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THE BERGMAN KERNEL FUNCTION OF SOME REINHARDT DOMAINS

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Abstract. The boundary behavior of the Bergman Kernel function of some Reinhardt domains is studied. Upper and lower bounds for the Bergman kernel function are found at the diagonal points $(z, \bar{z})$. Let D be the Reinhardt domain

$$D = \left\{ z \in \mathbf{C}^n \mid \|z\|_\alpha = \sum_{j=1}^n |z_j|^{2/\alpha_j} < 1 \right\}$$

where $\alpha_j > 0$, $j = 1, 2, \ldots, n$; and let $K(z, \bar{w})$ be the Bergman kernel function of D. Then there exist two positive constants m and M and a function F such that

$$mF(z, \bar{z}) \leq K(z, \bar{z}) \leq MF(z, \bar{z})$$

holds for every $z \in D$. Here

$$F(z, \bar{z}) = (-r(z))^{-n-1} \prod_{j=1}^n (-r(z) + |z_j|^{2/\alpha_j})^{1-\alpha_j}$$

and $r(z) = \|z\|_\alpha - 1$ is the defining function for D. The constants m and M depend only on $\alpha = (\alpha_1, \ldots, \alpha_n)$ and n, not on z.

1. Introduction

The Bergman kernel function $K(z, \bar{w}), z, w \in \Omega$ for a domain $\Omega \subset \mathbf{C}^n$ is the kernel of the Bergman projection operator, the operator projecting $L^2(\Omega)$ onto its holomorphic subspace. In this paper, we consider the Reinhardt domain

$$D = \left\{ z \in \mathbf{C}^n \mid \|z\|_\alpha = \sum_{j=1}^n |z_j|^{2/\alpha_j} < 1 \right\} \tag{1}$$

where $\alpha_j > 0$, $j = 1, 2, \ldots, n$. The purpose of this paper is to give an estimate of the kernel function $K(z, \bar{z})$ of D in a "small constant-large constant" sense. Precisely, we prove the following

Theorem 1. *There exist two positive constants m and M, which only depend on n and α_j, $j = 1, 2, \ldots, n$, such that*

$$mF(z, \bar{z}) \leq K(z, \bar{z}) \leq MF(z, \bar{z}) \tag{2}$$

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holds for every $z \in D$, where

$$F(z,\bar{z}) = (-r(z))^{-n-1}\prod_{j=1}^{n}(-r(z)+|z_j|^{2/\alpha_j})^{1-\alpha_j} \tag{3}$$

and $r(z) = \|z\|_\alpha - 1$ is the defining function for D.

The study of the boundary behavior of $K_\Omega(z,\bar{w})$ for a domain $\Omega \subset \mathbf{C}^n$ is quite old, going back to the original inquires of Bergman [1]. Both Fefferman [6] and later, Boutet de Monvel and Sjöstrand [2] obtained the asymptotic expansion of $K_\Omega(z,\bar{w})$, when $\Omega \Subset \mathbf{C}^n$ is strongly pseudo-convex. For domains in $\mathbf{C}^2$, Catlin [3] has given a precise description of the behavior of $K_\Omega(z,\bar{z})$ near finite-type points in $\partial\Omega$. Sharp estimates for $K_\Omega(z,\bar{w})$ for this class of domains were obtained by McNeal [7] and Nagel et al. [9]. For a "decoupled" class of $\Omega \Subset \mathbf{C}^n$, McNeal [8] described the exact estimates on $K_\Omega(z,\bar{w})$ for z near a point of finite type in $\partial\Omega$. For the Reinhardt domain (1), D'Angelo [4] gives the series form of the Bergman kernel function $K(z,\bar{w})$ as

$$\begin{aligned} K(z,\bar{w}) &= (\alpha_1\alpha_2\cdots\alpha_n\pi^n)^{-1} \\ &\times \sum_{m_1,m_2,\ldots,m_n\geq 0}\frac{\Gamma(\sum_{j=1}^{n}\alpha_j(m_j+1)+1)}{\Gamma(\alpha_1 m_1+\alpha_1)\cdots\Gamma(\alpha_n m_n+\alpha_n)}(z_1\bar{w}_1)^{m_1}\cdots(z_n\bar{w}_n)^{m_n}. \end{aligned} \tag{4}$$

We will use equation (4) to obtain the estimate of $K(z,\bar{z})$ given in inequalities (2).

It should be noted that D'Angelo [4, 5] has studied $K(z,\bar{z})$ for certain domains. He showed

$$K(z,\bar{z}) = \sum_{k=0}^{l+1} c_k\left(1-\sum_{j=1}^{l}|z_j|^2\right)^{-l-1+kp}\left(\left(1-\sum_{J=1}^{l}|z_j|^2\right)^{p}-\sum_{j=l+1}^{n}|z_j|^2\right)^{-n+l-k}$$

when the domain is $\Omega = \{z \in \mathbf{C}^n \mid \sum_{j=1}^{l}|z_j|^2 + (\sum_{j=l+1}^{n}|z_j|^2)^{1/p} < 1\}$, where $1 \leq l < n$, p is a positive real number, and the constants c_k depend on k, l, n and p only.

The sketch of the proof is now indicated. We will informally use the word "comparable" to mean the two functions or sequences of coefficients are related by inequalities such as those between K and F given in inequalities (2). The proof can be outlined as follows: We start with the expansion of the Bergman kernel function in (4). Then we show that the coefficients are comparable to certain coefficients A that are more tractable. The Main Lemma shows that these coefficients A are comparable to certain coefficients B. It follows that the coefficients in equation (4) are comparable to the coefficients B. Multiplying by the powers of z and summing, we find that the kernel function is comparable to a function F which has an expansion with the coefficients B.

Most of the lemmas are combinatorial. The Main Lemma which relates the coefficients A and B is stated in section 3. Its proof is carried out in sections 4 and 5. The proof of the Theorem is given in sections 6 and 7.

2. Preliminary lemmas

We start with some combinatorial lemmas proved using generating function. The routine proofs are not included.

Let $\mathbf{R}_+^n$ denote the set of vectors $x = (x_1, \ldots, x_n)$ in $\mathbf{R}^n$, such that all the components $x_j, j = 1, 2, \ldots, n$, are non-negative, and let $\sigma(x) = \sum_{i=1}^n x_i$. As usual we let e_j denote the unit vector of the j-th coordinate in $\mathbf{R}^n$, and let $x(j) = (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n)$, $\sigma(x(j)) = \sum_{i\neq j} x_i$ if $x = (x_1, \ldots, x_n)$.

Let a, t be real numbers, $|t| < 1$, and

$$(1-t)^{-a} = \sum_{k=0}^{\infty} b(a,k)t^k \tag{5}$$

Lemma 1.

$$b(a,k) = \frac{\Gamma(k+a)}{\Gamma(a)\Gamma(k+1)}, \quad \textit{when } a \neq 0,\ a > -1; \tag{6}$$

and $b(0,0) = 1, b(0,k) = 0$ *when* $k \neq 0$.

Let $u = (u_1, \ldots, u_n) \in \mathbf{R}_+^n$, $\sigma(u) = \sum_{i=1}^n u_i < 1$, and

$$(1-\sigma(u))^{-a} = \sum_{m\in\mathbf{Z}_+^n} C(a,m)u^m \tag{7}$$

where $m = (m_1, \ldots, m_n), u^m = u_1^{m_1} \cdots u_n^{m_n}$.

Lemma 2.

$$C(a,m) = \frac{\Gamma(a+\sigma(m))}{\Gamma(a)\prod_{j=1}^n \Gamma(1+m_j)}, \quad \textit{when } a \neq 0,\ a > -1; \tag{8}$$

and $C(0,0) = 1$, $C(0,m) = 0$ *when* $m \neq 0 = (0, \ldots, 0)$.

Lemma 3. *Let* $a_j,\ j = 1, \ldots, s$, *be real numbers,* $k \in \mathbf{Z}_+$, *then*

$$b(a_1 + \cdots + a_s, k) = \sum_{\sigma(l)=k} b(a_1,l_1)b(a_2,l_2)\cdots b(a_s,l_s) \tag{9}$$

where $b(a,k)$ *is defined by* (5), $l = (l_1, \ldots, l_s) \in \mathbf{Z}_+^s$, *and* $\sigma(l) = \sum_{j=1}^n l_j$.

Lemma 4. *Let* $a_j,\ j = 1, \ldots, s$, *be real numbers,* $l \in \mathbf{Z}_+^n$, *then*

$$C(a_1 + \cdots + a_s, l) = \sum_{m+p+\cdots+q=l} C(a_1,m)C(a_2,p)\cdots C(a_s,q) \tag{10}$$

where $C(a,l)$ *is defined by* (7) *and* $l, m, \ldots, q \in \mathbf{Z}_+^n$.

Lemma 5. *If* $a > -1$, $m = (m_1, \ldots, m_n) \in \mathbf{Z}_+^n$, $p = (p_1, \ldots, p_n) \in \mathbf{Z}_+^n, q = (q_1, \ldots, q_n) \in \mathbf{Z}_+^n$ *and* $m = p + q, p_j q_j = 0$ *for* $j = 1, 2, \ldots, n$, *then*

$$C(a,m) = C(a,p)C(a+\sigma(p),q) \tag{11}$$

where $C(a,l)$ *is defined by* (8).

Consider the special case that

$$p = m(j) = (m_1, \ldots, m_{j-1}, 0, m_{j+1}, \ldots, m_n),$$
$$q = m_j e_j = (0, \ldots, 0, m_j, 0, \ldots, 0).$$

Then (11) becomes

$$C(a,m) = C(a,m(j))C(a+\sigma(m(j)), m_j e_j) = C(a,m(j))b(a+\sigma(m(j)), m_j)$$

since

$$C(a+\sigma(m(j)), m_j e_j) = b(a+\sigma(m(j)), m_j).$$

If we take $p = m_j e_j,\ q = m(j)$, then (11) becomes

$$C(a, m) = b(a, m_j)C(a + m_j, m(j)).$$

Thus we have

$$C(a, m) = C(a, m(j))b(a + \sigma(m(j)), m_j) = C(a + m_j, m(j))b(a, m_j). \tag{12}$$

In particular, if $j = n$, then (12) becomes

$$C(a, m) = C(a, m(n))b(a + \sigma(m(n)), m_n) = C(a + m_n, m(n))b(a, m_n). \tag{13}$$

Lemma 6. *If a, m are two real numbers, $m \geq 1$, and $m + a > 0$, then there exist two positive numbers c and C, which are independent of m, and only depend on a, such that*

$$c\Gamma(m)m^a < \Gamma(m + a) < C\Gamma(m)m^a. \tag{14}$$

Proof. When m is a positive integer, then (14) is just the consequence of Euler-Gauss formula

$$\Gamma(a) = \lim_{m\to\infty} m^a \frac{1 \cdot 2 \cdots (m-1)}{a \cdot (a+1) \cdots (a+m)}.$$

When m is not a positive integer, we may prove it by using the Stirling's formula: For real $x > 0$,

$$\Gamma(x) = \sqrt{2\pi} x^{x-1/2} e^{-x} e^{\theta(x)/12x}$$

with $0 < \theta(x) < 1$. Using the Stirling formula, we have

$$\frac{\Gamma(m+a)}{\Gamma(m)m^a} = \left(1 + \frac{a}{m}\right)^{m+a-1/2} e^{-a} \exp\left\{\frac{\theta_1(m+a)}{12(m+a)} - \frac{\theta_2(m)}{12m}\right\}$$

where $0 < \theta_1 < 1, 0 < \theta_2 < 1$. It is easy to verify that the right hand side of the previous equality is bounded above by $\exp\{1 - a + \frac{1}{12(1+a)}\} \max[1, (1+a)^{a-1/2}]$, and bounded below by $\exp\{-\frac{1}{12} - a\} \min[1, e^a] \min[1, (1+a)^{a-1/2}]$. Thus we prove (14).

Actually, we have $\lim_{m\to\infty} \frac{\Gamma(m+a)}{\Gamma(m)m^a} = 1$, for any positive real number sequence $\{m\}$. $\square$

3. The Main Lemma

Lemma 7 (Main Lemma)**.** *Let $u = (u_1, \ldots, u_n) \in \mathbf{R}_+^n, \sigma(u) < 1,\ a > 0,\ \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \mathbf{R}^n,\ \varepsilon_j > -1,\ j = 1, 2, \ldots, n$. Further, if all $\varepsilon_j,\ j = 1, 2, \ldots, n$, are non-positive, we assume that $a + \sum_{j=1}^n \varepsilon_j > 1$. If some of the $\varepsilon_j,\ j = 1, 2, \ldots, n$, are non-negative and the others are non-positive, then we assume that $a + \sum_{j=1}^n \varepsilon_j > \sum_{j=1}^n \{\varepsilon_j\} + 1$, where $\{\varepsilon_j\}$ means the least integer which is equal to or greater than ε_j. Let*

$$F_n^{a,\varepsilon}(u) = (1 - \sigma(u))^{-a} \prod_{j=1}^n (1 + u_j - \sigma(u))^{-\varepsilon_j}, \tag{15}$$

and let the Taylor expansion of $F_n^{a,\varepsilon}(u)$ be

$$F_n^{a,\varepsilon}(u) = \sum_{s \in \mathbf{Z}_+^n} B_{s,n}^{a,\varepsilon} u^s \tag{16}$$

where $s = (s_1, s_2, \dots, s_n) \in \mathbf{Z}_+^n$, $u^s = u_1^{s_1} \cdots u_n^{s_n}$, *then there exist two positive constants* m_n *and* M_n, *which are independent of* s, *and depend on* a, n, *and* ε *only, such that*

$$m_n A_{s,n}^{a,\varepsilon} \le B_{s,n}^{a,\varepsilon} \le M_n A_{s,n}^{a,\varepsilon} \tag{17}$$

where

$$A_{s,n}^{a,\varepsilon} = \frac{\Gamma(a+\sigma(\varepsilon)+\sigma(s))}{\Gamma(a+\sigma(\varepsilon))\prod_{j=1}^n \Gamma(1+s_j)(1+s_j)^{\varepsilon_j}} = \frac{C(a+\sigma(\varepsilon), s)}{\prod_{j=1}^n (1+s_j)^{\varepsilon_j}}, \tag{18}$$

and $\sigma(\varepsilon) = \sum_{j=1}^n \varepsilon_j$, $\sigma(s) = \sum_{j=1}^n s_j$.

We prove the Main Lemma by induction for n.

When $n = 1$,

$$F_1^{a,\varepsilon}(u) = (1-u_1)^{-a} = \sum b(a, m_1) u_1^{m_1},$$

that is, $B_{m,1}^{a,\varepsilon} = b(a, m_1), (m = m_1)$; and

$$A_{m,1}^{a,\varepsilon} = \frac{\Gamma(a+\varepsilon_1+m_1)}{\Gamma(a+\varepsilon_1)\Gamma(1+m_1)(1+m_1)^{\varepsilon_1}} = b(a, m_1) I_1$$

where

$$I_1 = \frac{\Gamma(a+\varepsilon_1+m_1)\Gamma(a)}{\Gamma(a+\varepsilon_1)\Gamma(a+m_1)(1+m_1)^{\varepsilon_1}}.$$

Obviously (17) is true when $m_1 = 0$. If $m_1 \ne 0$, then by Lemma 6, there exist c and C, which depends only on a and ε, such that

$$I_1 \le C \frac{\Gamma(a)}{\Gamma(a+\varepsilon_1)} \left(\frac{m_1}{1+m_1}\right)^{\varepsilon_1} < C \frac{\Gamma(a)}{\Gamma(a+\varepsilon_1)}$$

if $\varepsilon_1 \ge 0$; and

$$I_1 \le C 2^{-\varepsilon_1} \frac{\Gamma(a)}{\Gamma(a+\varepsilon_1)}$$

if $\varepsilon_1 < 0$. Similarly,

$$I_1 > c \frac{\Gamma(a)}{\Gamma(a+\varepsilon_1)}$$

if $\varepsilon_1 < 0$; and

$$I_1 > c 2^{-\varepsilon_1} \frac{\Gamma(a)}{\Gamma(a+\varepsilon_1)}$$

if $\varepsilon_1 \ge 0$. We have proved (17) in the case $n = 1$.

