sqrt{47}$	$48^2, -4, -1$	1.95362
$\sqrt{3}, \sqrt{26}$	$5^2, 1, 2$	1.84700	$\sqrt{34}, \sqrt{35}$	$6^2, -2, -1$	1.81853
$\sqrt{3}, \sqrt{29}$	$5^2, 2, 4$	1.93291	$\sqrt{34}, \sqrt{38}$	$6^2, -2, 2$	1.89037
$\sqrt{3}, \sqrt{35}$	$71^2, -1, 2$	1.83947	$\sqrt{35}, \sqrt{37}$	$6^2, -1, 1$	1.81607
$\sqrt{3}, \sqrt{47}$	$7^2, -2, -1$	1.79307	$\sqrt{37}, \sqrt{38}$	$6^2, 1, 2$	1.81372
$\sqrt{5}, \sqrt{11}$	$199^2, -1, 4$	1.90617	$\sqrt[3]{3}, \sqrt[3]{21}$	$3^3, -6, -3$	2.57831
$\sqrt{5}, \sqrt{47}$	$7^2, -4, -2$	1.85791	$\sqrt[3]{3}, \sqrt[3]{30}$	$3^3, -3, 3$	2.50876
$\sqrt{5}, \sqrt{79}$	$9^2, -2, -1$	1.75964	$\sqrt[3]{5}, \sqrt[3]{9}$	$3^3, -3, -2$	2.52360
$\sqrt{5}, \sqrt{82}$	$9^2, -1, 1$	1.75893	$\sqrt[3]{7}, \sqrt[3]{9}$	$2^3, -1, 1$	2.57831
$\sqrt{5}, \sqrt{93}$	$29^2, -4, 4$	1.82863	$\sqrt[3]{22}, \sqrt[3]{43}$	$14^3, 6, 8$	2.22145
$\sqrt{6}, \sqrt{23}$	$5^2, -2, -1$	1.85558	$\sqrt[3]{23}, \sqrt[3]{25}$	$3^3, -4, -2$	2.78961
$\sqrt{6}, \sqrt{26}$	$5^2, -1, 1$	1.85115	$\sqrt[3]{25}, \sqrt[3]{26}$	$3^3, -2, -1$	2.13916
$\sqrt{6}, \sqrt{59}$	$169^2, -5, 5$	1.79760	$\sqrt[3]{25}, \sqrt[3]{29}$	$3^3, -2, 2$	2.73786
$\sqrt{7}, \sqrt{38}$	$37^2, -1, 3$	1.99312	$\sqrt[3]{26}, \sqrt[3]{28}$	$3^3, -1, 1$	2.13097
$\sqrt{7}, \sqrt{55}$	$37^2, 3, 6$	1.84857	$\sqrt[3]{28}, \sqrt[3]{29}$	$3^3, 1, 2$	2.12328
$\sqrt{7}, \sqrt{62}$	$8^2, -2, -1$	1.77425	$\sqrt[3]{29}, \sqrt[3]{30}$	$3^3, 2, 3$	2.48614
$\sqrt{7}, \sqrt{65}$	$8^2, -1, 1$	1.77324	$\sqrt[3]{29}, \sqrt[3]{31}$	$3^3, 2, 4$	2.69341
$\sqrt{7}, \sqrt{83}$	$82^2, -1, 3$	1.88907	$\sqrt[3]{30}, \sqrt[3]{33}$	$3^3, 3, 6$	2.45379

for  $q \geq q_0$  effective. If we instead consider  $n = 3, m = 1, N = 101847558$  and  $a_1 = 5$ , we find (see Chudnovsky [6, Table 1])

$$\left| \sqrt[3]{6} - \frac{p}{q} \right| > q^{-2.32056}$$

for  $q \geq q_1$  effectively computable, so that (8.1) is in fact weaker than the latter result.

