

## LINEAR ADDITIVE FUNCTIONALS OF SUPERDIFFUSIONS AND RELATED NONLINEAR P.D.E.

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ABSTRACT. Let  $L$  be a second order elliptic differential operator in a bounded smooth domain  $D$  in  $\mathbb{R}^d$  and let  $1 < \alpha \leq 2$ . We get necessary and sufficient conditions on measures  $\eta, \nu$  under which there exists a positive solution of the boundary value problem

$$(*) \quad \begin{aligned} -Lv + v^\alpha &= \eta && \text{in } D, \\ v &= \nu && \text{on } \partial D. \end{aligned}$$

The conditions are stated both analytically (in terms of capacities related to the Green's and Poisson kernels) and probabilistically (in terms of branching measure-valued processes called  $(L, \alpha)$ -superdiffusions).

We also investigate a closely related subject — linear additive functionals of superdiffusions. For a superdiffusion in an arbitrary domain  $E$  in  $\mathbb{R}^d$ , we establish a 1-1 correspondence between a class of such functionals and a class of  $L$ -excessive functions  $h$  (which we describe in terms of their Martin integral representation). The Laplace transform of  $A$  satisfies an integral equation which can be considered as a substitute for (\*).

### 1. INTRODUCTION

**1.1. Boundary value problem with measures.** We start from a differential operator

$$(1.1) \quad Lu = \sum_{i,j} a_{ij} \nabla_i \nabla_j u + \sum_i b_i \nabla_i u$$

( $\nabla_i$  stands for the partial derivative with respect to  $x_i$ ) in a bounded smooth domain  $D$  of  $\mathbb{R}^d$  with coefficients subject to conditions:

1.1.A. (Uniform ellipticity) There exists a constant  $\varkappa > 0$  such that

$$\sum_{i,j} a_{ij} \lambda_i \lambda_j \geq \varkappa \sum_i \lambda_i^2 \quad \text{for all } x \in D, \lambda_1, \dots, \lambda_d \in \mathbb{R},$$

1.1.B.  $a_{ij} \in C^{2,\lambda}(\bar{D})$ ,  $b_i \in C^{1,\lambda}(\bar{D})$ .<sup>1</sup>

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Received by the editors March 29, 1995.

1991 *Mathematics Subject Classification*. Primary 60J60, 35J65; Secondary 60J80, 31C15, 60J25, 60J55, 31C45, 35J60.

Partially supported by National Science Foundation Grant DMS-9301315

<sup>1</sup>We follow standard notation in P.D.E. (see, e.g., [21]). [Smooth domain means a domain of class  $C^{2,\lambda}$ .]

The classical boundary value problem

$$(1.2) \quad \begin{aligned} -Lv + v^\alpha &= \rho && \text{in } D, \\ v &= \sigma && \text{on } \partial D \end{aligned}$$

(with Hölder continuous  $\rho$  and continuous  $\sigma$ ) is equivalent to an integral equation

$$(1.3) \quad v(x) + \int_D g(x, y)v(y)^\alpha dy = h(x)$$

where

$$(1.4) \quad h(x) = \int_D g(x, y)\rho(y)dy + \int_{\partial D} k(x, y)\sigma(y)a(dy),$$

$g(x, y)$  is Green's function,  $k(x, y)$  is the Poisson kernel of  $L$  in  $D$  and  $a(dy)$  is the surface area on  $\partial D$ . We interpret  $v$  as a (generalized) solution of the problem

$$(1.5) \quad \begin{aligned} -Lv + v^\alpha &= \eta && \text{on } D, \\ v &= \nu && \text{on } \partial D \end{aligned}$$

involving two measures  $\eta$  and  $\nu$  if the equation (1.3) holds with

$$(1.6) \quad h(x) = \int_D g(x, y)\eta(dy) + \int_{\partial D} k(x, y)\nu(dy).$$

In Theorem 1.1, we establish sufficient conditions on  $\eta$  and  $\nu$  under which problem (1.5) has a solution. Necessary conditions are established in Theorem 1.2. The equivalence of both sets of conditions follows from results in [17]. [Theorems 1.1 and 1.2 are still valid if  $D$  is not smooth. However, in general, the equivalence of conditions imposed on  $\nu$  in the two theorems is not proved.]

Particular cases of problem (1.5) have been studied before. The case  $\nu = 0$  was treated in [2] and the case  $\eta = 0$  was considered in [18]. Even earlier, Gmira and Véron [22] have investigated a class of functions  $\psi$  such that the problem

$$\begin{aligned} \Delta v &= \psi(v) && \text{on } D, \\ v &= \nu && \text{on } \partial D \end{aligned}$$

has a solution for every finite measure  $\nu$ . This class contains  $\psi(v) = v^\alpha$  with  $(\alpha + 1)/(\alpha - 1) > d$ .

**1.2.  $L$ -diffusions.** Suppose  $D$  is a bounded smooth domain and that  $L$  satisfies conditions 1.1.A,B. Then there exists<sup>2</sup> a strictly positive function  $p_t(x, y), t > 0, x, y \in D$  such that:

1.2.A. If  $f$  is a continuous function on  $D$  with compact support and if

$$(1.7) \quad u_t(x) = \int_D p_t(x, y)f(y)dy,$$

<sup>2</sup>This is proved (under weaker conditions on  $L$ ) in Chapter 1 of [19].

then

$$(1.8) \quad \frac{\partial u_t(x)}{\partial t} = Lu_t(x),$$

$$(1.9) \quad u_t(x) \rightarrow f(x) \quad \text{as } t \rightarrow 0$$

and

$$(1.10) \quad u_t(x) \rightarrow 0 \quad \text{as } x \rightarrow z \in \partial D.$$

(All partial derivatives of  $p$  which appear in (1.8) are continuous in  $(t, x, y)$ .)

Function  $p_t(x, y)$  has the following properties:

1.2.B. For all  $s, t > 0, x, z \in D$ ,

$$\int_D p_s(x, y) dy p_t(y, z) = p_{s+t}(x, z).$$

1.2.C. For all  $t > 0, x \in D$ ,

$$\int_D p_t(x, y) dy \leq 1.$$

Therefore  $p_t(x, dy) = p_t(x, y)dy$  is a Markov transition function. It is well-known (see, e.g., [6]) that there exists a continuous Markov process  $\xi = (\xi_t, \Pi_x)$  in  $D$  with this transition function. We call it an  $L$ -diffusion. If  $\zeta$  is the life time of  $\xi$ , then  $\xi_{\zeta-}$  belongs to  $\partial D$ . By setting  $\xi_t = \xi_{\zeta-}$  for  $t \geq \zeta$ , we define an  $L$ -diffusion stopped at the exit from  $D$ . Note that  $\zeta$  can be interpreted as the first exit time of this process from  $D$ ; often we use the notation  $\tau$  for it.

Now suppose that  $E$  is an arbitrary domain in  $\mathbb{R}^d$  and that  $L$  is a differential operator in  $E$  which satisfies conditions 1.1.A, B in each bounded domain  $D$  with  $\bar{D} \subset E$ . Consider a sequence of bounded smooth domains  $D_n$  such that  $\bar{D}_n \subset D_{n+1}$  and  $\bigcup D_n = E$ . The corresponding functions  $p_t^n(x, y)$  increase monotonically and they tend to a limit  $p_t(x, y)$  which does not depend on the choice of  $D_n$  (this follows from [19, Ch. 1]). There exists a continuous Markov process  $\xi$  in  $E$  with the transition function  $p_t(x, dy) = p_t(x, y)dy$  (see, e.g., [6]). We call it an  $L$ -diffusion in  $E$ .

**1.3.  $G$ -equation.** Markov semigroup, Green's function  $g$  and Green's operator  $G$  for an  $L$ -diffusion  $\xi$  are defined by the formulae

$$(1.11) \quad T_t f(x) = \int_E p_t(x, dy) f(y),$$

$$(1.12) \quad g(x, y) = \int_0^\infty p_t(x, y) dt$$

and

$$(1.13) \quad Gf(x) = \int_0^\infty T_t f(x) dt = \int_E g(x, y) f(y) dy.$$

A positive Borel function  $h$  is called *excessive* if, for all  $x \in E$ ,  $T_t h(x) \leq h(x)$  and  $T_t h(x) \rightarrow h(x)$  as  $t \rightarrow 0$ . The case  $h(x) = \infty$  for all  $x$  is excluded. Since  $p_t(x, y) > 0$ , the set  $\{x : h(x) = \infty\}$  has the Lebesgue measure 0. There exist only two possibilities: either  $g(x, y) = \infty$  for all  $x, y \in E$  or  $g(x, y) < \infty$  for  $x \neq y$ . In the first case, constants are the only excessive functions and all problems treated in this paper are trivial. Therefore we concentrate on the second case.

Let  $1 < \alpha \leq 2$ . One of our goals is to find for which excessive functions  $h$  the equation

$$(1.14) \quad v + G(v^\alpha) = h$$

(we call it *G-equation*) has a solution.<sup>3</sup> Note that if (1.14) holds almost everywhere, then

$$(1.15) \quad \tilde{v} = \begin{cases} h - G(v^\alpha) & \text{on } \{h < \infty\}, \\ \infty & \text{on } \{h = \infty\} \end{cases}$$

satisfies (1.14) everywhere.

Fix an arbitrary point  $c \in E$  and put

$$(1.16) \quad k(x, y) = \begin{cases} \frac{g(x, y)}{g(c, y)} & \text{if } y \neq c, \\ 0 & \text{otherwise.} \end{cases}$$

There exist [see, e.g., [7]] a continuous injective mapping from  $E$  to a compact metrizable space  $\hat{E}$  and an extension of  $k(x, y)$  to  $E \times \hat{E}$  such that:

- 1.3.A. For every  $x \in E$ ,  $k(x, y) \rightarrow k(x, z)$  as  $y \rightarrow z \in \hat{E} \setminus E$ .
- 1.3.B. If  $k(\cdot, y_1) = k(\cdot, y_2)$ , then  $y_1 = y_2$ .

We call  $\hat{E}$  the *Martin space*. The set  $\partial E = \hat{E} \setminus E$  is called the *Martin boundary*. For every  $y \in E$ ,  $h(x) = g(x, y)$  is an extremal excessive function.<sup>4</sup> We denote by  $E^*$  the set of all  $y \in \partial E$  such that  $h(x) = k(x, y)$  is an extremal excessive function. ( $E^*$  is a Borel subset of  $\partial E$ .) Every excessive function  $h$  has a unique representation

$$(1.17) \quad h = G\eta + K\nu$$

where  $\eta$  is a  $\sigma$ -finite measure on  $E$ ,  $\nu$  is a finite measure on  $E^*$  and

$$(1.18) \quad G\eta(x) = \int_E g(x, y)\eta(dy), \quad K\nu(x) = \int_{E^*} k(x, y)\nu(dy)$$

(cf. (1.6)). Note that  $\eta(\Gamma) < \infty$  for every compact  $\Gamma \subset E$ . Indeed, if  $h(x_0) < \infty$ , then  $a\eta(\Gamma) \leq G\eta(x_0) < \infty$  where  $a = \inf_{y \in \Gamma} g(x_0, y) > 0$ .

Function  $f = K\nu$  is  $L$ -harmonic, that is it satisfies equation  $Lf = 0$ .  $L$ -harmonic functions can be also characterized by the following mean value property: for every bounded open set  $D$  such that  $\bar{D} \subset E$ ,

$$(1.19) \quad \Pi_x f(\xi_\tau) = f(x) \quad \text{for all } x \in E$$

where  $\tau$  is the first exit time from  $D$ .

<sup>3</sup>(Cf. (1.3).) When speaking about solutions of  $G$ -equation, we always mean positive solutions.  
<sup>4</sup>This means if  $h = h_1 + h_2$  and if  $h_1, h_2$  are excessive, then  $h_1, h_2$  are proportional to  $h$ .

