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LINEAR ADDITIVE FUNCTIONALS OF SUPERDIFFUSIONS AND RELATED NONLINEAR P.D.E.

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ABSTRACT. Let L be a second order elliptic differential operator in a bounded smooth domain D in $\mathbb{R}^d$ and let $1 < \alpha \le 2$. We get necessary and sufficient conditions on measures η, ν under which there exists a positive solution of the boundary value problem

$$(*) \qquad \begin{aligned} -Lv + v^\alpha &= \eta \quad \text{in } D,\\ v &= \nu \quad \text{on } \partial D. \end{aligned}$$

The conditions are stated both analytically (in terms of capacities related to the Green's and Poisson kernels) and probabilistically (in terms of branching measure-valued processes called (L,α)-superdiffusions).

We also investigate a closely related subject — linear additive functionals of superdiffusions. For a superdiffusion in an arbitrary domain E in $\mathbb{R}^d$, we establish a 1-1 correspondence between a class of such functionals and a class of L-excessive functions h (which we describe in terms of their Martin integral representation). The Laplace transform of A satisfies an integral equation which can be considered as a substitute for (*).

1. Introduction

1.1. Boundary value problem with measures. We start from a differential operator

$$(1.1) \qquad Lu = \sum_{i,j} a_{ij} \nabla_i \nabla_j u + \sum_i b_i \nabla_i u$$

(∇_i stands for the partial derivative with respect to x_i) in a bounded smooth domain D of $\mathbb{R}^d$ with coefficients subject to conditions:

1.1.A. (Uniform ellipticity) There exists a constant $\varkappa > 0$ such that

$$\sum_{i,j} a_{ij}\lambda_i\lambda_j \ge \varkappa \sum_i \lambda_i^2 \qquad \text{for all } x \in D, \lambda_1, \dots, \lambda_d \in \mathbb{R},$$

1.1.B. $a_{ij} \in C^{2,\lambda}(\bar{D})$, $b_i \in C^{1,\lambda}(\bar{D})$.[1]

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[1] We follow standard notation in P.D.E. (see, e.g., [21]). [Smooth domain means a domain of class $C^{2,\lambda}$.]

The classical boundary value problem

$$\begin{aligned} -Lv + v^\alpha &= \rho \quad \text{in } D, \\ v &= \sigma \quad \text{on } \partial D \end{aligned} \tag{1.2}$$

(with Hölder continuous ρ and continuous σ) is equivalent to an integral equation

$$v(x) + \int_D g(x,y) v(y)^\alpha dy = h(x) \tag{1.3}$$

where

$$h(x) = \int_D g(x,y)\rho(y)dy + \int_{\partial D} k(x,y)\sigma(y)a(dy), \tag{1.4}$$

$g(x,y)$ is Green's function, $k(x,y)$ is the Poisson kernel of L in D and $a(dy)$ is the surface area on ∂D. We interpret v as a (generalized) solution of the problem

$$\begin{aligned} -Lv + v^\alpha &= \eta \quad \text{on } D, \\ v &= \nu \quad \text{on } \partial D \end{aligned} \tag{1.5}$$

involving two measures η and ν if the equation (1.3) holds with

$$h(x) = \int_D g(x,y)\eta(dy) + \int_{\partial D} k(x,y)\nu(dy). \tag{1.6}$$

In Theorem 1.1, we establish sufficient conditions on η and ν under which problem (1.5) has a solution. Necessary conditions are established in Theorem 1.2. The equivalence of both sets of conditions follows from results in [17]. [Theorems 1.1 and 1.2 are still valid if D is not smooth. However, in general, the equivalence of conditions imposed on ν in the two theorems is not proved.]

Particular cases of problem (1.5) have been studied before. The case $\nu = 0$ was treated in [2] and the case $\eta = 0$ was considered in [18]. Even earlier, Gmira and Véron [22] have investigated a class of functions ψ such that the problem

$$\begin{aligned} \Delta v &= \psi(v) \quad \text{on } D, \\ v &= \nu \quad \text{on } \partial D \end{aligned}$$

has a solution for every finite measure ν. This class contains $\psi(v) = v^\alpha$ with $(\alpha+1)/(\alpha-1) > d$.

1.2. ***L*-diffusions.** Suppose D is a bounded smooth domain and that L satisfies conditions 1.1.A,B. Then there exists[2] a strictly positive function $p_t(x,y), t > 0, x, y \in D$ such that:

1.2.A. If f is a continuous function on D with compact support and if

$$u_t(x) = \int_D p_t(x,y) f(y) dy, \tag{1.7}$$

[2]This is proved (under weaker conditions on L) in Chapter 1 of [19].

then

$$\frac{\partial u_t(x)}{\partial t} = Lu_t(x), \tag{1.8}$$

$$u_t(x) \to f(x) \quad \text{as } t \to 0 \tag{1.9}$$

and

$$u_t(x) \to 0 \quad \text{as } x \to z \in \partial D. \tag{1.10}$$

(All partial derivatives of p which appear in (1.8) are continuous in (t, x, y).)

Function $p_t(x, y)$ has the following properties:

1.2.B. For all $s, t > 0, x, z \in D$,

$$\int_D p_s(x,y)dy\, p_t(y,z) = p_{s+t}(x,z).$$

1.2.C. For all $t > 0, x \in D$,

$$\int_D p_t(x,y)dy \le 1.$$

Therefore $p_t(x, dy) = p_t(x, y)dy$ is a Markov transition function. It is well-known (see, e.g., [6]) that there exists a continuous Markov process $\xi = (\xi_t, \Pi_x)$ in D with this transition function. We call it an *L-diffusion*. If ζ is the life time of ξ, then $\xi_{\zeta-}$ belongs to ∂D. By setting $\xi_t = \xi_{\zeta-}$ for $t \ge \zeta$, we define an *L-diffusion stopped at the exit from D*. Note that ζ can be interpreted as the first exit time of this process from D; often we use the notation τ for it.

Now suppose that E is an arbitrary domain in $\mathbb{R}^d$ and that L is a differential operator in E which satisfies conditions 1.1.A, B in each bounded domain D with $\bar{D} \subset E$. Consider a sequence of bounded smooth domains D_n such that $\bar{D}_n \subset D_{n+1}$ and $\bigcup D_n = E$. The corresponding functions $p_t^n(x, y)$ increase monotonically and they tend to a limit $p_t(x, y)$ which does not depend on the choice of D_n (this follows from [19, Ch. 1]). There exists a continuous Markov process ξ in E with the transition function $p_t(x, dy) = p_t(x, y)dy$ (see, e.g., [6]). We call it an *L-diffusion in E*.

1.3. G-equation. Markov semigroup, Green's function g and Green's operator G for an L-diffusion ξ are defined by the formulae

$$T_t f(x) = \int_E p_t(x,dy)f(y), \tag{1.11}$$

$$g(x,y) = \int_0^\infty p_t(x,y)dt \tag{1.12}$$

and

$$Gf(x) = \int_0^\infty T_t f(x)dt = \int_E g(x,y)f(y)dy. \tag{1.13}$$

A positive Borel function h is called *excessive* if, for all $x \in E$, $T_t h(x) \leq h(x)$ and $T_t h(x) \to h(x)$ as $t \to 0$. The case $h(x) = \infty$ for all x is excluded. Since $p_t(x, y) > 0$, the set $\{x : h(x) = \infty\}$ has the Lebesgue measure 0. There exist only two possibilities: either $g(x, y) = \infty$ for all $x, y \in E$ or $g(x, y) < \infty$ for $x \neq y$. In the first case, constants are the only excessive functions and all problems treated in this paper are trivial. Therefore we concentrate on the second case.

Let $1 < \alpha \leq 2$. One of our goals is to find for which excessive functions h the equation

$$v + G(v^\alpha) = h \tag{1.14}$$

(we call it *G-equation*) has a solution.[3] Note that if (1.14) holds almost everywhere, then

$$\tilde{v} = \begin{cases} h - G(v^\alpha) & \text{on } \{h < \infty\}, \\ \infty & \text{on } \{h = \infty\} \end{cases} \tag{1.15}$$

satisfies (1.14) everywhere.

Fix an arbitrary point $c \in E$ and put

$$\begin{aligned} k(x, y) &= \frac{g(x, y)}{g(c, y)} \quad \text{if } y \neq c, \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{1.16}$$

There exist [see, e.g., [7]] a continuous injective mapping from E to a compact metrizable space $\hat{E}$ and an extension of $k(x, y)$ to $E \times \hat{E}$ such that:

1.3.A. For every $x \in E$, $k(x, y) \to k(x, z)$ as $y \to z \in \hat{E} \setminus E$.

1.3.B. If $k(\cdot, y_1) = k(\cdot, y_2)$, then $y_1 = y_2$.

We call $\hat{E}$ the *Martin space*. The set $\partial E = \hat{E} \setminus E$ is called the *Martin boundary*. For every $y \in E$, $h(x) = g(x, y)$ is an extremal excessive function.[4] We denote by E^* the set of all $y \in \partial E$ such that $h(x) = k(x, y)$ is an extremal excessive function. (E^* is a Borel subset of ∂E.) Every excessive function h has a unique representation

$$h = G\eta + K\nu \tag{1.17}$$

where η is a σ-finite measure on E, ν is a finite measure on E^* and

$$G\eta(x) = \int_E g(x, y)\eta(dy), \quad K\nu(x) = \int_{E^*} k(x, y)\nu(dy) \tag{1.18}$$

(cf. (1.6)). Note that $\eta(\Gamma) < \infty$ for every compact $\Gamma \subset E$. Indeed, if $h(x_0) < \infty$, then $a\eta(\Gamma) \leq G\eta(x_0) < \infty$ where $a = \inf_{y \in \Gamma} g(x_0, y) > 0$.

Function $f = K\nu$ is L-harmonic, that is it satisfies equation $Lf = 0$. L-harmonic functions can be also characterized by the following mean value property: for every bounded open set D such that $\bar{D} \subset E$,

$$\Pi_x f(\xi_\tau) = f(x) \qquad \text{for all } x \in E \tag{1.19}$$

where τ is the first exit time from D.

[3](Cf. (1.3).) When speaking about solutions of G-equation, we always mean positive solutions.

[4]This means if $h = h_1 + h_2$ and if h_1, h_2 are excessive, then h_1, h_2 are proportional to h.

We fix $\alpha \in (1,2]$. *Green's capacity* CG is defined on compact subsets of E by the formula

$$CG(\Gamma) = \sup\{\eta(\Gamma) : \int_E g(c,x)dx \left[\int_\Gamma g(x,y)\eta(dy)\right]^\alpha \le 1\}. \tag{1.20}$$

Analogously, the *Martin capacity* CK is defined on compact subsets of ∂E by the formula

$$CK(\Gamma) = \sup\{\nu(\Gamma) : \int_E g(c,x)dx \left[\int_\Gamma k(x,y)\nu(dy)\right]^\alpha \le 1\}. \tag{1.21}$$

[By a Choquet theorem [3], CG and CK can be extended to all analytic subsets of E and E^*.] If η is a measure on E, then writing $\eta \prec CG$ means that $\eta(\Gamma) = 0$ if $CG(\Gamma) = 0$. Writing $\nu \prec CK$ has an analogous meaning.