In order to complete the induction process, we will need the following lemmas.

Lemma 8. *The assumptions are as in the Main Lemma. Moreover, we assume for a fixed* k, $1 \le k \le n$, $\varepsilon_k > 0$, *then*

$$B_{s,n}^{a,\varepsilon} = \sum_{l(k)+\eta(k)=s(k)} C(\varepsilon_k, \eta(k)) b(a+\sigma(\varepsilon(k))+\sigma(l(k)), s_k) B_{l(k),n-1}^{a,\varepsilon(k)}; \tag{19}$$

if we assume $\varepsilon_k = 0$, *then*

$$B_{s,n}^{a,\varepsilon} = b(a+\sigma(\varepsilon)) + \sigma(s(k)), s_k) B_{s(k),n-1}^{a,\varepsilon(k)}; \tag{20}$$

where $l = (l_1, \ldots, l_n) \in \mathbf{Z}_+^n$, $\eta = (\eta_1, \ldots, \eta_n) \in \mathbf{Z}_+^n$, $(s_1, \ldots, s_n) \in \mathbf{Z}_+^n$ *and* $B_{s(k),n-1}^{a,\varepsilon(k)}$ *is the coefficient of the Taylor expansion of the function*

$$F_{n-1}^{a,\varepsilon(k)}(u(k)) = (1 - \sigma(u(k))^{-a} \prod_{j \neq k} (1 - \sigma(u(k)) + u_j)^{-\varepsilon_j}. \tag{21}$$

Lemma 9. *The assumptions are as in the Main Lemma. Fix any* k, $1 \leq k \leq n$, *then*

$$\begin{aligned} B_{a,n}^{a,\varepsilon} = \sum_{l+\xi=s} & b(a_k, \xi_k) C(a_k + \varepsilon_k + \xi_k, \xi(k)) \\ & \times b(a - a_k + \sigma(\varepsilon(k)) + \sigma(l(k)), l_k) B_{l(k),n-1}^{a-a_k,\varepsilon(k)} \end{aligned} \tag{22}$$

where $a = a_1 + a_2 + \cdots + a_n$, *such that* $a_j > 0$, $a_j + \varepsilon_j > 0$, $j = 1, 2, \ldots, n$, *and* $l = (l_1, l_2, \ldots, l_n) \in \mathbf{Z}_+^n, \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{Z}_+^n$; $s = (s_1, s_2, \ldots, s_n) \in \mathbf{Z}_+^n$ *and* $B_{l(k),n-1}^{a-a_k,\varepsilon(k)}$ *is the coefficient of the Taylor expansion of the function*

$$F_{n-1}^{a-a_k,\varepsilon(k)}(u(k)) = (1 - \sigma(u(k)))^{a_k - a} \prod_{j \neq k} (1 - \sigma(u(k)) + u_j)^{-\varepsilon_j}.$$

Lemma 10. *Assumptions are as in the Main Lemma, then all the coefficients* $B_{s,n}^{a,\varepsilon}$ *in* (16) *are non-negative.*

Proof of Lemma 8. In the case $\varepsilon_k > 0$, we can expand each factor in (15) by power series, then

$$\begin{aligned} F_n^{a,\varepsilon}(u) = & \sum_{m,\beta(1),\ldots,\gamma(n-1),\delta(n) \in \mathbf{Z}_+^n} C(a,m) C(\varepsilon_1, \beta(1)) \cdots \\ & \qquad C(\varepsilon_{n-1}, \gamma(n-1)) C(\varepsilon_n, \delta(n)) u^{m+\beta(1)+\cdots+\delta(n)} \\ = & \sum_{s \in \mathbf{Z}_+^n} \sum_{m+\beta(1)+\cdots+\gamma(n-1)+\delta(n)=s} C(a,m) C(\varepsilon_1, \beta(1)) \cdots C(\varepsilon_n, \delta(n)) u^s. \end{aligned}$$

Comparing with (16), we get

$$B_{s,n}^{a,\varepsilon} = \sum_{m+\beta(1)+\cdots+\delta(n)=s} C(a,m) C(\varepsilon_1, \beta(1)) \cdots C(\varepsilon_n, \delta(n)).$$

Using Lemma 5, we can decompose each factor (except the factor $C(\varepsilon_k, \xi(k))$) on the right-hand side of this equality as

$$\begin{gathered} C(a,m) = C(a, m(k)) b(a + \sigma(m(k)), m_k), \\ C(\varepsilon_1, \beta(1)) = C(\varepsilon_1, \beta(1,k)) b(\varepsilon_1 + \sigma(\beta(1,k)), \beta_k), \\ \cdots\cdots\cdots \\ C(\varepsilon_n, \delta(n)) = C(\varepsilon_n, \delta(k,n)) b(\varepsilon_n + \sigma(\delta(k,n)), \delta_k). \end{gathered}$$

Then we have

$$
\begin{aligned}
B^{a,\varepsilon}_{s,n} = & \sum_{m+\beta(1)+\cdots+\delta(n)=s} C(\varepsilon_k,\xi(k))C(a,m(k))C(\varepsilon_1,\beta(1,k))\cdots \\
& [C(\varepsilon_k,\xi(k,k))]\cdots C(\varepsilon_n,\delta(k,n))b(a+\sigma(m(k)),m_k) \\
& \times b(\varepsilon_1+\sigma(\beta(1,k)),\beta_k)\cdots[b(\varepsilon_k+\sigma(\xi(k,k)),\beta_k)]\cdots b(\varepsilon_n+\sigma(\delta(k,n)),\delta_n) \\
= & \sum_{l(k)+\xi(k)=s(k)} C(\varepsilon_k,\xi(k)) \sum_{m(k)+\beta(1,k)+\cdots+\delta(k,n)=l(k)} C(a,m(k)) \\
& \times C(\varepsilon_1,\beta(1,k))\cdots[C(\varepsilon_k,\xi(k,k))]\cdots C(\varepsilon_n,\delta(k,n)) \\
& \times \sum_{m_k+\beta_k+\cdots+\delta_k=s_k} b(a+\sigma(m(k)),m_k)b(\varepsilon_1+\sigma(\beta(1,k)),\beta_k)\cdots \\
& [b(\varepsilon_k+\sigma(\xi(k,k)),\beta_k)]\cdots b(\varepsilon_n+\sigma(\delta(k,n)),\delta_n).
\end{aligned}
$$

where [ ] means that the term inside [ ] is deleted from the product. By Lemma 3, the last sum of the right hand side of this equality is equal to $b(a+\sigma(\varepsilon(k))+\sigma(l(k)),s_k)$, and the second sum is just the coefficient $B^{a,\varepsilon(k)}_{l(k),n-1}$ of the Taylor expansion of the function (21).

In the case $\varepsilon_k=0$, we have

$$
B^{a,\varepsilon}_{s,n} = \sum_{m+\beta(1)+\cdots+\delta(n)=s} C(a,m)C(\varepsilon_1,\beta(1))\cdots[C(\varepsilon_k,\xi(k))]\cdots C(\varepsilon_n,\delta(n)),
$$

and

$$
\begin{aligned}
B^{a,\varepsilon}_{s,n} = & \sum_{m(k)+\beta(1,k)+\cdots+\delta(k,n)=l(k)} C(a,m(k))C(\varepsilon_1,\beta(1,k))\cdots \\
& [C(\varepsilon_k,\xi(k,k))]\cdots C(\varepsilon_n,\delta(k,n)) \\
& \times \sum_{m_k+\beta_k+\cdots+\delta_k=s_k} b(a+\sigma(m(k)),m_k)b(\varepsilon_1+\sigma(\beta(1,k)),\beta_k)\cdots \\
& [b(\varepsilon_k+\sigma(\xi(k,k)),\xi_k)]\cdots b(\varepsilon_n+\sigma(\delta(k,n)),\delta_n).
\end{aligned}
$$

We get (20).

We have proved Lemma 8. $\square$

Proof of Lemma 9. We can express $F^{a,\varepsilon}_n(u)$ as

$$
F^{a,\varepsilon}_n = \prod_{j=1}^{n}[(1-\sigma(u))^{-a_j}(1-\sigma(u(j)))^{-\varepsilon_j}]. \tag{23}
$$

Since $\sigma(u)=\sigma(u(j))+u_j$, we get

$$
\begin{aligned}
&(1-\sigma(u))^{-a_j}(1-\sigma(u(j)))^{-\varepsilon_j} \\
&\quad = (1-u_j(1-\sigma(u(j))^{-1})^{-a_j}(1-\sigma(u(j)))^{-\varepsilon_j-a_j}.
\end{aligned}
$$

The assumption $\sigma(u)<1$ implies $u_j(1-\sigma(u(j)))^{-1}<1$,

$$
(1-u_j(1-\sigma(u(j)))^{-1})^{-a_j} = \sum_{k=0}^{\infty} b(a_j,k)u_j^k(1-\sigma(u(j)))^{-k}
$$

and

$$
\begin{aligned}
(1-\sigma(u))^{-a_j}(1-\sigma(u(j)))^{-\varepsilon_j} &= \sum_{k=0}^{\infty} b(a_j,k)u_j^k(1-\sigma(u(j)))^{-\varepsilon_j-a_j-k} \\
&= \sum_{k=0}^{\infty} b(a_j,k)u_j^k \sum_{m(j)\in\mathbf{Z}_+^n} C(\varepsilon_j+a_j+k,m(j))u^{m(j)} \qquad (24)\\
&= \sum_{m\in\mathbf{Z}_+^n} b(a_j,m_j)C(a_j+\varepsilon_j+m_j,m(j))u^m
\end{aligned}
$$

where $\sum_{m(j)\in\mathbf{Z}_+^n}$ means the sum taking for all lattice points in $\mathbf{Z}_+^n$, where the jth coordinate is zero.

Thus we can expand every factor in (23) by (24) as

$$
\begin{aligned}
F_n^{a,\varepsilon}(u) = &\sum_{\beta,\gamma,\dots,\delta,\eta\in\mathbf{Z}_+^n} b(a_1,\beta_1)C(a_1+\varepsilon_1+\beta_1,\beta(1))\cdots \\
&b(a_n,\eta_n)C(a_n+\varepsilon_n+\eta_n,\eta(n))u^{\beta+\gamma+\cdots+\delta+\eta} \\
= &\sum_{s\in\mathbf{Z}_+^n}\sum_{\beta+\gamma+\cdots+\delta+\eta=s} b(a_1,\beta_1)C(a_1+\varepsilon_1+\beta_1,\beta(1))\cdots \\
&b(a_n,\eta_n)C(a_n+\varepsilon_n+\eta_n,\eta(n))u^s.
\end{aligned}
$$

Comparing this formula with (16), we get

$$
\begin{aligned}
B_{s,n}^{a,\varepsilon} = &\sum_{\beta+\gamma+\cdots+\delta+\eta=s} b(a_1,\beta_1)C(a_1+\varepsilon_1+\beta_1,\beta(1))\cdots \qquad (25)\\
&b(a_n,\eta_n)C(a_n+\varepsilon_n+\eta_n,\eta(n)).
\end{aligned}
$$

Let

$$
\begin{aligned}
\beta(1,k) &= (0,\beta_2,\dots,\beta_{k-1},0,\beta_{k+1},\dots,\beta_n), \\
\gamma(2,k) &= (\gamma_1,0,\gamma_3,\dots,\gamma_{k-1},0,\gamma_{k+1},\dots,\gamma_n),
\end{aligned}
$$

etc., then we may decompose $C(a_1+\varepsilon_1+\beta_1,\beta(1))$, etc., by Lemma 5 as,

$$
\begin{aligned}
C(a_1+\varepsilon_1+\beta_1,\beta(1)) &= C(a_1+\varepsilon_1+\beta_1,\beta(1,k))b(a_1+\varepsilon_1+\sigma(\beta(k)),\beta_k), \\
C(a_2+\varepsilon_2+\gamma_2,\gamma(2)) &= C(a_2+\varepsilon_2+\gamma(2,k))b(a_2+\varepsilon_2+\sigma(\gamma(k)),\gamma_k), \\
&\cdots\cdots\cdots \\
C(a_n+\varepsilon_n+\eta_n,\eta(n)) &= C(a_n+\varepsilon+\eta_n,\eta(k,n))b(a_n+\varepsilon+\sigma(\eta(k)),\eta_k)
\end{aligned}
$$

except the term $C(a_k+\varepsilon_k+\xi_k,\xi(k))$.

Substituting the above equalities into (25), and let $l = s - \xi$, we get

$$\begin{aligned}
B_{s,n}^{a,\varepsilon} = &\sum_{l+\xi=s} b(a_k,\xi_k)C(a_k+\varepsilon_k+\xi_k,\xi(k)) \sum_{\beta+\gamma+\cdots+\eta=l} b(a_1,\beta_1) \\
&\times C(a_1+\varepsilon_1+\beta_1,\beta(1,k))b(a_2,\gamma_2)C(a_2+\varepsilon_2+\gamma_2,\gamma(2,k))\cdots \\
&[b(a_k,\xi_k)C(a_k+\varepsilon_k+\xi_k,\xi(k,k))]\cdots b(a_n,\eta_n)C(a_n+\varepsilon_n+\eta_n,\eta(k,n)) \\
&\times b(a_1+\varepsilon_1+\sigma(\beta(k)),\beta_k)b(a_2+\varepsilon+\sigma(\gamma(k)),\gamma_k) \\
&\cdots[b(a_k+\varepsilon_k+\sigma(\xi(k)),\xi_k)]\cdots b(a_n+\varepsilon_n+\sigma(\eta(k)),\eta_k) \\
= &\sum_{l+\xi=s} b(a_k,\xi_k)C(a_k+\varepsilon_k+\xi_k,\xi(k))b(a-a_k+\sigma(\varepsilon(k))+\sigma(l(k)),l_k) \\
&\times \sum_{\beta(k)+\gamma(k)+\cdots+\eta(k)=l(k)} b(a_1,\beta_1)C(a_1+\varepsilon+\beta_1,\beta(1,k))\cdots \\
&[b(a_k,\xi_k)C(a_k+\varepsilon_k+\xi_k,\xi(k,k))]\cdots b(a_k,\eta_k)C(a_n+\varepsilon_n+\eta_n,\eta(k,n))
\end{aligned}$$

by Lemma 3, where [ ] means that the term inside [ ] is deleted from the product. The last sum of the right-hand side of the previous equality is just the coefficient $B_{l(k),n-1}^{a-a_k,\varepsilon(k)}$ of the Taylor expansion of the function

$$\begin{aligned}
F_{n-1}^{a-a_k,\varepsilon(k)} &= (1-\sigma(u(k)))^{a_k-a}\prod_{j\neq k}(1+u_j-\sigma(u(k)))^{-\varepsilon_j} \\
&= \sum_{l(k)\in\mathbf{Z}_+^n} B_{l(k),n-1}^{a-a_k,\varepsilon(k)} u^{l(k)}.
\end{aligned}$$

We have proved Lemma 9. $\square$

Proof of Lemma 10. By the assumptions of the Main Lemma, we may choose $a_j > 0$, $j = 1, 2, \ldots, n$, such that $a = a_1 + \cdots + a_n$, and $a_j + \varepsilon_j > 0$, $j = 1, 2, \ldots, n$, then $B_{s,n}^{a,\varepsilon} \geq 0$ for all s follows by (25). $\square$

4. Proof of the Main Lemma, Part 1

Fix $s = (s_1, s_2, \ldots, s_n) \in \mathbf{Z}_+^n$, and let $s_k = \max_{1\leq j\leq n} s_j$. We may assume $s_k \geq 1$, otherwise $s = 0$, then the Main Lemma is true obviously. For that k, there are three possibilities of ε_k: (1) $\varepsilon_i > 0$; (2) $\varepsilon_k = 0$ and (3) $\varepsilon_k < 0$.

In this section, we prove the Main Lemma in the cases (1) $\varepsilon_k > 0$ holds or (2) $\varepsilon_k = 0$ holds.