We fix  $\alpha \in (1, 2]$ . *Green's capacity*  $CG$  is defined on compact subsets of  $E$  by the formula

$$(1.20) \quad CG(\Gamma) = \sup\{\eta(\Gamma) : \int_E g(c, x)dx \left[ \int_\Gamma g(x, y)\eta(dy) \right]^\alpha \leq 1\}.$$

Analogously, the *Martin capacity*  $CK$  is defined on compact subsets of  $\partial E$  by the formula

$$(1.21) \quad CK(\Gamma) = \sup\{\nu(\Gamma) : \int_E g(c, x)dx \left[ \int_\Gamma k(x, y)\nu(dy) \right]^\alpha \leq 1\}.$$

[By a Choquet theorem [3],  $CG$  and  $CK$  can be extended to all analytic subsets of  $E$  and  $E^*$ .] If  $\eta$  is a measure on  $E$ , then writing  $\eta \prec CG$  means that  $\eta(\Gamma) = 0$  if  $CG(\Gamma) = 0$ . Writing  $\nu \prec CK$  has an analogous meaning.

It follows from the results in Sections 2 and 3 that:

**Theorem 1.1.** *If  $h = G\eta + K\nu$  and if*

$$(1.22) \quad \eta \prec CG, \quad \nu \prec CK,$$

*then  $G$ -equation (1.14) has a solution  $v$  which is defined uniquely on the set  $E(h) = \{h < \infty\}$ .*

**1.4. Operators  $\mathcal{G}$  and  $\mathcal{K}$ .** Let  $\xi$  be an  $L$ -diffusion in a bounded smooth domain  $D$  stopped at the first exit time  $\tau$  from  $D$ . We introduce operators  $\mathcal{G}$  and  $\mathcal{K}$  acting on functions with the domain  $S = \mathbb{R}_+ \times E$  by the formulae

$$(1.23) \quad \mathcal{G}f(t, x) = \int_0^t ds \int_D p_s(x, dy) f(t - s, y) = \Pi_x \int_0^{\tau \wedge t} f(t - s, \xi_s) ds,$$

and<sup>5</sup>

$$(1.24) \quad \mathcal{K}f(t, x) = \Pi_x f(t - \tau, \xi_\tau).$$

If  $f(t, x) = f(x)$  does not depend on  $t$ , then

$$(1.25) \quad \begin{aligned} \mathcal{G}f(t, x) &= \Pi_x \int_0^t f(\xi_s) ds \rightarrow Gf(x), \\ \mathcal{K}f(t, x) &= \Pi_x f(\xi_\tau) 1_{\tau \leq t} \rightarrow Kf(x) \end{aligned}$$

as  $t \rightarrow \infty$ . Here  $G$  is defined by (1.13) and<sup>6</sup>

$$(1.26) \quad Kf(x) = \Pi_x f(\xi_\tau).$$

<sup>5</sup>We extend each function to  $\mathbb{R} \times E$  by setting it equal to zero for negative  $t$ .

<sup>6</sup>Operator (1.26) is a particular case of the operator  $K$  defined by (1.18): if  $E = D$  is a bounded smooth domain, then  $E^* = \partial D$  and  $\Pi_x f(\xi_\tau) = \int_{\partial D} k(x, y)\nu(dy)$  for  $\nu(dy) = f(y)a(dy)$  where  $k$  is the Poisson kernel and  $a$  is the surface area on  $\partial D$ . Writing the same letter for both operators should cause no confusion since one operator is applied only in the context of a smooth domain  $D$  and the second one only in the context of the Martin boundary of  $E$ .

The boundary of a cylinder  $Q = \mathbb{R}_+ \times D$  consists of the side surface  $A = (0, \infty) \times \partial D$  and the bottom  $B = \{0\} \times \bar{D}$ . Besides the boundary value problem (1.2), we consider also a boundary value problem for a parabolic equation

$$(1.27) \quad \begin{aligned} \frac{\partial u}{\partial t} - Lu + u^\alpha &= \rho && \text{in } Q, \\ u &= \sigma && \text{on } A, \\ u &= 0 && \text{on } B. \end{aligned}$$

If  $\rho$  and  $\sigma$  are Hölder continuous, then (1.27) is equivalent to the integral equation

$$(1.28) \quad u + \mathcal{G}(u^\alpha) = \mathcal{G}\rho + \mathcal{K}\sigma.$$

**1.5. Superdiffusions.** Let  $\xi = (\xi_t, \Pi_x)$  be a Markov process in a measurable space  $(E, \mathcal{B})$  and let  $\mathcal{M} = \mathcal{M}(E)$  be the space of all finite measures on  $\mathcal{B}$ . A  $(\xi, \alpha)$ -superprocess is a Markov process  $X = (X_t, P_\mu)$  in  $\mathcal{M}$  which satisfies the condition: for every  $\mu \in \mathcal{M}$  and every positive  $\mathcal{B}$ -measurable function  $f$ ,

$$(1.29) \quad \begin{aligned} P_\mu \exp\langle -f, X_t \rangle &= \exp\langle -u_t, \mu \rangle, \\ u_t(x) + \Pi_x \int_0^t u_{t-s}(\xi_s)^\alpha ds &= \Pi_x f(\xi_t). \end{aligned}$$

We say that  $X$  is an  $(L, \alpha)$ -superdiffusion if  $X$  is a right process and  $\xi$  is an  $L$ -diffusion. The existence of such processes for  $1 < \alpha \leq 2$  is proved, for instance, in [13] (we refer to [8] and [9] for the history of this subject starting from the pioneering work of Watanabe and Dawson).

In the theory of diffusion, a fundamental role is played by random points  $\xi_\tau$  corresponding to the first exit times from open sets  $D$ . An analogous role in the theory of superdiffusion is played by exit measures  $X_D$ . In contrast to  $\xi_\tau$  which can be defined through  $\xi_t$ , it is impossible, in general, to define  $X_D$  in terms of  $X_t$ . The probability distribution of  $X_D$  is defined by formulae similar to (1.29):

$$(1.30) \quad \begin{aligned} P_\mu \exp\langle -f, X_D \rangle &= \exp\langle -u, \mu \rangle, \\ u(x) + \Pi_x \int_0^\tau u(\xi_s)^\alpha ds &= \Pi_x f(\xi_\tau). \end{aligned}$$

The joint probability distribution of  $X_{t_1}, \dots, X_{t_n}$  is determined by (1.29) and the Markov property of  $X$ . Analogously, the joint probability distribution of  $X_{D_1}, \dots, X_{D_n}$  can be evaluated by using (1.30) and the following Markov property: for every positive  $\mathcal{F}_{\supset D}$ -measurable  $Y$ ,

$$(1.31) \quad P_\mu \{Y | \mathcal{F}_{\supset D}\} = P_{X_D} Y$$

where  $\mathcal{F}_{\supset D}$  is the  $\sigma$ -algebra generated by  $X_{D'}$  with  $D' \subset D$  and  $\mathcal{F}_{\supset D}$  the  $\sigma$ -algebra generated by  $X_{D''}$  with  $D'' \supset D$ .

We need even a wider class of exit measures [for instance, measures corresponding to the exit from  $D$  before time  $t$ ]. We introduce a random measure  $(X_Q, P_\mu)$  for

an arbitrary open set  $Q$  in  $S = \mathbb{R}_+ \times E$  and an arbitrary finite measure  $\mu$  on the Borel  $\sigma$ -algebra in  $S$ . Its probability distribution is defined by the formulae

$$(1.32) \quad \begin{aligned} P_\mu \exp\langle -f, X_Q \rangle &= \exp\langle -u, \mu \rangle, \\ u(r, x) + \Pi_{r,x} \int_r^{\tau^r} u(s, \xi_s)^\alpha ds &= \Pi_{r,x} f(\tau^r, \xi_{\tau^r}) \end{aligned}$$

where

$$(1.33) \quad \tau^r = \inf\{t : t \geq r, (t, \xi_t) \notin Q\}$$

is the first, after  $r$ , exit time of  $\xi$  from  $Q$  and  $\Pi_{r,x} Y = \Pi_x \theta_{-r} Y$  describes a Markov process with transition function  $p_t(x, dy)$  which starts at time  $r$  from point  $x$ . The joint probability distribution of  $X_{Q_1}, \dots, X_{Q_n}$  is determined by (1.33) and by the property: for every positive  $\mathcal{F}_{\supset Q}$ -measurable  $Y$ ,

$$(1.34) \quad P_\mu\{Y | \mathcal{F}_{\subset Q}\} = P_{X_Q} Y$$

where  $\mathcal{F}_{\subset Q}$  is the  $\sigma$ -algebra generated by  $X_{Q'}$  with  $Q' \subset Q$  and  $\mathcal{F}_{\supset Q}$  the  $\sigma$ -algebra generated by  $X_{Q''}$  with  $Q'' \supset Q$ .

The existence of a family  $(X_Q, P_\mu)$  subject to conditions (1.32) and (1.34) is proved in [8].

Formula  $j_r(x) = (r, x)$  defines a mapping from  $E$  to  $S$ . If  $\mu$  is a measure on  $E$ , then  $j_r(\mu)$  is a measure on  $S$  concentrated on  $\{r\} \times E$ . We set  $P_{j_r(\mu)} = P_{r,\mu}$ . It follows from (1.32) that

$$(1.35) \quad \begin{aligned} P_{r,\mu} \exp\langle -f, X_Q \rangle &= \exp\left\{ - \int_E u(r, x) \mu(dx) \right\}, \\ u(r, x) + \Pi_x \int_0^\tau u(s+r, \xi_s)^\alpha ds &= \Pi_x f(\tau+r, \xi_\tau). \end{aligned}$$

Formulae (1.29) and (1.30) can be considered as special cases of (1.35) if we identify  $X_t$  and  $X_D$  with the exit measures from  $S_{<t} = [0, t) \times E$  and from  $\mathbb{R}_+ \times D$ , projected on  $E$ .

If  $\tau$  is the first exit time from  $D$ , then  $\tau(t) = \tau \wedge t$  is the first exit time from  $Q_t = [0, t) \times D$ . We call the process  $\tilde{X}_t = X_{Q_t}$  an  $(L, \alpha)$ -superdiffusion stopped at the exit from  $D$ . If  $f(t, x) = f(x)$  vanishes outside  $D$  and if  $v_t(x) = -\log P_x \exp\langle -f, \tilde{X}_t \rangle$ , then  $v_{t-r}(x) = -\log P_{r,x} \exp\langle -f, X_{Q_t} \rangle$  and (1.35) implies

$$(1.36) \quad \begin{aligned} P_\mu \exp\langle -f, \tilde{X}_t \rangle &= \exp\langle -v_t, \mu \rangle, \\ v_t(x) + \Pi_x \int_0^{\tau(t)} v_{t-s}(\xi_s)^\alpha ds &= \Pi_x f(\xi_{\tau(t)}). \end{aligned}$$

Formula (1.36) can be obtained from (1.29) by replacing  $\xi$  with an  $L$ -diffusion stopped at the exit from  $D$ .

The shift operators  $\theta_t$  of a time-homogeneous process  $\xi$  induce analogous operators for  $X$  (see [14, Section 1.12]). We have  $X_s(\theta_t \omega) = X_{s+t}(\omega)$  and, if  $Q = \mathbb{R}_+ \times D$ , then  $X_Q(\theta_t \omega, \Gamma) = X_{Q_t}(\omega, \Gamma+t)$  where  $Q_t = S_{<t} \cup \{\gamma_t(Q)\}$  with  $\gamma_t(r, x) = (r+t, x)$ .

It follows from (1.35) that

$$(1.37) \quad P_\mu \int_Q f(s, x) X_Q(ds, dx) = \int \mu(dx) \Pi_x f(\tau, \xi_\tau)$$

[it is sufficient to apply (1.35) to  $\lambda f$  and to take the derivative with respect to  $\lambda$  at  $\lambda = 0$ ].

The following result (see Theorem I.1.8 in [8]) provides a link between superprocesses and the  $G$ -equation.