It follows from the results in Sections 2 and 3 that:

Theorem 1.1. *If $h = G\eta + K\nu$ and if*

$$\eta \prec CG, \quad \nu \prec CK, \tag{1.22}$$

then G-equation (1.14) *has a solution v which is defined uniquely on the set $E(h) = \{h < \infty\}$.*

1.4. Operators $\mathcal{G}$ and $\mathcal{K}$. Let ξ be an L-diffusion in a bounded smooth domain D stopped at the first exit time τ from D. We introduce operators $\mathcal{G}$ and $\mathcal{K}$ acting on functions with the domain $S = \mathbb{R}_+ \times E$ by the formulae

$$\mathcal{G}f(t,x) = \int_0^t ds \int_D p_s(x,dy) f(t-s,y) = \Pi_x \int_0^{\tau\wedge t} f(t-s,\xi_s)ds, \tag{1.23}$$

and[5]

$$\mathcal{K}f(t,x) = \Pi_x f(t-\tau,\xi_\tau). \tag{1.24}$$

If $f(t,x) = f(x)$ does not depend on t, then

$$\begin{aligned} \mathcal{G}f(t,x) &= \Pi_x \int_0^t f(\xi_s)ds \to Gf(x), \\ \mathcal{K}f(t,x) &= \Pi_x f(\xi_\tau) 1_{\tau \le t} \to Kf(x) \end{aligned} \tag{1.25}$$

as $t \to \infty$. Here G is defined by (1.13) and[6]

$$Kf(x) = \Pi_x f(\xi_\tau). \tag{1.26}$$

[5]We extend each function to $\mathbb{R} \times E$ by setting it equal to zero for negative t.

[6]Operator (1.26) is a particular case of the operator K defined by (1.18): if $E = D$ is a bounded smooth domain, then $E^* = \partial D$ and $\Pi_x f(\xi_\tau) = \int_{\partial D} k(x,y)\nu(dy)$ for $\nu(dy) = f(y)a(dy)$ where k is the Poisson kernel and a is the surface area on ∂D. Writing the same letter for both operators should cause no confusion since one operator is applied only in the context of a smooth domain D and the second one only in the context of the Martin boundary of E.

The boundary of a cylinder $Q = \mathbb{R}_+ \times D$ consists of the side surface $A = (0,\infty) \times \partial D$ and the bottom $B = \{0\} \times \bar{D}$. Besides the boundary value problem (1.2), we consider also a boundary value problem for a parabolic equation

$$\begin{aligned} \frac{\partial u}{\partial t} - Lu + u^\alpha &= \rho \quad \text{in } Q, \\ u &= \sigma \quad \text{on } A, \\ u &= 0 \quad \text{on } B. \end{aligned} \tag{1.27}$$

If ρ and σ are Hölder continuous, then (1.27) is equivalent to the integral equation

$$u + \mathcal{G}(u^\alpha) = \mathcal{G}\rho + \mathcal{K}\sigma. \tag{1.28}$$

1.5. Superdiffusions. Let $\xi = (\xi_t, \Pi_x)$ be a Markov process in a measurable space $(E, \mathcal{B})$ and let $\mathcal{M} = \mathcal{M}(E)$ be the space of all finite measures on $\mathcal{B}$. A (ξ,α)-superprocess is a Markov process $X = (X_t, P_\mu)$ in $\mathcal{M}$ which satisfies the condition: for every $\mu \in \mathcal{M}$ and every positive $\mathcal{B}$-measurable function f,

$$\begin{aligned} &P_\mu \exp\langle -f, X_t\rangle = \exp\langle -u_t, \mu\rangle, \\ &u_t(x) + \Pi_x \int_0^t u_{t-s}(\xi_s)^\alpha ds = \Pi_x f(\xi_t). \end{aligned} \tag{1.29}$$

We say that X is an (L,α)-*superdiffusion* if X is a right process and ξ is an L-diffusion. The existence of such processes for $1 < \alpha \le 2$ is proved, for instance, in [13] (we refer to [8] and [9] for the history of this subject starting from the pioneering work of Watanabe and Dawson).

In the theory of diffusion, a fundamental role is played by random points ξ_τ corresponding to the first exit times from open sets D. An analogous role in the theory of superdiffusion is played by exit measures X_D. In contrast to ξ_τ which can be defined through ξ_t, it is impossible, in general, to define X_D in terms of X_t. The probability distribution of X_D is defined by formulae similar to (1.29):

$$\begin{aligned} &P_\mu \exp\langle -f, X_D\rangle = \exp\langle -u, \mu\rangle, \\ &u(x) + \Pi_x \int_0^\tau u(\xi_s)^\alpha ds = \Pi_x f(\xi_\tau). \end{aligned} \tag{1.30}$$

The joint probability distribution of $X_{t_1}, \dots, X_{t_n}$ is determined by (1.29) and the Markov property of X. Analogously, the joint probability distribution of $X_{D_1}, \dots, X_{D_n}$ can be evaluated by using (1.30) and the following Markov property: for every positive $\mathcal{F}_{\supset D}$-measurable Y,

$$P_\mu\{Y \big| \mathcal{F}_{\subset D}\} = P_{X_D} Y \tag{1.31}$$

where $\mathcal{F}_{\subset D}$ is the σ-algebra generated by $X_{D'}$ with $D' \subset D$ and $\mathcal{F}_{\supset D}$ the σ-algebra generated by $X_{D''}$ with $D'' \supset D$.

We need even a wider class of exit measures [for instance, measures corresponding to the exit from D before time t]. We introduce a random measure (X_Q, P_μ) for an arbitrary open set Q in $S = \mathbb{R}_+ \times E$ and an arbitrary finite measure μ on the Borel σ-algebra in S. Its probability distribution is defined by the formulae

$$P_\mu \exp\langle -f, X_Q\rangle = \exp\langle -u, \mu\rangle,$$
$$u(r,x) + \Pi_{r,x}\int_r^{\tau^r} u(s,\xi_s)^\alpha ds = \Pi_{r,x} f(\tau^r, \xi_{\tau^r}) \tag{1.32}$$

where

$$\tau^r = \inf\{t : t \ge r, (t,\xi_t) \notin Q\} \tag{1.33}$$

is the first, after r, exit time of ξ from Q and $\Pi_{r,x} Y = \Pi_x \theta_{-r} Y$ describes a Markov process with transition function $p_t(x, dy)$ which starts at time r from point x. The joint probability distribution of $X_{Q_1}, \dots, X_{Q_n}$ is determined by (1.33) and by the property: for every positive $\mathcal{F}_{\supset Q}$-measurable Y,

$$P_\mu\{Y \big| \mathcal{F}_{\subset Q}\} = P_{X_Q} Y \tag{1.34}$$

where $\mathcal{F}_{\subset Q}$ is the σ-algebra generated by $X_{Q'}$ with $Q' \subset Q$ and $\mathcal{F}_{\supset Q}$ the σ-algebra generated by $X_{Q''}$ with $Q'' \supset Q$.

The existence of a family (X_Q, P_μ) subject to conditions (1.32) and (1.34) is proved in [8].

Formula $j_r(x) = (r,x)$ defines a mapping from E to S. If μ is a measure on E, then $j_r(\mu)$ is a measure on S concentrated on $\{r\} \times E$. We set $P_{j_r(\mu)} = P_{r,\mu}$. It follows from (1.32) that

$$P_{r,\mu} \exp\langle -f, X_Q\rangle = \exp\left\{-\int_E u(r,x)\mu(dx)\right\},$$
$$u(r,x) + \Pi_x \int_0^\tau u(s+r, \xi_s)^\alpha ds = \Pi_x f(\tau + r, \xi_\tau). \tag{1.35}$$

Formulae (1.29) and (1.30) can be considered as special cases of (1.35) if we identify X_t and X_D with the exit measures from $S_{<t} = [0,t) \times E$ and from $\mathbb{R}_+ \times D$, projected on E.

If τ is the first exit time from D, then $\tau(t) = \tau \wedge t$ is the first exit time from $Q_t = [0,t) \times D$. We call the process $\tilde{X}_t = X_{Q_t}$ an (L,α)-*superdiffusion stopped at the exit from* D. If $f(t,x) = f(x)$ vanishes outside D and if $v_t(x) = -\log P_x \exp\langle -f, \tilde{X}_t\rangle$, then $v_{t-r}(x) = -\log P_{r,x} \exp\langle -f, X_{Q_t}\rangle$ and (1.35) implies

$$P_\mu \exp\langle -f, \tilde{X}_t\rangle = \exp\langle -v_t, \mu\rangle,$$
$$v_t(x) + \Pi_x \int_0^{\tau(t)} v_{t-s}(\xi_s)^\alpha ds = \Pi_x f(\xi_{\tau(t)}). \tag{1.36}$$

Formula (1.36) can be obtained from (1.29) by replacing ξ with an L-diffusion stopped at the exit from D.

The shift operators θ_t of a time-homogeneous process ξ induce analogous operators for X (see [14, Section 1.12]). We have $X_s(\theta_t \omega) = X_{s+t}(\omega)$ and, if $Q = \mathbb{R}_+ \times D$, then $X_Q(\theta_t\omega, \Gamma) = X_{Q_t}(\omega, \Gamma + t)$ where $Q_t = S_{<t} \cup \{\gamma_t(Q)\}$ with $\gamma_t(r,x) = (r+t, x)$.

It follows from (1.35) that

$$P_\mu \int_Q f(s,x) X_Q(ds,dx) = \int \mu(dx) \Pi_x f(\tau, \xi_\tau) \tag{1.37}$$

[it is sufficient to apply (1.35) to λf and to take the derivative with respect to λ at $\lambda = 0$].

The following result (see Theorem I.1.8 in [8]) provides a link between superprocesses and the G-equation.

Theorem A. *Suppose that $\tilde{X}$ is an (L,α)-superdiffusion stopped at the exit from D, ρ is a positive Borel function on $\bar{D}$ vanishing on ∂D and σ is a positive Borel function on ∂D. Then*

$$v(x) = -\log P_x \exp\left\{-\left[\int_0^\infty \langle \rho, \tilde{X}_t\rangle dt + \langle \sigma, X_D\rangle\right]\right\} \tag{1.38}$$

is a solution of the G-equation (1.14) *where G is Green's operator for L-diffusion in D, K is given by* (1.26) *and*[7]

$$h = G\rho + K\sigma. \tag{1.39}$$

Moreover, for every $\mu \in \mathcal{M}(D)$,

$$P_\mu \exp\left\{-\left[\int_0^\infty \langle \rho, \tilde{X}_t\rangle dt + \langle \sigma, X_D\rangle\right]\right\} = e^{-\langle v,\mu\rangle}. \tag{1.40}$$

We also need another implication of Theorem I.1.8 in [8] [cf. Theorem 1.1 in [16]].