If $\varepsilon_k > 0$, by the conditions of the Main Lemma, all the factors in the right-hand side of (19) are non-negative.

The induction process is as follows: to prove (17) is true if we assume

$$m_{n-1}(j)A_{s(j),n-1}^{a,\varepsilon(j)} \leq B_{s(j),n-1}^{a,\varepsilon(j)} \leq M_{n-1}(j)A_{s(j),n-1}^{a,\varepsilon(j)}$$

is true for $j = 1, 2, \ldots, n$ where $m_{n-1}(j)$ and $M_{n-1}(j)$ are two positive constants which are independent of $s(j)$, and only dependent on a and ε, and

$$A_{l(j),n-1}^{a,\varepsilon(j)} = C(a+\sigma(\varepsilon(j)),l(j))\prod_{p\neq j}(1+l_p)^{-\varepsilon_p}.$$

Let $m_{n-1} = \min_{1\leq j\leq n} m_{n-1}(j)$ and $M_{n-1} = \max_{1\leq j\leq n} M_{n-1}(j)$, then

$$m_{n-1}A_{l(j),n-1}^{a,\varepsilon(j)} \leq B_{l(j),n-1}^{a,\varepsilon(k)} \leq M_{n-1}A_{l(j),n-1}^{a,\varepsilon(j)}.$$

In particular, if we take $j=k$, we have

$$m_{n-1}A^{a,\varepsilon(k)}_{l(k),n-1} \le B^{a,\varepsilon(k)}_{l(k),n-1} \le M_{n-1}A^{a,\varepsilon(k)}_{l(k),n-1},$$

where

$$A^{a,\varepsilon(k)}_{l(k),n-1} = C(a+\sigma(\varepsilon(k)), l(k)) \prod_{j\neq k}(1+l_j)^{-\varepsilon_j}. \tag{26}$$

Let

$$I^{a,\varepsilon}_s = \sum_{l(k)+\eta(k)=s(k)} C(\varepsilon_k, \eta(k)) b(a+\sigma(\varepsilon(k))+\sigma(l(k)), s_k) A^{a,\varepsilon(k)}_{l(k),n-1},$$

then

$$m_{n-1}I^{a,\varepsilon}_s \le B^{a,\varepsilon}_{s,n} \le M_{n-1}I^{a,\varepsilon}_s$$

by (19).

We estimate $I^{a,\varepsilon}_s$.

Fix l_k. We define the set E as

$$E = \{l(k) = (l_1,\dots,l_{k-1},0,l_{k+1},\dots,l_n) \in \mathbf{Z}^n_+,\ 0\le l_j \le s_j,\ j\neq k\}.$$

We define the mapping $\phi: E\to E$ as

$$\phi(l(k)) = \bar l(k) = (\bar l_1,\dots,\bar l_{k-1},0,\bar l_{k+1},\dots,\bar l_n)$$

where $\bar l_j = l_j$, if $\frac12 s_j \le l_j \le s_j$; $\bar l_j = s_j - l_j$, if $0\le l_j < \frac12 s_j$; and define $\bar\eta(k)$ as $\bar l(k)+\bar\eta(k) = l(k)+\eta(k) = s(k)$.

It is easily verified that

$$\prod_{j\neq k}\Gamma(1+\bar\eta_j)\Gamma(1+\bar l_j) = \prod_{j\neq k}\Gamma(1+\eta_j)\Gamma(1+l_j). \tag{27}$$

Consider

$$I \equiv \frac{C(\varepsilon_k,\bar\eta(k)) b(a+\sigma(\varepsilon(k))+\sigma(\bar l(k)), s_k) A^{a,\varepsilon(k)}_{\bar l(k),n-1}}{C(\varepsilon_k,\eta(k)) b(a+\sigma(\varepsilon(k))+\sigma(l(k)), s_k) A^{a,\varepsilon(k)}_{l(k),n-1}}.$$

By (26) and (27), we have

$$I = \frac{\Gamma(\varepsilon_k+\sigma(\bar\eta(k)))\Gamma(a+\sigma(\varepsilon(k))+\sigma(\bar l(k))+s_k)}{\Gamma(\varepsilon_k+\sigma(\eta(k)))\Gamma(a+\sigma(\varepsilon(k))+\sigma(l(k))+s_k)} \prod_{j\neq k}\left(\frac{1+l_j}{1+\bar l_j}\right)^{\varepsilon_j}.$$

If all $\bar l_j = l_j$, $j\neq k$, then $I=1$. If at least one of $\bar l_j \neq l_j$, then $d = \sigma(\bar l(k)) - \sigma(l(k)) = \sigma(\eta(k)) - \sigma(\bar\eta(k)) \ge 1$. Let $d_j = \bar l_j - l_j$, $j\neq k$, then

$$I = \prod_{\nu=0}^{d-1}\frac{a+\sigma(\varepsilon(k))+s_k+\sigma(l(k))+\nu}{\varepsilon_k+\sigma(\bar\eta(k))+\nu}\prod_{j\neq k}\left(\frac{1+l_j}{1+l_j+d_j}\right)^{\varepsilon_j}.$$

The reciprocal of the first product in the previous equality is

$$\begin{aligned}
&\prod_{\nu=0}^{d-1} \frac{\varepsilon_k + \sigma(\bar\eta(k)) + \nu}{a + \sigma(\varepsilon(k)) + s_k + \sigma(l(k)) + \nu} \\
&\qquad = \prod_{\nu=0}^{d-1} \left(1 + \frac{\varepsilon_k + \sigma(\bar\eta(k)) - a - \sigma(\varepsilon(k)) - s_k - \sigma(l(k))}{a + \sigma(\varepsilon(k)) + s_k + \sigma(l(k)) + \nu}\right) \\
&\qquad \le \prod_{\nu=0}^{d-1} \exp\left\{\frac{\varepsilon_k + \sigma(\bar\eta(k)) - a - \sigma(\varepsilon(k)) - s_k - \sigma(l(k))}{a + \sigma(\varepsilon(k)) + s_k + \sigma(l(k)) + \nu}\right\}
\end{aligned}$$

since $e^x \ge 1 + x$ when $x \ge -1$. Then

$$I \ge \exp\left(\sum_{\nu=0}^{d-1} \frac{a + \sigma(\varepsilon(k)) + s_k + \sigma(l(k)) - \sigma(\bar\eta(k)) - \varepsilon_k}{a + \sigma(\varepsilon(k)) + s_k + \sigma(l(k)) + \nu}\right) \prod_{j \ne k} \left(\frac{1 + l_j}{1 + l_j + d_j}\right)^{\varepsilon_j}.$$

When ε is given and $s_k = \max_{1 \le j \le n} s_j$, there are only a finite number of $s \in \mathbf{Z}_+^n$ such that $s_k \le 2\varepsilon_k$. In this situation, we only have a finite number of $B_{s,n}^{a,\varepsilon}$ and $A_{s,n}^{a,\varepsilon}$, thus we just need to take

$$M_n = \max_{s_k = \max s_j, s_k \le 2\varepsilon_k} \frac{B_{s,n}^{a,\varepsilon}}{A_{s,n}^{a,\varepsilon}}, \quad m_n = \min_{s_k = \max s_j, s_k \le 2\varepsilon_k} \frac{B_{s,n}^{a,\varepsilon}}{A_{s,n}^{a,\varepsilon}}.$$

It only remains for us to consider the case $s_k \ge 2\varepsilon_k$.

From the definition of the mapping ϕ, we know $\bar\eta_j \le l_j$, $j \ne k$, and hence $\sigma(l(k)) \ge \sigma(\bar\eta(k))$. Then

$$\begin{aligned}
&\frac{a + \sigma(\varepsilon(k)) + s_k + \sigma(l(k)) - \sigma(\bar\eta(k)) - \varepsilon_k}{a + \sigma(\varepsilon(k)) + s_k + \sigma(l(k)) + \nu} \\
&\qquad > \frac{s_k + \sigma(l(k)) - \sigma(\bar\eta(k)) - \varepsilon_k}{s_k + \sigma(l(k)) + \nu} > \frac{\frac{1}{2} s_k}{s_k + \sigma(l(k)) + \nu}
\end{aligned}$$

since $s_k \ge 2\varepsilon_k$. We have

$$I > \exp\left\{\sum_{\nu=0}^{d-1} \frac{s_k}{2(s_k + \sigma(l(k)) + \nu)}\right\} \prod_{j \ne k} \left(\frac{1 + l_j}{1 + l_j + d_j}\right)^{\varepsilon_j},$$

thus

$$I > \exp\left\{\sum_{\nu=0}^{d-1} \frac{s_k}{2(s_k + \sigma(l(k)) + \nu)}\right\} \prod_{j \ne k} \left(\frac{1}{1 + d_j}\right)^{|\varepsilon_j|},$$

since

$$\left(\frac{1 + l_j}{1 + l_j + d_j}\right)^{\varepsilon_j} \ge \left(\frac{1}{1 + d_j}\right)^{|\varepsilon_j|}.$$

Moreover, since $0 \le \sigma(l(k)) \le \sigma(s(k)) \le (n-1)s_k$,

$$\nu \le d \le \sigma(\bar l(k)) - \sigma(l(k)) \le \sigma(\bar l(k)) \le (s(k)) \le (n-1)s_k,$$

we have

$$\frac{s_k}{2(s_k + \sigma(l(k)) + \nu)} \ge \frac{s_k}{2(s_k + (n-1)s_k + (n-1)s_k)} = \frac{1}{4n-2}.$$

Therefore, we obtain

$$I > \left(\frac{1}{1+d}\right)^{\sum_{j\neq k}|\varepsilon_j|} \exp\left(\frac{d}{4n-2}\right) \equiv f(d).$$

The function $f(d)$ takes the minimum value at $d = (4n-2)\sum_{j\neq k}|\varepsilon_j| - 1$, hence

$$I > e^{-1}\left((4n-2)\sum_{j\neq k}|\varepsilon_j|\right)^{-\sum_{j\neq k}|\varepsilon_j|} \equiv a_0,$$

where a_0 is a constant which only depends on ε and n. From the definition of I, the following inequality

$$\begin{aligned} &C(\varepsilon_k,\eta(k))b(a+\sigma(\varepsilon(k))+\sigma(l(k)),s_k)A^{a,\varepsilon(k)}_{l(k),n-1} \\ &\quad\le a_0^{-1}C(\varepsilon_k,\bar\eta(k))b(a+\sigma(\varepsilon(k))+\sigma(\bar l(k)),s_k)A^{a,\varepsilon(k)}_{\bar l(k),n-1} \end{aligned}$$

holds. According to the definition of the set E and the mapping ϕ, at most 2^{n-1} different $l(k)$ map to the same $\bar l(k)$ by ϕ. Let $\bar E = \phi(E)$, and

$$J = \sum_{l(k)+\eta(k)=s(k),\, l(k)\in\bar E} C(\varepsilon_k,\eta(k))b(a+\sigma(\varepsilon(k))+\sigma(l(k)),s_k)A^{a,\varepsilon(k)}_{l(k),n-1},$$

then

$$J \le I^{a,\varepsilon}_{s,n},$$

and

$$\begin{aligned} I^{a,\varepsilon}_{s,n} &= \sum_{l(k)+\eta(k)=s(k)} C(\varepsilon_k,\eta(k))b(a+\sigma(\varepsilon(k))+\sigma(l(k)),s_k)A^{a,\varepsilon(k)}_{l(k),n-1} \\ &\le 2^{n-1}a_0^{-1}\sum_{l(k)+\eta(k)=s(k),\, l(k)\in\bar E} C(\varepsilon_k,\eta(k)) \\ &\quad\times b(a+\sigma(\varepsilon(k))+\sigma(l(k)),s_k)A^{a,\varepsilon(k)}_{l(k),n-1} \\ &= 2^{n-1}a_0^{-1}J. \end{aligned}$$

Thus

$$J \le I^{a,\varepsilon}_{s} \le 2^{n-1}a_0^{-1}J.$$

Hence, we only need to estimate J.

Since $l \in \bar E$, $\frac{1}{2}s_j \le l_j \le s_j,\ j\neq k$, we have

$$2^{-|\varepsilon_j|}(1+s_j)^{-\varepsilon_j} \le (1+l_j)^{-\varepsilon_j} \le 2^{|\varepsilon_j|}(1+s_j)^{-\varepsilon_j} \tag{28}$$

for $j \neq k$. By (26) and (12),

$$\begin{aligned} J = &\sum_{l(k)+\eta(k)=s(k), l(k)\in \bar{E}} C(\varepsilon_k, \eta(k)) b(a+\sigma(\varepsilon(k))+\sigma(l(k)), s_k) \\ &\qquad \times C(a+\sigma(\varepsilon(k)), l(k)) \prod_{j\neq k} (1+l_j)^{-\varepsilon_j} \\ = &\sum_{l(k)+\eta(k)=s(k), l(k)\in \bar{E}} C(\varepsilon_k, \eta(k)) C(a+\sigma(\varepsilon(k))+s_k, l(k)) \\ &\qquad \times b(a+\sigma(\varepsilon(k)), s_k) \prod_{j\neq k} (1+l_j)^{-\varepsilon_j}. \end{aligned}$$

By (28), we get $2^{-\sum_{j\neq k}|\varepsilon_j|} L \leq J \leq 2^{\sum_{j\neq k}|\varepsilon_j|} L$, where

$$\begin{aligned} L = &\, b(a+\sigma(\varepsilon(k)), s_k) \prod_{j\neq k} (1+s_j)^{-\varepsilon_j} \\ &\times \sum_{l(k)+\eta(k)=s(k), l(k)\in \bar{E}} C(\varepsilon_k, \eta(k)) C(a+\sigma(\varepsilon(k))+s_k, l(k)). \end{aligned}$$

Using Lemma 4, we have

$$L \leq \prod_{j\neq k} (1+s_j)^{-\varepsilon_j} b(a+\sigma(\varepsilon(k)), s_k) C(a+\sigma(\varepsilon)+s_k, s(k)).$$

Using Lemma 6, we have

$$\begin{aligned} &\frac{b(a+\sigma(\varepsilon(k)), s_k)(1+s_k)^{\varepsilon_k}}{b(a+\sigma(\varepsilon), s_k)} \\ &\qquad = \frac{\Gamma(a+\sigma(\varepsilon(k))+s_k)\Gamma(a+\sigma(\varepsilon(k))+\varepsilon_k)(1+s_k)^{\varepsilon_k}}{\Gamma(a+\sigma(\varepsilon(k)))\Gamma(a+\sigma(\varepsilon(k))+\varepsilon_k+s_k)} \\ &\qquad \leq C\frac{\Gamma(a+\sigma(\varepsilon))}{\Gamma(a+\sigma(\varepsilon(k)))}\left(\frac{1+s_k}{a+\sigma(\varepsilon)+s_k}\right)^{\varepsilon_k} \\ &\qquad \leq C\frac{\Gamma(a+\sigma(\varepsilon))}{\Gamma(a+\sigma(\varepsilon(k)))} \max(1, (a+\sigma(\varepsilon))^{-\varepsilon_k}) = C_1. \end{aligned}$$

Of course, the constant C_1 depends only on a and ε. Thus

$$\begin{aligned} J &\leq 2^{\sum_{j\neq k}|\varepsilon_j|} C_1 \prod_{j=1}^{n} (1+s_j)^{-\varepsilon_j} C(a+\sigma(\varepsilon)+s_k, s(k)) b(a+\sigma(\varepsilon), s_k) \\ &= 2^{\sum_{j\neq k}|\varepsilon_j|} C_1 C(a+\sigma(\varepsilon), s) \prod_{j=1}^{n} (1+s_j)^{-\varepsilon_j} = 2^{\sum_{j\neq k}|\varepsilon_j|} C_1 A_{s,n}^{a,\varepsilon} \end{aligned}$$

by (12) and (18)

We have proved the right-hand inequality of (17) when $\varepsilon_k > 0$.

Now we are going to prove the left-hand inequality of (17) when $\varepsilon_k > 0$.