In Table 1, we collect examples of bounds of the form (0.1) for pairs of square roots and cube roots of fairly small integers. We find these examples by considering integers “close” to squares (or cubes) which themselves possess “large” square (or cubic) factors. For instance, we obtain a simultaneous measure for  $(\sqrt{3}, \sqrt{5})$  because  $123^2 = 3 \cdot 71^2 + 6 = 5 \cdot 55^2 + 4$ . It is not difficult to show that if Theorem 1.1 provides a bound for  $(\theta_1, \theta_2) = (\sqrt{A}, \sqrt{B})$ , then it yields like bounds for  $(\sqrt{A}, \sqrt{AB})$  and  $(\sqrt{B}, \sqrt{AB})$  with the same value for  $\lambda$ . For example, we have, taking  $N = 49$ , and  $a_i \in \{-1, 0, 1\}$ ,

$$\max \left\{ \left| \sqrt{2} - \frac{p_1}{q} \right|, \left| \sqrt{3} - \frac{p_2}{q} \right| \right\} > q^{-1.79155} \quad (q \geq q_0)$$

while  $N = 100$  and  $a_i \in \{-4, -2, 0\}$  gives

$$\max \left\{ \left| \sqrt{2} - \frac{p_1}{q} \right|, \left| \sqrt{6} - \frac{p_2}{q} \right| \right\} > q^{-1.79155} \quad (q \geq q_1)$$

and  $N = 144$  and  $a_i \in \{0, 3, 6\}$  yields

$$\max \left\{ \left| \sqrt{3} - \frac{p_1}{q} \right|, \left| \sqrt{6} - \frac{p_2}{q} \right| \right\} > q^{-1.79155} \quad (q \geq q_2).$$

We will restrict ourselves in Table 1 to listing only examples which are “primitive” in this sense (to avoid duplication). Analogous results to this hold for other values of  $n$  and  $m$  and it is of interest to note that while the measure produced from Theorem 1.1 (taking  $N = 8$  and  $a_i \in \{0, 1, 2\}$ ) for  $(\theta_1, \theta_2) = (\sqrt[3]{9}, \sqrt[3]{10})$  is weaker than that obtained for  $\sqrt[3]{10}$  alone, the same is not known to be the case for the related pair  $(\sqrt[3]{3}, \sqrt[3]{30})$ .

For larger values of  $m$ , a couple of interesting examples are

$$\max \left\{ \left| \sqrt{2} - \frac{p_1}{q} \right|, \left| \sqrt{3} - \frac{p_2}{q} \right|, \left| \sqrt{47} - \frac{p_3}{q} \right|, \left| \sqrt{51} - \frac{p_4}{q} \right| \right\} > q^{-1.67429}$$

for  $q \geq q_0$ , and

$$\max \left\{ \left| \sqrt[3]{25} - \frac{p_1}{q} \right|, \left| \sqrt[3]{26} - \frac{p_2}{q} \right|, \left| \sqrt[3]{28} - \frac{p_3}{q} \right|, \left| \sqrt[3]{29} - \frac{p_4}{q} \right| \right\} > q^{-1.86545}$$

for  $q \geq q_1$ , where both  $q_0$  and  $q_1$  are effective. These follow by taking  $N = 49, a_u \in \{-2, -1, 0, 1, 2\}, n = 2$  and  $N = 27, a_u \in \{-2, -1, 0, 1, 2\}, n = 3$ , respectively, in Theorem 1.1.