Theorem B. *Let $\tilde{X}, D$ and ρ be the same as in Theorem* A *and let σ be a positive Borel function on $\bar{D}$ vanishing on D. Then*

$$u(t,x) = -\log P_x \exp\left\{-\left[\int_0^t \langle \rho, \tilde{X}_s\rangle ds + \langle \sigma, \tilde{X}_t\rangle\right]\right\} \tag{1.41}$$

is a solution of the equation (1.28). *Moreover, for every $\mu \in \mathcal{M}(D)$,*[8]

$$P_\mu \exp\left\{-\left[\int_0^t \langle \rho, \tilde{X}_s\rangle ds + \langle \sigma, \tilde{X}_t\rangle\right]\right\} = \exp\langle -u^t, \mu\rangle. \tag{1.42}$$

The *range* $\mathcal{R}$ of a superprocess X is the smallest closed subset of E which supports all measures X_t (it supports, a.s., every exit measure X_D). We denote by $\mathcal{R}^*$ the minimal closed subset of the Martin space $\hat{E}$ which supports all measures X_t. A set $\Gamma \subset E$ is called $\mathcal{R}$-*polar* if $P_x\{\mathcal{R} \cap \Gamma \neq \emptyset\} = 0$ for all $x \notin \Gamma$. A subset Γ of the Martin boundary ∂E is called $\mathcal{R}^*$-*polar* if $P_x\{\mathcal{R}^* \cap \Gamma \neq \emptyset\} = 0$ for all $x \in E$.[9]

We prove in Section 4:

[7] For $x \in \partial D$, $u(x) = h(x) = \sigma(x)$.

[8] We set $u^t(x) = u(t,x)$.

[9] $\mathcal{R}^*$-polarity is introduced only on ∂E because $\mathcal{R}^* \cap \Gamma = \mathcal{R} \cap \Gamma$ for every compact $\Gamma \subset E$.

Theorem 1.2. *If $h = G\eta + K\nu$ and if the G-equation* (1.14) *has a solution, then η does not charge $\mathcal{R}$-polar sets and ν does not charge $\mathcal{R}^*$-polar sets.*

We say that $\Gamma \subset E$ is G-polar if $CG(\Gamma) = 0$ and that $\Gamma \subset \partial E$ is K-polar if $CK(\Gamma) = 0$. By Theorem 1.1 in [17] [cf. Theorem 1.6 in [12]], the classes $\mathcal{R}$-polar and G-polar sets coincide. Theorems 1.1 and 1.2 imply:

1.5.A. All $\mathcal{R}^*$-polar sets are K-polar.

Indeed, if Γ is compact and if $CK(\Gamma) > 0$, then by (1.21), there exists a measure $\nu \neq 0$ concentrated on Γ such that

$$\int_E g(c,x)dx \left[\int k(x,y)\nu(dy)\right]^\alpha < \infty$$

which implies

$$\int_E g(c,x)dx \left[\int_B k(x,y)\nu(dy)\right]^\alpha < \infty$$

for every B. Hence $\nu(B) = 0$ for all K-polar sets B. By Theorems 1.1 and 1.2, $\nu(B) = 0$ for all $\mathcal{R}^*$-polar sets B. Therefore Γ is not $\mathcal{R}^*$-polar.

If ξ is an L-diffusion in a bounded smooth domain of $\mathbb{R}^d$, then a stronger result than 1.5.A follows from Theorem 1.2 in [17]:

1.5.B. The classes of $\mathcal{R}^*$-polar and K-polar sets coincide.

It remains an open problem if 1.5.B holds in the general case. If it holds for a diffusion ξ and if X is the corresponding superdiffusion, then each of Theorems 1.1–1.2 gives necessary and sufficient conditions on h for the existence of a solution of (1.14).

1.6. Additive functionals. Let X be a superdiffusion. We denote by $\mathcal{F}_t$ the σ-algebra in Ω generated by the exit measures X_Q for all $Q \subset S_{<t}$. A function $A_t(\omega)$ from $[0,\infty] \times \Omega$ to $[0,\infty]$ is called an *additive functional of* X if:

1.6.A. For every ω, A_t is monotone increasing in t.

1.6.B. A_t is measurable with respect to the completion of $\mathcal{F}_t$ with respect to all measures P_μ, $\mu \in \mathcal{M}(E)$.

1.6.C. For every ω, A_t is left continuous in t.

1.6.D. $A_{s+t} = A_s + \theta_s A_t$ for all pairs s,t and all ω.[10]

All these conditions hold for

$$A_t = \int_0^t \langle \rho, X_s\rangle ds \tag{1.43}$$

where ρ is an arbitrary positive Borel function. By a limit procedure, we construct, starting from (1.43), a class of functionals for which a weaker form of condition 1.6.D holds.

We say that a set Λ is ξ-*polar* if $\Pi_x\{\xi_t \notin \Lambda \text{ for all } t > 0\} = 1$ for all x. All ξ-polar sets have the Lebesgue measure 0. A subset $\mathcal{N}$ of $\mathcal{M}(E)$ is called *exceptional* if the set $\{x : \delta_x \in \mathcal{N}\}$ is ξ-polar and if, for all stopped superdiffusions $\tilde{X}$ and for every $\mu \notin \mathcal{N}$, $P_\mu\{\tilde{X}_t \notin \mathcal{N} \text{ for all } t\} = 1$.

[10]Let $\beta(\omega) = \sup\{t : A_t(\omega) < \infty\}$. Then there exists a unique measure $A(\omega, dt)$ on $[0, \beta(\omega))$ such that $A[0,t) = A_t$ for all $t < \beta$.

If h is an arbitrary excessive function, then the set $\Lambda(h) = \{x : h(x) = \infty\}$ is ξ-polar and the set $\mathcal{N}(h) = \{\mu : \langle h, \mu\rangle = \infty\}$ is exceptional.

We say that A is an *additive functional with an exceptional set* $\mathcal{N}$ if A satisfies 1.6.A, B, C and:

1.6.D*. $A_{s+t} = A_s + \theta_s A_t$ for all $s, t, \omega \in \Omega_0$ and $P_\mu(\Omega_0) = 1$ for all $\mu \notin \mathcal{N}$.

Two additive functionals A and $\tilde{A}$ are called *equivalent* if there exists an exceptional set $\mathcal{N}$ such that $P_\mu\{A_t = \tilde{A}_t \text{ for all } t\} = 1$ for all $\mu \notin \mathcal{N}$.

Let h be an excessive function. An additive functional A with an exceptional set $\mathcal{N}$ is called a *linear additive functional with potential* h if, for every $\mu \notin \mathcal{N}$,

$$P_\mu A_\infty = \langle h, \mu\rangle. \tag{1.44}$$

If $G\rho(x) < \infty$ for some x, then the additive functional (1.43) is linear with potential $G\rho$ (condition (1.44) holds for every μ).

Theorem 1.3. *If $h = G\eta + K\nu$ and if $\eta \prec CG, \quad \nu \prec CK$, then h is the potential of a linear additive functional A of X with an exceptional set $\mathcal{N}$. For every $\mu \notin \mathcal{N}$,*

$$P_\mu e^{-A_\infty} = e^{-\langle v, \mu\rangle} \tag{1.45}$$

where v is a solution of the G-equation (1.14).

Theorem 1.3 is proved in Section 3. Theorem 1.1 follows immediately from Theorem 1.3 and a uniqueness Theorem 2.1.

Remark. The construction of A in Section 3 implies that A depends linearly on h. More precisely, if A^i corresponds to h^i, then, for every $c_1, c_2 \geq 0$, the functional A corresponding to $c_1 h^1 + c_2 h^2$ is equivalent to $c_1 A^1 + c_2 A^2$. Therefore, if $h, \tilde{h}$ and $h - \tilde{h}$ are excessive functions and if $v, \tilde{v}$ are the solutions of (1.14) corresponding to h and $\tilde{h}$, then $\tilde{v} \leq v$ outside a ξ-polar set.

In Section 4 we establish:

Theorem 1.4. *If h is the potential of a linear additive functional with an exceptional set $\mathcal{N}$, then $h = G\eta + K\nu$ with η vanishing on all $\mathcal{R}$-polar sets and ν vanishing on all $\mathcal{R}^*$-polar sets.*

Linear additive functionals of superprocesses have been introduced in [11] (in a time-inhomogeneous setting). There a linear additive functional corresponding to a *bounded* excessive function h was constructed for a $(\xi, 2)$-superprocess where ξ is an arbitrary right Markov process. (No exceptional set is needed in this case.)

The case of an (L, α)-superdiffusion with an arbitrary $\alpha \in (1, 2]$ was investigated in [15]. For $h = G\eta$ with $\eta \prec CG$, a functional A was constructed, subject to conditions 1.6.A, B with the property, for every $\mu \in \mathcal{M}^0$,

1.6.D**. $A_{s+t} = A_s + \theta_s A_t$ P_μ-a.s. for all s, t.

Here $\mathcal{M}^0$ is the set of measures of the form $\mu(dx) = \rho(x)dx$ with $\int \rho(x)^{\alpha'} dx < \infty$ where $\alpha' = \alpha/(\alpha - 1)$. Condition (1.44) was proved also only for $\mu \in \mathcal{M}^0$. (Note that $\mathcal{M}^0$ is not the complement of an exceptional set!)

Recent results of Le Gall [23] on additive functionals of the Brownian snake can be translated into our language as follows: if $h = K\nu$ with $\nu \prec CK$, then there exists a functional of an $(\Delta, 2)$-superdiffusion which satisfies conditions 1.6.A, B, C, (1.44) and 1.6.D** for P_x for almost all x.

Additive functionals with an exceptional set have been introduced, in a different context, by Fukushima [20]. In his setting, X is a symmetric Markov process associated with a Dirichlet form and an exceptional set is a polar subset of the state space (in the sense of theory of Dirichlet spaces).

1.7. We have the following logical implications: $\mathcal{A} \Longrightarrow \mathcal{B} \Longrightarrow \mathcal{C} \Longrightarrow \mathcal{D}$ where:

$\mathcal{A}$: $h = G\eta + K\nu$ with $\eta \prec CG, \nu \prec CK$;

$\mathcal{B}$: h is the potential of a linear additive functional A with an exceptional set $\mathcal{N}$. Moreover for every $\mu \notin \mathcal{N}$,

$$P_\mu e^{-A_\infty} = e^{-\langle v,\mu\rangle}$$

where v is a solution of the G-equation (1.14).

$\mathcal{C}$: h is the potential of a linear additive functional A.

$\mathcal{D}$: $h = G\eta + K\nu$ with η vanishing on all $\mathcal{R}$-polar sets and ν vanishing on all $\mathcal{R}^*$-polar sets.

We get $\mathcal{A} \Longrightarrow \mathcal{B}$ by Theorem 1.3 and $\mathcal{C} \Longrightarrow \mathcal{D}$ by Theorem 1.4. The implication $\mathcal{B} \Longrightarrow \mathcal{C}$ is trivial.