From the definition of I,

$$I = I_0 \prod_{j\neq k} \left(\frac{1+\bar{l}_j}{1+l_j}\right)^{-\varepsilon_j}$$

where I_0 is equal to

$$\frac{C(\varepsilon_k,\bar\eta(k))b(a+\sigma(\varepsilon(k))+\sigma(\bar l(k)),s_k)C(a+\sigma(\varepsilon(k)),\bar l(k))}{C(\varepsilon_k,\eta(k))b(a+\sigma(\varepsilon(k))+\sigma(l(k)),s_k)C(a+\sigma(\varepsilon(k)),l(k))}.$$

Then

$$\begin{aligned}I_0 &= \frac{\Gamma(\varepsilon_k+\sigma(\bar\eta(k)))\Gamma(a+\sigma(\varepsilon(k))+\sigma(\bar l(k)))}{\Gamma(\varepsilon_k+\sigma(\eta(k)))\Gamma(a+\sigma(\varepsilon(k))+s_k+\sigma(l(k)))}\\
&= \prod_{\nu=0}^{d-1}\frac{a+\sigma(\varepsilon(k))+s_k+\sigma(l(k))+\nu}{\varepsilon_k+\sigma(\bar\eta(k))+\nu}\\
&\quad \geq \exp\left\{\sum_{\nu=0}^{d-1}\frac{a+\sigma(\varepsilon(k))+s_k+\sigma(l(k))-\sigma(\bar\eta(k))-\varepsilon_k}{a+\sigma(\varepsilon(k))+s_k+\sigma(l(k))+\nu}\right\}\\
&\quad \geq \exp\left\{\sum_{\nu=0}^{d-1}\frac{s_k}{2(s_k+\sigma(l(k))+\nu)}\right\} > \exp\frac{d}{4n-2}\geq 1\end{aligned}$$

since $s_k\geq 2\varepsilon_k$ and $e^x\leq(1-x)^{-1}$ when $1>x\geq 0$.

Applying this inequality to the J, we obtain

$$\begin{aligned}J &= \sum_{l(k)+\eta(k)=s(k),l(k)\in\bar E} C(\varepsilon_k,\eta(k))C(a+\sigma(\varepsilon(k))+s_k,l(k))\\
&\quad\times b(a+\sigma(\varepsilon(k)),s_k)\prod_{j\neq k}(1+l_j)^{-\varepsilon_j}\\
&\geq 2^{-\sum_{j\neq k}|\varepsilon_j|}\sum_{l(k)+\eta(k)=s(k),l(k)\in\bar E} C(\varepsilon_k,\eta(k))\\
&\quad\times b(a+\sigma(\varepsilon(k))+\sigma(l(k)),s_k)C(a+\sigma(\varepsilon(k)),l(k))\prod_{j\neq k}(1+s_j)^{-\varepsilon_j}\\
&\geq 2^{1-n-\sum_{j\neq k}|\varepsilon_j|}\prod_{j\neq k}(1+s_j)^{-\varepsilon_j}b(a+\sigma(\varepsilon(k)),s_k)\\
&\quad\times\sum_{l(k)+\eta(k)=s(k)} C(\varepsilon_k,\eta(k))C(a+\sigma(\varepsilon(k))+s_k,l(k))\\
&= 2^{1-n-\sum_{j\neq k}|\varepsilon_j|}\prod_{j\neq k}(1+s_j)^{-\varepsilon_j}b(a+\sigma(\varepsilon(k)),s_k)\\
&\quad\times C(a+\sigma(\varepsilon)+s_k,s(k))\end{aligned}$$

by Lemma 4.

Using Lemma 6, we have

$$\begin{aligned}&\frac{b(a+\sigma(\varepsilon(k)),s_k)(1+s_k)^{\varepsilon_k}}{b(a+\sigma(\varepsilon),s_k)}\\
&\quad=\frac{\Gamma(a+\sigma(\varepsilon(k))+\varepsilon_k)\Gamma(a+\sigma(\varepsilon(k))+s_k)(1+s_k)^{\varepsilon_k}}{\Gamma(a+\sigma(\varepsilon(k)))\Gamma(a+\sigma(\varepsilon(k))+\varepsilon_k+s_k)}\\
&\quad\geq C\frac{\Gamma(a+\sigma(\varepsilon))}{\Gamma(a+\sigma(\varepsilon(k)))}\left(\frac{1+s_k}{a+\sigma(\varepsilon)+s_k}\right)^{\varepsilon_k}\\
&\quad\geq C\frac{\Gamma(a+\sigma(\varepsilon))}{\Gamma(a+\sigma(\varepsilon(k)))}\min\left\{1,\left(\frac{1+2\varepsilon_k}{a+\sigma(\varepsilon)+2\varepsilon_k}\right)\right\}\equiv c_1\end{aligned}$$

since $s_k > 2\varepsilon_k$, and c_1 is a constant which only depends on a and ε. Thus

$$\begin{aligned}J &\geq 2^{1-n-\sum_{j\neq k}|\varepsilon_k|}c_1\prod_{j=1}^{n}(1+s_j)^{-\varepsilon_j}C(a+\sigma(\varepsilon)+s_k, s(k))b(a+\sigma(\varepsilon), s_k)\\ &= 2^{1-n-\sum_{j\neq k}|\varepsilon_j|}c_1C(a+\sigma(\varepsilon), s)\prod_{j=1}^{n}(1+s_j)^{-\varepsilon_j}\\ &= 2^{1-n-\sum_{j\neq k}|\varepsilon_j|}c_1A^{a,\varepsilon}_{s,n}\end{aligned}$$

by (12) and (18).

We have proved the left-hand inequality of (17) when $\varepsilon_k > 0$.

We now consider the case 2) $\varepsilon_k = 0$. The proof of (17) is easy. By (20), we know

$$B^{a,\varepsilon}_{s,n} = b(a+\sigma(\varepsilon(k))+\sigma(s(k)), s_k)B^{a,\varepsilon(k)}_{s(k),n-1}.$$

By induction hypothesis, there exist two constants m_{n-1} and M_{n-1}, which are independent of s, such that

$$m_{n-1}A^{a,\varepsilon(k)}_{s(k),n-1} \leq B^{a,\varepsilon(k)}_{s(k),n-1} \leq M_{n-1}A^{a,\varepsilon(k)}_{s(k),n-1}.$$

Thus

$$\begin{aligned}B^{a,\varepsilon}_{s,n} &\leq M_{n-1}b(a+\sigma(\varepsilon(k))+\sigma(s(k)), s_k)A^{a,\varepsilon(k)}_{s(k),n-1}\\ &= M_{n-1}b(a+\sigma(\varepsilon(k))+\sigma(s(k)), s_k)C(a+\sigma(\varepsilon(k)), s(k))\prod_{j\neq k}(1+s_j)^{-\varepsilon_j}\\ &= M_{n-1}C(a+\sigma(\varepsilon), s)\prod_{j=1}^{n}(1+s_j)^{-\varepsilon_j} = M_{n-1}A^{a,\varepsilon}_{s,n}\end{aligned}$$

since $\varepsilon_k = 0$.

Similarly, we can prove $B^{a,\varepsilon}_{s,n} \geq m_{n-1}A^{a,\varepsilon}_{s,n}$.

5. Proof of the Main Lemma, Part 2

In this section we prove the Main Lemma in the final case 3) $\varepsilon_k < 0$.

We take $a_k = -\varepsilon_k$ at (22). By Lemma 10, we know $B^{a-a_k,\varepsilon(k)}_{l(k),n-1} \geq 0$. All the factors in each term of the right-hand side of the equality (22) are non-negative.

The induction process is as follows: to prove (17) is true if we assume

$$m'_{n-1}(j)A^{a-a_k,\varepsilon(j)}_{l(j),n-1} \leq B^{a-a_k,\varepsilon(j)}_{l(j),n-1} \leq M'_{n-1}(j)A^{a-a_k,\varepsilon(j)}_{l(j),n-1}$$

is true for $j = 1, 2, \ldots, n$, where $m'_{n-1}(j)$ and $M'_{n-1}(j)$ are two positive constants, which are independent of $s(j)$, and only depend on a and $\varepsilon(j)$, and

$$A^{a-a_j,\varepsilon(j)}_{l(j),n-1} = C(a-a_j+\sigma(\varepsilon(j)), l(j))\prod_{j\neq k}(1+l_j)^{-\varepsilon_j}.$$

Let

$$m'_{n-1} = \min_{1\leq j\leq n} m'_{m-1}(j), \quad M'_{n-1} = \max_{1\leq j\leq n} M'_{n-1}(j),$$

then

$$m'_{n-1}A^{a-a_j,\varepsilon(j)}_{l(j),n-1} \leq B^{a-a_k,\varepsilon(j)}_{l(j),n-1} \leq M'_{n-1}A^{a-a_j,\varepsilon(j)}_{l(j),n-1}.$$

In particular, if we take $j = k$, then

$$m'_{n-1} A^{a-a_k,\varepsilon(k)}_{l(k),n-1} \le B^{a-a_k,\varepsilon(k)}_{l(k),n-1} \le M'_{n-1} A^{a-a,\varepsilon(k)}_{l(k),n-1}.$$

Let

$$P^{a,\varepsilon}_s = \sum_{l+\xi=s} C(\xi_k, \xi(k)) b(a_k, \xi_k) b(a - a_k + \sigma(\varepsilon(k)) + \sigma(l(k)), l_k) A^{a-a_k,\varepsilon(k)}_{l(k),n-1},$$

then

$$m'_{n-1} P^{a,\varepsilon}_s \le B^{a,\varepsilon}_{s,n} \le M'_{n-1} P^{a,\varepsilon}_s$$

hold by (22).

We estimate $P^{a,\varepsilon}_s$.

Partition the sum $P^{a,\varepsilon}_s$ into two parts,

$$P^{a,\varepsilon}_s = \sum_{l+\xi=s} = \sum_{l_k+\xi_k=s_k, l_k \ge \frac{3}{4}s_k} \ \sum_{l(k)+\xi(k)=s(k)} + \sum_{l_k+\xi_k=s_k, l_k < \frac{3}{4}s_k} \ \sum_{l(k)+\xi(k)=s(k)} = P_1 + P_2.$$

Let

$$Q^{a,\varepsilon}_{l,\xi} = C(\xi_k, \xi(k)) b(a - a_k + \sigma(\varepsilon(k)) + \sigma(l(k)), l_k) A^{a-a_k,\varepsilon(k)}_{l(k),n-1},$$

then

$$Q^{a,\varepsilon}_{l,\xi} = C(\xi_k, \xi(k)) b(a - a_k + \sigma(\varepsilon(k)) + \sigma(l(k)), l_k) C(a - a_k + \sigma(\varepsilon(k)), l(k)) \times \prod_{j\ne k} (1 + l_j)^{-\varepsilon_j} = C(\xi_k, \xi(k)) C(a - a_k + \sigma(\varepsilon(k)), l) \prod_{j\ne k} (1 + l_j)^{-\varepsilon_j}$$

by (12), and

$$P^{a,\varepsilon}_s = \sum_{l+\xi=s} b(a_k, \xi_k) Q^{a,\varepsilon}_{l,\xi} = P_1 + P_2.$$

Let

$$E_0 = \{l = (l_1, \ldots, l_n) \in \mathbf{Z}^n_+, 0 \le l_j \le s_j,\ j = 1, \ldots, n\},$$

and define the mapping $\phi_0 \colon E_0 \to E_0$, to $\phi_0(l) = \bar{l} = (\bar{l}_1, \ldots, \bar{l}_n)$ where $\bar{l}_j = l_j$, if $\frac{1}{2}s_j \le l_j \le s_j$; $\bar{l}_j = s_j - l_j$, if $0 \le l_j < \frac{1}{2}s_j$, $j = 1, \ldots, n$; and define $\bar{\xi}$ as $\bar{\xi} + l = \xi + l = s$, $\bar{E}_0 = \phi_0(E_0)$.

Consider

$$G = \frac{Q^{a,\varepsilon}_{\bar{l},\bar{\xi}}}{Q^{a,\varepsilon}_{l,\xi}} = \frac{C(\bar{\xi}_k, \bar{\xi}(k)) C(a - a_k + \sigma(\varepsilon(k)), \bar{l})}{C(\xi_k, \xi(k)) C(a - a_k + \sigma(\varepsilon(k)), l)} \prod_{j\ne k} \left(\frac{1 + l_j}{1 + \bar{l}_j} \right)^{\varepsilon_j}.$$

When $l_k \ge \frac{3}{4}s_k$, we have $\bar{l}_k = l_k$, $\bar{\xi}_k = \xi_k$, and then

$$\begin{aligned} G &= \frac{\Gamma(\sigma(\bar{\xi})\Gamma(a - a_k + \sigma(\varepsilon(k)) + \sigma(\bar{l}))}{\Gamma(\sigma(\xi))\Gamma(a - a_k + \sigma(\varepsilon(k)) + \sigma(l))} \prod_{j\ne k} \left(\frac{1 + l_j}{1 + \bar{l}_j} \right)^{\varepsilon_j} \\ &= \prod_{\nu=0}^{d-1} \frac{a - a_k + \sigma(\varepsilon(k)) + \sigma(l) + \nu}{\sigma(\bar{\xi}) + \nu} \prod_{j\ne k} \left(\frac{1 + l_j}{1 + l_j + d_j} \right)^{\varepsilon_j} \end{aligned} \tag{29}$$

where $d = \sigma(\bar{l}(k)) - \sigma(l(k))$, $d_j = \bar{l}_j - l_j$, $j \ne k$.

By the definition of ϕ_0, we have $\sigma(l(k)) \geq \sigma(\bar{\xi}(k))$, and $l_k - \bar{\xi}_k \geq \frac{1}{2}s_k$ since $l_k \geq \frac{3}{4}s_k$. By the hypothesis of the Main Lemma and $a_k + \varepsilon_k = 0$, we have $a - a_k + \sigma(\varepsilon(k)) > 0$. The first product in the right-hand side of (29) is equal to

$$\prod_{\nu=0}^{d-1}\left(1 + \frac{a - a_k + \sigma(\varepsilon(k)) + \sigma(l) - \sigma(\bar{\xi})}{\sigma(\bar{\xi}) + \nu}\right) \geq \prod_{\nu=0}^{d-1}\left(1 + \frac{\frac{1}{2}s_k}{\sigma(\bar{\xi}) + \nu}\right)$$

since $a - a_k + \sigma(\varepsilon(k)) > 0$, and

$$\sigma(l) - \sigma(\bar{\xi}) = l_k - \bar{\xi}_k + \sigma(l(k)) - \sigma(\bar{\xi}(k)) > l_k - \bar{\xi}_k \geq \tfrac{1}{2}s_k.$$

Moreover,

$$\sigma(\bar{\xi}) + \nu \leq ns_k + (n-1)s_k \leq (2n-1)s_k$$

and

$$\left(\frac{1 + l_j}{1 + l_j + d_j}\right)^{\varepsilon_j} \geq (1 + d_j)^{-|\varepsilon_j|} \geq (1 + d)^{-|\varepsilon_j|},$$

hence

$$\begin{aligned} G &\geq \prod_{\nu=0}^{d-1}\left(\frac{4n-1}{4n-2}\right)\prod_{j\neq k}\left(\frac{1}{1+d}\right)^{|\varepsilon_j|} \\ &= \left(\frac{4n-1}{4n-2}\right)^{d}\left(\frac{1}{1+d}\right)^{\sum_{j\neq k}|\varepsilon_j|} \equiv f. \end{aligned}$$

Then function f take its minimum value at $d = (\log\frac{4n-1}{4n-2})^{-1}\sum_{j\neq k}|\varepsilon_j| - 1$, thus

$$\begin{aligned} G \geq &\left(\frac{4n-1}{4n-2}\right)^{\sum_{j\neq k}|\varepsilon_j|(\log(\frac{4n-1}{4n-2}))^{-1}-1} \\ &\times\left(\log\left(\frac{4n-1}{4n-2}\right)\right)^{\sum_{j\neq k}|\varepsilon_j|}\left(\sum_{j\neq k}|\varepsilon_j|\right)^{-\sum_{j\neq k}|\varepsilon_j|} \equiv a_0' \end{aligned}$$

if $\sum_{j\neq k}|\varepsilon_j| \neq 0$; and $G \geq 1$, if $\sum_{j\neq k}|\varepsilon_j| = 0$. Let $a_1' = \min(1, a_0')$, then $G \geq a_1'$. Obviously, a_1' is a constant which only depends on a and ε.