It appears to be rather harder to find “attractive” examples of the utility of Theorem 1.2. Part of the difficulty is that if  $\theta_1, \dots, \theta_m$  are elements of a real field of degree  $m + 1$  over  $\mathbb{Q}$  with  $1, \theta_1, \dots, \theta_m$  linearly independent over  $\mathbb{Q}$ , then

$$|x_0 + x_1\theta_1 + \dots + x_m\theta_m| > X^{-m}$$

for sufficiently large  $X = \max_{0 \leq j \leq m} |x_j|$ , via consideration of norms (see e.g. Cassels [5, Chapter V]). We are therefore only interested in bounds upon linear forms which sharpen this. One such result follows from Theorem 1.2 with  $n = m = 2, N = 816^2$  and  $a_i \in \{0, 2, 4\}$ , whence

$$|x + y\sqrt{2} + z\sqrt{985}| > X^{-2.89514}$$

for  $x, y$  and  $z$  integers,  $X = \max \{|x|, |y|, |z|\}$  and  $X \geq X_0$  effectively computable. We may also attain this bound via transference from Theorem 1.1 (from which one obtains  $\lambda = 1.59144$  as a simultaneous irrationality measure for  $(\sqrt{2}, \sqrt{985})$ ), with direct application of Theorem 1.2 providing an improvement only noticeable in the sixth decimal place of  $\lambda_1$ . In general, Theorem 1.2 represents only a slight sharpening of the corresponding transference result if the  $a_i$ ’s are “evenly” distributed, but is more effective otherwise.

The problem of solving simultaneous Pell’s equations is discussed in [14], motivated by the desire to find elliptic curves over  $\mathbb{Q}(\sqrt{5})$  with good reduction away from

the prime 2, and in [18], related to solutions of Thue equations. Results obtained from Theorem 1.1 may be applied to this problem. For instance, to solve

$$(8.2) \quad x^2 - 3z^2 = u, \quad y^2 - 5z^2 = v$$

we use the bound

$$\max \left\{ \left| \sqrt{3} - \frac{p_1}{q} \right|, \left| \sqrt{5} - \frac{p_2}{q} \right| \right\} > q^{-1.82227}$$

( $q \geq q_0$ ) from Table 1. Arguing as in [15], one may deduce that

$$\max\{|x|, |y|, |z|\} \leq k_0(\max\{|u|, |v|\})^{5.62652}$$

for an effectively computable absolute constant  $k_0$ . For fixed  $u$  and  $v$ , this enables us to find all solutions to (8.2). The examples considered by Rickert [15] are the pair of simultaneous equations

$$x^2 - 2z^2 = u, \quad y^2 - 3z^2 = v$$

and

$$x^2 - 2z^2 = u, \quad y^2 - 58z^2 = v.$$

For these, we obtain

$$\max\{|x|, |y|, |z|\} \leq k_1(\max\{|u|, |v|\})^{4.79732}$$

and

$$\max\{|x|, |y|, |z|\} \leq k_2(\max\{|u|, |v|\})^{2.67294}$$

respectively, where  $k_1$  and  $k_2$  are effective. Similar results for simultaneous Pell-type equations (of greater degree) may also be readily produced.

## 9. NORM FORM EQUATIONS

In [10], Fel'dman considers norm form equations

$$N_{K/\mathbb{Q}}(x_0w_0 + \cdots + x_mw_m) = f(x_0, \dots, x_m)$$

where  $w_0, \dots, w_m$  are certain algebraic numbers,  $f$  a polynomial of small degree and  $K = \mathbb{Q}(w_0, \dots, w_m)$ . To solve these, he essentially utilizes a version of Theorem 1.2 to simultaneously approximate the  $w_i$ 's. We will show that a slight refinement of his techniques, utilizing both Theorems 1.1 and 1.2, enables one to effectively solve the equation ( $u$  constant)

$$(9.1) \quad N_{K/\mathbb{Q}} \left( x + y \sqrt[4]{M^4 - 1} + z \sqrt[4]{M^4 + 1} \right) = u$$

for all  $M \geq 6$ . Similarly, we find all solutions to

$$(9.2) \quad N_{K/\mathbb{Q}} \left( x + y \sqrt[6]{M^6 - 1} + z \sqrt[6]{M^6 + 1} \right) = u$$

for  $M \geq 4$ .