**Theorem A.** *Suppose that  $\tilde{X}$  is an  $(L, \alpha)$ -superdiffusion stopped at the exit from  $D$ ,  $\rho$  is a positive Borel function on  $\bar{D}$  vanishing on  $\partial D$  and  $\sigma$  is a positive Borel function on  $\partial D$ . Then*

$$(1.38) \quad v(x) = -\log P_x \exp \left\{ - \left[ \int_0^\infty \langle \rho, \tilde{X}_t \rangle dt + \langle \sigma, X_D \rangle \right] \right\}$$

is a solution of the  $G$ -equation (1.14) where  $G$  is Green's operator for  $L$ -diffusion in  $D$ ,  $K$  is given by (1.26) and<sup>7</sup>

$$(1.39) \quad h = G\rho + K\sigma.$$

Moreover, for every  $\mu \in \mathcal{M}(D)$ ,

$$(1.40) \quad P_\mu \exp \left\{ - \left[ \int_0^\infty \langle \rho, \tilde{X}_t \rangle dt + \langle \sigma, X_D \rangle \right] \right\} = e^{-\langle v, \mu \rangle}.$$

We also need another implication of Theorem I.1.8 in [8] [cf. Theorem 1.1 in [16]].

**Theorem B.** *Let  $\tilde{X}$ ,  $D$  and  $\rho$  be the same as in Theorem A and let  $\sigma$  be a positive Borel function on  $\bar{D}$  vanishing on  $D$ . Then*

$$(1.41) \quad u(t, x) = -\log P_x \exp \left\{ - \left[ \int_0^t \langle \rho, \tilde{X}_s \rangle ds + \langle \sigma, \tilde{X}_t \rangle \right] \right\}$$

is a solution of the equation (1.28). Moreover, for every  $\mu \in \mathcal{M}(D)$ ,<sup>8</sup>

$$(1.42) \quad P_\mu \exp \left\{ - \left[ \int_0^t \langle \rho, \tilde{X}_s \rangle ds + \langle \sigma, \tilde{X}_t \rangle \right] \right\} = \exp\langle -u^t, \mu \rangle.$$

The range  $\mathcal{R}$  of a superprocess  $X$  is the smallest closed subset of  $E$  which supports all measures  $X_t$  (it supports, a.s., every exit measure  $X_D$ ). We denote by  $\mathcal{R}^*$  the minimal closed subset of the Martin space  $\hat{E}$  which supports all measures  $X_t$ . A set  $\Gamma \subset E$  is called  $\mathcal{R}$ -polar if  $P_x\{\mathcal{R} \cap \Gamma \neq \emptyset\} = 0$  for all  $x \notin \Gamma$ . A subset  $\Gamma$  of the Martin boundary  $\partial E$  is called  $\mathcal{R}^*$ -polar if  $P_x\{\mathcal{R}^* \cap \Gamma \neq \emptyset\} = 0$  for all  $x \in E$ .<sup>9</sup>

We prove in Section 4:

<sup>7</sup>For  $x \in \partial D$ ,  $u(x) = h(x) = \sigma(x)$ .

<sup>8</sup>We set  $u^t(x) = u(t, x)$ .

<sup>9</sup> $\mathcal{R}^*$ -polarity is introduced only on  $\partial E$  because  $\mathcal{R}^* \cap \Gamma = \mathcal{R} \cap \Gamma$  for every compact  $\Gamma \subset E$ .



**Theorem 1.2.** *If  $h = G\eta + K\nu$  and if the  $G$ -equation (1.14) has a solution, then  $\eta$  does not charge  $\mathcal{R}$ -polar sets and  $\nu$  does not charge  $\mathcal{R}^*$ -polar sets.*

We say that  $\Gamma \subset E$  is  $G$ -polar if  $CG(\Gamma) = 0$  and that  $\Gamma \subset \partial E$  is  $K$ -polar if  $CK(\Gamma) = 0$ . By Theorem 1.1 in [17] [cf. Theorem 1.6 in [12]], the classes  $\mathcal{R}$ -polar and  $G$ -polar sets coincide. Theorems 1.1 and 1.2 imply:

1.5.A. All  $\mathcal{R}^*$ -polar sets are  $K$ -polar.

Indeed, if  $\Gamma$  is compact and if  $CK(\Gamma) > 0$ , then by (1.21), there exists a measure  $\nu \neq 0$  concentrated on  $\Gamma$  such that

$$\int_E g(c, x) dx \left[ \int k(x, y) \nu(dy) \right]^\alpha < \infty$$

which implies

$$\int_E g(c, x) dx \left[ \int_B k(x, y) \nu(dy) \right]^\alpha < \infty$$

for every  $B$ . Hence  $\nu(B) = 0$  for all  $K$ -polar sets  $B$ . By Theorems 1.1 and 1.2,  $\nu(B) = 0$  for all  $\mathcal{R}^*$ -polar sets  $B$ . Therefore  $\Gamma$  is not  $\mathcal{R}^*$ -polar.

If  $\xi$  is an  $L$ -diffusion in a bounded smooth domain of  $\mathbb{R}^d$ , then a stronger result than 1.5.A follows from Theorem 1.2 in [17]:

1.5.B. The classes of  $\mathcal{R}^*$ -polar and  $K$ -polar sets coincide.

It remains an open problem if 1.5.B holds in the general case. If it holds for a diffusion  $\xi$  and if  $X$  is the corresponding superdiffusion, then each of Theorems 1.1–1.2 gives necessary and sufficient conditions on  $h$  for the existence of a solution of (1.14).

**1.6. Additive functionals.** Let  $X$  be a superdiffusion. We denote by  $\mathcal{F}_t$  the  $\sigma$ -algebra in  $\Omega$  generated by the exit measures  $X_Q$  for all  $Q \subset S_{<t}$ . A function  $A_t(\omega)$  from  $[0, \infty] \times \Omega$  to  $[0, \infty]$  is called an *additive functional* of  $X$  if:

1.6.A. For every  $\omega$ ,  $A_t$  is monotone increasing in  $t$ .

1.6.B.  $A_t$  is measurable with respect to the completion of  $\mathcal{F}_t$  with respect to all measures  $P_\mu$ ,  $\mu \in \mathcal{M}(E)$ .

1.6.C. For every  $\omega$ ,  $A_t$  is left continuous in  $t$ .

1.6.D.  $A_{s+t} = A_s + \theta_s A_t$  for all pairs  $s, t$  and all  $\omega$ .<sup>10</sup>

All these conditions hold for

$$(1.43) \quad A_t = \int_0^t \langle \rho, X_s \rangle ds$$

where  $\rho$  is an arbitrary positive Borel function. By a limit procedure, we construct, starting from (1.43), a class of functionals for which a weaker form of condition 1.6.D holds.

We say that a set  $\Lambda$  is  $\xi$ -polar if  $\Pi_x \{ \xi_t \notin \Lambda \text{ for all } t > 0 \} = 1$  for all  $x$ . All  $\xi$ -polar sets have the Lebesgue measure 0. A subset  $\mathcal{N}$  of  $\mathcal{M}(E)$  is called *exceptional* if the set  $\{ x : \delta_x \in \mathcal{N} \}$  is  $\xi$ -polar and if, for all stopped superdiffusions  $\tilde{X}$  and for every  $\mu \notin \mathcal{N}$ ,  $P_\mu \{ \tilde{X}_t \notin \mathcal{N} \text{ for all } t \} = 1$ .

<sup>10</sup>Let  $\beta(\omega) = \sup \{ t : A_t(\omega) < \infty \}$ . Then there exists a unique measure  $A(\omega, dt)$  on  $[0, \beta(\omega))$  such that  $A[0, t) = A_t$  for all  $t < \beta$ .

If  $h$  is an arbitrary excessive function, then the set  $\Lambda(h) = \{x : h(x) = \infty\}$  is  $\xi$ -polar and the set  $\mathcal{N}(h) = \{\mu : \langle h, \mu \rangle = \infty\}$  is exceptional.

We say that  $A$  is an *additive functional with an exceptional set  $\mathcal{N}$*  if  $A$  satisfies 1.6.A, B, C and:

$$1.6.D^*. A_{s+t} = A_s + \theta_s A_t \text{ for all } s, t, \omega \in \Omega_0 \text{ and } P_\mu(\Omega_0) = 1 \text{ for all } \mu \notin \mathcal{N}.$$

Two additive functionals  $A$  and  $\tilde{A}$  are called *equivalent* if there exists an exceptional set  $\mathcal{N}$  such that  $P_\mu\{A_t = \tilde{A}_t \text{ for all } t\} = 1$  for all  $\mu \notin \mathcal{N}$ .

Let  $h$  be an excessive function. An additive functional  $A$  with an exceptional set  $\mathcal{N}$  is called a *linear additive functional with potential  $h$*  if, for every  $\mu \notin \mathcal{N}$ ,

$$(1.44) \quad P_\mu A_\infty = \langle h, \mu \rangle.$$

If  $G\rho(x) < \infty$  for some  $x$ , then the additive functional (1.43) is linear with potential  $G\rho$  (condition (1.44) holds for every  $\mu$ ).

**Theorem 1.3.** *If  $h = G\eta + K\nu$  and if  $\eta \prec CG, \nu \prec CK$ , then  $h$  is the potential of a linear additive functional  $A$  of  $X$  with an exceptional set  $\mathcal{N}$ . For every  $\mu \notin \mathcal{N}$ ,*

$$(1.45) \quad P_\mu e^{-A_\infty} = e^{-\langle v, \mu \rangle}$$

where  $v$  is a solution of the  $G$ -equation (1.14).

Theorem 1.3 is proved in Section 3. Theorem 1.1 follows immediately from Theorem 1.3 and a uniqueness Theorem 2.1.

*Remark.* The construction of  $A$  in Section 3 implies that  $A$  depends linearly on  $h$ . More precisely, if  $A^i$  corresponds to  $h^i$ , then, for every  $c_1, c_2 \geq 0$ , the functional  $A$  corresponding to  $c_1 h^1 + c_2 h^2$  is equivalent to  $c_1 A^1 + c_2 A^2$ . Therefore, if  $h, \tilde{h}$  and  $h - \tilde{h}$  are excessive functions and if  $v, \tilde{v}$  are the solutions of (1.14) corresponding to  $h$  and  $\tilde{h}$ , then  $\tilde{v} \leq v$  outside a  $\xi$ -polar set.

In Section 4 we establish:

**Theorem 1.4.** *If  $h$  is the potential of a linear additive functional with an exceptional set  $\mathcal{N}$ , then  $h = G\eta + K\nu$  with  $\eta$  vanishing on all  $\mathcal{R}$ -polar sets and  $\nu$  vanishing on all  $\mathcal{R}^*$ -polar sets.*

Linear additive functionals of superprocesses have been introduced in [11] (in a time-inhomogeneous setting). There a linear additive functional corresponding to a *bounded* excessive function  $h$  was constructed for a  $(\xi, 2)$ -superprocess where  $\xi$  is an arbitrary right Markov process. (No exceptional set is needed in this case.)

The case of an  $(L, \alpha)$ -superdiffusion with an arbitrary  $\alpha \in (1, 2]$  was investigated in [15]. For  $h = G\eta$  with  $\eta \prec CG$ , a functional  $A$  was constructed, subject to conditions 1.6.A, B with the property, for every  $\mu \in \mathcal{M}^0$ ,

$$1.6.D^{**}. A_{s+t} = A_s + \theta_s A_t \text{ } P_\mu\text{-a.s. for all } s, t.$$

Here  $\mathcal{M}^0$  is the set of measures of the form  $\mu(dx) = \rho(x)dx$  with  $\int \rho(x)^{\alpha'} dx < \infty$  where  $\alpha' = \alpha/(\alpha - 1)$ . Condition (1.44) was proved also only for  $\mu \in \mathcal{M}^0$ . (Note that  $\mathcal{M}^0$  is not the complement of an exceptional set!)

Recent results of Le Gall [23] on additive functionals of the Brownian snake can be translated into our language as follows: if  $h = K\nu$  with  $\nu \prec CK$ , then there exists a functional of an  $(\Delta, 2)$ -superdiffusion which satisfies conditions 1.6.A, B, C, (1.44) and 1.6.D<sup>\*\*</sup> for  $P_x$  for almost all  $x$ .

Additive functionals with an exceptional set have been introduced, in a different context, by Fukushima [20]. In his setting,  $X$  is a symmetric Markov process associated with a Dirichlet form and an exceptional set is a polar subset of the state space (in the sense of theory of Dirichlet spaces).