Using the inequality $G \geq a_1'$ and (29), we have

$$\begin{aligned} P_1 &= \sum_{l_k+\xi_k=s_k, l_k\geq\frac{3}{4}s_k}\ \sum_{l(k)+\xi(k)=s(k)} b(a_k,\xi_k)Q_{l,\xi}^{a,\varepsilon} \\ &\leq \frac{2^{n-1}}{a_1'}\sum_{l_k+\xi_k=s_k, l_k\geq\frac{3}{4}s_k}\ \sum_{l(k)+\xi(k)=s(k), l\in\bar{E}_0} b(a_k,\xi_k)Q_{l,\xi}^{a,\varepsilon}. \end{aligned}$$

By the definition of $Q_{l,\xi}^{a,\varepsilon}$, and we observe (28) holds true if $l \in \bar{E}_0$. Then

$$2^{-\sum_{j\neq k}|\varepsilon_j|}Q_0 \leq Q_{l,\xi}^{a,\varepsilon} \leq 2^{\sum_{j\neq k}|\varepsilon_j|}Q_0$$

where

$$Q_0 = C(\xi_k, \xi(k))C(a - a_k + \sigma(\varepsilon(k)), l)\prod_{j\neq k}(1 + s_j)^{-\varepsilon_j}.$$

The right-hand inequality implies

$$
\begin{aligned}
P_1 \le C_2 & \sum_{l_k+\xi_k=s_k, l_k\ge\frac{3}{4}s_{s_k}} \ \sum_{l(k)+\xi(k)=s(k), l\in\bar{E}_0} b(a_k,\xi_k)C(\xi_k,\xi(k)) \\
& \times C(a-a_k+\sigma(\varepsilon(k)),l)\prod_{j\ne k}(1+s_j)^{-\varepsilon_j} \\
& \le C_2\prod_{j\ne k}(1+s_j)^{-\varepsilon_j}\sum_{l+\xi=s} b(a_k,\xi_k)C(\xi_k,\xi(k))C(a-a_k+\sigma(\varepsilon(k)),l) \\
& = C_2\prod_{j\ne k}(1+s_j)^{-\varepsilon_j}\sum_{l_k+\xi_k=s_k} b(a_k,\xi_k)b(a-a_k+\sigma(\varepsilon(k)),l_k) \\
& \quad \times \sum_{l(k)+\xi(k)=s(k)} C(\xi_k,\xi(k))C(a-a_k+\sigma(\varepsilon(k))+l_k,l(k)) \\
& = C_2\prod_{j\ne k}(1+s_j)^{-\varepsilon_j}\sum_{l_k+\xi_k=s_k} b(a_k,\xi_k)b(a-a_k+\sigma(\varepsilon(k)),l_k) \\
& \quad \times C(a+\sigma(\varepsilon)+s_k,s(k)) \\
& = C_2\prod_{j\ne k}(1+s_j)^{-\varepsilon_j} b(a+\sigma(\varepsilon(k)),s_k)C(a+\sigma(\varepsilon)+s_k,s(k))
\end{aligned}
$$

where $C_2 = 2^{n-1+\sum_{j\ne k}|\varepsilon_j|}/a_1'$.

Using Lemma 6, and a similar process as we used in section 4, we may prove

$$
\frac{b(a+\sigma(\varepsilon(k)),s_k)(1+s_k)^{\varepsilon_k}}{b(a+\sigma(\varepsilon),s_k)} \le C_3
$$

where C_3 is a constant, which only depends on a and ε. So we have

$$
\begin{aligned}
P_1 &\le C_2C_3\prod_{j=1}^{n}(1+s_j)^{-\varepsilon_j} b(a+\sigma(\varepsilon),s_k)C(a+\sigma(\varepsilon)+s_k,s(k)) \\
&= C_2C_3A_{s,n}^{a,\varepsilon}.
\end{aligned}
$$

Now we estimate the upper bound of P_2.

P_2 is defined as

$$
\begin{aligned}
P_2 = & \sum_{l_k+\xi_k=s_k, l_k<\frac{3}{4}s_k} \ \sum_{l(k)+\xi(k)=s(k)} b(a_k,\xi_k)C(\xi_k,\xi(k)) \\
& \times C(a-a_k+\sigma(\varepsilon(k)),l)\prod_{j\ne k}(1+l_j)^{-\varepsilon_j}.
\end{aligned}
$$

It is equal to

$$
\begin{aligned}
& \sum_{l_k+\xi_k=s_k, l_k<\frac{3}{4}s_k} \ \sum_{l(k)+\xi(k)=s(k)} \frac{C(\xi_k,\xi(k))}{c(a_k+\xi_k,\xi(k))} b(a-a_k+\sigma(\varepsilon(k))+\sigma(l(k)),l_k) \\
& \times b(a_k+\sigma(\xi(k)),\xi_k)C(a_k,\xi(k))C(a-a_k+\sigma(\varepsilon(k)),l(k))\prod_{j\ne k}(1+l_j)^{-\varepsilon_j}
\end{aligned}
$$

since

$$
b(a_k,\xi_k)C(a_k+\xi_k,\xi(k)) = b(a_k+\sigma(\xi(k)),\xi_k)C(a_k,\xi(k))
$$

by (12). Let $r_j = \{\varepsilon_j\}$, $j = 1, \ldots, n$, where $\{x\}$ means the least integer X, such that $X \geq x$, and $r = (r_1, \ldots, r_n), \bar{\varepsilon}_j = \varepsilon_j - \{\varepsilon_j\}$, then $-1 < \bar{\varepsilon}_j \leq 0$, $j \neq k$, and $\bar{\varepsilon}_k = \varepsilon_k$ since $-1 < \varepsilon_k \leq 0$.

Using Lemma 6, we have

$$\begin{aligned}\frac{\Gamma(1+l_j+r_j)(1+l_j+r_j)^{\bar{\varepsilon}_j}}{\Gamma(1+l_j)(1+l_j)^{\varepsilon_j}} &\leq C_{4j}\frac{(1+l_j)^{r_j}(1+l_j+r_j)^{\bar{\varepsilon}_j}}{(1+l_j)^{\varepsilon_j}} \\ &= C_{4j}\left(\frac{1+l_j+r_j}{1+l_j}\right)^{\bar{\varepsilon}_j} \leq C_{4j},\end{aligned}$$

and

$$\begin{aligned}\frac{\Gamma(1+l_j+r_j)(1+l_j+r_j)^{\bar{\varepsilon}_j}}{\Gamma(1+l_j)(1+l_j)^{\varepsilon_j}} &\geq c_{3j}\left(\frac{1+l_j+r_j}{1+l_j}\right)^{\bar{\varepsilon}_j} \\ &\geq c_{3j}(1+r_j)^{\bar{\varepsilon}_j} = c_{4j}\end{aligned}$$

where C_{4j}, c_{3j} and c_{4j} are constants, which only depend on ε_j. Let

$$C_4 = \max_{j\neq k} C_{4j}, \quad c_4 = \min_{j\neq k} c_{4j}.$$

Thus we have

$$\begin{aligned}c_4(1+l_j+r_j)^{-\bar{\varepsilon}_j}\frac{\Gamma(1+l_j)}{\Gamma(1+l_j+r_j)} &\leq (1+l_j)^{-\varepsilon_j} \\ &\leq C_4(1+l_j+r_j)^{-\bar{\varepsilon}_j}\frac{\Gamma(1+l_j)}{\Gamma(1+l_j+r_j)}.\end{aligned}$$

Let $q = l + r$, then

$$\begin{aligned}&C(a-a_k+\sigma(\varepsilon(k)), l)\prod_{j\neq k}(1+l_j)^{-\varepsilon_j} \\ &\quad\leq C_4\frac{\Gamma(a+\sigma(\varepsilon)+\sigma(l))}{\Gamma(a+\sigma(\varepsilon))}\prod\frac{(1+q_j)^{-\bar{\varepsilon}_j}}{\Gamma(1+q_j)} \\ &\quad= C_5\frac{\Gamma(a+\sigma(\varepsilon)+\sigma(q)-\sigma(r))}{\Gamma(a+\sigma(\varepsilon)-\sigma(r))\prod\Gamma(1+q_j)}\prod_{j\neq k}(1+q_j)^{-\bar{\varepsilon}_j} \\ &\quad= C_5 C(a+\sigma(\varepsilon)-\sigma(r), q)\prod_{j\neq k}(1+q_j)^{-\bar{\varepsilon}_j}\end{aligned}$$

and

$$C(a+\sigma(\varepsilon), l)\prod_{j\neq k}(1+l_j)^{-\varepsilon_j} \geq c_5 C(a+\sigma(\varepsilon)-\sigma(r), q)\prod_{j\neq k}(1+q_j)^{-\bar{\varepsilon}_j} \tag{30}$$

where

$$C_5 = C_4\frac{\Gamma(a+\sigma(\varepsilon)-\sigma(r))}{\Gamma(a+\sigma(\varepsilon))}, \quad c_5 = c_4\frac{\Gamma(a+\sigma(\varepsilon)-\sigma(r))}{\Gamma(a+\sigma(\varepsilon))}.$$

Define the mapping $\phi^*\colon E \to E$, by $l(k) \to \bar{l}(k) = \phi^*(l(k))$, where $\bar{l}_j = l_j$, if $\frac{1}{2}(s_j - r_j) < l_j \leq s_j$; and $\bar{l}_j = s_j - l_j$, if $0 \leq l_j \leq \frac{1}{2}(s_j - r_j)$.

Let $E^* = \phi^*(E)$. Define $\xi, \bar{l}$ and $\bar{\xi}$ by $l + \xi = s$, $\bar{l} = \phi^*(l)$ and $\bar{l} + \bar{\xi} = l + \xi = s$. Consider

$$R = \frac{C(a_k, \bar{\xi})C(a + \sigma(\varepsilon), \bar{l})}{C(a_k, \xi)C(a + \sigma(\varepsilon), l)} \prod_{j \neq k} \left(\frac{1 + l_j}{1 + \bar{l}_j} \right)^{\varepsilon_j}.$$

Then

$$R \geq \frac{c_5}{C_5} \frac{C(a_k, \bar{\xi})C(a + \sigma(\varepsilon) - \sigma(r), \bar{q})}{C(a_k, \xi)C(a + \sigma(\varepsilon) - \sigma(r), q)} \prod_{j \neq k} \left(\frac{1 + \bar{q}_j}{1 + q_j} \right)^{-\bar{\varepsilon}_j}$$

where

$$\bar{q} + \bar{\xi} = q + \xi = l + r + \xi = s + r = \bar{l} + r + \bar{\xi}.$$

By the definition of ϕ^* and $\bar{q}$, we have $q_j \leq \bar{q}_j$, hence

$$\prod_{j \neq k} \left(\frac{1 + \bar{q}_j}{1 + q_j} \right)^{-\bar{\varepsilon}_j} \geq 1,$$

and

$$\begin{aligned} R &\geq c_6 \frac{C(a_k, \bar{\xi})C(a + \sigma(\varepsilon) - \sigma(r), \bar{q})}{C(a_k, \xi)C(a + \sigma(\varepsilon) - \sigma(r), q)} \\ &= c_6 \frac{\Gamma(\sigma(\bar{\xi}) + a_k)\Gamma(a + \sigma(\varepsilon) - \sigma(r) + \sigma(\bar{q}))}{\Gamma(\sigma(\xi) + a_k)\Gamma(a + \sigma(\varepsilon) - \sigma(r) + \sigma(q))} \\ &\geq c_6 \frac{\Gamma(\sigma(\bar{\xi} + a_k))\Gamma(a + \sigma(\varepsilon) - \sigma(r) + \sigma(\bar{q}))}{\Gamma(\sigma(\xi) + a_k)\Gamma(a + \sigma(\varepsilon) - \sigma(r) + \sigma(q))} \\ &= c_6 \prod_{\nu=1}^{d-1} \frac{a + \sigma(\varepsilon) - \sigma(r) + \sigma(q) + \nu}{a_k + \sigma(\bar{\xi}) + \nu} \geq c_6 \end{aligned}$$

since $\sigma(q) \geq \sigma(\bar{\xi})$ and $a + \sigma(\varepsilon) > \sigma(r) + a_k$, where $c_6 = c_5/C_5$. Thus

$$P_2 \leq c_6^{-1} 2^{n-1} b_1^{-1} \sum_{l_k + \xi_k = s_k, l_k < \frac{3}{4}s_k} \sum_{l(k) + \xi(k) = s(k), l(k) \in E^*} b(a_k, \xi_k)$$

$$\times b(a + \sigma(\varepsilon) + \sigma(l(k)), l_k) C(\xi_k, \xi(k)) C(a + \sigma(\varepsilon), l(k)) \prod_{j \neq k} (1 + l_j)^{-\varepsilon_j}$$

where

$$0 < b_1 \leq C(\xi_k, \xi(k))(C(a_k + \xi_k, \xi(k))^{-1} \leq 1.$$

Since $l(k) \in E^*$, $s_j \geq l_j > \frac{1}{2}(s_j - r_j)$, we can find a constant $p_j > 0$, which only depends on ε_j, such that $(1 + l_j)^{-\varepsilon_j} \leq p_j^{|\varepsilon_j|}(1 + s_j)^{-\varepsilon_j}$ hold for all but a finite number of values of s_j. Therefore,

$$\begin{aligned}
P_2 &\le C_6 \prod_{j\neq k}(1+s_j)^{-\varepsilon_j} \sum_{l_k+\xi_k=s_k, l_k<\frac{3}{4}s_k} \ \sum_{l(k)+\xi(k)=s(k), l(k)\in E^*} b(a_k,\xi_k) \\
&\quad \times b(a+\sigma(\varepsilon)+\sigma(l(k)), l_k)C(\xi_k,\xi(k))C(a+\sigma(\varepsilon),l(k)) \\
&\le C_6 \prod_{j\neq k}(1+s_j)^{-\varepsilon_j} \sum_{l_k+\xi_k=s_k} \ \sum_{l(k)+\xi(k)=s(k)} b(a_k,\xi_k)b(a+\sigma(\varepsilon) \\
&\quad + \sigma(l(k), l_k)C(\xi_k,\xi(k))C(a+\sigma(\varepsilon),l(k)) \\
&= C_6 \prod_{j\neq k}(1+s_j)^{-\varepsilon_j} \sum_{l_k+\xi_k=s_k} \ \sum_{l(k)+\xi(k)=s(k)} b(a_k,\xi_k) \\
&\quad \times C(a+\sigma(\varepsilon)+l_k, l(k))b(a+\sigma(\varepsilon),l_k)C(\xi_k,\xi(k)) \\
&= C_6 \prod_{j\neq k}(1+s_j)^{-\varepsilon_j} \sum_{l_k+\xi_k=s_k} b(a_k,\xi_k)b(a+\sigma(\varepsilon),l_k) \\
&\quad \times C(a+\sigma(\varepsilon)+s_k, s(k)) \\
&= C_6 \prod_{j\neq k}(1+s_j)^{-\varepsilon_j} b(a+\sigma(\varepsilon(k)), s_k)C(a+\sigma(\varepsilon)+s_k,s(k)) \\
&\le C_6C_3 \prod_{j=1}^{n}(1+s_j)^{-\varepsilon_j} b(a+\sigma(\varepsilon), s_k)C(a+\sigma(\varepsilon)+s_k,s(k)) \\
&= C_6C_3A^{a,\varepsilon}_{s,n}
\end{aligned}$$

where $C_6 = c_6^{-1}2^{n-1}b_1^{-1}\prod_{j\neq k} p_j^{|\varepsilon_j|}$.

We have proved the right-hand inequality of (17) when $\varepsilon_k < 0$.

Now we are going to prove the left-hand inequality of (17) when $\varepsilon_k < 0$. We just need to estimate the lower bound of $P^{a,\varepsilon}_s$ when $\varepsilon_k < 0$.

