

FRACTAL DIMENSIONS AND RANDOM TRANSFORMATIONS

YURI KIFER

ABSTRACT. I start with random base expansions of numbers from the interval $[0, 1]$ and, more generally, vectors from $[0, 1]^d$, which leads to random expanding transformations on the d -dimensional torus \mathbb{T}^d . As in the classical deterministic case of Besicovitch and Eggleston I find the Hausdorff dimension of random sets of numbers with given averages of occurrences of digits in these expansions, as well as of general closed sets “invariant” with respect to these random transformations, generalizing the corresponding deterministic result of Furstenberg. In place of the usual entropy which emerges (as explained in Billingsley’s book) in the Besicovitch-Eggleston and Furstenberg cases, the relativised entropy of random expanding transformations comes into play in my setup. I also extend to the case of random transformations the Bowen-Ruelle formula for the Hausdorff dimension of repellers.

1. INTRODUCTION

Suppose we throw a fair die repeatedly and obtain a sequence of outcomes $m_1, m_2, \dots, m_i \in \{1, 2, \dots, 6\}$. Then any $x \in [0, 1]$ has a “random base” expansion

$$x = \sum_{i=1}^{\infty} \frac{x_{i-1}}{m_1 m_2 \cdots m_i}, \text{ where } x_i \in \{0, 1, \dots, m_{i+1} - 1\}.$$

It is not difficult to observe that with probability one for almost all x with respect to the Lebesgue measure on $[0, 1]$ the frequency of a digit $k \in \{0, 1, \dots, 5\}$ in this expansion will be equal to $\frac{1}{6} \sum_{k+1 \leq l \leq 6} 1/l$. Thus the subsets of $[0, 1]$ which have other frequencies of digits in the above expansion will have Lebesgue measure zero, and one may be interested in computing the Hausdorff dimension of such sets, which in the deterministic case was done many years ago in the classical papers [Be] and [Eg] of Besicovitch and Eggleston. More generally, we can simultaneously throw d dice each time, write d respective “random base” expansions, and study the dimensions of subsets of $[0, 1]^d$ corresponding to different frequencies of vectors of d digits in these expansions.

The general setup includes an ergodic measure preserving transformation θ of a probability space (Ω, P) and a \mathbb{Z}_+^d -valued random vector $m(\omega) = (m_1(\omega), \dots, m_d(\omega))$, $\omega \in \Omega$, $m_i(\omega) \in \mathbb{Z}_+ = \{1, 2, \dots\}$, such that for all $i = 1, \dots, d$,

$$(1.1) \quad 0 < \int \log m_i dP = \int \log m_1 dP < \infty.$$

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For any $x = (x_1, \dots, x_d) \in [0, 1]^d$ one can write

$$(1.2) \quad x = \sum_{j=0}^{\infty} x_j(\omega) (M(\theta^j \omega) \cdots M(\theta \omega) M(\omega))^{-1},$$

where $M(\omega)$ is the $d \times d$ diagonal matrix with the diagonal elements $m_1(\omega), \dots, m_d(\omega)$ and $x_j(\omega) = (x_{1j}(\omega), \dots, x_{dj}(\omega))$ are row vectors with $x_{ij}(\omega) \in \{0, 1, \dots, m_i(\theta^j \omega) - 1\}$. This expansion is always possible since for $x_i \in [0, 1]$ we can set $x_{i0}(\omega) = [x_i m_i(\omega)]$, where $[\cdot]$ denotes the integer part, and

$$(1.3) \quad x_{ij}(\omega) = \left[m_i(\omega) m_i(\theta \omega) \cdots m_i(\theta^j \omega) \left(x - \sum_{n=0}^{j-1} \frac{x_{in}(\omega)}{m_i(\omega) \cdots m_i(\theta^n \omega)} \right) \right]$$

after $x_{i0}(\omega), \dots, x_{i,j-1}(\omega)$ have already been defined. Thus zero tails are permitted but the tails $x_{ij}(\omega) = m_i(\theta^j \omega) - 1$, for all $j \geq n$ are not. For any $k, l \in \mathbb{Z}_+^d$ set

$$(1.4) \quad \begin{aligned} N_{kl}^\omega(x, n) &= |\{j \geq 0, j < n : m(\theta^j \omega) = k, x_j(\omega) = l - \mathbf{1}\}|, \\ N_l^\omega(x, n) &= \sum_{k \in \mathbb{Z}_+^d} N_{kl}^\omega(x, n), \end{aligned}$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{Z}_+^d$ and $|\{\cdot\}|$ denotes the number of elements in a set $\{\cdot\}$. Let $r = (r_k, k \in \mathbb{Z}_+^d)$ be an infinite probability vector and $A = (a_{kl}, k, l \in \mathbb{Z}_+^d)$ be an infinite probability matrix such that $a_{kl} = 0$ unless $l \leq k$, where $l = (l_1, \dots, l_d) \leq k = (k_1, \dots, k_d)$ means that $l_i \leq k_i, \forall i = 1, \dots, d$. I shall compute for P -almost all (a.a.) $\omega \in \Omega$ the Hausdorff dimensions (which will be denoted by HD) of the sets