**1.7.** We have the following logical implications:  $\mathcal{A} \implies \mathcal{B} \implies \mathcal{C} \implies \mathcal{D}$  where:

$\mathcal{A}$  :  $h = G\eta + K\nu$  with  $\eta \prec CG, \nu \prec CK$ ;

$\mathcal{B}$  :  $h$  is the potential of a linear additive functional  $A$  with an exceptional set  $\mathcal{N}$ .  
 Moreover for every  $\mu \notin \mathcal{N}$ ,

$$P_\mu e^{-A_\infty} = e^{-\langle v, \mu \rangle}$$

where  $v$  is a solution of the  $G$ -equation (1.14).

$\mathcal{C}$  :  $h$  is the potential of a linear additive functional  $A$ .

$\mathcal{D}$  :  $h = G\eta + K\nu$  with  $\eta$  vanishing on all  $\mathcal{R}$ -polar sets and  $\nu$  vanishing on all  $\mathcal{R}^*$ -polar sets.

We get  $\mathcal{A} \implies \mathcal{B}$  by Theorem 1.3 and  $\mathcal{C} \implies \mathcal{D}$  by Theorem 1.4. The implication  $\mathcal{B} \implies \mathcal{C}$  is trivial.

If 1.5.B holds for a diffusion  $\xi$  and if  $X$  is the corresponding superdiffusion, then  $\mathcal{D} \implies \mathcal{A}$  and all four statements  $\mathcal{A}, \mathcal{B}, \mathcal{C}$  and  $\mathcal{D}$  are equivalent. In particular, this is true if  $\xi$  is an  $L$ -diffusion in a bounded smooth domain  $D$ . This also is true for an arbitrary domain  $E$  if we consider only excessive functions  $h = G\eta$  (in other words if we set  $\nu = 0$ ).

**Acknowledgments.** The authors are greatly indebted to N. V. Krylov. The first author, during his visit to the University of Minnesota in January 1995, had a number of stimulating conversations with Krylov on the subject of quasi-linear PDEs. A result of these discussions was a considerable simplification of proofs in Theorems 2.1 and 2.2.

## 2. G-EQUATION

### 2.1. Monotonicity and uniqueness.

**Theorem 2.1.** *Let  $\xi$  be an  $L$ -diffusion stopped at the first exit time  $\tau$  from an open set  $D$ , and let  $G, K$  be given by (1.13),(1.26). Suppose that  $\eta$  is a measure on  $D$ ,  $u, \hat{u}, \sigma \geq 0$  and, for almost all  $x$ ,*

$$(2.1) \quad \hat{u} + G(\hat{u}^\alpha) = u + G(u^\alpha) + G\eta + K\sigma < \infty.$$

*Then  $\hat{u} \geq u$  at every point of the set (2.1). If  $\eta = 0$  and  $\sigma = 0$ , then  $\hat{u} = u$  on the same set.*

An analogous result holds for the  $\mathcal{G}$ -equation. For every measure  $\eta$  on  $S$  we put

$$(2.2) \quad \mathcal{G}\eta(t, x) = \int_0^t \int_D p_{t-s}(x, y)\eta(ds, dy)$$

(cf. (1.23)). If  $\eta(ds, dy) = ds\eta(dy)$ , then

$$(2.3) \quad \mathcal{G}\eta(t, x) = \int_0^t ds \int_D p_s(x, y)\eta(dy) \rightarrow G\eta(x)$$

as  $t \rightarrow \infty$ .

**Theorem 2.1\*.** *Let  $\xi$  be the same as in Theorem 2.1 and let  $\mathcal{G}, \mathcal{K}$  be given by (2.2), (1.24). Suppose that  $\eta$  is a measure on  $S$ ,  $u, \hat{u}, \sigma \geq 0$  and, for almost all  $t, x$ ,*

$$(2.4) \quad \hat{u} + \mathcal{G}(\hat{u}^\alpha) = u + \mathcal{G}(u^\alpha) + \mathcal{G}\eta + \mathcal{K}\sigma < \infty.$$

*Then  $\hat{u} \geq u$  at every point of the set (2.4). If  $\eta = 0$  and  $\sigma = 0$ , then  $\hat{u} = u$  on the same set.*

We use as a tool a process  $(\xi_s, \Pi_x^{t,y})$  with  $x, y \in D$ . Its finite-dimensional distributions are given by the formula

$$(2.5) \quad \begin{aligned} & \Pi_x^{t,y} \{ \xi_{t_1} \in dy_1, \dots, \xi_{t_n} \in dy_n, t_n < t < \tau \} \\ & = p_{t_1}(x, dy_1) p_{t_2-t_1}(y_1, dy_2) \dots p_{t_n-t_{n-1}}(y_{n-1}, dy_n) p_{t-t_{n-1}}(y_n, y) \end{aligned}$$

for all  $0 < t_1 < \dots < t_n < t$ . (Here  $p_t(x, dy)$  is the transition function and  $p_t(x, y)$  is the transition density of the part of  $\xi$  in  $D$ .)<sup>11</sup>

Let  $f$  be a positive Borel function. Formula

$$(2.6) \quad p_t^\varphi(x, y) = \Pi_x^{t,y} \left\{ \exp \left\{ - \int_0^t \varphi(\xi_s) ds \right\} \right\}$$

defines the transition density of a Markov process obtained from  $\xi$  by killing with rate  $f(x)$  at point  $x$ .

Operator  $G_\varphi$  corresponding to  $p^\varphi$  by (2.2) can be expressed by formula

$$(2.7) \quad \mathcal{G}_\varphi \rho(t, x) = \Pi_x \int_0^{\tau \wedge t} ds \rho(t-s, \xi_s) \exp \left\{ - \int_0^s \varphi(t-r, \xi_r) dr \right\}.$$

We prove Theorem 2.1\*. (Proof of Theorem 2.1 is similar but simpler.) We need two lemmas.

**Lemma 2.1.** <sup>12</sup> (i) *Let  $\rho$  be a Borel function on  $S$ . Equation*

$$(2.8) \quad \mathcal{G}\rho - \mathcal{G}_\varphi \rho = \mathcal{G}_\varphi(\varphi \mathcal{G}\rho)$$

*holds on the set  $\{\mathcal{G}|\rho| < \infty\}$ .*

(ii) *If  $\eta$  is a measure on  $S$ , then*

$$(2.9) \quad \mathcal{G}\eta - \mathcal{G}_\varphi \eta = \mathcal{G}_\varphi(\varphi \mathcal{G}\eta)$$

*on the set  $\{\mathcal{G}\eta < \infty\}$ .*

(iii) *For every positive Borel  $\sigma$ , equation*

$$(2.10) \quad \mathcal{K}\sigma - \mathcal{K}_\varphi \sigma = \mathcal{G}_\varphi(\varphi \mathcal{K}\sigma)$$

*holds on the set  $\{\mathcal{K}\sigma < \infty\}$ .*

<sup>11</sup>Normalized measure  $\Pi_x^{t,y}$  can be obtained by conditioning the diffusion  $\xi$  started from point  $x$  to come at point  $y$  at time  $t$ .

<sup>12</sup>Cf. [5]. This result can be interpreted as the resolvent form of the Feynman-Kac formula.

*Proof.* 1°. It is sufficient to check (2.8) for  $\rho \geq 0$ . We use (1.23) and (2.7), the Markov property of  $\xi$ , Fubini's theorem and relation

$$\int_0^s da Y_a \exp \left\{ - \int_0^a Y_r dr \right\} = 1 - \exp \left\{ - \int_0^s Y_r dr \right\}$$

which we apply to  $Y_s = \varphi(t - s, \xi_s)$ .

2°. Put

$$(2.11) \quad \rho_\varepsilon(x) = \int_D p_\varepsilon(x, y) \eta(dy).$$

Note that

$$(2.12) \quad G\rho_\varepsilon(x) = \int_\varepsilon^\infty dt \int_D p_t(x, y) \eta(dy).$$

We get (2.9) by applying (2.8) to  $\rho_\varepsilon$  and by passing to the limit as  $\varepsilon \rightarrow 0$ .

3°. Formula (2.10) can be proved in the same way as (2.8).

**Lemma 2.2.** *Suppose that  $\varphi, \sigma \geq 0$  and that, for almost all  $t, x$ ,*

$$(2.13) \quad \mathcal{G}\eta + \mathcal{K}\sigma + \mathcal{G}|\varphi w| < \infty$$

and

$$(2.14) \quad w + \mathcal{G}(\varphi w) = \mathcal{G}\eta + \mathcal{K}\sigma.$$

Then

$$(2.15) \quad w = \mathcal{G}_\varphi\eta + \mathcal{K}_\varphi\sigma$$

at every point  $(t, x)$  where (2.13) and (2.14) hold.

*Proof.* We have

$$(2.16) \quad \mathcal{G}_\varphi(\varphi w) + \mathcal{G}_\varphi[\varphi\mathcal{G}(\varphi w)] = \mathcal{G}_\varphi(\varphi\mathcal{G}\eta) + \mathcal{G}_\varphi(\varphi\mathcal{K}\sigma).$$

On the set defined by (2.13) and (2.14), the left side in (2.16) is equal to  $\mathcal{G}(\varphi w)$  by (2.8) and, the right side is equal to  $\mathcal{G}\eta + \mathcal{K}\sigma - \mathcal{G}_\varphi\eta - \mathcal{K}_\varphi\sigma$  by (2.8) and (2.10). Therefore  $\mathcal{G}(\varphi w) = \mathcal{G}\eta + \mathcal{K}\sigma - \mathcal{G}_\varphi\eta - \mathcal{K}_\varphi\sigma$  and (2.15) follows from (2.14).

*Proof of Theorem 2.1\*.* Denote by  $\hat{S}$  the set defined by (2.4). Put  $w = \hat{u} - u$  on  $\hat{S}$  and  $w = 0$  on  $E \setminus \hat{S}$ . There exists a function  $\varphi \geq 0$  such that  $\hat{u}^\alpha - u^\alpha = \varphi w$  a.e. Equation (2.4) implies (2.14). Since  $G|\varphi w| \leq G(u^\alpha) + G(\hat{u}^\alpha) < \infty$  on  $\hat{S}$ , Theorem 2.1\* follows from Lemma 2.2.

**2.2. Properties of  $G$  and  $\mathcal{G}$ .** In this subsection we deal with operators corresponding to an  $L$ -diffusion  $\xi$  in a bounded smooth domain  $D$ . We denote by  $\|u\|$  the norm of  $u$  in  $L^1(D)$ . For a function  $f$  on  $S = \mathbb{R}_+ \times D$  and for  $b \in \mathbb{R}_+$ , we set

$$\ell_b(f) = \int_0^b \int_D |f(r, x)| dr dx.$$

We need the following results.

2.2.A. There is a constant  $C$  such that

$$\int_D g(x, y) dx \leq C \quad \text{for all } y \in D.$$

2.2.A\*. For every  $b > 0$ , there exists a constant  $C$  such that

$$\int_D p_t(x, y) dx \leq C \quad \text{for all } y \in D, 0 < t \leq b.$$

2.2.B. If  $f_n$  is a sequence of functions such that  $\ell_b(f_n)$  are bounded for every  $b$ , then the sequence  $\mathcal{G}f_n$  contains a subsequence which converges a.e. (relative to  $dr dx$ ).

2.2.C. Let

$$\theta = \sup_{x \in D} c^*(x)$$

where

$$(2.17) \quad c^* = \sum_{i,j=1}^d \nabla_i \nabla_j a_{ij} - \sum_{i=1}^d \nabla_i b_i.$$

Then

$$(2.18) \quad \int_D f \operatorname{sign} Gf dx \geq -\theta \|Gf\|$$

for all  $f \in L^1(D)$ .

Properties 2.2.A and 2.2.A\* follow from well-known bounds for  $g(x, y)$  ([24, Chapter 3]) and  $p_t(x, y)$  ([19, Chapter 1]).

*Proof of 2.2.B.* Denote by  $\varphi_\delta$  a function equal to 0 for  $|t| < \delta/2$ , equal to 1 for  $|t| > \delta$  and linear on  $[-\delta, -\delta/2]$  and on  $[\delta/2, \delta]$ . Formula

$$\mathcal{G}_\delta f(t, x; s, y) = \varphi_\delta(t - s) p_{t-s}(x, y)$$

defines a continuous kernel on  $S_b = [0, b] \times \bar{D}$ . The corresponding operator  $\mathcal{G}_\delta$  is compact in  $L^1(S_b)$  because functions  $\mathcal{G}_\delta f_n$  are equicontinuous for every sequence  $f_n$  bounded in  $L^1(S_b)$ .