We already know

$$\begin{aligned}
P^{a,\varepsilon}_s &= \sum_{l+\xi=s} b(a_k,\xi_k)Q^{a,\varepsilon}_{l,\xi} \\
&= \sum_{l_k+\xi_k=s_k} \ \sum_{l(k)+\xi(k)=s(k)} b(a_k,\xi_k)C(\xi_k,\xi(k)) \\
&\quad \times C(a+\sigma(\varepsilon),l)\prod_{j\neq k}(1+l_j)^{-\varepsilon_j} \\
&\ge \sum_{l_k+\xi_k=s_k} \ \sum_{l(k)+\xi(k)=s(k), l(k)\in\bar{E}} b(a_k,\xi_k)C(\xi_k,\xi(k)) \\
&\quad \times C(a+\sigma(\varepsilon),l)\prod_{j\neq k}(1+l_j)^{-\varepsilon_j}
\end{aligned}$$

where $\bar{E} = \phi(E)$, and ϕ, E are defined in section 4. If we use inequality (28) and the equality

$$C(a+\sigma(\varepsilon),l) = b(a+\sigma(\varepsilon),l_k)C(a+\sigma(\varepsilon)+l_k,l(k)),$$

we have

$$P_s^{a,\varepsilon} \geq 2^{-\sum_{j\neq k}|\varepsilon_j|}\prod_{j\neq k}(1+s_j)^{-\varepsilon_j}\sum_{l_k+\xi_k=s_k} b(a_k,\xi_k)$$
$$\times\, b(a+\sigma(\varepsilon),l_k)\sum_{l(k)+\xi(k)=s(k),l(k)\in\bar{E}} C(\xi_k,\xi(k))C(a+\sigma(\varepsilon)+l_k,l(k)).$$

Let $\bar{l}(k)=\phi(l(k)), \bar{l}(k)+\bar{\xi}(k)=l(k)+\xi(k)=s(k)$, and consider

$$\begin{aligned} H &= \frac{C(\xi_k,\bar{\xi}(k))C(a+\sigma(\varepsilon),\bar{l})}{C(\xi_k,\xi(k))C(a+\sigma(\varepsilon),l)} \\ &= \frac{\Gamma(\sigma(\bar{\xi}))\Gamma(a+\sigma(\varepsilon)+\sigma(\bar{l}))}{\Gamma(\sigma(\xi))\Gamma(a+\sigma(\varepsilon)+\sigma(l))} \\ &= \prod_{\nu=0}^{d-1}\frac{a+\sigma(\varepsilon)+\sigma(l)+\nu}{\sigma(\bar{\xi})+\nu} \end{aligned}$$

where $d=\sigma(\bar{l}(k))-\sigma(l(k))$. Then $H\geq 1$ since $a+\sigma(\varepsilon)>0$, and $\sigma(l)\geq\sigma(\bar{\xi})$. Hence

$$\begin{aligned} P_s^{a,\varepsilon} &\geq c_7\prod_{j\neq k}(1+s_j)^{-\varepsilon_j}\sum_{l_k+\xi_k=s_k} b(a_k,\xi_k) \\ &\quad\times b(a+\sigma(\varepsilon),l_k)\sum_{l(k)+\xi(k)=s(k)} C(\xi_k,\xi(k))C(a+\sigma(\varepsilon)+l_k,l(k)) \\ &= c_7\prod_{j\neq k}(1+s_j)^{-\varepsilon_j}\sum_{l_k+\xi_k=s_k} b(a_k,\xi_k) \\ &\quad\times b(a+\sigma(\varepsilon),l_k)C(a+\sigma(\varepsilon)+s_k,s(k)) \\ &= c_7\prod_{j\neq k}(1+s_j)^{-\varepsilon_j} b(a+\sigma(\varepsilon(k)),s_k)C(a+\sigma(\varepsilon)+s_k,s(k)) \\ &\geq c_7c_1\prod_{j=1}^{n}(1+s_j)^{-\varepsilon_j} b(a+\sigma(\varepsilon),s_k)C(a+\sigma(\varepsilon)+s_k,s(k)) \\ &= c_7c_1A_{s,n}^{a\varepsilon} \end{aligned}$$

where $c_7=2^{1-n-\sum_{j\neq k}|\varepsilon_j|}$ and c_1 is defined in section 4.

We have proved the left-hand inequality of (17) when $\varepsilon_k<0$. $\square$

6. Proof of the Theorem, Part 1

Let D be defined as in (1), and its Bergman kernel function be

$$K(z,\bar{w})=\sum_{m\in\mathbf{Z}_+^n} C_m z^m\bar{w}^m$$

where $m=(m_1,\ldots,m_n), z=(z_1,\ldots,z_n)\in D, w=(w_1,\ldots,w_n)\in D$, $z^m=z_1^{m_1}\cdots z_n^{m_n}$ and $w^m=w_1^{m_1}\cdots w_n^{m_n}$. We know

$$C_m=\frac{1}{\pi^n(\prod_{j=1}^n\alpha_j)}\frac{\Gamma(\sigma(\alpha m)+\sigma(\alpha)+1)}{\prod_{j=1}^n\Gamma(\alpha_j m_j+\alpha_j)} \tag{31}$$

by (4).

Lemma 11. *There exist two constants c_8 and C_8, which are independent of m, and depend only on α and n, such that*

$$c_8 B^{n+1,\alpha-1}_{[\alpha m],n} \prod_{j=1}^{n} \left(\frac{1+\sigma(\alpha)+\sigma([\alpha m])}{1+[\alpha_j m_j]} \right)^{\varepsilon_j} \leq C_m$$
$$\leq C_8 B^{n+1,\alpha-1}_{[\alpha m],n} \prod_{j=1}^{n} \left(\frac{1+\sigma(\alpha)+\sigma([\alpha m])}{1+[\alpha_j m_j]} \right)^{\varepsilon_j}$$

where $[x]$ means the largest integer X, such that $X \leq x$, $\varepsilon_j = \alpha_j m_j - [\alpha_j m_j]$, and $\sigma([\alpha m]) = \sum_{j=1}^{n} [\alpha_j m_j]$, and hence $0 \leq \varepsilon_j < 1$.

Proof. By Lemma 6, there exist c_8', c_8'', C_8' and C_8'', which are independent of m, and only depend on α and n, such that

$$c_8' \Gamma(\sigma([\alpha m]) + \sigma(\alpha) + 1)(\sigma([\alpha m]) + \sigma(\alpha) + 1)^{\sigma(\varepsilon)}$$
$$\leq \Gamma(\sigma([\alpha m]) + \sigma(\varepsilon) + \sigma(\alpha) + 1)$$
$$\leq C_8' \Gamma(\sigma([\alpha m]) + \sigma(\alpha) + 1)(\sigma([\alpha m]) + \sigma(\alpha) + 1)^{\sigma(\varepsilon)};$$

and

$$c_8'' \Gamma([\alpha_j m_j] + 1)([\alpha_j m_j] + 1)^{\varepsilon_j + \alpha_j - 1} \leq \Gamma([\alpha_j m_j] + \varepsilon_j + \alpha_j)$$
$$\leq C_8'' \Gamma([\alpha_j m_j] + 1)([\alpha_j m_j] + 1)^{\varepsilon_j + \alpha_j - 1}.$$

Thus we have

$$\frac{c_8'}{C_8''} \frac{1}{\pi^n (\prod_{j+1}^{n} \alpha_j)} \frac{\Gamma(\sigma([\alpha m]) + \sigma(\alpha) + 1)(\sigma([\alpha m]) + \sigma(\alpha) + 1)^{\sigma(\varepsilon)}}{\prod_{j=1}^{n} \Gamma([\alpha_j m_j] + 1)([\alpha_j m_j] + 1)^{\varepsilon_j + \alpha_j - 1}} \leq C_m$$
$$\leq \frac{C_8'}{c_8''} \frac{1}{\pi^n (\prod_{j=1}^{n} \alpha_j)} \frac{\Gamma(\sigma([\alpha m]) + \sigma(\alpha) + 1)(\sigma([\alpha m]) + \sigma(\alpha) + 1)^{\sigma(\varepsilon)}}{\prod_{j=1}^{n} \Gamma([\alpha_j m_j] + 1)([\alpha_j m_j] + 1)^{\varepsilon_j + \alpha_j - 1}}.$$

By the Main Lemma, we know

$$m_n A^{n+1,\alpha-1}_{[\alpha m],n} \leq B^{n+1,\alpha-1}_{[\alpha m],n} \leq M_n A^{n+1,\alpha-1}_{[\alpha m],n},$$

where $\alpha - 1 = (\alpha_1 - 1, \ldots, \alpha_n - 1)$, m_n and M_n are two constants, which are independent of m, and only depend on α, n. According to the definition (18),

$$A^{n+1,\alpha-1}_{[\alpha m],n} = \frac{\Gamma(\sigma([\alpha m]) + \sigma(\alpha) + 1)}{\Gamma(1+\sigma(\alpha)) \prod_{j=1}^{n} \Gamma(1 + [\alpha_j m_j])(1 + [\alpha_j m_j])^{\alpha_j - 1}},$$

we have

$$m_n \frac{\Gamma(\sigma([\alpha m]) + \sigma(\alpha) + 1)(\sigma([\alpha m]) + \sigma(\alpha) + 1)^{\sigma(\varepsilon)}}{\Gamma(1+\sigma(\alpha)) \prod_{j=1}^{n} \Gamma(1 + [\alpha_j m_j])(1 + [\alpha_j m_j])^{\varepsilon_j + \alpha_j - 1}}$$
$$\leq B^{n+1,\alpha-1}_{[\alpha m],n} \prod_{j=1}^{n} \left(\frac{1+\sigma(\alpha)+\sigma([\alpha m])}{1+[\alpha_j m_j]} \right)^{\varepsilon_j}$$
$$\leq M_n \frac{\Gamma(\sigma([\alpha m]) + \sigma(\alpha) + 1)(\sigma([\alpha m]) + \sigma(\alpha) + 1)^{\sigma(\varepsilon)}}{\Gamma(1+\sigma(\alpha)) \prod_{j=1}^{n} \Gamma(1 + [\alpha_j m_j])(1 + [\alpha_j m_j])(1 + [\alpha_j m_j])^{\varepsilon_j + \alpha_j - 1}}.$$

Thus we prove the lemma. □

Lemma 12. *If $\delta = (\delta_1, \ldots, \delta_n)$, $\delta_j = 0$ or $1, j = 1, 2, \ldots, n$, then there exist two positive constants c_9 and C_9, which are independent of m, and only depend on α and n, such that*

$$c_9 B^{n+1,\alpha-1}_{[\alpha m],n} \prod_{j=1}^{n} \left(\frac{1+\sigma(\alpha)+\sigma([\alpha m])}{1+[\alpha_j m_j]}\right)^{\delta_j} \leq B^{n+1,\alpha-1+\delta}_{[\alpha m],n}$$
$$\leq C_9 B^{n+1,\alpha-1}_{[\alpha m],n} \prod_{j=1}^{n} \left(\frac{1+\sigma(\alpha)+\sigma([\alpha m])}{1+[\alpha_j m_j]}\right)^{\delta_j}$$

where $\alpha - 1 + \delta = (\alpha_1 - 1 + \delta_1, \ldots, \alpha_n - 1 + \delta_n)$.

Proof. By the Main Lemma, there exist two positive constants c_9' and C_9', which are independent of m and δ, and only depend on α and n, such that

$$c_9' A^{n+1,\alpha-1+\delta}_{[\alpha m],n} \leq B^{n+1,\alpha-1+\delta}_{[\alpha m],n} \leq C_9' A^{n+1,\alpha-1+\delta}_{[\alpha m],n}.$$

According to the definition (18),

$$A^{n+1,\alpha-1+\delta}_{[\alpha m],n} = \frac{\Gamma(1+\sigma(\alpha)+\sigma([\alpha m])+\sigma(\delta))}{\Gamma(1+\sigma(\alpha)+\sigma(\delta))\prod_{j=1}^{n}\Gamma(1+[\alpha_j m_j])(1+[\alpha_j m_j])^{\alpha_j-1+\delta_j}},$$

where $\sigma(\delta) = \sum_{j=1}^{n} \delta_j$. By Lemma 6, there exist two positive constants c_9'' and C_9'', which are independent of m and δ, and only depend on α and n, such that

$$c_9'' \frac{\Gamma(\sigma([\alpha m])+\sigma(\alpha)+1)(\sigma([\alpha m])+\sigma(\alpha)+1)^{\sigma(\delta)}}{\Gamma(1+\sigma(\alpha))(1+\sigma(\delta))^{\sigma(\delta)}\prod_{j=1}^{n}\Gamma(1+[\alpha_j m_j])(1+[\alpha_j m_j])^{\alpha_j-1+\delta_j}}$$
$$\leq A^{n+1,\alpha-1+\delta}_{[\alpha m],n}$$
$$\leq C_9'' \frac{\Gamma(\sigma([\alpha m])+\sigma(\alpha)+1)(\sigma([\alpha m])+\sigma(\alpha)+1)^{\sigma(\delta)}}{\Gamma(1+\sigma(\alpha))(1+\sigma(\alpha))^{\sigma(\delta)}\prod_{j=1}^{n}\Gamma(1+[\alpha_j m_j])(1+[\alpha_j m_j])^{\alpha_j-1+\delta_j}}.$$

That is

$$c_9'' B^{n+1,\alpha-1}_{[\alpha m],n} \prod_{j=1}^{n} \left(\frac{\sigma([\alpha m])+\sigma(\alpha)+1}{[\alpha_j m_j]+1}\right)^{\delta_j} (1+\sigma(\alpha))^{-\sigma(\delta)} \leq A^{n+1,\alpha-1+\delta}_{[\alpha m],n}$$
$$\leq C_9'' B^{n+1,\alpha-1}_{[\alpha m],n} \prod_{j=1}^{n} \left(\frac{\sigma([\alpha m])+\sigma(\alpha)+1}{[\alpha_j m_j]+1}\right)^{\delta_j} (1+\sigma(\alpha))^{-\sigma(\delta)}.$$

Thus we have proved the lemma. □

From (4), we know

$$K(z,\bar{z}) = \sum_{z\in\mathbf{Z}^n_+} C_m |z|^{2m}$$

where C_m is given by (31), and $|z|^{2m} = |z_1|^{2m_1}\cdots|z_n|^{2m_n}$. Let $u_j = |z_j|^{2/\alpha_j}$, $j = 1, 2, \ldots, n$, $\|z\|_\alpha = \sum_{j=1}^{n} |z_j|^{2/\alpha_j}$, then $D = \{z \in \mathbf{C}^n \mid \|z\|_\alpha < 1\}$ and

$$K(z,\bar{z}) = \sum_{m\in\mathbf{Z}^n_+} C_m u^{m\alpha}$$

where $u^{m\alpha} = u_1^{m_1\alpha_1}\cdots u_n^{m_n\alpha_n}$.

Let $K_\alpha(u) = \sum_{m\in\mathbf{Z}_+^n} C_m u^{m\alpha}$, then

$$K_\alpha(u) = \sum_{m\in\mathbf{Z}_+^n} C_m u_1^{[\alpha_1 m_1]}\cdots u_n^{[\alpha_n m_n]} u_1^{\varepsilon_1}\cdots u_n^{\varepsilon_n}$$

where $\varepsilon_j = \alpha_j m_j - [\alpha_j m_j]$ and hence $1 < \varepsilon_j \geq 0$. By Lemma 11,

$$\begin{aligned} K_\alpha(u) &\leq C_8 \sum_{m\in\mathbf{Z}_+^n} B_{[\alpha m],n}^{n+1,\alpha-1} \prod_{j=1}^n \left(\frac{1+\sigma(\alpha)+\sigma([\alpha m])}{1+[\alpha_j m_j]} u_j\right)^{\varepsilon_j} u_1^{[\alpha_1 m_1]}\cdots u_n^{[\alpha_n m_n]} \\ &\leq C_8 \sum_{m\in\mathbf{Z}_+^n} B_{[\alpha m],n}^{n+1,\alpha-1} \prod_{j=1}^n \left(1+\frac{1+\sigma(\alpha)+\sigma([\alpha m])}{1+[\alpha_j m_j]} u_j\right) u_1^{[\alpha_1 m_1]}\cdots u_n^{[\alpha_n m_n]} \end{aligned}$$

since $1+x > x^a$ if $x>0$ and $0\leq a\leq 1$.