Let us concentrate on equation (9.1); the argument is essentially unchanged for (9.2). We will require the observation that Theorems 1.1 and 1.2 remain valid if  $p_0, \dots, p_m, q$  and  $x_0, \dots, x_m$  are taken to be integers in a fixed imaginary quadratic field (see Fel'dman [9]). Let

$$L = x + y \sqrt[4]{M^4 - 1} + z \sqrt[4]{M^4 + 1}$$

for  $x, y$  and  $z$  integers, not all zero, and let  $L_{s,t}$  ( $0 \leq s, t \leq 3$ ) be the conjugates of  $L$  associated with the extension  $K = \mathbb{Q}(\sqrt[4]{M^4 - 1}, \sqrt[4]{M^4 + 1})$ , as given by

$$(9.3) \quad L_{s,t} = x + y \sqrt[4]{M^4 - 1} i^s + z \sqrt[4]{M^4 + 1} i^t.$$

Reasoning as in [10, §3], all but four of these conjugates satisfy

$$(9.4) \quad |L_{s,t}| \geq c_0 X$$

for  $c_0$  effective,  $X = \max\{|x|, |y|, |z|\}$ . Let  $L_i$  ( $1 \leq i \leq 4$ ) be the remaining conjugates, ordered so that

$$(9.5) \quad |L_1| \leq |L_2| \leq |L_3| \leq |L_4|.$$

We apply Theorem 1.2 (with  $x, y, z \in \mathbb{Z}[i]$ ) to estimate the smallest conjugate, finding that

$$(9.6) \quad |L_1| > X^{-6.35740}$$

for  $X \geq X_0$  and  $M \geq 6$ . It follows from (9.3) that

$$(9.7) \quad |L_2| \geq \frac{1}{2}|L_2 - L_1| = \frac{1}{2}|y \sqrt[4]{M^4 - 1} (i^{s_1} - i^{s_2}) + z \sqrt[4]{M^4 + 1} (i^{t_1} - i^{t_2})|.$$

Applying Theorem 1.1 (more specifically, Theorem 7.1 with  $p, q \in \mathbb{Z}[i]$ ), we have

$$\left| \sqrt[4]{1 + \frac{2}{M^4 - 1}} - \frac{p}{q} \right| > |q|^{-2.49951}$$

for  $|q| \geq q_0$  and  $M \geq 6$ . Thus (9.7) implies

$$(9.8) \quad |L_2| > X^{-1.49952}$$

for  $X \geq X_1$  and  $M \geq 6$ . Inequalities (9.4), (9.5), (9.6) and (9.8) combine to yield

$$|N_{K/\mathbb{Q}}(x + y \sqrt[4]{M^4 - 1} + z \sqrt[4]{M^4 + 1})| = \left| \prod_{0 \leq s, t \leq 3} L_{s,t} \right| > X^{1.14405}$$

for  $X \geq X_2$  and  $M \geq 6$ , which enables one to solve (9.1) as desired.

If we replace the constant  $u$  in equation (9.1) or (9.2) by a polynomial in  $\mathbb{Z}[x, y, z]$  of fixed degree, then the preceding argument may permit solution of the resulting equation, provided  $M$  is large enough. For example, to solve

$$N_{K/\mathbb{Q}}(x + y \sqrt[4]{M^4 - 1} + z \sqrt[4]{M^4 + 1}) = f(x, y, z)$$

we require only that  $M$  be bounded as follows.

degree $f$	0	$M \geq$	6
	1		6
	2		8
	3		10
	4		16
	5		41
	6		777

In essence, in this situation, we are dealing with a Diophantine inequality of the form

$$\left| N_{K/\mathbb{Q}}(x + y \sqrt[4]{M^4 - 1} + z \sqrt[4]{M^4 + 1}) \right| < X^\delta$$

which we may solve provided  $\delta < 7$ .