If 1.5.B holds for a diffusion ξ and if X is the corresponding superdiffusion, then $\mathcal{D} \Longrightarrow \mathcal{A}$ and all four statements $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ are equivalent. In particular, this is true if ξ is an L-diffusion in a bounded smooth domain D. This also is true for an arbitrary domain E if we consider only excessive functions $h = G\eta$ (in other words if we set $\nu = 0$).

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2. G-equation

2.1. Monotonicity and uniqueness.

Theorem 2.1. *Let ξ be an L-diffusion stopped at the first exit time τ from an open set D, and let G, K be given by* (1.13),(1.26). *Suppose that η is a measure on D, $u, \hat{u}, \sigma \geq 0$ and, for almost all x,*

$$\hat{u} + G(\hat{u}^\alpha) = u + G(u^\alpha) + G\eta + K\sigma < \infty. \tag{2.1}$$

Then $\hat{u} \geq u$ at every point of the set (2.1). *If $\eta = 0$ and $\sigma = 0$, then $\hat{u} = u$ on the same set.*

An analogous result holds for the $\mathcal{G}$-equation. For every measure η on S we put

$$\mathcal{G}\eta(t,x) = \int_0^t \int_D p_{t-s}(x,y)\eta(ds,dy) \tag{2.2}$$

(cf. (1.23)). If $\eta(ds,dy) = ds\eta(dy)$, then

$$\mathcal{G}\eta(t,x) = \int_0^t ds \int_D p_s(x,y)\eta(dy) \to G\eta(x) \tag{2.3}$$

as $t \to \infty$.

Theorem 2.1*. *Let ξ be the same as in Theorem 2.1 and let $\mathcal{G}, \mathcal{K}$ be given by (2.2),(1.24). Suppose that η is a measure on S, $u, \hat{u}, \sigma \geq 0$ and, for almost all t, x,*

$$\hat{u} + \mathcal{G}(\hat{u}^\alpha) = u + \mathcal{G}(u^\alpha) + \mathcal{G}\eta + \mathcal{K}\sigma < \infty. \tag{2.4}$$

Then $\hat{u} \geq u$ at every point of the set (2.4). *If $\eta = 0$ and $\sigma = 0$, then $\hat{u} = u$ on the same set.*

We use as a tool a process $(\xi_s, \Pi_x^{t,y})$ with $x, y \in D$. Its finite-dimensional distributions are given by the formula

$$\begin{aligned} &\Pi_x^{t,y}\{\xi_{t_1} \in dy_1, \ldots, \xi_{t_n} \in dy_n, t_n < t < \tau\} \\ &= p_{t_1}(x, dy_1) p_{t_2 - t_1}(y_1, dy_2) \ldots p_{t_n - t_{n-1}}(y_{n-1}, dy_n) p_{t - t_{n-1}}(y_n, y) \end{aligned} \tag{2.5}$$

for all $0 < t_1 < \cdots < t_n < t$. (Here $p_t(x, dy)$ is the transition function and $p_t(x, y)$ is the transition density of the part of ξ in D.)[11]

Let f be a positive Borel function. Formula

$$p_t^\varphi(x, y) = \Pi_x^{t,y}\left\{\exp\Big\{-\int_0^t \varphi(\xi_s) ds\Big\}\right\} \tag{2.6}$$

defines the transition density of a Markov process obtained from ξ by killing with rate $f(x)$ at point x.

Operator G_φ corresponding to p^φ by (2.2) can be expressed by formula

$$\mathcal{G}_\varphi \rho(t, x) = \Pi_x \int_0^{\tau \wedge t} ds \rho(t - s, \xi_s) \exp\left\{-\int_0^s \varphi(t - r, \xi_r) dr\right\}. \tag{2.7}$$

We prove Theorem 2.1*. (Proof of Theorem 2.1 is similar but simpler.) We need two lemmas.

Lemma 2.1. [12] (i) *Let ρ be a Borel function on S. Equation*

$$\mathcal{G}\rho - \mathcal{G}_\varphi \rho = \mathcal{G}_\varphi(\varphi \mathcal{G}\rho) \tag{2.8}$$

holds on the set $\{\mathcal{G}|\rho| < \infty\}$.

(ii) *If η is a measure on S, then*

$$\mathcal{G}\eta - \mathcal{G}_\varphi \eta = \mathcal{G}_\varphi(\varphi \mathcal{G}\eta) \tag{2.9}$$

on the set $\{\mathcal{G}\eta < \infty\}$.

(iii) *For every positive Borel σ, equation*

$$\mathcal{K}\sigma - \mathcal{K}_\varphi \sigma = \mathcal{G}_\varphi(\varphi \mathcal{K}\sigma) \tag{2.10}$$

holds on the set $\{\mathcal{K}\sigma < \infty\}$.

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[11] Normalized measure $\Pi_x^{t,y}$ can be obtained by conditioning the diffusion ξ started from point x to come at point y at time t.

[12] Cf. [5]. This result can be interpreted as the resolvent form of the Feynman-Kac formula.

Proof. 1°. It is sufficient to check (2.8) for $\rho \geq 0$. We use (1.23) and (2.7), the Markov property of ξ, Fubini's theorem and relation

$$\int_0^s daY_a \exp\left\{-\int_0^a Y_r dr\right\} = 1 - \exp\left\{-\int_0^s Y_r dr\right\}$$

which we apply to $Y_s = \varphi(t-s, \xi_s)$.

2°. Put

$$\rho_\varepsilon(x) = \int_D p_\varepsilon(x,y)\eta(dy). \tag{2.11}$$

Note that

$$G\rho_\varepsilon(x) = \int_\varepsilon^\infty dt \int_D p_t(x,y)\eta(dy). \tag{2.12}$$

We get (2.9) by applying (2.8) to ρ_ε and by passing to the limit as $\varepsilon \to 0$.

3°. Formula (2.10) can be proved in the same way as (2.8).

Lemma 2.2. *Suppose that $\varphi, \sigma \geq 0$ and that, for almost all t, x,*

$$\mathcal{G}\eta + \mathcal{K}\sigma + \mathcal{G}|\varphi w| < \infty \tag{2.13}$$

and

$$w + \mathcal{G}(\varphi w) = \mathcal{G}\eta + \mathcal{K}\sigma. \tag{2.14}$$

Then

$$w = \mathcal{G}_\varphi \eta + \mathcal{K}_\varphi \sigma \tag{2.15}$$

at every point (t,x) *where* (2.13) *and* (2.14) *hold.*

Proof. We have

$$\mathcal{G}_\varphi(\varphi w) + \mathcal{G}_\varphi[\varphi\mathcal{G}(\varphi w)] = \mathcal{G}_\varphi(\varphi\mathcal{G}\eta) + \mathcal{G}_\varphi(\varphi\mathcal{K}\sigma). \tag{2.16}$$

On the set defined by (2.13) and (2.14), the left side in (2.16) is equal to $\mathcal{G}(\varphi w)$ by (2.8) and, the right side is equal to $\mathcal{G}\eta + \mathcal{K}\sigma - \mathcal{G}_\varphi\eta - \mathcal{K}_\varphi\sigma$ by (2.8) and (2.10). Therefore $\mathcal{G}(\varphi w) = \mathcal{G}\eta + \mathcal{K}\sigma - \mathcal{G}_\varphi\eta - \mathcal{K}_\varphi\sigma$ and (2.15) follows from (2.14).

Proof of Theorem 2.1*. Denote by $\hat{S}$ the set defined by (2.4). Put $w = \hat{u} - u$ on $\hat{S}$ and $w = 0$ on $E \setminus \hat{S}$. There exists a function $\varphi \geq 0$ such that $\hat{u}^\alpha - u^\alpha = \varphi w$ a.e. Equation (2.4) implies (2.14). Since $G|\varphi w| \leq G(u^\alpha) + G(\hat{u}^\alpha) < \infty$ on $\hat{S}$, Theorem 2.1* follows from Lemma 2.2.

2.2. Properties of G and $\mathcal{G}$. In this subsection we deal with operators corresponding to an L-diffusion ξ in a bounded smooth domain D. We denote by $\|u\|$ the norm of u in $L^1(D)$. For a function f on $S = \mathbb{R}_+ \times D$ and for $b \in \mathbb{R}_+$, we set

$$\ell_b(f) = \int_0^b \int_D |f(r,x)| dr dx.$$

We need the following results.

2.2.A. There is a constant C such that

$$\int_D g(x,y)dx \leq C \qquad \text{for all} \quad y \in D.$$

2.2.A*. For every $b > 0$, there exists a constant C such that

$$\int_D p_t(x,y)dx \leq C \qquad \text{for all} \quad y \in D, 0 < t \leq b.$$

2.2.B. If f_n is a sequence of functions such that $\ell_b(f_n)$ are bounded for every b, then the sequence $\mathcal{G}f_n$ contains a subsequence which converges a.e. (relative to $drdx$).

2.2.C. Let

$$\theta = \sup_{x \in D} c^*(x)$$

where

$$c^* = \sum_{i,j=1}^{d} \nabla_i \nabla_j a_{ij} - \sum_{i=1}^{d} \nabla_i b_i. \tag{2.17}$$

Then

$$\int_D f \operatorname{sign} Gf dx \geq -\theta \|Gf\| \tag{2.18}$$

for all $f \in L^1(D)$.

Properties 2.2.A and 2.2.A* follow from well-known bounds for $g(x,y)$ ([24, Chapter 3]) and $p_t(x,y)$ ([19, Chapter 1]).

Proof of 2.2.B. Denote by φ_δ a function equal to 0 for $|t| < \delta/2$, equal to 1 for $|t| > \delta$ and linear on $[-\delta, -\delta/2]$ and on $[\delta/2, \delta]$. Formula

$$\mathcal{G}_\delta f(t,x;s,y) = \varphi_\delta(t-s)p_{t-s}(x,y)$$

defines a continuous kernel on $S_b = [0,b] \times \bar{D}$. The corresponding operator $\mathcal{G}_\delta$ is compact in $L^1(S_b)$ because functions $\mathcal{G}_\delta f_n$ are equicontinuous for every sequence f_n bounded in $L^1(S_b)$.

By 2.2.A* and Fubini's theorem,

$$\begin{aligned}\ell_b(\mathcal{G}f - \mathcal{G}_\delta f) &= \int_{S_b} dtdx \int_{S_b} [1 - \varphi_\delta(t-s)] p_{t-s}(x,y)|f(s,y)|dsdy \\ &\leq \int_{S_b} dsdy|f(s,y)| \int_s^{(s+\delta)\wedge b} dtdx p_{t-s}(x,y) \leq C\delta\ell_b(f).\end{aligned}$$

Therefore $\mathcal{G}$ is a compact operator in $L^1(S_b)$. We get 2.2.B by the diagonal procedure.