By 2.2.A\* and Fubini's theorem,

$$\begin{aligned} \ell_b(\mathcal{G}f - \mathcal{G}_\delta f) &= \int_{S_b} dt dx \int_{S_b} [1 - \varphi_\delta(t - s)] p_{t-s}(x, y) |f(s, y)| ds dy \\ &\leq \int_{S_b} ds dy |f(s, y)| \int_s^{(s+\delta) \wedge b} dt dx p_{t-s}(x, y) \leq C \delta \ell_b(f). \end{aligned}$$

Therefore  $\mathcal{G}$  is a compact operator in  $L^1(S_b)$ . We get 2.2.B by the diagonal procedure.

*Proof of 2.2.C.* 1°. Suppose that  $\varphi$  is a bounded increasing continuously differentiable function on  $\mathbb{R}$  such that  $\varphi(0) = 0$ . Suppose that

$$(2.19) \quad u \in C^2(\bar{D}), \quad u = 0 \quad \text{on } \partial D.$$

Put  $\Phi(t) = \int_0^t \varphi(s) ds$ . By integration by parts, we get

$$(2.20) \quad \begin{aligned} - \int_D \varphi(u) Lu dx &= \int_D \left[ \sum_{i,j} a_{ij} \varphi'(u) \nabla_i u \nabla_j u + \sum_i \left( \sum_j \nabla_j a_{ij} + b_i \right) \varphi(u) \nabla_i u \right] dx \\ &= \int_D \left[ \sum a_{ij} \varphi'(u) \nabla_i u \nabla_j u - c^* \Phi(u) \right] dx \end{aligned}$$

and therefore

$$(2.21) \quad - \int_D dx \varphi(u) Lu \geq -\theta \int_D \Phi(u) dx.$$

2°. Suppose  $u = Gf$  with  $f \in C^2$ . Then  $u$  satisfies (2.19) and  $Lu = -f$ . By (2.21),

$$(2.22) \quad \int_D \varphi(u) f dx \geq -\theta \int_D \Phi(u) dx.$$

An arbitrary  $f \in L^1(D)$  is the strong limit of a sequence  $f_n \in L^1(D) \cap C^2$ . Let  $u_n = Gf_n, u = Gf$ . We have

$$(2.23) \quad \int \varphi(u) f dx - \int \varphi(u_n) f_n dx = \int \varphi(u_n) (f - f_n) dx + \int (\varphi(u) - \varphi(u_n)) f dx.$$

By 2.2.A,  $u_n \rightarrow u$  in  $L^1(D)$ . Therefore a subsequence  $u_{n_k}$  converges to  $u$  a.e. and the second term in the right side of (2.23) converges to 0 along this subsequence. The first term also converges to 0. Since (2.22) holds for  $f_n$ , it holds also for  $f$ .

3°. By applying (2.22) to a sequence of functions  $\varphi_n$  which converge boundedly to sign  $u$  and by passing to the limit, we get

$$\int_D f \text{sign } u \, dx \geq -\theta \int_D |u| dx$$

which is equivalent to (2.18).

**2.3. Existence.** Suppose that  $\xi$  is an  $L$ -diffusion stopped at the first exit time  $\tau$  from a bounded smooth domain  $D$ ,  $L$  satisfies conditions 1.1.A–B,  $p$  is defined by condition 1.2.A and  $g$  is the corresponding Green’s function defined by (1.12). We consider a function in  $D$  defined by the formula

$$(2.24) \quad h = G\eta + K\sigma$$

where  $\eta$  is a finite measure on  $D$  and  $\sigma$  is a positive bounded Borel function on  $\partial D$ . Put  $D(h) = \{h < \infty\}$ ,  $D(h, \alpha) = \{h + G(h^\alpha) < \infty\}$  and  $\mathcal{N}(h, \alpha) = \{\mu : \langle h + G(h^\alpha), \mu \rangle = \infty\}$ . Note that  $D(h, \alpha)$  is either empty or is the complement of a  $\xi$ -polar set. Let  $Q(h) = \mathbb{R}_+ \times D(h)$  and  $Q(h, \alpha) = \mathbb{R}_+ \times D(h, \alpha)$ .

**Theorem 2.2.** *Suppose that  $D(h, \alpha)$  is nonempty. Then there exists  $v \geq 0$  such that*

$$(2.25) \quad v + G(v^\alpha) = h \quad \text{on } D(h, \alpha).$$

Equation (2.25) determines  $v$  uniquely on  $D(h, \alpha)$ . We have

$$(2.26) \quad \|v^\alpha\| \leq 2C\eta(D) + C_1(\sigma)$$

where  $C$  is defined in 2.2.A and  $C_1(\sigma)$  does not depend on  $\eta$ . Let  $\tilde{X}$  be an  $(L, \alpha)$ -superdiffusion stopped at exit from  $D$  and let

$$(2.27) \quad v_\varepsilon(x) = -\log P_x \exp \left\{ - \left[ \int_0^\infty \langle \rho_\varepsilon, \tilde{X}_t \rangle dt + \langle \sigma, X_D \rangle \right] \right\},$$

$$(2.28) \quad u_\varepsilon(t, x) = -\log P_x \exp \left\{ - \left[ \int_0^t \langle \rho_\varepsilon, \tilde{X}_s \rangle ds + \langle \sigma, \tilde{X}_t \rangle \right] \right\}$$

where  $\sigma = 0$  in  $D$ ,  $\rho_\varepsilon$  is given by (2.11) in  $D$  and it is equal to 0 on  $\partial D$ .

We have

$$(2.29) \quad \lim_{\varepsilon \rightarrow 0} v_\varepsilon(x) = v(x) \quad \text{on } D(h, \alpha),$$

$$(2.30) \quad \lim_{\varepsilon \rightarrow 0} u_\varepsilon(t, x) = u(t, x) \quad \text{on } Q(h, \alpha)$$

where  $v$  is the solution of (2.25) and  $u$  is the solution of the equation

$$(2.31) \quad u + \mathcal{G}(u^\alpha) = \mathcal{G}\eta + \mathcal{K}\sigma \quad \text{on } Q(h, \alpha).$$

Moreover, if  $\mu \notin \mathcal{N}(h, \alpha)$ , then

$$(2.32) \quad \langle v_\varepsilon, \mu \rangle \rightarrow \langle v, \mu \rangle$$

and

$$(2.33) \quad \langle u_\varepsilon^t, \mu \rangle \rightarrow \langle u^t, \mu \rangle$$

for all  $t$ .

Finally,

$$(2.34) \quad u(t, x) \uparrow v(x) \quad \text{as } t \rightarrow \infty \quad \text{on } Q(h, \alpha).$$

*Proof.* By Theorems 2.1 and 2.1\*, each of the equations (2.25) and (2.31) has no more than one solution. We split the proof of Theorem 2.2 into three steps. First, we establish a bound for  $\|v_\varepsilon^\alpha\|$ . Then we use this bound to prove formulae (2.30) and (2.33). Finally, we establish (2.34), (2.29), (2.32), (2.25), (2.31) and (2.26).



1°. It follows from (2.12) that

$$(2.35) \quad h_\varepsilon \leq h \quad \text{and} \quad h_\varepsilon \uparrow h \quad \text{as} \quad \varepsilon \rightarrow 0$$

where  $h_\varepsilon = G\rho_\varepsilon + K\sigma$ . By Theorem A and (1.19),  $v_\varepsilon$  given by (2.27) satisfies equation

$$(2.36) \quad v_\varepsilon + G([v_\varepsilon]^\alpha) = h_\varepsilon$$

and

$$(2.37) \quad w(x) = -\log P_x \exp\{-\langle \sigma, X_D \rangle\}$$

satisfies equation

$$(2.38) \quad w + G(w^\alpha) = K\sigma.$$

Note that functions  $\rho_\varepsilon, w$  and  $K\sigma$  are bounded and

$$(2.39) \quad v_\varepsilon - w = G(F_\varepsilon)$$

where

$$(2.40) \quad F_\varepsilon = \rho_\varepsilon - v_\varepsilon^\alpha + w^\alpha.$$

By 2.2.C,

$$\int F_\varepsilon \operatorname{sign}(v_\varepsilon - w) dx = \int F_\varepsilon \operatorname{sign} GF_\varepsilon dx \geq -\theta \|v_\varepsilon - w\|$$

and, since  $\operatorname{sign}(v_\varepsilon^\alpha - w^\alpha) = \operatorname{sign}(v_\varepsilon - w)$ , we have

$$(2.41) \quad \|v_\varepsilon^\alpha - w^\alpha\| = \int (v_\varepsilon^\alpha - w^\alpha) \operatorname{sign}(v_\varepsilon^\alpha - w^\alpha) dx \leq \|\rho_\varepsilon\| + \theta \|v_\varepsilon - w\|.$$

By 2.2.A\* and (2.11),

$$(2.42) \quad \|\rho_\varepsilon\| \leq C\eta(D).$$

Note that, if  $\alpha > 1$ , then for every  $\delta > 0$ , there exists a constant  $C_\delta$  such that

$$(2.43) \quad |b - a| \leq \delta |b^\alpha - a^\alpha| + C_\delta$$

for all reals  $a, b$ . It follows from (2.41), (2.42) and (2.43) that

$$(2.44) \quad \|v_\varepsilon^\alpha - w^\alpha\| \leq \theta\delta \|v_\varepsilon^\alpha - w^\alpha\| + C\eta(D) + \theta C_\delta.$$

If  $\delta\theta \leq 1/2$ , then

$$(2.45) \quad \|v_\varepsilon^\alpha - w^\alpha\| \leq 2C\eta(D) + 2\theta C_\delta.$$

Since  $w \leq K\sigma$  and  $\sigma$  is bounded, (2.45) implies

$$(2.46) \quad \|v_\varepsilon^\alpha\| \leq 2C\eta(D) + C_1(\sigma)$$

where  $C_1(\sigma) = 2\theta C_\delta + \|(K\sigma)^\alpha\|$ .

2°. By (1.23), (2.12) and (2.24),

$$(2.47) \quad \mathcal{G}\rho_\varepsilon + \mathcal{K}\sigma \leq G\rho_\varepsilon + K\sigma \leq h.$$

By Theorem B,

$$(2.48) \quad u_\varepsilon + \mathcal{G}(u_\varepsilon^\alpha) = \mathcal{G}\rho_\varepsilon + \mathcal{K}\sigma$$

and

$$W(t, x) = -\log P_x \exp\{-\langle \sigma, \tilde{X}_t \rangle\}$$

is a solution of the equation

$$W + \mathcal{G}[W^\alpha] = \mathcal{K}\sigma.$$

We have

$$(2.49) \quad u_\varepsilon - W = \mathcal{G}(F_\varepsilon)$$

where

$$(2.50) \quad F_\varepsilon = \rho_\varepsilon - u_\varepsilon^\alpha + W^\alpha.$$

By (2.28) and (2.27),

$$(2.51) \quad u_\varepsilon(t, x) \leq v_\varepsilon(x) \quad \text{for all } t, x.$$

For every  $b$ , by (2.46) and (2.51),

$$(2.52) \quad b^{-1}\ell_b[(u_\varepsilon)^\alpha] \leq \|v_\varepsilon^\alpha\| \leq 2C\eta(D) + C_1(\sigma).$$

It follows from (2.11) and 1.2.B that

$$(2.53) \quad \mathcal{G}\rho_\varepsilon(t, x) = \int_\varepsilon^{t+\varepsilon} ds \int_D p_s(x, y)\eta(dy) \leq h(x)$$

and therefore

$$(2.54) \quad \mathcal{G}\rho_\varepsilon \rightarrow \mathcal{G}\eta \quad \text{as } \varepsilon \rightarrow 0 \quad \text{on } D(h).$$

For every  $b$ ,  $\ell_b(F_\varepsilon)$  are bounded by (2.50), (2.42) and (2.52). By (2.49) and 2.2.B, every sequence  $u_{\varepsilon_n}$  contains a subsequence which converges, a.e., to a limit  $u$ . Suppose  $u_{\varepsilon_n} \rightarrow u$  a.e. By (2.48) and (2.47),  $u_\varepsilon \leq h$ . It follows from (1.23) and the dominated convergence theorem that

$$(2.55) \quad \mathcal{G}[(u_{\varepsilon_n})^\alpha] \rightarrow \mathcal{G}[(u)^\alpha] \quad \text{on } Q(h, \alpha).$$

By (2.55) and (2.48),  $u$  satisfies (2.31) a.s. Formula (2.30) holds because, otherwise,  $|u_{\varepsilon_n} - u| > \delta$  for some  $(r, x) \in Q(h, \alpha), \delta > 0$  and for some sequence  $\varepsilon_n \rightarrow 0$ . By applying once more the dominated convergence theorem, we get (2.33).