$$(1.5) \quad U_r^\omega = \left\{ x \in [0, 1]^d : \lim_{n \rightarrow \infty} \frac{1}{n} N_l^\omega(x, n) = r_l \text{ for all } l \in \mathbb{Z}_+^d \right\}$$

and

$$(1.6) \quad V_A^\omega = \left\{ x \in [0, 1]^d : \lim_{n \rightarrow \infty} \frac{1}{n} N_{kl}^\omega(x, n) = q_k a_{kl} \text{ for all } k, l \in \mathbb{Z}_+^d \right\}$$

where $q_k = P\{m = k\}$. By ergodicity of θ it follows that $HD(U_r^\omega)$ and $HD(V_A^\omega)$ are the same for P -a.a. ω . I obtain these Hausdorff dimensions in Theorem 2.1 of the next section. If $d = 1$ and m is constant with probability one, then we get the results from [Be] and [Eg]. B. Mandelbrot pointed out to me the paper of Peyrière [Pey], who computed the Hausdorff dimension of sets similar to V_A^ω with $d = 1$, which correspond to expansions of numbers from $[0, 1]$ with respect to nonrandom sequences of positive integers satisfying certain conditions which, in general, do not hold true with probability one for the sequence $m_1(\omega), m_1(\theta \omega), m_1(\theta^2 \omega), \dots$ assuming only (1.1).

Define the transformations $f(\omega) : [0, 1]^d \rightarrow [0, 1]^d$ and $\tau : [0, 1]^d \times \Omega \rightarrow [0, 1]^d \times \Omega$ by the formulas

$$(1.7) \quad (f(\omega)x)_i = m_i(\omega)x_i - [m_i(\omega)x_i] \quad \forall i = 1, \dots, d \text{ and } x = (x_1, \dots, x_d) \in [0, 1]^d$$

and

$$(1.8) \quad \tau(x, \omega) = (f(\omega)x, \theta\omega).$$

It is convenient to identify $[0, 1]^d$ with the d -dimensional torus \mathbb{T}^d , and from now on I assume that the expansion (1.2) is given for points of \mathbb{T}^d . Thus $f(\omega)$ becomes a random smooth expanding transformation of \mathbb{T}^d which I use frequently in this paper.

For any $x \in \mathbb{T}^d$ and $\omega \in \Omega$ set $\phi(x, \omega) = x_0(\omega) \in \mathbb{Z}_+^d$. Then $x_1(\omega) = (f(\omega)x)_0(\theta\omega) = (\phi \circ \tau)(x, \omega)$, and so by induction, assuming that $(\phi \circ \tau^{i-1})(x, \omega) = x_{i-1}(\omega)$, I obtain

$$(1.9) \quad (\phi \circ \tau^i)(x, \omega) = (\phi \circ \tau^{i-1})(f(\omega)x, \theta\omega) = (f(\omega)x)_{i-1}(\theta\omega) = x_i(\omega).$$

This connection between the expansion (1.2) and the skew product transformation τ enables me to compute $HD(U_r^\omega)$ and $HD(V_A^\omega)$ by modifying Billingsley's entropy approach to the results from [Be] and [Eg] given in [Bi1]-[Bi3]. Employing the same machinery I obtain in Section 3 Hausdorff dimensions of more general sets, relax the condition (1.1), and also derive almost sure large deviations bounds for $\frac{1}{n}N_{kl}^\omega(x, n)$ and $\frac{1}{n}N_l^\omega(x, n)$.

Let $K^\omega, \omega \in \Omega$, be a measurable family of compact subsets of \mathbb{T}^d such that

$$(1.10) \quad f(\omega)K^\omega = K^{\theta\omega}.$$

One obtains such sets, for instance, by considering all $x \in \mathbb{T}^d$ whose expansion (1.2) does not contain certain prescribed "vector-digits" which can be called random Cantor sets. Generalizing Furstenberg's deterministic formula from [Fu], I shall show in Section 4 that for compact K^ω satisfying (1.10) with probability one,

$$(1.11) \quad HD(K^\omega) = \frac{h_{\text{top}}^{(r)}(\tau, K)}{\int m_1 dP}$$

where $h_{\text{top}}^{(r)}(\tau, K)$ is the relativized topological entropy of τ restricted to the invariant set $K = \{(x, \omega) : x \in K^\omega\}$. Note that the sets U_r^ω and V_A^ω also satisfy (1.10), but they are usually everywhere dense in \mathbb{T}^d , and so they should be treated differently. The Hausdorff and Minkowski dimensions of random Cantor sets K^ω will be computed also in the case when the integrals in (1.1) are not the same for different i 's; this yields random versions of results from [Mc] and [KP2]. Finally, I shall obtain the Hausdorff dimension of general "random invariant repellers" for random conformal maps, which generalizes the corresponding deterministic Bowen-Ruelle formula (see [Ru]). In this case the relativised topological pressure comes into the picture in place of the relativized topological entropy above. I also define random iterated function systems and explain that some of my models can be studied also via this setup.