Proof of 2.2.C. 1°. Suppose that φ is a bounded increasing continuously differentiable function on $\mathbb{R}$ such that $\varphi(0) = 0$. Suppose that

$$u \in C^2(\bar{D}), \qquad u = 0 \quad \text{on} \quad \partial D. \tag{2.19}$$

Put $\Phi(t) = \int_0^t \varphi(s)ds$. By integration by parts, we get

$$\begin{aligned} -\int_D \varphi(u)Ludx &= \int_D \Big[\sum_{i,j} a_{ij}\varphi'(u)\nabla_i u\nabla_j u + \sum_i \big(\sum_j \nabla_j a_{ij} + b_i\big)\varphi(u)\nabla_i u\Big]dx \\ &= \int_D \big[\sum a_{ij}\varphi'(u)\nabla_i u\nabla_j u - c^*\Phi(u)\big]dx \end{aligned} \tag{2.20}$$

and therefore

$$-\int_D dx\varphi(u)Lu \geq -\theta\int_D \Phi(u)dx. \tag{2.21}$$

2°. Suppose $u = Gf$ with $f \in C^2$. Then u satisfies (2.19) and $Lu = -f$. By (2.21),

$$\int_D \varphi(u)fdx \geq -\theta\int_D \Phi(u)dx. \tag{2.22}$$

An arbitrary $f \in L^1(D)$ is the strong limit of a sequence $f_n \in L^1(D) \cap C^2$. Let $u_n = Gf_n, u = Gf$. We have

$$\int \varphi(u)fdx - \int \varphi(u_n)f_ndx = \int \varphi(u_n)(f - f_n)dx + \int (\varphi(u) - \varphi(u_n))fdx. \tag{2.23}$$

By 2.2.A, $u_n \to u$ in $L^1(D)$. Therefore a subsequence u_{n_k} converges to u a.e. and the second term in the right side of (2.23) converges to 0 along this subsequence. The first term also converges to 0. Since (2.22) holds for f_n, it holds also for f.

3°. By applying (2.22) to a sequence of functions φ_n which converge boundedly to $\operatorname{sign} u$ and by passing to the limit, we get

$$\int_D f \operatorname{sign} u \; dx \geq -\theta\int_D |u|dx$$

which is equivalent to (2.18).

2.3. Existence. Suppose that ξ is an L-diffusion stopped at the first exit time τ from a bounded smooth domain D, L satisfies conditions 1.1.A–B, p is defined by condition 1.2.A and g is the corresponding Green's function defined by (1.12). We consider a function in D defined by the formula

$$h = G\eta + K\sigma \tag{2.24}$$

where η is a finite measure on D and σ is a positive bounded Borel function on ∂D. Put $D(h) = \{h < \infty\}$, $D(h,\alpha) = \{h + G(h^\alpha) < \infty\}$ and $\mathcal{N}(h,\alpha) = \{\mu : \langle h + G(h^\alpha), \mu\rangle = \infty\}$. Note that $D(h,\alpha)$ is either empty or is the complement of a ξ-polar set. Let $Q(h) = \mathbb{R}_+ \times D(h)$ and $Q(h,\alpha) = \mathbb{R}_+ \times D(h,\alpha)$.

Theorem 2.2. *Suppose that $D(h,\alpha)$ is nonempty. Then there exists $v \geq 0$ such that*

$$v + G(v^\alpha) = h \quad \text{on } D(h,\alpha). \tag{2.25}$$

Equation (2.25) *determines v uniquely on $D(h,\alpha)$. We have*

$$\|v^\alpha\| \leq 2C\eta(D) + C_1(\sigma) \tag{2.26}$$

where C is defined in 2.2.A *and $C_1(\sigma)$ does not depend on η. Let $\tilde{X}$ be an (L,α)-superdiffusion stopped at exit from D and let*

$$v_\varepsilon(x) = -\log P_x \exp\left\{-\left[\int_0^\infty \langle \rho_\varepsilon, \tilde{X}_t\rangle dt + \langle \sigma, X_D\rangle\right]\right\}, \tag{2.27}$$

$$u_\varepsilon(t,x) = -\log P_x \exp\left\{-\left[\int_0^t \langle \rho_\varepsilon, \tilde{X}_s\rangle ds + \langle \sigma, \tilde{X}_t\rangle\right]\right\} \tag{2.28}$$

where $\sigma = 0$ in D, ρ_ε is given by (2.11) *in D and it is equal to 0 on ∂D .*

We have

$$\lim_{\varepsilon \to 0} v_\varepsilon(x) = v(x) \quad \text{on } D(h,\alpha), \tag{2.29}$$

$$\lim_{\varepsilon \to 0} u_\varepsilon(t,x) = u(t,x) \quad \text{on } Q(h,\alpha) \tag{2.30}$$

where v is the solution of (2.25) *and u is the solution of the equation*

$$u + \mathcal{G}(u^\alpha) = \mathcal{G}\eta + \mathcal{K}\sigma \quad \text{on } Q(h,\alpha). \tag{2.31}$$

Moreover, if $\mu \notin \mathcal{N}(h,\alpha)$, then

$$\langle v_\varepsilon, \mu\rangle \to \langle v, \mu\rangle \tag{2.32}$$

and

$$\langle u_\varepsilon^t, \mu\rangle \to \langle u^t, \mu\rangle \tag{2.33}$$

for all t.

Finally,

$$u(t,x) \uparrow v(x) \quad \text{as} \quad t \to \infty \quad \text{on } Q(h,\alpha). \tag{2.34}$$

Proof. By Theorems 2.1 and 2.1*, each of the equations (2.25) and (2.31) has no more than one solution. We split the proof of Theorem 2.2 into three steps. First, we establish a bound for $\|v_\varepsilon^\alpha\|$. Then we use this bound to prove formulae (2.30) and (2.33). Finally, we establish (2.34), (2.29), (2.32), (2.25), (2.31) and (2.26).

1°. It follows from (2.12) that

$$h_\varepsilon \le h \quad \text{and} \quad h_\varepsilon \uparrow h \quad \text{as} \ \ \varepsilon \to 0 \tag{2.35}$$

where $h_\varepsilon = G\rho_\varepsilon + K\sigma$. By Theorem A and (1.19), v_ε given by (2.27) satisfies equation

$$v_\varepsilon + G([v_\varepsilon]^\alpha) = h_\varepsilon \tag{2.36}$$

and

$$w(x) = -\log P_x \exp\{-\langle \sigma, X_D\rangle\} \tag{2.37}$$

satisfies equation

$$w + G(w^\alpha) = K\sigma. \tag{2.38}$$

Note that functions ρ_ε, w and $K\sigma$ are bounded and

$$v_\varepsilon - w = G(F_\varepsilon) \tag{2.39}$$

where

$$F_\varepsilon = \rho_\varepsilon - v_\varepsilon^\alpha + w^\alpha. \tag{2.40}$$

By 2.2.C,

$$\int F_\varepsilon \operatorname{sign}(v_\varepsilon - w)dx = \int F_\varepsilon \operatorname{sign} GF_\varepsilon dx \ge -\theta\|v_\varepsilon - w\|$$

and, since $\operatorname{sign}(v_\varepsilon^\alpha - w^\alpha) = \operatorname{sign}(v_\varepsilon - w)$, we have

$$\|v_\varepsilon^\alpha - w^\alpha\| = \int (v_\varepsilon^\alpha - w^\alpha)\operatorname{sign}(v_\varepsilon^\alpha - w^\alpha)dx \le \|\rho_\varepsilon\| + \theta\|v_\varepsilon - w\|. \tag{2.41}$$

By 2.2.A* and (2.11),

$$\|\rho_\varepsilon\| \le C\eta(D). \tag{2.42}$$

Note that, if $\alpha > 1$, then for every $\delta > 0$, there exists a constant C_δ such that

$$|b - a| \le \delta|b^\alpha - a^\alpha| + C_\delta \tag{2.43}$$

for all reals a, b. It follows from (2.41), (2.42) and (2.43) that

$$\|v_\varepsilon^\alpha - w^\alpha\| \le \theta\delta\|v_\varepsilon^\alpha - w^\alpha\| + C\eta(D) + \theta C_\delta. \tag{2.44}$$

If $\delta\theta \le 1/2$, then

$$\|v_\varepsilon^\alpha - w^\alpha\| \le 2C\eta(D) + 2\theta C_\delta. \tag{2.45}$$

Since $w \le K\sigma$ and σ is bounded, (2.45) implies

$$\|v_\varepsilon^\alpha\| \le 2C\eta(D) + C_1(\sigma) \tag{2.46}$$

where $C_1(\sigma) = 2\theta C_\delta + \|(K\sigma)^\alpha\|$.

2°. By (1.23), (2.12) and (2.24),

$$\mathcal{G}\rho_\varepsilon + \mathcal{K}\sigma \le G\rho_\varepsilon + K\sigma \le h. \tag{2.47}$$

By Theorem B,

$$u_\varepsilon + \mathcal{G}(u_\varepsilon^\alpha) = \mathcal{G}\rho_\varepsilon + \mathcal{K}\sigma \tag{2.48}$$

and

$$W(t,x) = -\log P_x \exp\{-\langle \sigma, \tilde{X}_t\rangle\}$$

is a solution of the equation

$$W + \mathcal{G}[W^\alpha] = \mathcal{K}\sigma.$$

We have

$$u_\varepsilon - W = \mathcal{G}(F_\varepsilon) \tag{2.49}$$

where

$$F_\varepsilon = \rho_\varepsilon - u_\varepsilon^\alpha + W^\alpha. \tag{2.50}$$

By (2.28) and (2.27),

$$u_\varepsilon(t,x) \le v_\varepsilon(x) \qquad \text{for all } \ t, x. \tag{2.51}$$

For every b, by (2.46) and (2.51),

$$b^{-1}\ell_b[(u_\varepsilon)^\alpha] \le \|v_\varepsilon^\alpha\| \le 2C\eta(D) + C_1(\sigma). \tag{2.52}$$

It follows from (2.11) and 1.2.B that

$$\mathcal{G}\rho_\varepsilon(t,x) = \int_\varepsilon^{t+\varepsilon} ds \int_D p_s(x,y)\eta(dy) \le h(x) \tag{2.53}$$

and therefore

$$\mathcal{G}\rho_\varepsilon \to \mathcal{G}\eta \quad \text{as } \ \varepsilon \to 0 \quad \text{on } D(h). \tag{2.54}$$

For every b, $\ell_b(F_\varepsilon)$ are bounded by (2.50), (2.42) and (2.52). By (2.49) and 2.2.B, every sequence u_{ε_n} contains a subsequence which converges, a.e., to a limit u. Suppose $u_{\varepsilon_n} \to u$ a.e. By (2.48) and (2.47), $u_\varepsilon \le h$. It follows from (1.23) and the dominated convergence theorem that

$$\mathcal{G}[(u_{\varepsilon_n})^\alpha] \to \mathcal{G}[(u)^\alpha] \quad \text{on } Q(h,\alpha). \tag{2.55}$$

By (2.55) and (2.48), u satisfies (2.31) a.s. Formula (2.30) holds because, otherwise, $|u_{\varepsilon_n} - u| > \delta$ for some $(r,x) \in Q(h,\alpha), \delta > 0$ and for some sequence $\varepsilon_n \to 0$. By applying once more the dominated convergence theorem, we get (2.33).