3°. It is clear from (2.28) that  $u_\varepsilon(t, x)$  is monotone increasing in  $t$ . Therefore for every  $x \in D(h, \alpha)$ ,  $u(t, x)$  is also monotone increasing in  $t$ . By the monotone convergence theorem,  $v(x) = \lim_{t \rightarrow \infty} u(t, x)$  satisfies (2.25). Formula (2.26) follows from (2.52) and (2.34).

Note that  $u_\varepsilon \leq v_\varepsilon$  and, by (2.30) and (2.34),  $\liminf_{\varepsilon \rightarrow 0} v_\varepsilon \geq v$  on  $D(h, \alpha)$ . On the other hand, it follows from (1.25) and (1.13) that  $\mathcal{G}(u_\varepsilon^\alpha) \leq \mathcal{G}(v_\varepsilon^\alpha) \leq G(v_\varepsilon^\alpha)$  and, by (2.48), (2.36), (2.12), (2.53), (1.25) and (1.26),

$$0 \leq v_\varepsilon - u_\varepsilon^t \leq \int_{t+\varepsilon}^\infty ds \int_D p_s(x, y) \eta(dy) + \Pi_x \sigma(\xi_\tau) 1_{\tau > t}$$

and therefore  $\limsup_{\varepsilon \rightarrow 0} v_\varepsilon \leq v$  on  $D(h, \alpha)$ . Clearly, this implies (2.29). Formula (2.32) can be deduced from (2.33) in an analogous way.

### 3. PROOF OF THEOREM 1.3

**3.1.** We use several times a property of exit measures which will be established in Lemma 3.1. We start from a functional

$$(3.1) \quad B_t(\varepsilon) = \int_0^t \langle \rho_\varepsilon, \tilde{X}_s \rangle ds + C_t$$

where  $\rho_\varepsilon$  is given by (2.11) and  $C_t$  is a left continuous modification of  $\langle \sigma, \tilde{X}_t \rangle$  which we define in Lemma 3.2. Put

$$(3.2) \quad B_t = \lim_{k \rightarrow \infty} \text{med } B_t(1/k) \quad \text{for all } t > 0$$

where  $\lim \text{med}$  is Mokobodzki's medial limit. It is defined for every sequence  $a_n \in [0, \infty]$  and it takes values in  $[0, \infty]$ . We need the following properties of this limit (see, e.g., [4, X.56, X.57]):

- 3.1.A.  $\liminf a_n \leq \lim \text{med } a_n \leq \limsup a_n$ ;
- 3.1.B.  $\lim \text{med}(a_n + b_n) = \lim \text{med } a_n + \lim \text{med } b_n$ ;
- 3.1.C. If  $a_n \leq b_n$  for all  $n$ , then  $\lim \text{med } a_n \leq \lim \text{med } b_n$ ;
- 3.1.D. Let  $Z_n$  be measurable mappings from a measurable space  $(\Omega, \mathcal{F})$  to  $[0, \infty]$ . Then  $Z(\omega) = \lim \text{med } Z_n(\omega)$  is measurable with respect to the universal completion of  $\mathcal{F}$ . If  $P$  is a probability measure on  $(\Omega, \mathcal{F})$  and if  $Z_n \rightarrow Y$  in  $P$ -probability, then  $Y = Z$   $P$ -a.s.

In Theorem 3.1, we construct a functional  $B$  of an  $(L, \alpha)$ -superdiffusion  $\tilde{X}$  stopped at the exit from a bounded smooth domain  $D$  which satisfies conditions 1.6.A, B and the following condition:

$$(3.3) \quad B_{s+t} \leq B_s + \theta_s B_t \quad \text{a.s. for every } s, t.$$

Moreover, for every  $\mu \notin \mathcal{N}(h, \alpha)$ :

$$(3.4) \quad B_t = \lim_{\varepsilon \rightarrow 0} B_t(\varepsilon) \quad \text{in } P_\mu\text{-probability for all } t \in \mathbb{R}_+$$

and

$$(3.5) \quad -\log P_\mu e^{-Bt} = \langle u^t, \mu \rangle$$

where  $u$  satisfies (2.31).

The next step is a passage to the limit from bounded smooth domains to an arbitrary domain  $E$ . We assume that  $h$  is given by (1.17) and that  $E(h, \alpha) = \{h < \infty, G(h^\alpha) < \infty\}$  is nonempty. We consider a sequence of bounded smooth domains  $D_n$  which approximate  $E$  and we denote by  $G^n, K^n$  the Green and Poisson operators corresponding to the  $L$ -diffusion stopped at the exit from  $D_n$ . Put  $\sigma_n = 1_{E \setminus D_n} K\nu$  and denote by  $B^n$  the function corresponding to

$$(3.6) \quad h_n(x) = \int_{D_n} g^n(x, y)\eta(dy) + K^n\sigma_n(x)$$

by Theorem 3.1. By 3.1.C, D, function

$$(3.7) \quad B_t = \lim_{n \rightarrow \infty} \text{med } B_t^n$$

satisfies 1.6.A, B. We show that, for every  $\mu \notin \mathcal{N}(h, \alpha)$  and every  $t$ ,

$$(3.8) \quad B_t = \lim_{n \rightarrow \infty} B_t^n \quad P_\mu - \text{a.s.}$$

Function  $A_t = B_{t-}$  satisfies 1.6.A-C and 1.6.D\*\* with  $\mathcal{N} = \mathcal{N}(h, \alpha)$ .

At the final stage, we use Lemma 3.3 to decompose measures  $\eta, \nu$ , subject to condition (1.22), into series of measures  $\eta_n, \nu_n$  for which  $E(h_n, \alpha) \neq \emptyset$  (here  $h_n = G\eta_n + K\nu_n$ ). The functional corresponding to  $\eta, \nu$  is defined as the sum of functionals corresponding to  $\eta_n, \nu_n$ .

This way we obtain a functional of  $X$ , subject to conditions 1.6.A, B, C, for which 1.6.D\*\* and (1.45) hold for all  $\mu$  outside of an exceptional set  $\mathcal{N}$ . Then we refer to a result in [4] to prove the existence of an equivalent functional which satisfies 1.6.D\*.

### 3.2. A property of exit measures.

**Lemma 3.1.** *Suppose that  $Q_1 \supset Q_2$  are open subsets of  $S$  and  $\Gamma \cap Q_1 = \emptyset$ . Then  $X_{Q_1}(\Gamma) \geq X_{Q_2}(\Gamma)$  a.s.*

*Proof.* For every  $\nu \in \mathcal{M}(E)$ ,  $P_\nu\{X_{Q_1}(\Gamma) \geq \nu(\Gamma)\} = 1$ . Indeed,  $\Pi_{r,x}\{\tau^r = r\} = 1$  for every  $(r, x) \notin Q_1$  and, by (1.32), for every  $\lambda > 0$ ,

$$P_\nu e^{-\lambda X_{Q_1}(\Gamma)} = e^{-\langle v_\lambda, \nu \rangle}$$

with  $v_\lambda = \lambda 1_\Gamma$  on  $\Gamma$ . Hence,

$$(3.9) \quad P_\nu e^{-\lambda X_{Q_1}(\Gamma)} \leq e^{-\lambda \nu(\Gamma)}.$$

Put  $Y = X_{Q_1}(\Gamma) - \nu(\Gamma)$ . By (3.9),  $P_\nu e^{-\lambda Y} \leq 1$  for all  $\lambda > 0$  and therefore  $Y \geq 0$   $P_\nu$ -a.e.

It follows from (1.34) that, for every positive measurable  $f$ ,  $P_\mu f(X_{Q_2}, X_{Q_1}) = P_\mu F(X_{Q_2})$  where  $F(\nu) = P_\nu f(\nu, X_{Q_1})$ . If  $f(\nu_1, \nu_2) = 1_{\nu_1(\Gamma) \leq \nu_2(\Gamma)}$ , then  $F(\nu) = P_\nu\{\nu(\Gamma) \leq X_{Q_1}(\Gamma)\} = 1$ .

**3.3. Regularization of  $\langle \sigma, \tilde{X}_t \rangle$ .**

**Lemma 3.2.** *Let  $\tilde{X}$  be an  $(L, \alpha)$ -superdiffusion stopped at the exit from a bounded smooth domain  $D$  and let  $\sigma$  be a positive Borel function on  $\bar{D}$  which vanishes on  $D$ . There exists a function  $C_t$  subject to conditions 1.6.A–C such that, for every  $t$ ,*

$$(3.10) \quad C_t = \langle \sigma, \tilde{X}_t \rangle \quad \text{a.s.}$$

*Proof.* Put  $Y_t = \langle \sigma, \tilde{X}_t \rangle$ . Recall (see Section 1.5) that  $\tilde{X}_t = X_{Q_t}$  where  $Q_t = [0, t) \times D$ . It follows from Lemma 3.1 that  $\tilde{X}_r(\Gamma) \leq \tilde{X}_s(\Gamma)$  a.s. if  $r < s$  and  $\Gamma \cap Q_s = \emptyset$ . Since  $\sigma = 0$  in  $Q_s$ ,  $Y_r \leq Y_s$  a.s. Denote by  $\mathbb{Q}_+$  the set of positive rationals. The set

$$\Omega_t = \{Y_r \leq Y_s \text{ for all } r < s \in \mathbb{Q}_+ \cap [0, t)\}$$

belongs to  $\mathcal{F}_t$  and  $P_\mu\{\Omega_t\} = 1$  for all  $\mu \in \mathcal{M}(D)$ . Function

$$C_t = \begin{cases} \lim_{s \uparrow t, s \in \mathbb{Q}_+} Y_s & \text{on } \Omega_t, \\ \infty & \text{otherwise} \end{cases}$$

satisfies conditions 1.6.A–C. It remains to prove that  $Y_t = C_t$  a.s. By Theorem B,

$$(3.11) \quad -\log P_\mu e^{-Y_t} = \langle u^t, \mu \rangle$$

where  $u$  is a solution of the equation

$$(3.12) \quad u + \mathcal{G}[u^\alpha] = \mathcal{K}\sigma.$$

By (3.11),  $u(t, x)$  is monotone increasing in  $t$ . Put  $u_-(t, x) = u(t-, x)$ . Since  $\Pi_x\{\tau = t\} = 0$  for all  $t$ , function  $\mathcal{K}\sigma$  is continuous in  $t$ . By passing to the limit in (3.12), we get

$$u_- + \mathcal{G}[u_-^\alpha] = \mathcal{K}\sigma.$$

By (1.25), functions  $\mathcal{K}\sigma \leq K\sigma$  are bounded and, by Theorem 2.1\*,  $u_- = u$ . By (3.11),  $P_\mu e^{-Y_t} = P_\mu e^{-C_t}$ . Since  $Y_t \leq C_t$ , this implies (3.10).

**3.4.**

**Lemma 3.3.** *Let  $\eta$  and  $\nu$  satisfy condition (1.22). Then there exist measures  $\eta_n, \nu_n$  such that*

$$\eta = \eta_1 + \dots + \eta_n + \dots, \quad \nu = \nu_1 + \dots + \nu_n + \dots$$

and

$$(3.13) \quad G(h_n^\alpha)(c) < \infty$$

where  $h_n = G\eta_n + \mathcal{K}\nu_n$  and  $c$  is the same as in formula (1.16).