2. RANDOM BASE EXPANSIONS

By (1.4)-(1.6) and (1.9) I can write

$$(2.1) \quad U_r^\omega = \left\{ x \in \mathbb{T}^d : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{l-1}(\phi \circ \tau^i(x, \omega)) = r_l \forall l \in \mathbb{Z}^d \right\},$$

where $\chi_l(k) = 1$ if $l = k$ and $\chi_l(k) = 0$ otherwise, and

$$(2.2) \quad V_A^\omega = \left\{ x \in \mathbb{T}^d : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_k(m \circ \theta^i(\omega)) \chi_{l-1}(\phi \circ \tau^i(x, \omega)) = q_k a_{kl} \text{ for all } l, k \in \mathbb{Z}_+^d, l \leq k \right\}.$$

Therefore, with probability one,

$$(2.3) \quad f(\omega)V_A^\omega = \left\{ y \in \mathbb{T}^d : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-2} \chi_k(m \circ \theta^i(\theta\omega)) \chi_{l-1}(\phi \circ \tau^i(y, \theta\omega)), = q_k a_{kl} \text{ for all } l, k \in \mathbb{Z}_+^d, l \leq k \right\} = V_A^{\theta\omega}$$

and similarly,

$$(2.4) \quad f(\omega)U_r^\omega = U_r^{\theta\omega}.$$

Thus, clearly,

$$(2.5) \quad HD(U_r^\omega) = HD(U_r^{\theta\omega}) \quad \text{and} \quad HD(V_A^\omega) = HD(V_A^{\theta\omega}),$$

and since θ is ergodic and $HD(U_r^\omega), HD(V_A^\omega)$ depend measurably on ω , I conclude that with probability one,

$$(2.6) \quad HD(U_r^\omega) = \text{const} \quad \text{and} \quad HD(V_A^\omega) = \text{const}.$$

The main result of this section is the following

2.1. Theorem. *With probability one,*

$$(2.7) \quad HD(V_A^\omega) = \frac{-\sum_{k \in \mathbb{Z}_+^d} q_k \sum_{l \leq k} a_{kl} \log a_{kl}}{\int \log m_1 dP} \stackrel{\text{def}}{=} H_A,$$

and so $HD(V_A^\omega) = d$ if and only if $a_{kl} = \prod_{i=1}^d k_i^{-1}$ for all $l \leq k$ and any $k = (k_1, \dots, k_d) \in \mathbb{Z}_+^d$ such that $q_k \neq 0$. In the last case with probability one V_A^ω also has Lebesgue measure one. The sets U_r^ω have Lebesgue measure one for P -a.a. ω if and only if $r_l = \sum_{k \geq l} q_k \prod_{i=1}^d k_i^{-1}$ for all $l \in \mathbb{Z}_+^d$. Furthermore, for P -a.a. ω ,

$$(2.8) \quad HD(U_r^\omega) = \sup_{A \in \mathcal{A}_{qr}} H_A \stackrel{\text{def}}{=} H,$$

where the supremum in (2.8) is taken over the set \mathcal{A}_{qr} of all infinite probability matrices $A = (a_{kl})$ such that $a_{kl} = 0$ unless $l \leq k$ and $qA = r$ with q and r considered

as row vectors (i.e. $\sum_{k \in \mathbb{Z}_+^d} q_k a_{kl} = r_l, \forall l \in \mathbb{Z}_+^d$). The set \mathcal{A}_{qr} is nonempty if and only if

$$(2.9) \quad \sum_{l \in F} q_l \geq \sum_{l \in F} r_l$$

for any $F \in \mathcal{F}$, where \mathcal{F} is the collection of all filters in \mathbb{Z}_+^d , i.e. the subsets $F \subset \mathbb{Z}^d$ such that if $l \in F$ and $l \leq k$, then $k \in F$. If (2.9) does not hold true for some $F \in \mathcal{F}$, then with probability one U_r^ω is empty.

Proof. By the definition (1.4),

$$(2.10) \quad \sum_{l \in F} N_l^\omega(x, n) = \sum_{l \in F} \sum_{k \in \mathbb{Z}_+^d} N_{kl}^\omega(x, n) = \sum_{k \in F} \sum_{l \in F} N_{kl}^\omega(x, n) \leq \sum_{k \in F} M_k^\omega(n),$$

where

$$M_k^\omega(n) = |I_k^\omega(n)| \quad \text{and} \quad I_k^\omega(n) = \{j \geq 0 : m(\theta^j \omega) = k, j < n\},$$

and so by Birkhoff's ergodic theorem, with probability one,

$$(2.11) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{l \in F} N_l^\omega(x, n) \leq \sum_{k \in F} q_k.$$

Hence, if (2.9) does not hold true then U_r^ω is empty with probability one.

For each $\kappa > 0$ and a set $M \subset \mathbb{T}^d$ put

$$(2.12) \quad \ell_\kappa(M, \rho) = \inf \sum_i |v_i|^\kappa,$$

where the infimum is taken over all covers of M by cubes v_i with sides of length not exceeding ρ , where $|v_i|$ denotes the volume of v_i . Let $\alpha_0 = (\alpha_{10}, \dots, \alpha_{d0}), \dots, \alpha_{n-1} = (\alpha_{1,n-1}, \dots, \alpha_{d,n-1})$ be some integer vectors such that $0 \leq \alpha_{ij} < m_i(\theta^j \omega)$. Set

$$C_{\alpha_0, \dots, \alpha_{n-1}}^\omega = \{x \in \mathbb{T}^d : x_j(\omega) = \alpha_j, \forall j = 0, 1, \dots, n-1\},$$

which I shall call an (ω, n) -cylinder or just an ω -cylinder if n is not fixed. Note that each such cylinder is a rectangular parallelepiped with the volume

$$(2.13) \quad |C_{a_0, \dots, a_{n-1}}^\omega| = \left(\prod_{i=1}^d \left(\prod_{j=0}^{n-1} m_i(\theta^j \omega) \right) \right)^{-1}.$$