3°. It is clear from (2.28) that $u_\varepsilon(t,x)$ is monotone increasing in t. Therefore for every $x \in D(h,\alpha)$, $u(t,x)$ is also monotone increasing in t. By the monotone convergence theorem, $v(x) = \lim_{t\to\infty} u(t,x)$ satisfies (2.25). Formula (2.26) follows from (2.52) and (2.34).

Note that $u_\varepsilon \le v_\varepsilon$ and, by (2.30) and (2.34), $\liminf_{\varepsilon\to 0} v_\varepsilon \ge v$ on $D(h,\alpha)$. On the other hand, it follows from (1.25) and (1.13) that $\mathcal{G}(u_\varepsilon^\alpha) \le \mathcal{G}(v_\varepsilon^\alpha) \le G(v_\varepsilon^\alpha)$ and, by (2.48), (2.36), (2.12), (2.53), (1.25) and (1.26),

$$0 \le v_\varepsilon - u_\varepsilon^t \le \int_{t+\varepsilon}^\infty ds \int_D p_s(x,y)\eta(dy) + \Pi_x \sigma(\xi_\tau) 1_{\tau > t}$$

and therefore $\limsup_{\varepsilon\to 0} v_\varepsilon \le v$ on $D(h,\alpha)$. Clearly, this implies (2.29). Formula (2.32) can be deduced from (2.33) in an analogous way.

3. Proof of Theorem 1.3

3.1. We use several times a property of exit measures which will be established in Lemma 3.1. We start from a functional

$$B_t(\varepsilon) = \int_0^t \langle \rho_\varepsilon, \tilde{X}_s \rangle ds + C_t \tag{3.1}$$

where ρ_ε is given by (2.11) and C_t is a left continuous modification of $\langle \sigma, \tilde{X}_t \rangle$ which we define in Lemma 3.2. Put

$$B_t = \operatorname*{lim\,med}_{k\to\infty} B_t(1/k) \qquad \text{for all} \quad t > 0 \tag{3.2}$$

where lim med is Mokobodzki's medial limit. It is defined for every sequence $a_n \in [0,\infty]$ and it takes values in $[0,\infty]$. We need the following properties of this limit (see, e.g., [4, X.56, X.57]):

3.1.A. $\liminf a_n \le \operatorname{lim\,med} a_n \le \limsup a_n$;

3.1.B. $\operatorname{lim\,med}(a_n + b_n) = \operatorname{lim\,med} a_n + \operatorname{lim\,med} b_n$;

3.1.C. If $a_n \le b_n$ for all n, then $\operatorname{lim\,med} a_n \le \operatorname{lim\,med} b_n$;

3.1.D. Let Z_n be measurable mappings from a measurable space $(\Omega, \mathcal{F})$ to $[0,\infty]$. Then $Z(\omega) = \operatorname{lim\,med} Z_n(\omega)$ is measurable with respect to the universal completion of $\mathcal{F}$. If P is a probability measure on $(\Omega,\mathcal{F})$ and if $Z_n \to Y$ in P-probability, then $Y = Z$ P-a.s.

In Theorem 3.1, we construct a functional B of an (L,α)-superdiffusion $\tilde{X}$ stopped at the exit from a bounded smooth domain D which satisfies conditions 1.6.A, B and the following condition:

$$B_{s+t} \le B_s + \theta_s B_t \quad \text{a.s. for every } s,t. \tag{3.3}$$

Moreover, for every $\mu \notin \mathcal{N}(h,\alpha)$:

$$B_t = \lim_{\varepsilon\to 0} B_t(\varepsilon) \quad \text{in } P_\mu\text{-probability for all } t \in \mathbb{R}_+ \tag{3.4}$$

and

$$-\log P_\mu e^{-B_t} = \langle u^t, \mu\rangle \tag{3.5}$$

where u satisfies (2.31).

The next step is a passage to the limit from bounded smooth domains to an arbitrary domain E. We assume that h is given by (1.17) and that $E(h,\alpha) = \{h < \infty, G(h^\alpha) < \infty\}$ is nonempty. We consider a sequence of bounded smooth domains D_n which approximate E and we denote by G^n, K^n the Green and Poisson operators corresponding to the L-diffusion stopped at the exit from D_n. Put $\sigma_n = 1_{E\setminus D_n} K\nu$ and denote by B^n the function corresponding to

$$h_n(x) = \int_{D_n} g^n(x,y)\eta(dy) + K^n\sigma_n(x) \tag{3.6}$$

by Theorem 3.1. By 3.1.C, D, function

$$B_t = \lim_{n\to\infty} \operatorname{med} B_t^n \tag{3.7}$$

satisfies 1.6.A, B. We show that, for every $\mu \notin \mathcal{N}(h,\alpha)$ and every t,

$$B_t = \lim_{n\to\infty} B_t^n \quad P_\mu - \text{a.s.} \tag{3.8}$$

Function $A_t = B_{t-}$ satisfies 1.6.A-C and 1.6.D** with $\mathcal{N} = \mathcal{N}(h,\alpha)$.

At the final stage, we use Lemma 3.3 to decompose measures η,ν, subject to condition (1.22), into series of measures η_n,ν_n for which $E(h_n,\alpha) \neq \emptyset$ (here $h_n = G\eta_n + K\nu_n$). The functional corresponding to η,ν is defined as the sum of functionals corresponding to η_n,ν_n.

This way we obtain a functional of X, subject to conditions 1.6.A, B, C, for which 1.6.D** and (1.45) hold for all μ outside of an exceptional set $\mathcal{N}$. Then we refer to a result in [4] to prove the existence of an equivalent functional which satisfies 1.6.D*.

3.2. A property of exit measures.

Lemma 3.1. *Suppose that $Q_1 \supset Q_2$ are open subsets of S and $\Gamma \cap Q_1 = \emptyset$. Then $X_{Q_1}(\Gamma) \geq X_{Q_2}(\Gamma)$ a.s.*

Proof. For every $\nu \in \mathcal{M}(E)$, $P_\nu\{X_{Q_1}(\Gamma) \geq \nu(\Gamma)\} = 1$. Indeed, $\Pi_{r,x}\{\tau^r = r\} = 1$ for every $(r,x) \notin Q_1$ and, by (1.32), for every $\lambda > 0$,

$$P_\nu e^{-\lambda X_{Q_1}(\Gamma)} = e^{-\langle v_\lambda,\nu\rangle}$$

with $v_\lambda = \lambda 1_\Gamma$ on Γ. Hence,

$$P_\nu e^{-\lambda X_{Q_1}(\Gamma)} \leq e^{-\lambda\nu(\Gamma)}. \tag{3.9}$$

Put $Y = X_{Q_1}(\Gamma) - \nu(\Gamma)$. By (3.9), $P_\nu e^{-\lambda Y} \leq 1$ for all $\lambda > 0$ and therefore $Y \geq 0$ P_ν-a.e.

It follows from (1.34) that, for every positive measurable f, $P_\mu f(X_{Q_2}, X_{Q_1}) = P_\mu F(X_{Q_2})$ where $F(\nu) = P_\nu f(\nu, X_{Q_1})$. If $f(\nu_1,\nu_2) = 1_{\nu_1(\Gamma)\leq\nu_2(\Gamma)}$, then $F(\nu) = P_\nu\{\nu(\Gamma) \leq X_{Q_1}(\Gamma)\} = 1$.

3.3. Regularization of $\langle \sigma, \tilde{X}_t \rangle$.

Lemma 3.2. *Let $\tilde{X}$ be an (L, α)-superdiffusion stopped at the exit from a bounded smooth domain D and let σ be a positive Borel function on $\bar{D}$ which vanishes on D. There exists a function C_t subject to conditions* 1.6.A–C *such that, for every t,*

$$C_t = \langle \sigma, \tilde{X}_t \rangle \quad a.s. \tag{3.10}$$

Proof. Put $Y_t = \langle \sigma, \tilde{X}_t \rangle$. Recall (see Section 1.5) that $\tilde{X}_t = X_{Q_t}$ where $Q_t = [0, t) \times D$. It follows from Lemma 3.1 that $\tilde{X}_r(\Gamma) \le \tilde{X}_s(\Gamma)$ a.s. if $r < s$ and $\Gamma \cap Q_s = \emptyset$. Since $\sigma = 0$ in Q_s, $Y_r \le Y_s$ a.s. Denote by $\mathbb{Q}_+$ the set of positive rationals. The set

$$\Omega_t = \{Y_r \le Y_s \text{ for all } r < s \in \mathbb{Q}_+ \cap [0, t)\}$$

belongs to $\mathcal{F}_t$ and $P_\mu\{\Omega_t\} = 1$ for all $\mu \in \mathcal{M}(D)$. Function

$$C_t = \begin{cases} \lim\limits_{s \uparrow t, s \in \mathbb{Q}_+} Y_s & \text{on } \Omega_t, \\ \infty & \text{otherwise} \end{cases}$$

satisfies conditions 1.6.A–C. It remains to prove that $Y_t = C_t$ a.s. By Theorem B,

$$-\log P_\mu e^{-Y_t} = \langle u^t, \mu \rangle \tag{3.11}$$

where u is a solution of the equation

$$u + \mathcal{G}[u^\alpha] = \mathcal{K}\sigma. \tag{3.12}$$

By (3.11), $u(t, x)$ is monotone increasing in t. Put $u_-(t, x) = u(t-, x)$. Since $\Pi_x\{\tau = t\} = 0$ for all t, function $\mathcal{K}\sigma$ is continuous in t. By passing to the limit in (3.12), we get

$$u_- + \mathcal{G}[u_-^\alpha] = \mathcal{K}\sigma.$$

By (1.25), functions $\mathcal{K}\sigma \le K\sigma$ are bounded and, by Theorem 2.1*, $u_- = u$. By (3.11), $P_\mu e^{-Y_t} = P_\mu e^{-C_t}$. Since $Y_t \le C_t$, this implies (3.10).

3.4.

Lemma 3.3. *Let η and ν satisfy condition* (1.22)*. Then there exist measures η_n, ν_n such that*

$$\eta = \eta_1 + \cdots + \eta_n + \cdots, \quad \nu = \nu_1 + \cdots + \nu_n + \cdots$$

and

$$G(h_n^\alpha)(c) < \infty \tag{3.13}$$

where $h_n = G\eta_n + \mathcal{K}\nu_n$ and c is the same as in formula (1.16)*.*

Proof. Since $(\frac{a+b}{2})^\alpha \le \frac{1}{2}(a^\alpha + b^\alpha)$ for all $\alpha, a, b \ge 0$, we can assume that $\eta = 0$ or $\nu = 0$. We refer to [18, Theorem 2.2] in the first case and [2, Lemma 4.2] in the second case.

3.5.