Let

$$\prod_{j=1}^n (1+x_j) = \sum_{\delta\in Q} x^\delta = \sum_{\delta\in Q} x_1^{\delta_1}\cdots x_n^{\delta_n}$$

where $\delta = (\delta_1,\ldots,\delta_n)$, $\delta_j = 0$ or 1, $j=1,2,\ldots,n$, and Q is a set of lattice points in $\mathbf{R}_+^n$, which contains all the lattice points whose components are only 0 or 1, then

$$\begin{aligned} K_\alpha(u) \leq C_8 &\sum_{m\in\mathbf{Z}_+^n} B_{[\alpha m],n}^{n+1,\alpha-1} u_1^{[\alpha_1 m_1]}\cdots u_n^{[\alpha_n m_n]} \\ &\times \sum_{\delta\in Q}\prod_{j=1}^n \left(\frac{1+\sigma(\alpha)+\sigma([\alpha m])}{1+[\alpha_j m_j]}\right)^{\delta_j} u^{\delta_j}. \end{aligned}$$

By Lemma 12, we have

$$K_\alpha(u) \leq \frac{C_8}{c_9}\sum_{\delta\in Q} u^\delta \sum_{m\in\mathbf{Z}_+^n} B_{[\alpha m],n}^{n+1,\alpha-1+\delta} u_1^{[\alpha_1 m_1]}\cdots u_n^{[\alpha_n m_n]}$$

where $u^\delta = u_1^{\delta_1}\cdots u_n^{\delta_n}$.

Let $h=(h_1,\ldots,h_n)\in\mathbf{Z}_+^n$ be given. Then as an equation of $m_j, [\alpha_j m_j] = h_j$ at most has $\{\frac{1}{\alpha_j}\}$ solutions, $j=1,2,\ldots,n$, where $\{x\}$ means the least integer X such that $x\leq X$. Thus

$$K_\alpha(u) \leq \frac{C_8}{c_9}\prod_{j=1}^n \left\{\frac{1}{\alpha_j}\right\}\sum_{\delta\in Q} u^\delta \sum_{h\in\mathbf{Z}_+^n} B_{h,n}^{n+1,\alpha-1+\delta} u_1^{h_1}\cdots u_n^{h_n}.$$

By the definition of F, (15) and (16), we have

$$\begin{aligned} \sum_{h\in\mathbf{Z}_+^n} B_{h,n}^{n+1,\alpha-1+\delta} u_1^{h_1}\cdots u_n^{h_n} &= F_n^{n+1,\alpha-1+\delta}(u) \\ &= (1-\sigma(u))^{-n-1}\prod_{j=1}^n (1+u_j-\sigma(u))^{-\alpha_j+1-\delta_j}. \end{aligned}$$

Since $\frac{u_j}{1+u_j-\sigma(u)} < 1$ and $\delta_j = 0$ or 1, we have

$$\begin{aligned}K_\alpha(u) &\le C_{10} \sum_{\delta\in Q} u^\delta F_n^{n+1,\alpha-1+\delta}(u)\\ &= C_{10} \sum_{\delta\in Q} (1-\sigma(u))^{-n-1} \prod_{j=1}^n \left(\frac{u_j}{1+u_j-\sigma(u)}\right)^{\delta_j}\\ &\quad \times \prod_{j=1}^n (1+u_j-\sigma(u))^{1-\alpha_j}\\ &= C_{10} \sum_{\delta\in Q} F_n^{n+1,\alpha-1}(u) \prod_{j=1}^n \left(\frac{u_j}{1+u_j-\sigma(u)}\right)^{\delta_j}\\ &\le 2^n C_{10} F_n^{n+1,\alpha-1}(u) = 2^n C_{10} F(z,\bar{z})\end{aligned}$$

where

$$C_{10} = \frac{C_8}{c_9} \prod_{j=1}^n \left\{\frac{1}{\alpha_j}\right\}.$$

We have proved the right-hand inequality of (2).

7. Proof of Theorem, Part 2

The proof of the left-hand inequality of (2) is little bit difficult.

For any $\delta = (\delta_1, \dots, \delta_n) \in Q$, we define

$$\phi_\delta(m) = (\phi_{\delta,1}(m_1), \dots, \phi_{\delta,n}(m_n))$$

as $\phi_{\delta,j}(m_j) = [m_j\alpha_j]$ when $\delta_j = 1$; and $\phi_{\delta,j}(m_j) = \{m_j\alpha_j\}$ when $\delta_j = 0$; $j = 1, 2, \dots, n$; where Q is defined in section 6.

For any $h \in \mathbf{Z}_+^n$, $m \in \mathbf{Z}_+^n$, and $\delta \in Q$, we define $N_\delta(m,h) = 1$, if $h = \phi_\delta(m)$; and $N_\delta(m,h) = 0$ if $h \ne \phi_\delta(m)$, then we have

(1) for a fixed $m \in \mathbf{Z}_+^n$,

$$\sum_{h\in\mathbf{Z}_+^n} N_\delta(m,h) = 1; \tag{32}$$

(2) for a fixed $h \in \mathbf{Z}_+^n$,

$$0 \le \sum_{m\in\mathbf{Z}_+^n} N_\delta(m,h) \equiv N_\delta(h) \le \prod_{j=1}^n \{\frac{1}{\alpha_j}\}$$

since each of the equations $[\alpha_j m_j] = h_j$ and $\{\alpha_j m_j\} = h_j$ has at most $\{1/\alpha_j\}$ integer solutions for m_j.

For any $m \in \mathbf{Z}_+^n$ and $\delta \in Q$, we may decompose αm as

$$\begin{aligned}\alpha m &= (\alpha_1 m_1, \dots, \alpha_n m_n)\\ &= (\phi_{\delta,1}(m_1) + \varepsilon_1(\delta_1, m_1), \dots, \phi_{\delta,n}(m_n) + \varepsilon_n(\delta_n, m_n)),\end{aligned}$$

where

$$\varepsilon_j(\delta_j, m_j) = \alpha_j m_j - \phi_{\delta,j}(m_j) = \alpha_j m_j - [\alpha_j m_j]$$

if $\delta_j = 1$; and

$$\varepsilon_j(\delta_j, m_j) = \alpha_j m_j - \phi_{\delta,j}(m_j) = \alpha_j m_j - \{\alpha_j m_j\}$$

if $\delta_j = 0$; $j = 1, 2, \ldots, n$.

Let $\phi_{\delta,j}(m_j) = h_j, \varepsilon_j = \varepsilon_j(\delta_j, m_j)$, then

$$\alpha_j m_j = \phi_{\delta,j}(m_j) + \varepsilon_j(\delta_j, m_j) = h_j + \varepsilon_j.$$

We may extend Lemma 11 as

Lemma 13. *If $\phi_\delta(h) = m$, then there exist two positive constants c_{11} and C_{11}, which are independent of m and δ, and only depend on α and n, such that*

$$c_{11} B_{h,n}^{n+1,\alpha-1} \prod_{j=1}^{n} \left(\frac{1 + \sigma(\alpha) + \sigma(h)}{h_j + 1} \right)^{\varepsilon_j} \leq C_m$$
$$\leq C_{11} B_{h,n}^{n+1,\alpha-1} \prod_{j=1}^{n} \left(\frac{1 + \sigma(\alpha) + \sigma(h)}{h_j + 1} \right)^{\varepsilon_j} .$$

Since the proof of Lemma 13 is similar to that of Lemma 11, we omit it.

By (32), we have

$$K_\alpha(u) = \sum_{m \in \mathbf{Z}_+^n} \sum_{h \in \mathbf{Z}_+^n} N_\delta(m, h) C_m u_1^{h_1} \cdots u_n^{h_n} u_1^{\varepsilon_1} \cdots u_n^{\varepsilon_n}.$$

Using Lemma 13, we find that

$$K_\alpha \geq c_{11} \sum_{m,h \in \mathbf{Z}_+^n} N_\delta(m, h) B_{h,n}^{n+1,\alpha-1} \prod_{j=1}^{n} \left(\frac{\sigma(h) + \sigma(\alpha) + 1}{h_j + 1} \right)^{\varepsilon_j} u^h$$

holds for any $\delta \in Q$. Since the total number of lattice points in Q is 2^n, we have

$$K_\alpha(u) \geq 2^{-n} c_{11} \sum_{h \in \mathbf{Z}_+^n} B_{h,n}^{n+1,\alpha-1} u^h \sum_{\delta \in Q}$$
$$\times \left(\sum_{m \in \mathbf{Z}_+^n} N_\delta(m, h) \prod_{j=1}^{n} \left(\frac{\sigma(h) + \sigma(\alpha) + 1}{h_j + 1} u_j \right)^{\varepsilon_j(\delta_j, m_j)} \right).$$

We consider the last bracket on the right-hand side of the previous inequality. For a fixed $h \in \mathbf{Z}_+^n$, and a fixed $\delta \in Q$, as an equation in m, $\phi_\delta(m) = h$ may have solutions, that means, $h \in \phi_\delta(\mathbf{Z}_+^n)$; or may not have a solution, that means, h does not belong to $\phi_\delta(\mathbf{Z}_+^n)$. We define $\bar{N}_\delta(h)$ as the characteristic function of $\phi_\delta(\mathbf{Z}_+^n)$ when δ is fixed, that is, $\bar{N}_\delta(h) = 1$ if $h \in \phi_\delta(\mathbf{Z}_+^n)$; and $\bar{N}_\delta(h) = 0$, otherwise, Then

$$K_\alpha(u) > 2^{-n} c_{11} \sum_{h \in \mathbf{Z}_+^N} B_{h,n}^{n+1,\alpha-1} \sum_{\delta \in Q} \bar{N}_\delta(h) \prod_{j=1}^{n} \left(\frac{\sigma(h) + \sigma(\alpha) + 1}{h_j + 1} u_j \right)^{\varepsilon_j(\delta_j, h_j)}$$

where $\varepsilon_j(\delta_j, h_j) = \varepsilon_j(\delta_j, m_j)$ if m_j is a solution of $h_j = \phi_{\delta,j}(m_j)$, $j = 1, 2, \ldots, n$.

Obviously, $\phi(\mathbf{Z}_+^n)$ has the following properties:

(1) If $h \in \phi_\delta(\mathbf{Z}_+^n)$, then $h(j) = (h_1, \ldots, h_{j-1}, 0, h_{j+1}, \ldots, h_n) \in \phi_\delta(\mathbf{Z}_+^n)$;

(2) If $h \in \phi_\delta(\mathbf{Z}_+^n)$, then there exists an integer k_j, $1 \leq k_j \leq \{\alpha_j\}$, such that $h + k_j e_j \in \phi_\delta(\mathbf{Z}_+^n)$.

Fix $h(1) \in \phi_\delta(\mathbf{Z}_+^n)$, then there exists an integer sequence

$$n_{1,0} = 0 < n_{1,1} < n_{1,2} < \cdots < n_{1,k} < \cdots, \qquad k = 0, 1, 2, \ldots,$$

which is independent of the choice of $h(1)$, such that $h(1) + n_{1,k} e_1 \in \phi_\delta(\mathbf{Z}_+^n)$, and $1 \leq n_{1,k+1} - n_{1,k} \leq \{\alpha_1\}$. Moreover, if $n_{1,k+1} \geq n_{1,k} + 2$ holds, then $h(1) + (n_{1,k} + s) e_1$ does not belong to $\phi_\delta(\mathbf{Z}_+^n)$, when $1 \leq s \leq n_{1,k+1} - n_{1,k} - 1$.

Fix $\delta \in Q$, and consider a subset $D_1(\delta) \subset \mathbf{Z}_+^n$ as

$$D_1(\delta) = \{h(1) + ke_1 \in \mathbf{Z}_+^n,\ \text{for all } h(1) \in \phi_\delta(\mathbf{Z}_+^n),\ k = 0, 1, \dots\},$$

then $D_1(\delta) \supset \phi_\delta(\mathbf{Z}_+^n)$, and

$$K_\alpha(u) > c_{11} 2^{-n} \sum_{\delta \in Q} \sum_{h \in D_1(\delta)} \bar{N}_\delta(h) B_{h,n}^{n+1,\alpha-1} u^h \prod_{j=1}^n \left(\frac{\sigma(h) + \sigma(\alpha) + 1}{h_j + 1} u_j \right)^{\varepsilon_j(\delta_j, h_j)}.$$

Let $\chi = h(1) + n_{1,k} e_1 \in \phi_\delta(\mathbf{Z}_+^n)$, $\tau = h(1) + n_{1,k+1} e_1 \in \phi_\delta(\mathbf{Z}_+^n)$ and $l = n_{1,k+1} - n_{1,k}$, $1 \le s \le n_{1,k+1} - n_{1,k} - 1 = l - 1$ if $l \ge 2$, then

(33)

$$\begin{aligned}
&\frac{s}{l} B_{\tau,n}^{n+1,\alpha-1} u^\tau \prod_{j=1}^n \left(\frac{\sigma(\tau) + \sigma(\alpha) + 1}{\tau_j + 1} u_j \right)^{\varepsilon_j(\delta_j, \tau_j)} \\
&\qquad + \frac{l-s}{l} B_{\chi,n}^{n+1,\alpha-1} u^\chi \prod_{j=1}^n \left(\frac{\sigma(\chi) + \sigma(\alpha) + 1}{\chi_j + 1} u_j \right)^{\varepsilon_j(\delta_j, \chi_j)} \\
&\quad \ge (B_{\tau,n}^{n+1,\alpha-1})^{s/l} (B_{\chi,n}^{n+1,\alpha-1})^{(l-s)/l} u^{(\tau + \varepsilon(\delta,\tau)) s/l + (\chi + \varepsilon(\delta,\chi))^{(l-s)/s}} \\
&\qquad \times \prod_{j=1}^n \left(\frac{\sigma(\tau) + \sigma(\alpha) + 1}{\tau_j + 1} \right)^{s/l \varepsilon_j(\delta_j, \tau_j)} \left(\frac{\sigma(\chi) + \sigma(\alpha) + 1}{\chi + 1} \right)^{(l-s)/s \varepsilon_j(\delta_j, \chi_j)},
\end{aligned}$$

since the arithmetic mean is greater or equal to the geometric mean.