Set also

$$(2.14) \quad \lambda_\kappa^\omega(M, \rho) = \inf \sum_i |C_i|^\kappa,$$

where the infimum is taken over all covers of $M \subset \mathbb{T}^d$ by ω -cylinders C_i whose maximal side does not exceed ρ , where $|C_i|$ is the volume of C_i . The following result is a generalization of an argument from [Bi 2].

2.2. Lemma. For any $\varepsilon > 0$ and P -a.a. ω there exists $K_\varepsilon(\omega) > 0$ such that for any set $M \subset \mathbb{T}^d$, $\rho > 0$, and $\kappa > \varepsilon$,

$$(2.15) \quad K_\varepsilon^{-1}(\omega)\ell_{\kappa+\varepsilon}(M, \rho) \leq \lambda_\kappa^\omega(M, \rho) \leq K_\varepsilon(\omega)\ell_{\kappa-\varepsilon}(M, \rho).$$

Proof. Take a cube $u \subset \mathbb{Z}^d$ and let n be minimal such that $u \supset C_{\alpha_0, \dots, \alpha_{n-1}}^\omega$ (i.e. u does not contain any $(\omega, n-1)$ -cylinder $C_{\beta_0, \dots, \beta_{n-2}}^\omega$). Then it is easy to see that u is covered by no more than $2^d \prod_{i=1}^d m_i(\theta^{n-1}\omega)$ (ω, n) -cylinders (whose maximal side obviously does not exceed the side of u). By (2.13) I derive that

$$(2.16) \quad |C_{\alpha_0, \dots, \alpha_{n-1}}^\omega|^\kappa \leq |u|^\kappa \leq \left((2^d \prod_{i=1}^d m_i(\theta^{n-1}\omega))^\kappa |C_{\alpha_0, \dots, \alpha_{n-1}}^\omega|^\kappa \right) \leq K_\varepsilon^{(n)}(\omega) |u|^{\kappa-\varepsilon}$$

where

$$K_\varepsilon^{(n)} = 2^{d(\kappa-\varepsilon)} \left(\prod_{i=1}^d m_i(\theta^{n-1}\omega) \right)^\kappa \left(\prod_{i=1}^d \left(\prod_{j=0}^{n-2} m_i(\theta^j\omega) \right) \right)^{-1}.$$

Next, observe that any cylinder $C_{\alpha_0, \dots, \alpha_{n-1}}^\omega$ contains a cube u with side equal to the minimal side of $C_{\alpha_0, \dots, \alpha_{n-1}}^\omega$, and so each such cylinder can be covered by no more than

$$k_n(\omega) = \left(1 + \left[\max_{i,j} \left(\prod_{a=0}^{n-1} \frac{m_i(\theta^a\omega)}{m_j(\theta^a\omega)} \right) \right] \right)^d$$

of such cubes u , where $[\cdot]$ denotes the integer part. Then

$$(2.17) \quad k_n(\omega) |u|^{\kappa+\varepsilon} \leq L_\varepsilon^{(n)}(\omega) |C_{\alpha_0, \dots, \alpha_{n-1}}^\omega|^\kappa,$$

where

$$L_\varepsilon^{(n)}(\omega) = k_n(\omega) |u|^\varepsilon = k_n(\omega) \min_i \left(\prod_{j=0}^{n-1} m_i(\theta^j\omega) \right)^{-1}.$$

Observe that, by (2.1) and the Birkhoff ergodic theorem, with probability one for each $i = 1, \dots, d$,

$$(2.18) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log m_i(\theta^j\omega) = \int \log m_1 dP > 0,$$

and by the ergodic theorem, with probability one,

$$(2.19) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log m_i(\theta^n\omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \left(\sum_{j=0}^n \log m_i(\theta^j\omega) - \sum_{j=0}^{n-1} \log m_i(\theta^j\omega) \right) = 0.$$

It follows that with probability one

$$(2.20) \quad \lim_{n \rightarrow \infty} K_\varepsilon^{(n)} = \lim_{n \rightarrow \infty} L_\varepsilon^{(n)} = 0.$$

Then with probability one $K_\varepsilon = \sup_{n \geq 2} \max(K_\varepsilon^{(n)}, L_\varepsilon^{(n)}) < \infty$, and I derive (2.15) with such K_ε in view of (2.16) and (2.17). \square