Theorem 3.1. *Let $\tilde{X}$ be an (L,α)-superdiffusion stopped at the exit from a bounded smooth domain D and let h,η,ρ_ε and σ be as in Theorem* 2.2. *If $B_t(\varepsilon)$ is defined by* (3.1)*, then function B_t given by* (3.2) *satisfies conditions* 1.6.A, B*,* (3.3)*,* (3.4) *and* (3.5)*.*

Proof. Properties 1.6.A, B follow from 3.1.C, D. By Theorem B,

$$u_{\delta\varepsilon}(t,x) = -\log P_x \exp\{-\frac{1}{2}(B_t(\delta)+B_t(\varepsilon))\} \tag{3.14}$$

satisfies the equation

$$u_{\delta\varepsilon} + \mathcal{G}[u^\alpha_{\delta\varepsilon}] = \frac{1}{2}(\mathcal{G}\rho_\delta + \mathcal{G}\rho_\varepsilon) + \mathcal{K}\sigma.$$

The same arguments as in the proof of Theorem 2.2 show that, for all $\mu \notin \mathcal{N}(h,\alpha)$ and all t,

$$\langle u^t_{\delta\varepsilon},\mu\rangle \to \langle u^t,\mu\rangle \quad \text{as} \quad \delta,\varepsilon\to 0 \tag{3.15}$$

where u is the unique solution of (2.31).

By Theorem B,

$$P_\mu\left[e^{-B_t(\varepsilon)/2} - e^{-B_t(\delta)/2}\right]^2 = e^{-\langle u^t_{\varepsilon\varepsilon},\mu\rangle} + e^{-\langle u^t_{\delta\delta},\mu\rangle} - 2e^{-\langle u^t_{\delta\varepsilon},\mu\rangle} \tag{3.16}$$

for every $\mu\in\mathcal{M}(D)$. If $\mu\notin\mathcal{N}(h,\alpha)$, then, by (3.15), the right side in (3.16) tends to 0 as $\delta,\varepsilon\to 0$. Hence $e^{-B_t(\varepsilon)}$ converges in $L^2(P_\mu)$ as $\varepsilon\to 0$ which implies that $B_t(\varepsilon)$ converges in P_μ-probability to a limit B^μ_t. It follows from 3.1.D that $B_t = B^\mu_t$ P_μ-a.s. which implies (3.4). To prove (3.3), we note that $\tilde{X}_t = X_{Q(t)}$ where $Q(t) = [0,t)\times D$. Therefore (see Section 1.5)) $\theta_s\tilde{X}_t = X_{Q(s,t)}$ where $Q(s,t) = S_{<s}\cup Q(s+t)$ and, by Lemma 3.1, $\langle\rho_\varepsilon,\theta_s\tilde{X}_t\rangle \ge \langle\rho_\varepsilon,\tilde{X}_{s+t}\rangle$ and $\langle\sigma,\theta_s\tilde{X}_t\rangle\ge\langle\sigma,\tilde{X}_{s+t}\rangle$ a.s. Clearly, restrictions of measures $\tilde{X}_s$ and $\tilde{X}_{s+t}$ to $[0,s)\times\partial D$ coincide, and, by (3.1),

$$B_s(\varepsilon) + \theta_s B_t(\varepsilon) \ge B_{s+t}(\varepsilon) \quad \text{a.s.} \tag{3.17}$$

and (3.3) follows from (3.2).

Let $\mu\notin\mathcal{N}(h,\alpha)$. By (1.42),

$$-\log P_\mu e^{-B_t(\varepsilon)} = \langle u^t_\varepsilon,\mu\rangle$$

where u_ε is given by (2.28). By (3.4) and (2.33), this implies (3.5).

3.6. The next step in the program outlined in Section 3.1 is a passage to the limit from bounded smooth domains to an arbitrary domain E. Recall that, according to Section 1.2, L-diffusion ξ in E can be constructed by using a sequence of bounded smooth domains D_n such that $\bar{D}_n\subset D_{n+1}$ and $E=\bigcup D_n$: the transition density $p_t(x,y)$ of ξ is the limit of monotone increasing sequence $p^n_t(x,y)$ defined in 1.2.A (it is convenient to set $p^n_t(x,y)=0$ if $x\notin D_n$ or $y\notin D_n$). Green's functions $g^n(x,y)$,

$g(x,y)$ and Green's operators G^n, G corresponding to p^n, p are determined by (1.12) and (1.13). Operators K^n correspond by (1.26) to the first exit times τ_n from D_n.

Let X be an (L,α)-superdiffusion in E and let X^n be an (L,α)-superdiffusion stopped at the exit from D_n. Denote by Y_t^n the restriction of X_t^n to D_n. By Lemma 3.1, for every t and every n,

$$Y_t^n \le Y_t^{n+1} \quad \text{a.s.} \tag{3.18}$$

By (1.37),

$$P_\mu Y_t^n(B) = \int_E \mu(dx) \int_B p_t^n(x,y)dy \uparrow \int_E \mu(dx) \int_B p_t(x,y)dy = P_\mu X_t(B)$$

and therefore

$$Y_t^n \uparrow X_t \quad \text{a.s.} \tag{3.19}$$

3.7. Let h be given by (1.17) with finite measures η and ν. Suppose that $E(h,\alpha) \neq \emptyset$. Put $f = K\nu$. By (1.19) and (1.26),

$$f(x) = \Pi_x \sigma_n(\xi_{\tau_n}) = K^n \sigma_n(x)$$

where $\sigma_n = 1_{E \setminus D_n} f$. We define B^n and B as in Section 3.1. By 3.1.D, to prove formula (3.8), we need only show that B_t^n converges P_μ-a.s. as $n \to \infty$. Put

$$Z_t^n(\varepsilon) = \int_0^t \langle \rho_\varepsilon^n, Y_r^n \rangle dr. \tag{3.20}$$

By 3.1.B,

$$B_t^n = Z_t^n + C_t^n$$

where

$$Z_t^n = \lim_{k\to\infty} \operatorname{med} Z_t^n(1/k).$$

For every n, $\rho_\varepsilon^{n+1} \ge \rho_\varepsilon^n$ and, by (3.20) and (3.18), $Z_t^n(\varepsilon)$ is, a.s., monotone increasing in n. By 3.1.C, sequence Z_t^n has the same property and therefore it converges P_μ-a.s.

On the other hand, since f is L-harmonic, it follows from the Markov property (1.34) that the sequence $W_n = \langle f, X_t^n \rangle$ is a martingale with respect to P_μ. Therefore C_t^n also converges a.s.

3.8. Put $S(h,\alpha) = \mathbb{R}_+ \times E(h,\alpha)$. By Theorem 2.2, for every $\mu \notin \mathcal{N}(h,\alpha)$,

$$-\log P_\mu e^{-B_t^n} = \int_{D_n} u_n(t,x)\mu(dx)$$

where u_n satisfies the equation

$$u_n(t,x) + \int_0^t ds \int_{D_n} p_{t-s}^n(x,y) u_n(s,y)^\alpha dy = H_n(t,x) \quad \text{on } S(h,\alpha) \tag{3.21}$$

with

$$H_n(t,x) = \int_0^t ds \int_{D_n} p^n_{t-s}(x,y)\eta(dy) + \Pi_x f(\xi_{\tau_n}) 1_{\tau_n<t}. \tag{3.22}$$

Moreover, by (2.28), (2.30) and (3.20),

$$u_n(t,x) = -\lim_{\varepsilon\to 0} \log P_x \exp\{-B^n_t(\varepsilon)\} \quad \text{on } S(h,\alpha). \tag{3.23}$$

By (3.4),

$$u_n(t,x) = -\log P_x e^{-B^n_t} \quad \text{on } S(h,\alpha).$$

By (3.8),

$$u_n(t,x) \to u(t,x) = -\log P_x e^{-B_t} \quad \text{on } S(h,\alpha). \tag{3.24}$$

Note that

$$\Pi_x f(\xi_{\tau_n}) 1_{\tau_n<t} = f(x) - \Pi_x f(\xi_t) 1_{\tau_n\geq t}$$

and therefore H_n converges to

$$H(t,x) = \int_0^t ds \int_E p_{t-s}(x,y)\eta(dy) + F(t,x) \tag{3.25}$$

where

$$F(t,x) = f(x) - \Pi_x f(\xi_t). \tag{3.26}$$

By (3.21), (3.22) and (1.17), $u_n \leq h$. The second term in (3.21) converges to $\mathcal{G}[u^\alpha]$ by (2.2) and the dominated convergence theorem. Hence, (3.21) implies

$$u + \mathcal{G}[u^\alpha] = H \quad \text{on } S(h,\alpha). \tag{3.27}$$

3.9. Note that $u(t,x)$ increases in t by (3.24) and 1.6.A. Put $u_-(t,x) = u(t-,x)$. An L-excessive function f has a representation

$$f = f_0 + \int_0^\infty \varphi_s ds$$

where $T_t f_0 = f_0$ and $T_t \varphi_s = \varphi_{s+t}$ for all t, s (see [10, Section 2.8]). Therefore

$$H(t,x) = \int_0^t ds[\int_E p_s(x,y)\eta(dy) + \varphi_s]$$

is increasing and continuous in t. By passing to the limit in (3.24) and (3.27), we get

$$\begin{gathered} u_-(t,x) = -\log P_x e^{-B_{t-}}, \\ u_- + \mathcal{G}[u_-^\alpha] = H \quad \text{on } S(h,\alpha). \end{gathered} \tag{3.28}$$

By Theorem 2.1*, this implies $u_- = u$. Since $B_{t-} \le B_t$, (3.24) and (3.28) yield $B_{t-} = B_t$ a.s. Function $A_t = B_{t-}$ satisfies conditions 1.6.A–C.

We claim that 1.6.D** holds for $\mathcal{N} = \mathcal{N}(h, \alpha)$. Indeed, if $\mu \notin \mathcal{N}(h, \alpha)$, then

$$B_t^n(\varepsilon) = Z_t^n(\varepsilon) + C_t^n$$

converges in P_μ-probability to B_t^n by (3.4). By the Markov property (1.34),

$$P_\mu e^{-B_{s+t}^n(\varepsilon)} = P_\mu \left[e^{-B_s^n(\varepsilon)} P_{Y_s^n} e^{-B_t^n(\varepsilon)} \right]$$

for all $s, t > 0$. This implies

$$P_\mu e^{-B_{s+t}^n} = P_\mu \left[e^{-B_s^n} P_{Y_s^n} e^{-B_t^n} \right]$$

and therefore

$$|P_\mu e^{-B_{s+t}^n} - P_\mu \left[e^{-B_s^n} P_{X_s} e^{-B_t^n} \right] | \le P_\mu | P_{Y_s^n} e^{-B_t^n} - P_{X_s} e^{-B_t^n} | \tag{3.29}$$

By (3.5), the right side is equal to

$$P_\mu | e^{-\langle v_n^t, Y_s^n \rangle} - e^{-\langle v_n^t, X_s \rangle} |$$

where $v_n^t(x) = u_n^t(0, x)$. By (3.7), $v_n^t \le h$ and therefore (3.29) does not exceed

$$P_\mu | 1 - e^{-\langle h, X_s - Y_s^n \rangle} |.$$

By (3.19), this tends to 0 and we conclude from (3.29) and the Markov property of X that

$$P_\mu e^{-B_{s+t}} = P_\mu e^{-B_s} P_{X_s} e^{-B_t} = P_\mu e^{-(B_s + \theta_s B_t)}. \tag{3.30}$$

By (3.1), $B_{s+t} \le B_s + \theta_s B_t$ and (3.30) implies 1.6.D**.