*Proof.* Since  $(\frac{a+b}{2})^\alpha \leq \frac{1}{2}(a^\alpha + b^\alpha)$  for all  $\alpha, a, b \geq 0$ , we can assume that  $\eta = 0$  or  $\nu = 0$ . We refer to [18, Theorem 2.2] in the first case and [2, Lemma 4.2] in the second case.

**3.5.**

**Theorem 3.1.** *Let  $\tilde{X}$  be an  $(L, \alpha)$ -superdiffusion stopped at the exit from a bounded smooth domain  $D$  and let  $h, \eta, \rho_\varepsilon$  and  $\sigma$  be as in Theorem 2.2. If  $B_t(\varepsilon)$  is defined by (3.1), then function  $B_t$  given by (3.2) satisfies conditions 1.6.A, B, (3.3), (3.4) and (3.5).*

*Proof.* Properties 1.6.A, B follow from 3.1.C, D. By Theorem B,

$$(3.14) \quad u_{\delta\varepsilon}(t, x) = -\log P_x \exp\left\{-\frac{1}{2}(B_t(\delta) + B_t(\varepsilon))\right\}$$

satisfies the equation

$$u_{\delta\varepsilon} + \mathcal{G}[u_{\delta\varepsilon}^\alpha] = \frac{1}{2}(\mathcal{G}\rho_\delta + \mathcal{G}\rho_\varepsilon) + \mathcal{K}\sigma.$$

The same arguments as in the proof of Theorem 2.2 show that, for all  $\mu \notin \mathcal{N}(h, \alpha)$  and all  $t$ ,

$$(3.15) \quad \langle u_{\delta\varepsilon}^t, \mu \rangle \rightarrow \langle u^t, \mu \rangle \quad \text{as } \delta, \varepsilon \rightarrow 0$$

where  $u$  is the unique solution of (2.31).

By Theorem B,

$$(3.16) \quad P_\mu \left[ e^{-B_t(\varepsilon)/2} - e^{-B_t(\delta)/2} \right]^2 = e^{-\langle u_{\varepsilon\varepsilon}^t, \mu \rangle} + e^{-\langle u_{\delta\delta}^t, \mu \rangle} - 2e^{-\langle u_{\delta\varepsilon}^t, \mu \rangle}$$

for every  $\mu \in \mathcal{M}(D)$ . If  $\mu \notin \mathcal{N}(h, \alpha)$ , then, by (3.15), the right side in (3.16) tends to 0 as  $\delta, \varepsilon \rightarrow 0$ . Hence  $e^{-B_t(\varepsilon)}$  converges in  $L^2(P_\mu)$  as  $\varepsilon \rightarrow 0$  which implies that  $B_t(\varepsilon)$  converges in  $P_\mu$ -probability to a limit  $B_t^\mu$ . It follows from 3.1.D that  $B_t = B_t^\mu$   $P_\mu$ -a.s. which implies (3.4). To prove (3.3), we note that  $\tilde{X}_t = X_{Q(t)}$  where  $Q(t) = [0, t] \times D$ . Therefore (see Section 1.5))  $\theta_s \tilde{X}_t = X_{Q(s,t)}$  where  $Q(s, t) = S_{<s} \cup Q(s+t)$  and, by Lemma 3.1,  $\langle \rho_\varepsilon, \theta_s \tilde{X}_t \rangle \geq \langle \rho_\varepsilon, \tilde{X}_{s+t} \rangle$  and  $\langle \sigma, \theta_s \tilde{X}_t \rangle \geq \langle \sigma, \tilde{X}_{s+t} \rangle$  a.s. Clearly, restrictions of measures  $\tilde{X}_s$  and  $\tilde{X}_{s+t}$  to  $[0, s] \times \partial D$  coincide, and, by (3.1),

$$(3.17) \quad B_s(\varepsilon) + \theta_s B_t(\varepsilon) \geq B_{s+t}(\varepsilon) \quad \text{a.s.}$$

and (3.3) follows from (3.2).

Let  $\mu \notin \mathcal{N}(h, \alpha)$ . By (1.42),

$$-\log P_\mu e^{-B_t(\varepsilon)} = \langle u_\varepsilon^t, \mu \rangle$$

where  $u_\varepsilon$  is given by (2.28). By (3.4) and (2.33), this implies (3.5).

**3.6.** The next step in the program outlined in Section 3.1 is a passage to the limit from bounded smooth domains to an arbitrary domain  $E$ . Recall that, according to Section 1.2,  $L$ -diffusion  $\xi$  in  $E$  can be constructed by using a sequence of bounded smooth domains  $D_n$  such that  $\bar{D}_n \subset D_{n+1}$  and  $E = \bigcup D_n$ : the transition density  $p_t(x, y)$  of  $\xi$  is the limit of monotone increasing sequence  $p_t^n(x, y)$  defined in 1.2.A (it is convenient to set  $p_t^n(x, y) = 0$  if  $x \notin D_n$  or  $y \notin D_n$ ). Green's functions  $g^n(x, y)$ ,

$g(x, y)$  and Green's operators  $G^n, G$  corresponding to  $p^n, p$  are determined by (1.12) and (1.13). Operators  $K^n$  correspond by (1.26) to the first exit times  $\tau_n$  from  $D_n$ .

Let  $X$  be an  $(L, \alpha)$ -superdiffusion in  $E$  and let  $X^n$  be an  $(L, \alpha)$ -superdiffusion stopped at the exit from  $D_n$ . Denote by  $Y_t^n$  the restriction of  $X_t^n$  to  $D_n$ . By Lemma 3.1, for every  $t$  and every  $n$ ,

$$(3.18) \quad Y_t^n \leq Y_t^{n+1} \quad \text{a.s.}$$

By (1.37),

$$P_\mu Y_t^n(B) = \int_E \mu(dx) \int_B p_t^n(x, y) dy \uparrow \int_E \mu(dx) \int_B p_t(x, y) dy = P_\mu X_t(B)$$

and therefore

$$(3.19) \quad Y_t^n \uparrow X_t \quad \text{a.s.}$$

**3.7.** Let  $h$  be given by (1.17) with finite measures  $\eta$  and  $\nu$ . Suppose that  $E(h, \alpha) \neq \emptyset$ . Put  $f = K\nu$ . By (1.19) and (1.26),

$$f(x) = \Pi_x \sigma_n(\xi_{\tau_n}) = K^n \sigma_n(x)$$

where  $\sigma_n = 1_{E \setminus D_n} f$ . We define  $B^n$  and  $B$  as in Section 3.1. By 3.1.D, to prove formula (3.8), we need only show that  $B_t^n$  converges  $P_\mu$ -a.s. as  $n \rightarrow \infty$ . Put

$$(3.20) \quad Z_t^n(\varepsilon) = \int_0^t \langle \rho_\varepsilon^n, Y_r^n \rangle dr.$$

By 3.1.B,

$$B_t^n = Z_t^n + C_t^n$$

where

$$Z_t^n = \lim_{k \rightarrow \infty} \text{med } Z_t^n(1/k).$$

For every  $n$ ,  $\rho_\varepsilon^{n+1} \geq \rho_\varepsilon^n$  and, by (3.20) and (3.18),  $Z_t^n(\varepsilon)$  is, a.s., monotone increasing in  $n$ . By 3.1.C, sequence  $Z_t^n$  has the same property and therefore it converges  $P_\mu$ -a.s.

On the other hand, since  $f$  is  $L$ -harmonic, it follows from the Markov property (1.34) that the sequence  $W_n = \langle f, X_t^n \rangle$  is a martingale with respect to  $P_\mu$ . Therefore  $C_t^n$  also converges a.s.

**3.8.** Put  $S(h, \alpha) = \mathbb{R}_+ \times E(h, \alpha)$ . By Theorem 2.2, for every  $\mu \notin \mathcal{N}(h, \alpha)$ ,

$$-\log P_\mu e^{-B_t^n} = \int_{D_n} u_n(t, x) \mu(dx)$$

where  $u_n$  satisfies the equation

$$(3.21) \quad u_n(t, x) + \int_0^t ds \int_{D_n} p_{t-s}^n(x, y) u_n(s, y)^\alpha dy = H_n(t, x) \quad \text{on } S(h, \alpha)$$

with

$$(3.22) \quad H_n(t, x) = \int_0^t ds \int_{D_n} p_{t-s}^n(x, y) \eta(dy) + \Pi_x f(\xi_{\tau_n}) 1_{\tau_n < t}.$$

Moreover, by (2.28), (2.30) and (3.20),

$$(3.23) \quad u_n(t, x) = - \lim_{\varepsilon \rightarrow 0} \log P_x \exp\{-B_t^n(\varepsilon)\} \quad \text{on } S(h, \alpha).$$

By (3.4),

$$u_n(t, x) = - \log P_x e^{-B_t^n} \quad \text{on } S(h, \alpha).$$

By (3.8),

$$(3.24) \quad u_n(t, x) \rightarrow u(t, x) = - \log P_x e^{-B_t} \quad \text{on } S(h, \alpha).$$

Note that

$$\Pi_x f(\xi_{\tau_n}) 1_{\tau_n < t} = f(x) - \Pi_x f(\xi_t) 1_{\tau_n \geq t}$$

and therefore  $H_n$  converges to

$$(3.25) \quad H(t, x) = \int_0^t ds \int_E p_{t-s}(x, y) \eta(dy) + F(t, x)$$

where

$$(3.26) \quad F(t, x) = f(x) - \Pi_x f(\xi_t).$$

By (3.21), (3.22) and (1.17),  $u_n \leq h$ . The second term in (3.21) converges to  $\mathcal{G}[u^\alpha]$  by (2.2) and the dominated convergence theorem. Hence, (3.21) implies

$$(3.27) \quad u + \mathcal{G}[u^\alpha] = H \quad \text{on } S(h, \alpha).$$

**3.9.** Note that  $u(t, x)$  increases in  $t$  by (3.24) and 1.6.A. Put  $u_-(t, x) = u(t-, x)$ . An  $L$ -excessive function  $f$  has a representation

$$f = f_0 + \int_0^\infty \varphi_s ds$$

where  $T_t f_0 = f_0$  and  $T_t \varphi_s = \varphi_{s+t}$  for all  $t, s$  (see [10, Section 2.8]). Therefore

$$H(t, x) = \int_0^t ds \left[ \int_E p_s(x, y) \eta(dy) + \varphi_s \right]$$

is increasing and continuous in  $t$ . By passing to the limit in (3.24) and (3.27), we get

$$(3.28) \quad \begin{aligned} u_-(t, x) &= - \log P_x e^{-B_{t-}}, \\ u_- + \mathcal{G}[u_-^\alpha] &= H \quad \text{on } S(h, \alpha). \end{aligned}$$



By Theorem 2.1\*, this implies  $u_- = u$ . Since  $B_{t-} \leq B_t$ , (3.24) and (3.28) yield  $B_{t-} = B_t$  a.s. Function  $A_t = B_{t-}$  satisfies conditions 1.6.A–C.

We claim that 1.6.D\*\* holds for  $\mathcal{N} = \mathcal{N}(h, \alpha)$ . Indeed, if  $\mu \notin \mathcal{N}(h, \alpha)$ , then

$$B_t^n(\varepsilon) = Z_t^n(\varepsilon) + C_t^n$$

converges in  $P_\mu$ -probability to  $B_t^n$  by (3.4). By the Markov property (1.34),

$$P_\mu e^{-B_{s+t}^n(\varepsilon)} = P_\mu \left[ e^{-B_s^n(\varepsilon)} P_{Y_s^n} e^{-B_t^n(\varepsilon)} \right]$$

for all  $s, t > 0$ . This implies

$$P_\mu e^{-B_{s+t}^n} = P_\mu \left[ e^{-B_s^n} P_{Y_s^n} e^{-B_t^n} \right]$$

and therefore

$$(3.29) \quad |P_\mu e^{-B_{s+t}^n} - P_\mu \left[ e^{-B_s^n} P_{X_s} e^{-B_t^n} \right]| \leq P_\mu |P_{Y_s^n} e^{-B_t^n} - P_{X_s} e^{-B_t^n}|$$

By (3.5), the right side is equal to

$$P_\mu |e^{-\langle v_n^t, Y_s^n \rangle} - e^{-\langle v_n^t, X_s \rangle}|$$

where  $v_n^t(x) = u_n^t(0, x)$ . By (3.7),  $v_n^t \leq h$  and therefore (3.29) does not exceed

$$P_\mu |1 - e^{-\langle h, X_s - Y_s^n \rangle}|.$$

By (3.19), this tends to 0 and we conclude from (3.29) and the Markov property of  $X$  that

$$(3.30) \quad P_\mu e^{-B_{s+t}} = P_\mu e^{-B_s} P_{X_s} e^{-B_t} = P_\mu e^{-(B_s + \theta_s B_t)}.$$

By (3.1),  $B_{s+t} \leq B_s + \theta_s B_t$  and (3.30) implies 1.6.D\*\*.