Next, similarly to [Bi1] and [Bi2], for any nonatomic probability measure μ on \mathbb{T}^d set

$$\mu_\kappa^\omega(M, \rho) = \inf \sum_i \mu(C_i)^\kappa,$$

where the infimum is taken over all covers by ω -cylinders C_i such that $\mu(C_i) < \rho$. Note that if μ is the Lebesgue measure on \mathbb{T}^d , then $\mu_\kappa^\omega(M, \rho) = \lambda_\kappa^\omega(M, \rho)$. By the monotonicity of $\ell_\kappa(M, \rho)$, $\lambda_\kappa^\omega(M, \rho)$, and $\mu_\kappa^\omega(M, \rho)$ in ρ there exist the limits

$$(2.21) \quad \begin{aligned} \ell_\kappa(M) &= \lim_{\rho \rightarrow 0} \ell_\kappa(M, \rho), & \lambda_\kappa^\omega(M) &= \lim_{\rho \rightarrow 0} \lambda_\kappa^\omega(M, \rho), \\ \mu_\kappa^\omega(M) &= \lim_{\rho \rightarrow 0} \mu_\kappa^\omega(M, \rho). \end{aligned}$$

By Lemma 2.2, for P -a.a. ω ,

$$HD(M) = \kappa_0 \quad \text{iff} \quad \lambda_\kappa^\omega(M) = \infty, \forall \kappa < \kappa_0 \text{ and } \lambda_\kappa^\omega(M) = 0, \forall \kappa > \kappa_0.$$

Set

$$HD_\mu^\omega(M) = \inf\{\kappa > 0 : \mu_\kappa^\omega(M) = 0\};$$

then it is easy to check (see [Bi1]) that $\mu_\kappa^\omega(M) = \infty$ for all $\kappa < HD_\mu^\omega(M)$. By Lemma 2.2,

$$(2.22) \quad HD_{\text{Leb}}^\omega(M) = \inf\{\kappa > 0 : \lambda_\kappa^\omega(M) = 0\} = d^{-1}HD(M),$$

where Leb denotes the Lebesgue measure on \mathbb{T}^d . For $x \in \mathbb{T}^d$ denote $C_n(x) = C_{x_0(\omega), \dots, x_{n-1}(\omega)}^\omega$. Let μ and ν be nonatomic probability measures on \mathbb{T}^d and $\delta = \delta^\omega$ a nonnegative random variable. Set

$$\underline{X}_{\mu, \nu}^\omega(\delta) = \left\{ x : \delta^\omega \leq \liminf_{n \rightarrow \infty} \frac{\log \nu(C_n^\omega(x))}{\log \mu(C_n^\omega(x))} \right\}$$

and

$$\overline{X}_{\mu, \nu}^\omega(\delta) = \left\{ x : \limsup_{n \rightarrow \infty} \frac{\log \nu(C_n^\omega(x))}{\log \mu(C_n^\omega(x))} \leq \delta^\omega \right\}$$

with the convention that $\log a / \log 0 = \log 1 / \log 0 = 0$, $\log 0 / \log a = \log a / \log 1 = \log 0 / \log 1 = \infty$ and $\log 0 / \log 0 = \log 1 / \log 1 = 1$ for any $a \in (0, 1)$. The following is a version of Theorems 2.2 and 2.3 from [Bi1,II] (see also Theorem 14.1 from [Bi2], Section 5 in [Bi3], and Proposition 2.1 in [Yo]).

2.3. Proposition. *Let M, N be subsets of \mathbb{T}^d and $\underline{\delta}, \overline{\delta}$ nonnegative random variables. Then for all $\omega \in \Omega$,*

$$(2.23) \quad \text{if } M \subset \underline{X}_{\mu, \nu}^\omega(\underline{\delta}), \text{ then } HD_\mu^\omega(M) \geq \underline{\delta}^\omega HD_\nu^\omega(M)$$

and

$$(2.24) \quad \text{if } N \subset \overline{X}_{\mu, \nu}^\omega(\overline{\delta}), \text{ then } HD_\mu^\omega(N) \leq \overline{\delta}^\omega HD_\nu^\omega(N).$$

Moreover,

$$(2.25) \quad HD_\mu^\omega \left\{ x : \liminf_{n \rightarrow \infty} \frac{\log \nu(C_n^\omega(x))}{\log \mu(C_n^\omega(x))} \leq \bar{\delta}^\omega \right\} \leq \bar{\delta}^\omega.$$

Proof. For each $\omega \in \Omega$ I can define the stochastic process $y_n^\omega(x) = x_n(\omega)$ on the probability space (\mathbb{T}^d, μ) , and this is precisely the setup of Theorems 2.2 and 2.3 from [Bi1,II] which imply Proposition 2.3 above. \square

Next, I shall apply Proposition 2.3 with $\mu = \text{Leb}$, taking into account that, by (2.13),

$$\text{Leb}(C_n^\omega(x)) = \left(\prod_{i=1}^d \left(\prod_{j=0}^{n-1} m_i(\theta^j \omega) \right) \right)^{-1},$$

and so, by Birkhoff's ergodic theorem,

$$(2.26) \quad Y_\nu^\omega(\gamma) \stackrel{\text{def}}{=} \left\{ x : \lim_{n \rightarrow \infty} \left(-\frac{1}{n} \log \nu(C_n^\omega(x)) \right) = \gamma \right\} \\ = \left\{ x : \lim_{n \rightarrow \infty} \left(\frac{\frac{1}{n} \log \nu(C_n^\omega(x))}{\frac{1}{n} \log \text{Leb}(C_n^\omega(x))} \right) = \frac{\gamma}{\int \log m_1 dP} \right\}.$$