## ACKNOWLEDGEMENTS

The author would like to thank Professor Cameron Stewart for many helpful discussions on the contents of this paper. Thanks are also due to Carolyn Jackson for her precise typing and to Ryan English for preparation of Figure 1.

$$\max \left\{ \left| \theta_1 - \frac{p_1}{q} \right|, \left| \theta_2 - \frac{p_2}{q} \right| \right\} > q^{-\lambda}$$

for  $q \geq q_0(\theta_1, \theta_2, \lambda)$ .

*Note added in proof.* It has come to the author's attention that an analogous result to our Theorem 1.2 has appeared in a paper by A. Dubitskas (*Estimation of a linear form in algebraic numbers*, Lithuanian Math. J. **31** (1991), 56–61).

## REFERENCES

1. A. Baker, *Rational approximations to certain algebraic numbers*, Proc. London Math. Soc. (3) **14** (1964), 385–398. MR **28**:5029
2. ———, *Rational approximations to  $\sqrt[3]{2}$  and other algebraic numbers*. Quart. J. Math. Oxford Ser. (2) **15** (1964), 375–383. MR **30**:1977
3. ———, *Simultaneous rational approximations to certain algebraic numbers*, Proc. Cambridge Philos. Soc. **63** (1967), 693–702. MR **35**:4167
4. H. Bateman and A. Erdélyi, *Higher Transcendental Functions*, McGraw-Hill, 1953, v.1.
5. J.W.S. Cassels, *An Introduction to Diophantine Approximation*, Cambridge University Press, Cambridge, 1957. MR **19**:396h
6. G.V. Chudnovsky, *On the method of Thue-Siegel*, Ann. of Math., (2) **117** (1983), 325–382. MR **85g**:11058
7. J. Dieudonné, *Calcul infinitésimal*, Hermann, Paris, 1968. MR **37**:2557
8. D. Easton, *Effective irrationality measures for certain algebraic numbers*, Math. Comp. **46**, no. 174 (1986), 613–622. MR **87f**:11047
9. N.I. Fel'dman, *An estimate of an incomplete linear form in several algebraic variables*, Math. Notes **7** (1970), 343–349. MR **42**:3026
10. ———, *Effective bounds for the number of solutions of certain diophantine equations*. Math. Notes **8** (1970), 674–679. MR **42**:7590
11. M. Hata, *Rational approximation to  $\pi$  and some other numbers*, Acta Arith. **63** (1993), 335–349. MR **94e**:11082
12. A. Heimonen, T. Matala-aho and K. Väänänen, *An application of Jacobi type polynomials to irrationality measures*, Bull. Austral. Math. Soc. **50** (1994), 225–243. MR **95**:02
13. C.F. Osgood, *The simultaneous diophantine approximation of certain  $k$ th roots*, Proc. Cambridge Philos. Soc. **67** (1970), 75–86. MR **43**:1927
14. R.G.E. Pinch, *Simultaneous Pellian equations*, Proc. Cambridge Philos. Soc. **103** (1988), 35–46. MR **89a**:11029
15. J.H. Rickert, *Simultaneous rational approximations and related diophantine equations*, Proc. Cambridge Philos. Soc. **113** (1993), 461–472. MR **94i**:11043
16. K.R. Roth, *Rational approximations to algebraic numbers*, Mathematika **2** (1955), 1–20: corrigendum, ibid. 168. MR **17**:1060d; MR **17**:242d
17. W.M. Schmidt, *Simultaneous approximation to algebraic numbers by rationals*, Acta Math. **125** (1970), 189–201. MR **42**:3028
18. N. Tzanakis, *Explicit solution of a class of quartic Thue equations*, Acta Arith. **64** (1993), 271–283. MR **94e**:11022
19. L. Washington, *Introduction to cyclotomic fields*, Springer-Verlag, GTM 83, 1982. MR **85g**:11001

DEPARTMENT OF PURE MATHEMATICS, THE UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, CANADA, N2L 3G1

*Current address:* Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

*E-mail address:* mabennet@math.lsa.umich.edu