We get (1.45) by passing to the limit in (3.5) and (3.27) as  $t \rightarrow \infty$ .

**3.10.** Let  $h$  be an arbitrary function of the form (1.17) with  $\eta$  and  $\nu$  subject to conditions (1.22). Consider measures  $\eta_n$  and  $\nu_n$  defined in Lemma 3.3. Denote by  $A^n$  the functional of  $X$  corresponding to  $h_n(x) = G\eta_n + K\nu_n$  by Section 3.9 and put

$$A = A_1 + \dots + A_n + \dots.$$

Clearly, conditions 1.6.A,B,C and 1.6.D\*\* hold for  $A$ . Formula (1.45) holds if  $\mu \notin \mathcal{N} = \bigcup \mathcal{N}(h_n, \alpha)$  and  $v$  satisfies (1.14) on  $E = \bigcap E(h_n, \alpha)$ . Function  $\tilde{v}$  defined by (1.15) is a solution of (1.14) everywhere. It also satisfies (1.45).

Formula (1.44) can be obtained from (1.45) in the same way as (1.37) was deduced from (1.35).

By the “perfection” theorem [4, 15.8], there exists a functional equivalent to  $A$  which satisfies 1.6.D\*. (In [4] functionals without an exceptional set are considered, but the proof is applicable without any change to our case.)

## 4. PROOF OF THEOREMS 1.2 AND 1.4

## 4.1.

**Lemma 4.1.** *Let an excessive function be given by formula (1.17). If there exists  $u$  such that*

$$(4.1) \quad u + G(u^\alpha) = h,$$

*then there exists  $v$  such that*

$$(4.2) \quad v + G(v^\alpha) = K\nu.$$

*Proof.* Let  $D_n, G^n$  and  $K^n$  have the same meaning as in Section 3.6. By the strong Markov property of  $\xi$ , (4.1) implies

$$(4.3) \quad u + G^n(u^\alpha) = G^n\eta + K^n u \quad \text{on } D_n.$$

By Theorem A,

$$v_n(x) = -\log P_x e^{-\langle u, X_{D_n} \rangle}$$

satisfies the equation

$$(4.4) \quad v_n + G^n(v_n^\alpha) = K^n u \quad \text{on } D_n.$$

We use again the strong Markov property of  $\xi$  to get from here that, for each  $m > n$ ,

$$(4.5) \quad v_m + G^n(v_m^\alpha) = K^n v_m \quad \text{on } D_n.$$

We conclude from Theorem 2.1, by comparing (4.3) and (4.4), that  $v_n \leq u$  in  $D_n \cap E(h)$ , and, by comparing (4.4) and (4.5), that  $v_m \leq v_n$  in  $D_n \cap E(h)$ . Therefore there exists a limit

$$v = \lim_{n \rightarrow \infty} v_n \quad \text{on } E(h).$$

It follows from (4.3) by monotone convergence theorem that

$$u + G(u^\alpha) = G\eta + \lim K^n u.$$

In combination with (4.1), this yields  $\lim K^n u = K\nu$  on  $E(h)$ . By (4.1),  $G(u^\alpha) < \infty$  on  $E(h)$  and, by the dominated convergence theorem,  $\lim G^n(v_n^\alpha) = G(v^\alpha)$ . Therefore (4.4) implies that (4.2) holds on  $E(h)$ . It holds everywhere for a function  $v$  modified by formula (1.15).

**4.2. Proof of Theorem 1.2.** Suppose that  $u$  is a solution of (4.1). By Lemma 4.1, equation (4.2) has a solution and  $\nu$  does not charge  $\mathcal{R}^*$ -polar sets by Theorem 3.1 in [18].

It remains to prove that  $\eta(\Gamma) = 0$  for  $\mathcal{R}$ -polar sets  $\Gamma$ . We can assume that  $\Gamma$  is compact. Let  $D$  be a bounded smooth domain such that  $\Gamma \subset D$  and  $\bar{D} \subset E$ . Equation (4.1) implies

$$(4.6) \quad u + G_D(u^\alpha) = G_D\eta + K_D u \quad \text{in } D$$

(cf. (4.3)). By Theorem E° in [17],  $\text{Cap}_{2,\alpha'}(\Gamma) = 0$ . We use the following fact (see Lemma 4.1 in [2]): a signed measure  $\gamma$  does not charge sets  $\Gamma$  with  $\text{Cap}_{2,\alpha'}(\Gamma) = 0$  if

$$\int_D \varphi(x)\gamma(dx) \leq \text{const} \cdot \|\varphi\|_{2,\alpha'} \quad \text{for all } \varphi \in C_0^\infty(D)$$

(here  $\|\varphi\|_{2,\alpha'}$  is the norm of  $\varphi$  in the Sobolev space  $W^{2,\alpha'}(D)$ ).

By Lemma 4.1, there exists  $v \geq 0$  such that

$$(4.7) \quad v + G_D(v^\alpha) = K_D u \quad \text{in } D.$$

By Theorem 2.1,  $w = u - v \geq 0$ . There exists a function  $q \geq 0$  such that  $u^\alpha - v^\alpha = qw$  a.e. (cf. proof of Theorem 2.1\*). It follows from (4.6) and (4.7) that

$$(4.8) \quad w + G_D(qw) = G_D \eta.$$

Let  $\gamma(dx) = \eta(dx) - (qw)(x)dx$  and  $\varphi \in C_0^\infty(D)$ . Put  $\psi = -L^* \varphi$ . Note that  $\|\psi\|_{\alpha'} \leq \|\varphi\|_{2,\alpha'}$  and

$$\varphi(y) = \int_D dx \psi(x) g_D(x, y).$$

By (4.8),  $G_D \gamma = w$  and therefore

$$\begin{aligned} \int_D \varphi(x)\gamma(dx) &= \int_{D \times D} dx \psi(x) g_D(x, y) \gamma(dy) = \int_D w(x)\psi(x) dx \\ &\leq \|w\|_\alpha \|\psi\|_{\alpha'} \leq \|w\|_\alpha \|\varphi\|_{2,\alpha'}. \end{aligned}$$

If  $h(c) < \infty$ , then  $G(u^\alpha)(c) < \infty$  by (4.1) and  $u \in L^\alpha(D)$  because  $\inf_D g(c, y) > 0$ .

We have  $0 \leq w \leq u$  and therefore  $w \in L^\alpha(D)$ . Hence  $\gamma(\Gamma) = 0$ . Since  $\text{Cap}_{2,\alpha'}(\Gamma) = 0$  implies that the Lebesgue measure of  $\Gamma$  is equal to 0, we get  $\eta(\Gamma) = 0$ .

**4.3. Localization.** To prove Theorem 1.4, we need some preparations. Suppose that  $h$  is the potential of a linear additive functional  $A$  with exceptional set  $\mathcal{N}$  and let  $\eta, \nu$  correspond to  $h$  by (1.17). For every positive bounded continuous function  $\varphi$  on  $E \cup E^*$ , we put  $h^\varphi(x) = G(\eta^\varphi) + K(\nu^\varphi)$  where  $\eta^\varphi(dx) = \varphi(x)\eta(dx)$ ,  $\nu^\varphi(dx) = \varphi(x)\nu(dx)$ . It follows from 1.6.D\*, the strong Markov property of  $X$  and (1.44) that

$$\langle h, X_T \rangle = P_\mu \{A_\infty | \mathcal{F}_T\} - A_T \quad P_\mu\text{-a.s.}$$

for every  $\mathcal{F}_t$ -stopping time  $T$  and for every  $\mu \notin \mathcal{N}$ . It is easy to see from here that  $\langle h, X_t \rangle$  is a supermartingale of class (D) relative to  $P_\mu$  (cf. [4, V.15]). Since  $\langle h^\varphi, X_t \rangle \leq \text{const} \langle h, X_t \rangle$ , the same is true for  $\langle h^\varphi, X_t \rangle$ . By [4, Th. XV.6] or [25, Th. 38.1]<sup>13</sup>, there exists a *natural* additive functional  $A^\varphi$ <sup>14</sup> such that:

4.3.A.  $P_\mu A_\infty^\varphi = \langle h^\varphi, \mu \rangle$  for all  $\mu \notin \mathcal{N}$ .

We call it the  $\varphi$ -localization of  $A$ . In the same way as in Theorem 3.3 of [17], we establish:

4.3.B. If  $\varphi_1 \leq \varphi_2$ , then  $A^{\varphi_1} \leq A^{\varphi_2}$   $P_\mu$ -a.s. for all  $\mu \notin \mathcal{N}$ .

4.3.C. If  $\varphi = 0$  on  $D$ , then  $\{A^\varphi = 0\} \supset \{\mathcal{R} \subset \bar{D}\}$   $P_\mu$ -a.s. and  $\{A^\varphi = 0\} \supset \{\mathcal{R}^* \subset \bar{D}\}$   $P_\mu$ -a.s. for all  $\mu \notin \mathcal{N}$ .

<sup>13</sup>As in the case of "perfection", these theorems can be easily extended to functionals with an exceptional set.

<sup>14</sup>An additive functional  $A$  is natural if the process  $A_{t+}$  is predictable (cf. [4, IV.61] or the Appendix to [9]). We believe that functional  $A$  constructed in Theorem 1.3 is natural but this is not proved in the present paper.

**4.4. Proof of Theorem 1.4.** This proof is similar to that of Theorem 3.3 in [18]. Let, for instance,  $\Gamma \subset E$  be a compact  $\mathcal{R}$ -polar set. Put

$$D_n = \{x \in E : d(x, \Gamma) > \frac{1}{n}\}$$

where  $d$  is the distance in the Martin space  $\hat{E}$ . Bounded positive continuous functions

$$\varphi_n(x) = (1 - nd(x, \Gamma))_+$$

vanish on  $D_n$ . Consider the corresponding localizations  $A^{\varphi_n}$ . For every  $\mu \notin \mathcal{N}$ ,

$$A^1 \geq A^{\varphi_1} \geq \dots \geq A^{\varphi_n} \geq \dots, \quad P_\mu\text{-a.s.}$$

by 4.3.B and

$$\{\mathcal{R} \subset D_n\} \subset \{A_\infty^{\varphi_n} = 0\}, \quad P_\mu\text{-a.s.}$$

by 4.3.C. Let  $\mu(\Gamma) = 0$ . Since  $\Gamma$  is  $\mathcal{R}$ -polar,  $1_{\mathcal{R} \subset D_n} \uparrow 1$   $P_\mu$ -a.s. and therefore  $A_\infty^{\varphi_n} \rightarrow 0$   $P_\mu$ -a.s. By the dominated convergence theorem,

$$(4.9) \quad \lim P_\mu A_\infty^{\varphi_n} = 0$$

On the other hand, by 4.3.A,

$$\begin{aligned} P_\mu A_\infty^{\varphi_n} &= \int \mu(dx) \int_E g(x, y) \varphi_n(y) \eta(dy) \\ &\quad + \int \mu(dx) \int_{E^*} \mu(dx) \int_{E^*} k(x, y) \varphi_n(y) \nu(dy) \downarrow \int \mu(dx) \int_\Gamma g(x, y) \eta(dy). \end{aligned}$$

In combination with (4.9), this implies  $\eta(\Gamma) = 0$ . The case of  $\mathcal{R}^*$ -polar set  $\Gamma \subset E^*$  can be treated in a similar way.

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