Then, by (2.22) and Proposition 2.3,

$$(2.27) \quad \text{if } M \subset Y_\nu^\omega(\gamma), \text{ then } HD(M) = \frac{\gamma}{\int \log m_1 dP} HD_\nu^\omega(M).$$

By (2.18) for each x and P -a.a. ω , $C_n^\omega(x) \downarrow \{x\}$ as $n \uparrow \infty$, and so for any nonatomic measure ν with probability one $\nu(C_n^\omega(x)) \rightarrow 0$ as $n \rightarrow \infty$. It is easy to see (cf. [Bi1,2]) that for such ν if $\nu(M) > 0$ then $HD_\nu^\omega(M) = 1$ with probability one. Thus by (2.27), for P -a.a. ω ,

$$(2.28) \quad \text{if } M \subset Y_\nu^\omega(\gamma) \text{ and } \nu(M) > 0, \text{ then } HD(M) = \frac{\gamma}{\int \log m_1 dP}.$$

Next, in order to obtain (2.7) via (2.28) I should construct an appropriate family of measures ν^ω , $\omega \in \Omega$. Consider a measurable family p^ω , $\omega \in \Omega$, of infinite probability vectors $p^\omega = (p_l^\omega; l \in \mathbb{Z}_+^d)$ such that $\sum_{l \leq m(\omega)} p_l^\omega = 1$. Define

$$(2.29) \quad \nu_p^\omega(C_{\alpha_0, \dots, \alpha_{n-1}}^\omega) = \prod_{j=0}^{n-1} p_{\alpha_j+1}^{\theta^j \omega}.$$

Since for P -a.a. ω the ω -cylinders generate the whole Borel σ -algebra on \mathbb{T}^d , then by the Kolmogorov extension theorem for P -a.a. ω there exists a unique (and so measurably depending on $\omega \in \Omega$) probability measure ν_p^ω on \mathbb{T}^d satisfying (2.29). Observe that if $p_l^\omega = (m_1(\omega) \cdots m_d(\omega))^{-1}$ for all $l \leq m(\omega)$, $l \in \mathbb{Z}_+^d$ and $\omega \in \Omega$, then

$\nu_p^\omega = \text{Leb}$ for all ω , since then $\nu_p(C_{\alpha_0, \dots, \alpha_{n-1}}^\omega)$ equals the right hand side of (2.13). Next,

$$\begin{aligned} (f(\omega)\nu_p^\omega)(C_{\alpha_0, \dots, \alpha_{n-1}}^{\theta\omega}) &= \nu_p^\omega(f^{-1}(\omega)C_{\alpha_0, \dots, \alpha_{n-1}}^{\theta\omega}) = \sum_{\beta \leq m(\omega)-1} \nu_p^\omega(C_{\beta, \alpha_0, \dots, \alpha_{n-1}}^\omega) \\ &= \sum_{\beta \leq m(\omega)-1} p_{\beta+1}^\omega \prod_{j=0}^{n-1} p_{\alpha_j+1}^{\theta^{j+1}\omega} = \prod_{j=0}^{n-1} p_{\alpha_j+1}^{\theta^{j+1}\omega} = \nu_p^{\theta\omega}(C_{\alpha_0, \dots, \alpha_{n-1}}^{\theta\omega}). \end{aligned}$$

Since this is true for any ω -cylinder, I derive that for P -a.a. ω ,

$$(2.30) \quad f(\omega)\nu_p^\omega = \nu_p^{\theta\omega}.$$

Define the probability measure ν_p on $\mathbb{T}^d \times \Omega$ by $\nu_p(W) = \int \nu_p^\omega(W^\omega) dP(\omega)$, where $W^\omega = \{x \in \mathbb{T}^d : (x, \omega) \in W\}$ and $W \subset \mathbb{T}^d \times \Omega$. Then (see [Ki2]) (2.30) implies that ν_p is invariant with respect to the skew product transformation τ given by (1.8). Moreover, it turns out that ν_p is ergodic (see [KK] and Remark 2.8 below) since it is the unique equilibrium state corresponding to the function $\psi(x, \omega) = \log p_{x_0(\omega)+1}^\omega$ (such ν_p is naturally called “a random Bernoulli measure”). This argument requires an additional justification in the case when some of the p_l^ω are equal to 0, and so the function ψ may take the value $-\infty$. Namely, I construct the measures ν_p^ω only on the subset of x 's such that $x_i(\omega) \in \{0, 1, \dots, m(\theta^i\omega)\}$ and $p_{x_i(\omega)+1}^{\theta^i\omega} > 0$. In the symbolic representation this corresponds to the sequences $(\alpha_0, \alpha_1, \dots)$ with $p_{\alpha_i}^{\theta^i\omega} > 0$ which determines a random subshift of finite type (see [BG] and [KK]). Again, the corresponding measure ν_p will be ergodic (see [KK]). The measure ν_p^ω will be nonatomic except for the special case when all the p_l^ω equal zero or one, only. Observe also that except for the last case, with probability one,

$$\nu_p^\omega \left(\bigcup_{n; \alpha_0, \dots, \alpha_{n-1}} \partial C_{\alpha_0, \dots, \alpha_{n-1}}^\omega \right) = 0,$$

which is a partial case of the general statement saying that the boundaries of random Markov partitions for random expanding transformations have zero measure for any Gibbs measure (equilibrium state) constructed by a function $\psi(x, \omega)$ Hölder continuous in x (see [KK]).