We get (1.45) by passing to the limit in (3.5) and (3.27) as $t \to \infty$.

3.10. Let h be an arbitrary function of the form (1.17) with η and ν subject to conditions (1.22). Consider measures η_n and ν_n defined in Lemma 3.3. Denote by A^n the functional of X corresponding to $h_n(x) = G\eta_n + K\nu_n$ by Section 3.9 and put

$$A = A_1 + \cdots + A_n + \cdots .$$

Clearly, conditions 1.6.A,B,C and 1.6.D** hold for A. Formula (1.45) holds if $\mu \notin \mathcal{N} = \bigcup \mathcal{N}(h_n, \alpha)$ and v satisfies (1.14) on $E = \bigcap E(h_n, \alpha)$. Function $\tilde{v}$ defined by (1.15) is a solution of (1.14) everywhere. It also satisfies (1.45).

Formula (1.44) can be obtained from (1.45) in the same way as (1.37) was deduced from (1.35).

By the "perfection" theorem [4, 15.8], there exists a functional equivalent to A which satisfies 1.6.D*. (In [4] functionals without an exceptional set are considered, but the proof is applicable without any change to our case.)

4. Proof of Theorems 1.2 and 1.4

4.1.

Lemma 4.1. *Let an excessive function be given by formula* (1.17)*. If there exists* u *such that*

$$u + G(u^\alpha) = h, \tag{4.1}$$

then there exists v *such that*

$$v + G(v^\alpha) = K\nu. \tag{4.2}$$

Proof. Let D_n, G^n and K^n have the same meaning as in Section 3.6. By the strong Markov property of ξ, (4.1) implies

$$u + G^n(u^\alpha) = G^n\eta + K^n u \quad \text{on } D_n. \tag{4.3}$$

By Theorem A,

$$v_n(x) = -\log P_x e^{-\langle u, X_{D_n}\rangle}$$

satisfies the equation

$$v_n + G^n(v_n^\alpha) = K^n u \quad \text{on } D_n. \tag{4.4}$$

We use again the strong Markov property of ξ to get from here that, for each $m > n$,

$$v_m + G^n(v_m^\alpha) = K^n v_m \quad \text{on } D_n. \tag{4.5}$$

We conclude from Theorem 2.1, by comparing (4.3) and (4.4), that $v_n \le u$ in $D_n \cap E(h)$, and, by comparing (4.4) and (4.5), that $v_m \le v_n$ in $D_n \cap E(h)$. Therefore there exists a limit

$$v = \lim_{n\to\infty} v_n \quad \text{on } E(h).$$

It follows from (4.3) by monotone convergence theorem that

$$u + G(u^\alpha) = G\eta + \lim K^n u.$$

In combination with (4.1), this yields $\lim K^n u = K\nu$ on $E(h)$. By (4.1), $G(u^\alpha) < \infty$ on $E(h)$ and, by the dominated convergence theorem, $\lim G^n(v_n^\alpha) = G(v^\alpha)$. Therefore (4.4) implies that (4.2) holds on $E(h)$. It holds everywhere for a function v modified by formula (1.15).

4.2. Proof of Theorem 1.2. Suppose that u is a solution of (4.1). By Lemma 4.1, equation (4.2) has a solution and ν does not charge $\mathcal{R}^*$-polar sets by Theorem 3.1 in [18].

It remains to prove that $\eta(\Gamma) = 0$ for $\mathcal{R}$-polar sets Γ. We can assume that Γ is compact. Let D be a bounded smooth domain such that $\Gamma \subset D$ and $\bar{D} \subset E$. Equation (4.1) implies

$$u + G_D(u^\alpha) = G_D\eta + K_D u \qquad \text{in } D \tag{4.6}$$

(cf. (4.3)). By Theorem E° in [17], $\mathrm{Cap}_{2,\alpha'}(\Gamma) = 0$. We use the following fact (see Lemma 4.1 in [2]): a signed measure γ does not charge sets Γ with $\mathrm{Cap}_{2,\alpha'}(\Gamma) = 0$ if

$$\int_D \varphi(x)\gamma(dx) \le \text{const.}\|\varphi\|_{2,\alpha'} \qquad \text{for all} \quad \varphi \in C_0^\infty(D)$$

(here $\|\varphi\|_{2,\alpha'}$ is the norm of φ in the Sobolev space $W^{2,\alpha'}(D)$).

By Lemma 4.1, there exists $v \ge 0$ such that

$$v + G_D(v^\alpha) = K_D u \qquad \text{in} \quad D. \tag{4.7}$$

By Theorem 2.1, $w = u - v \ge 0$. There exists a function $q \ge 0$ such that $u^\alpha - v^\alpha = qw$ a.e. (cf. proof of Theorem 2.1*). It follows from (4.6) and (4.7) that

$$w + G_D(qw) = G_D\eta. \tag{4.8}$$

Let $\gamma(dx) = \eta(dx) - (qw)(x)dx$ and $\varphi \in C_0^\infty(D)$. Put $\psi = -L^*\varphi$. Note that $\|\psi\|_{\alpha'} \le \|\varphi\|_{2,\alpha'}$ and

$$\varphi(y) = \int_D dx\psi(x)g_D(x,y).$$

By (4.8), $G_D\gamma = w$ and therefore

$$\begin{aligned}\int_D \varphi(x)\gamma(dx) = \int_{D\times D} dx\psi(x)g_D(x,y)\gamma(dy) = \int_D w(x)\psi(x)dx \\ \le \|w\|_\alpha\|\psi\|_{\alpha'} \le \|w\|_\alpha\|\varphi\|_{2,\alpha'}.\end{aligned}$$

If $h(c) < \infty$, then $G(u^\alpha)(c) < \infty$ by (4.1) and $u \in L^\alpha(D)$ because $\inf_D g(c,y) > 0$. We have $0 \le w \le u$ and therefore $w \in L^\alpha(D)$. Hence $\gamma(\Gamma) = 0$. Since $\mathrm{Cap}_{2,\alpha'}(\Gamma) = 0$ implies that the Lebesgue measure of Γ is equal to 0, we get $\eta(\Gamma) = 0$.

4.3. Localization. To prove Theorem 1.4, we need some preparations. Suppose that h is the potential of a linear additive functional A with exceptional set $\mathcal{N}$ and let η, ν correspond to h by (1.17). For every positive bounded continuous function φ on $E \cup E^*$, we put $h^\varphi(x) = G(\eta^\varphi) + K(\nu^\varphi)$ where $\eta^\varphi(dx) = \varphi(x)\eta(dx), \nu^\varphi(dx) = \varphi(x)\nu(dx)$. It follows from 1.6.D*, the strong Markov property of X and (1.44) that

$$\langle h, X_T\rangle = P_\mu\{A_\infty|\mathcal{F}_T\} - A_T \quad P_\mu\text{-a.s.}$$

for every $\mathcal{F}_t$-stopping time T and for every $\mu \notin \mathcal{N}$. It is easy to see from here that $\langle h, X_t\rangle$ is a supermartingale of class (D) relative to P_μ (cf. [4, V.15]). Since $\langle h^\varphi, X_t\rangle \le \mathrm{const}\langle h, X_t\rangle$, the same is true for $\langle h^\varphi, X_t\rangle$. By [4, Th. XV.6] or [25, Th. 38.1][13], there exists a *natural* additive functional A^φ[14] such that:

4.3.A. $P_\mu A^\varphi_\infty = \langle h^\varphi, \mu\rangle$ for all $\mu \notin \mathcal{N}$.

We call it the *φ-localization of A*. In the same way as in Theorem 3.3 of [17], we establish:

4.3.B. If $\varphi_1 \le \varphi_2$, then $A^{\varphi_1} \le A^{\varphi_2}$ P_μ-a.s. for all $\mu \notin \mathcal{N}$.

4.3.C. If $\varphi = 0$ on D, then $\{A^\varphi = 0\} \supset \{\mathcal{R} \subset \bar{D}\}$ P_μ-a.s. and $\{A^\varphi = 0\} \supset \{\mathcal{R}^* \subset \bar{D}\}$ P_μ-a.s. for all $\mu \notin \mathcal{N}$.

[13] As in the case of "perfection", these theorems can be easily extended to functionals with an exceptional set.

[14] An additive functional A is natural if the process A_{t+} is predictable (cf. [4,IV.61] or the Appendix to [9]). We believe that functional A constructed in Theorem 1.3 is natural but this is not proved in the present paper.

4.4. Proof of Theorem 1.4. This proof is similar to that of Theorem 3.3 in [18]. Let, for instance, $\Gamma \subset E$ be a compact $\mathcal{R}$-polar set. Put

$$D_n = \{x \in E : d(x, \Gamma) > \frac{1}{n}\}$$

where d is the distance in the Martin space $\hat{E}$. Bounded positive continuous functions

$$\varphi_n(x) = (1 - nd(x, \Gamma))_+$$

vanish on D_n. Consider the corresponding localizations A^{φ_n}. For every $\mu \notin \mathcal{N}$,

$$A^1 \geq A^{\varphi_1} \geq \cdots \geq A^{\varphi_n} \geq \cdots, \quad P_\mu\text{-a.s.}$$

by 4.3.B and

$$\{\mathcal{R} \subset D_n\} \subset \{A^{\varphi_n}_\infty = 0\}, \quad P_\mu\text{-a.s.}$$

by 4.3.C. Let $\mu(\Gamma) = 0$. Since Γ is $\mathcal{R}$-polar, $1_{\mathcal{R} \subset D_n} \uparrow 1$ P_μ-a.s. and therefore $A^{\varphi_n}_\infty \to 0$ P_μ-a.s. By the dominated convergence theorem,

$$\lim P_\mu A^{\varphi_n}_\infty = 0 \tag{4.9}$$

On the other hand, by 4.3.A,

$$\begin{aligned} P_\mu A^{\varphi_n}_\infty = & \int \mu(dx) \int_E g(x,y)\varphi_n(y)\eta(dy) \\ & + \int \mu(dx) \int \mu(dx) \int_{E^*} k(x,y)\varphi_n(y)\nu(dy) \downarrow \int \mu(dx) \int_\Gamma g(x,y)\eta(dy). \end{aligned}$$

In combination with (4.9), this implies $\eta(\Gamma) = 0$. The case of $\mathcal{R}^*$-polar set $\Gamma \subset E^*$ can be treated in a similar way.

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