According to the Main Lemma, there exist two constants c_{12} and C_{12}, which are independent of τ and χ, and only depend on n and α, such that

$$c_{12} A_{\tau,n}^{n+1,\alpha-1} \le B_{\tau,n}^{n+1,\alpha-1} \le C_{12} A_{\tau,n}^{n+1,\alpha-1},$$

and

$$c_{12} A_{\chi,n}^{n+1,\alpha-1} \le B_{\chi,n}^{n+1,\alpha-1} \le C_{12} A_{\tau,n}^{n+1,\alpha-1}.$$

Let $h_1 = n_{1,k} + s$, then

$$\begin{aligned}
A_{\tau,n}^{n+1,\alpha-1} &= \frac{\Gamma(\sigma(\alpha) + \sigma(h(1)) + n_{1,k+1} + 1)}{\Gamma(\sigma(\alpha) + 1) \prod_{j=2}^n \Gamma(1 + h_j)(1 + h_j)^{\alpha_j - 1} \Gamma(1 + n_{1,k+1})(1 + n_{1,k+1})^{\alpha_1 - 1}} \\
&= \frac{\Gamma(\sigma(\alpha) + \sigma(h) + l - s + 1)}{\Gamma(\sigma(\alpha) + 1) \prod_{j=2}^n \Gamma(1 + h_j)(1 + h_j)^{\alpha_j - 1} \Gamma(h_1 + l - s + 1)(h_1 + l - s + 1)^{\alpha_1 - 1}} \\
&= A_{h,n}^{n+1,\alpha-1} \frac{(\sigma(\alpha) + \sigma(h) + 1) \cdots (\sigma(\alpha) + \sigma(h) + l - s)(1 + h_1)^{\alpha_1 - 1}}{(h_1 + 1) \cdots (h_1 + l - s)(h_1 + l - s + 1)^{\alpha_1 - 1}},
\end{aligned}$$

and

$$\begin{aligned}
A_{\chi,n}^{n+1,\alpha-1} &= \frac{\Gamma(\sigma(\alpha) + \sigma(h(1)) + n_{1,k} + 1)}{\Gamma(\sigma(\alpha) + 1) \prod_{j=2}^n \Gamma(1 + h_j)(1 + h_j)^{\alpha_j - 1} \Gamma(1 + n_{1,k})(1 + n_{1,k})^{\alpha_1 - 1}} \\
&= \frac{\Gamma(\sigma(\alpha) + \sigma(h) + 1)}{\Gamma(\sigma(\alpha) + 1) \prod_{j=2}^n \Gamma(1 + h_j)(1 + h_j)^{\alpha_j - 1} \Gamma(1 + h_1)(1 + n_{1,k})^{\alpha_1 - 1}} \\
&\quad \times \frac{(1 + n_{1,k}) \cdots (s + n_{1,k})}{(\sigma(\alpha) + \sigma(h(1)) + n_{1,k} + 1) \cdots (\sigma(\alpha) + \sigma(h(1)) + n_{1,k} + s)} \\
&= A_{h,n}^{n+1,\alpha-1} \frac{(1 + n_{1,k}) \cdots (s + n_{1,k})(1 + h_1)^{\alpha_1 - 1}}{(\sigma(\alpha) + \sigma(h(1)) + n_{1,k} + 1) \cdots (\sigma(\alpha) + \sigma(h(1)) + n_{1,k} + s)(1 + n_{1,k})^{\alpha_1 - 1}}.
\end{aligned}$$

Thus

$$\begin{aligned}
&(A_{\tau,n}^{n+1,\alpha-1})^{s/l}(A_{\chi,n}^{n+1,\alpha-1})^{(l-s)/l} = A_{h,n}^{n+1,\alpha-1}(1+h_1)^{\alpha_1-1} \\
&\quad\times\left(\frac{\sigma(\alpha)+\sigma(h)+1)\cdots(\sigma(\alpha)+\sigma(h)+l-s)}{(h_1+1)\cdots(h_1+l-s)(h_1+l-s+1)^{\alpha_1-1}}\right)^{s/l} \\
&\quad\times\left(\frac{(1+n_{1,k})\cdots(s+n_{1,k})}{(\sigma(\alpha)+\sigma(h(1))+n_{1,k}+1)\cdots(\sigma(\alpha)+\sigma(h(1))+n_{1,k}+s)(1+n_{1,k})^{\alpha_1-1}}\right)^{(l-s)/l} \\
&= A_{h,n}^{n+1,\alpha-1}I_1I_2I_3,
\end{aligned}$$

where

$$I_1 = \frac{(1+h_1)^{\alpha_1-1}}{((h_1+l-s+1)^{s/l}(h_1-s+1)^{(l-s)/l})^{\alpha_1-1}},$$

$$I_2 = \frac{((\sigma(\alpha)+\sigma(h)+1)\cdots(\sigma(\alpha)+\sigma(h)+l-s))^{s/l}}{((\sigma(\alpha)+\sigma(h)-s+1)\cdots(\sigma(\alpha)+\sigma(h)))^{(l-s)/l}}$$

and

$$I_3 = \frac{(h_1(h_1-1)\cdots(h_1-s+1))^{(l-s)/l}}{((h_1+1)\cdots(h_1+l-s))^{s/l}}.$$

Obviously, I_1, I_2 and I_3 are bounded above and below since $l \le \{\alpha_1\}$ and $s \le l-1 \le \{\alpha_1\}-1$. Thus there exist two positive constants c_{14} and C_{14}, which are independent of τ, χ and h, and only depend on α and n, such that

$$c_{14}B_{h,n}^{n+1,\alpha-1} \le (B_{\tau,n}^{n+1,\alpha-1})^{s/l}(B_{\chi,n}^{n+1,\alpha-1})^{(l-s)/l} \le C_{14}B_{h,n}^{n+1,\alpha-1}$$

where $1 \le s \le n_{1,k+1} - n_{1,k} - 1$.

Moreover, we have

$$\begin{aligned}
\tau\frac{s}{l} + \chi\frac{l-s}{l} &= (h(1)+n_{1,k+1}e_1)\frac{s}{l} + (h(1)+n_{1,k}e_1)\frac{l-s}{l} \\
&= h(1) + (n_{1,k}+s)e_1 = h,
\end{aligned}$$

and

$$\frac{\sigma(\tau)+\sigma(\alpha)+1}{\tau_j+1} = \frac{\sigma(h)+\sigma(\alpha)+1+l-s}{h_j+1}$$

if $j \ne 1$, then there exist two positive constants c'_{15} and C'_{15}, which are independent of h, and only depend on α and n, such that

$$c'_{15}\frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1} \le \frac{\sigma(\tau)+\sigma(\alpha)+1}{\tau_j+1} \le C'_{15}\frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1}$$

when $j \ne 1$, since $1 \le l-s \le \{\alpha_1\}-1$. When $j=1$, we have

$$\frac{\sigma(\tau)+\sigma(\alpha)+1}{\tau_1+1} = \frac{n_{1,k+1}+\sigma(h(1))+\sigma(\alpha)+1}{n_{1,k+1}+1} = \frac{\sigma(h)+\sigma(\alpha)+1+l-s}{h_1+1+l-s},$$

then there exist two positive constants c''_{15} and C''_{15}, which are independent of h and τ, and only depend on α and n, such that

$$c''_{15}\frac{\sigma(h)+\sigma(\alpha)+1}{h_1+1} \le \frac{\sigma(\tau)+\sigma(\alpha)+1}{\tau_1+1} \le C''_{15}\frac{\sigma(h)+\sigma(\alpha)+1}{h_1+1}.$$

Thus

$$c_{15}\frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1} \le \frac{\sigma(\tau)+\sigma(\alpha)+1}{\tau_j+1} \le C_{15}\frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1}$$

hold for $j = 1, 2, \ldots, n$, where $c_{15} = \min(c'_{15}, c''_{15})$, and $C_{15} = \max(C'_{15}, C''_{15})$. Similarly, the inequalities

$$c_{16}\frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1} \le \frac{\sigma(\chi)+\sigma(\alpha)+1}{\chi_j+1} \le C_{16}\frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1}$$

holds for $j = 1, 2, \ldots, n$, where c_{16} and C_{16} are two positive constants, which are independent of h, and only depend on α and n.

Combining all these results, we get the right-hand side of (33) is not less than

$$c_{17}B_{h,n}^{n+1,\alpha-1}u^h \prod_{j=2}^{n}\left(\frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1}u_j\right)^{\varepsilon_j(\delta_j,h_j)} \times \left(\frac{\sigma(h)+\sigma(\alpha)+1}{h_1+1}u_1\right)^{\varepsilon'_1(\delta_1,h_1)}$$

where $c_{17} = c_{14}c_{15}^{s/l}c_{16}^{(l-s)/l}$, and $\varepsilon'_1(\delta_1, h_1) = \frac{s}{l}\varepsilon_1(\delta_1,\tau_1) + \frac{l-s}{l}\varepsilon_1(\delta_1,\chi_1)$.

Let

$$T(y) = B_{y,n}^{n+1,\alpha-1}u^y \prod_{j=1}^{n}\left(\frac{\sigma(y)+\sigma(\alpha)+1}{y_j+1}u_j\right)^{\varepsilon_j(\delta_j,y_i)}$$

where $y = \tau, \chi$ or h. Then

$$\begin{aligned}
T(\tau)+T(\chi) &= \frac{2}{l-1}\sum_{s=1}^{l-1}\frac{1}{l}(sT(\tau)+(l-s)T(\chi)) \\
&= \frac{1}{2}(T(\tau)+T(\chi)) + \frac{1}{4}(T(\tau)+T(\chi)) + \frac{1}{2(l-1)}\sum_{s=1}^{l-1}\frac{1}{l}(sT(\tau)+(l-s)T(\chi)) \\
&\ge \frac{1}{2}(T(\tau)+T(\chi)) + \frac{1}{4}(T(\tau)+T(\chi)) \\
&\quad + \frac{c_{17}}{2(l-1)}\sum_{s=1}^{l-1}T(h)\left(\frac{\sigma(h)+\sigma(\alpha)+1}{h_1+1}u_1\right)^{\varepsilon'_1(\delta_1,h_1)-\varepsilon_1(\delta_1,h_1)} \\
&\ge \frac{1}{2}(T(\tau)+T(\chi)) + c_{18}\sum_{s=0}^{l}T(h)\left(\frac{\sigma(h)+\sigma(\alpha)+1}{h_1+1}u_1\right)^{\varepsilon'_1(\delta_1,h_1)-\varepsilon_1(\delta_1,h_1)}
\end{aligned}$$

where $c_{18} = \min(\frac{1}{4}, \frac{c_{17}}{2(\{\alpha_1\}-1)})$. Thus, we have

$$\begin{aligned}
K_\alpha(u) > 2^{-n}c_{11}c_{18}&\sum_{\delta\in Q}\sum_{h\in\mathbf{Z}_+^n}\bar{N}_\delta(1,h)B_{h,n}^{n+1,\alpha-1}u^h \\
&\times\prod_{j=2}^{n}\left(\frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1}u_j\right)^{\varepsilon_j,(\delta_j,h_j)}\left(\frac{\sigma(h)+\sigma(\alpha)+1}{h_1+1}u_1\right)^{\varepsilon'_1(\delta_1,h_1)}
\end{aligned} \tag{34}$$

where $\bar{N}_\delta(1,h)$ is the characteristic function of $D_1(\delta)$, that is, $\bar{N}_\delta(1,h) = 1$, if $h \in D_1$; $\bar{N}_\delta(1,h) = 0$, if h does not belong to D_1.

If $\delta_1 = 1$, then there exist two positive integers m_1 and m'_1, such that $[\alpha_1 m'_1] = n_{1,k+1}$, and $[\alpha_1 m_1] = n_{1,k}$. Hence

$$\varepsilon_1(1,\tau_1) = \alpha_1 m'_1 - [\alpha_1 m'_1] = \alpha_1 m'_1 - n_{1,k+1} \ge 0,$$

and

$$\varepsilon_1(1,\chi_1) = \alpha_1 m_1 - [\alpha_1 m_1] = \alpha_1 m_1 - n_{1,k} \geq 0.$$

Since by the definition of ε_1',

$$\varepsilon_1'(\delta_1, h_1) = \frac{s}{l}\varepsilon_1(\delta_1,\tau_1) + \frac{l-s}{l}\varepsilon_1(\delta_1,\chi_1),$$

we have $\varepsilon_1'(1,h_1) \geq 0$. Similarly, we can prove that $\varepsilon_1'(0,h_1) \leq 0$.

Fix $h(2) \in D_1(\delta)$, then there exists a sequence of integers

$$n_{2,0} = 0 < n_{2,1} < \cdots < n_{2,k} < \cdots,$$

which is independent of the choice of $h(2) \in D_1(\delta)$, such that $h(2)+n_{2,k}e_2 \in D_1(\delta)$, and $1 \leq n_{2,k+1} - n_{2,k} \leq \{\alpha_2\}$ and $h(2) + (n_{2,k}+s)$ does not belong to $D_1(\delta)$ when $1 \leq s \leq n_{2,k+1} - n_{2,k} - 1$ and $n_{2,k+1} \geq n_{2,k} + 2$.

Define $D_2 = D_2(\delta)$ by

$$D_2(\delta) = \{h(2) + ke_2 \in \mathbf{Z}_+^n, \text{ for all } h(2) \in D_1(\delta),\ k = 0,1,2,\dots\}.$$

Then

$$D_2(\delta) \supset D_1(\delta) \supset \phi_\delta(\mathbf{Z}_+^n),$$

and $D_2(\delta)$ has the following properties:

(1) $h(j) \in D_2(\delta),\ j = 1,2,\dots,n$ if $h \in D_2(\delta)$, in particular, $h(1,2) \in D_2(\delta)$;

(2) $h(1,2) + ke_1 + se_2 \in D_2(\delta)$, $k,s = 0,1,2,\dots$, if $h(1,2) \in D_2(\delta)$;

(3) If $h \in D_2(\delta)$, then there exists k_j, $1 \leq k_j \leq \{\alpha_j\}$, such that $h + k_je_j \in D_2(\delta)$, $j = 3,4,\dots,n$.

Using the same method as we used to prove (34), we can prove

$$K_\alpha(u) > c_{19} \sum_{\delta\in Q}\sum_{h\in\mathbf{Z}_+^n} \bar{N}_\delta(2,h) B_{h,n}^{n+1,\alpha-1} u^h \prod_{j=3}^n \left(\frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1}u_j\right)^{\varepsilon_j(\delta_j,h_j)}$$
$$\times \left(\frac{\sigma(h)+\sigma(\alpha)+1}{h_1+1}u_1\right)^{\varepsilon_1'(\delta_1,h_1)} \left(\frac{\sigma(h)+\sigma(\alpha)+1}{h_2+1}u_2\right)^{\varepsilon_2'(\delta_2,h_2)}$$

where c_{19} is a positive constant, which only depends on α and n, and $\bar{N}_\delta(2,h)$ is the characteristic function of $D_2(\delta)$, that is, $\bar{N}_\delta(2,h) = 1$ if $h \in D_2$; $\bar{N}_\delta(2,h) = 0$ if h does not belong to D_2; and $\varepsilon_2'(\delta_2,h_2)$ has the following property:

$$\varepsilon_2'(1,h_2) \geq 0, \quad \varepsilon_2'(0,h_2) \leq 0.$$

We can repeat this process again and again, finally, we have

$$K_\alpha(u) > c_{20} \sum_{\delta\in Q}\sum_{h\in\mathbf{Z}_+^n} \bar{N}_\delta(n,h) B_{h,n}^{n+1,\alpha-1} u^h$$
$$\times \prod_{j=1}^n \left(\frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1}u_j\right)^{\varepsilon_j'(\delta_j,h_j)}$$

where c_{20} is a positive constant, which only depends on α and n, and $\bar{N}_\delta(n,h)$ is the characteristic function of $D_n(\delta)$, by the definition of $D_j(\delta)$, $j = 1,2,\dots,n$, we have $D_n(\delta) = \mathbf{Z}_+^n$, thus $\bar{\mathbf{N}}_\delta(n,h) \equiv 1$, when $h \in \mathbf{Z}_+^n$; and $\varepsilon_j'(\delta_j,h_j)$ has the properties:

$$\varepsilon_j'(1,h_j)\geq 0,\quad \varepsilon_j'(0,h_j)\leq 0,\qquad j=1,2,\dots,n.$$

Thus

$$K_\alpha(u)>c_{20}\sum_{h\in\mathbf{Z}_+^n}B_{h,n}^{n+1,\alpha-1}u^h\sum_{s\in Q}\prod_{j=1}^n\left(\frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1}u_j\right)^{\varepsilon_j'(\delta_j,h_j)}.$$

For a fixed $h\in\mathbf{Z}_+^n$, we may take $\delta=(\delta_1,\dots,\delta_n)\in Q$, such that

$$\delta_j=0\quad\text{if } \frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1}u_j\leq 1;$$
$$\delta_j=1\quad\text{if } \frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1}u_j\geq 1;$$

then

$$\prod_{j=1}^n\left(\frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1}u_j\right)^{\varepsilon_j'(\delta_j,h_j)}\geq 1,$$

and hence

$$\sum_{\delta\in Q}\prod_{j=1}^n\left(\frac{\sigma(h)+\sigma(\alpha)+1}{h_j+1}u_j\right)^{\varepsilon_j'(\delta_j,h_j)}\geq 1.$$

Finally we obtain

$$K_\alpha(u)>c_{20}\sum_{h\in\mathbf{Z}_+^n}B_{h,n}^{n+1,\alpha-1}u^h=c_{20}F_n^{n+1,\alpha-1}(u)=c_{20}F(z,\bar z).$$

We have completed the proof of the Theorem. □

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