Now let the probability vectors $p_A^\omega = (p_l^\omega, l \in \mathbb{Z}_+^d)$ be constructed by the matrix A appearing in (1.6) so that $p_l^\omega = a_{kl}$ for $\omega \in \{\omega : m(\omega) = k\}$, $k \in \mathbb{Z}_+^d$. Assume first that $0 < a_{kl} < 1$ for all $l \leq k$ and $k \in \mathbb{Z}_+^d$. Then by (1.4),

$$-\frac{1}{n} \log \nu_{p_A}^\omega(C_n^\omega(x)) = -\frac{1}{n} \sum_{j=0}^{n-1} \log p_{x_j(\omega)+1}^{\theta^j\omega} = -\frac{1}{n} \sum_{k \in \mathbb{Z}_+^d} \sum_{l \leq k} N_{kl}^\omega(x, n) \log a_{kl},$$

provided $\nu_{p_A}^\omega(C_n^\omega(x)) \neq 0$, and so, by the definition (1.6),

$$(2.31) \quad V_A^\omega \subset \left\{ x : \lim_{n \rightarrow \infty} \left(-\frac{1}{n} \log \nu_{p_A}^\omega(C_n^\omega(x)) \right) = - \sum_{k \in \mathbb{Z}_+^d} q_k \sum_{l \leq k} a_{kl} \log a_{kl} \right\}.$$

As I observed above, ν_{p_A} is an ergodic invariant measure of the skew product transformation τ , and so, by Birkhoff's ergodic theorem for ν_{p_A} -a.a. (x, ω) ,

$$(2.32) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_k(m \circ \theta^j(\omega)) \chi_{l-1}(\phi \circ \tau^j(x, \omega)) \\ &= \int \chi_k(m(\omega)) \left(\int \chi_{l-1}(x_0(\omega)) d\nu_{p_A}^\omega(x) \right) dP(\omega) = q_k a_{kl}. \end{aligned}$$

By the Fubini theorem I conclude from (2.2) and (2.32) that $\nu_{p_A}^\omega(V_A^\omega) = 1$ for P -a.a. ω . This together with (2.31) enables me to apply (2.28), which yields (2.7). Observe that the function $-t \log t$ is strictly convex on $[0, 1]$, and so $-\sum_{l \leq k} a_{kl} \log a_{kl} \leq \log \prod_{i=1}^d k_i$ if $k = (k_1, \dots, k_d) \in \mathbb{Z}_+^d$, with equality if and only if $a_{kl} = (\prod_{i=1}^k k_i)^{-1}$ for all $l \leq k$, $l \in \mathbb{Z}_+^d$. For such A one has $\nu_{p_A}^\omega = \text{Leb}$ P -a.s., and so $\text{Leb}(V_A^\omega) = 1$ P -a.s. It follows also by (1.4), (2.1), and (2.32) that for $\text{Leb } P$ - a.a. (x, ω) ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} N_l^\omega(x, n) = \sum_{k \geq l} \left(\prod_{i=1}^d k_i \right)^{-1} q_k,$$

and so $\text{Leb}(U_r^\omega) = 1$ with probability one if and only if $r_l = \sum_{k \geq l} (\prod_{i=1}^d k_i)^{-1} q_k$, for all $l \in \mathbb{Z}_+^d$. If some a_{kl} equal 0 for k with $q_k = 0$, I just redefine a_{kl} to make them positive without any influence on the final result. If $q_k > 0$ and some $a_{kl} = 0$, then the measure $\nu_{p_A}^\omega$ is still supported by those x for which $p_{x_i(\omega)+1}^{\theta^i \omega} > 0$, i.e. the set \tilde{V}_A^ω of such x 's has $\nu_{p_A}^\omega$ -measure equal to 1. In general, \tilde{V}_A^ω is a proper subset of V_A^ω , and so I obtain only the lower bound $H_A = HD(\tilde{V}_A^\omega) \leq HD(V_A^\omega)$. In order to obtain the upper bound in this case one can employ either direct combinatorial estimates as (2.38)-(2.39) below estimating the number of cylinder sets intersecting V_A , which resemble Eggleston's original estimates from [Eg], or proceed by an approximation argument described in Remark 2.5 below. Observe that even in the deterministic case, if some of the prescribed frequencies are zero the proof of the upper bound using (2.24) (or versions of it) does not work, and one should employ either combinatorial upper bound estimates from [Eg] or the argument from Remark 2.5 below. This point was overlooked in [Bi1-3] and in [Fa], and, it seems, nobody noticed it for more than 30 years since this proof first appeared.

Next, I shall derive (2.8). First, observe that by (1.4) if $A = (a_{kl}) \in \mathcal{A}_{qr}$ and $\lim_{n \rightarrow \infty} \frac{1}{n} N_{kl}^\omega(x, n) = q_k a_{kl}$ for all $l \leq k$ and $k, l \in \mathbb{Z}_+^d$, then $\lim_{n \rightarrow \infty} \frac{1}{n} N_l^\omega(x, n) = r_l$ for all $l \in \mathbb{Z}_+^d$. Thus $V_A^\omega \subset U_r^\omega$ for any $A \in \mathcal{A}_{qr}$, and so

$$(2.33) \quad HD(U_r^\omega) \geq \sup_{A \in \mathcal{A}_{qr}} HD(V_A^\omega) = H.$$

P. Billingsley indicated to me that in order to obtain the inequality in the other direction one can employ a modification of the argument of Theorem 6 from [Bi3] based on a stronger version of Proposition 2.3 above which does not require some of the computations below. Still, this argument requires some additional work in order to adapt it to my circumstances, so I shall proceed by directly estimating the number of (ω, n) -cylinders needed to cover the set U_r^ω . Put $r_l^\omega(x, n) = \frac{1}{n} N_l^\omega(x, n)$,

$q_k^\omega(n) = \frac{1}{n}M_k^\omega(n)$ (where $M_k^\omega(n)$ is the same as in (2.10)), $d_{kl}^\omega(x, n) = \frac{1}{n}N_{kl}^\omega(x, n)$, and let $r^\omega(x, n) = (r_l^\omega(x, n))$, $q^\omega(n) = (q_k^\omega(n))$, $D^\omega(x, n) = (d_{kl}^\omega(x, n))$ be the corresponding vectors and matrix. Since $\sum_{l \leq k} N_{kl}^\omega(x, n) = M_k^\omega(n)$, taking into account (1.4) I can choose an infinite probability matrix $A^\omega(x, n) = (a_{kl}^\omega(x, n))$ so that $a_{kl}^\omega(x, n) = 0$ unless $l \leq k$, and

$$(2.34) \quad d_{kl}^\omega(x, n) = q_k^\omega(n)a_{kl}^\omega(x, n) \quad \text{and} \quad q^\omega(n)A^\omega(x, n) = r^\omega(x, n).$$

Let ρ be a metric on the compact convex space of infinite probability matrices so that for $B = (b_{kl})$ and $C = (c_{kl})$,

$$\rho(B, C) = \sum_{k, l \in \mathbb{Z}_+^d} 2^{-\|k\|} |b_{kl} - c_{kl}|,$$

where $\|k\| = |k_1| + \dots + |k_d|$. Denote also

$$\mathcal{D}_{qr} = \{D = (d_{kl}) : d_{kl} = q_k a_{kl} \text{ for some } A = (a_{kl}) \in \mathcal{A}_{qr}\}.$$

By Birkhoff's ergodic theorem with probability one $q^\omega(n) \rightarrow q$ as $n \rightarrow \infty$, and by the definition (1.5) $r^\omega(x, n) \rightarrow r$ as $n \rightarrow \infty$ for any $x \in U_r^\omega$. This together with (2.34) yields that with probability one

$$(2.35) \quad U_r^\omega \subset \{x : \lim_{n \rightarrow \infty} \rho(D^\omega(x, n), \mathcal{D}_{qr}) = 0\}$$

and, for $A = (a_{kl})$,

$$(2.36) \quad V_A^\omega = \{x : \lim_{n \rightarrow \infty} D^\omega(x, n) = D = (d_{kl}), d_{kl} = q_k a_{kl}\}.$$

Since \mathcal{A}_{qr} is compact, for each small $\varepsilon > 0$ I can choose matrices $A^{\varepsilon, j} = (a_{k, l}^{\varepsilon, j}) \in \mathcal{A}_{qr}$, $j = 1, \dots, J_\varepsilon$, such that for any $A \in \mathcal{A}_{qr}$ there exists j with $\rho(A, A^{\varepsilon, j}) < \varepsilon$. Let $D^{\varepsilon, j} = (d_{kl}^{\varepsilon, j})$; where $d_{kl}^{\varepsilon, j} = q_k a_{kl}^{\varepsilon, j}$ then also for any $D \in \mathcal{D}_{qr}$ there exists j with $\rho(D, D^{\varepsilon, j}) < \varepsilon$. Set

$$G_{\varepsilon, j}^\omega(n) = \{x \in \mathbb{T}^d : \rho(D^\omega(x, n), D^{\varepsilon, j}) < \varepsilon\} \quad \text{and} \quad G_\varepsilon^\omega(n) = \bigcup_{1 \leq j \leq J_\varepsilon} G_{\varepsilon, j}^\omega(n).$$

By (2.35), for P -a.a. ω ,

$$(2.37) \quad U_r^\omega \subset \bigcup_{i=1}^\infty \bigcap_{n \geq i} G_\varepsilon^\omega(n).$$

It is clear that $G_{\varepsilon, j}^\omega(n)$ and $G_\varepsilon^\omega(n)$ are unions of collections $\mathcal{G}_{\varepsilon, j}^\omega(n)$ and $\mathcal{G}_\varepsilon^\omega(n) = \bigcup_{1 \leq j \leq J_\varepsilon} \mathcal{G}_{\varepsilon, j}^\omega(n)$ of (ω, n) -cylinders so that

$$(2.38) \quad \mathcal{G}_{\varepsilon, j}^\omega(n) = \left\{ C_{\alpha_0, \dots, \alpha_{n-1}}^\omega : \left| \frac{1}{n} \sum_{i \in I_k^\omega(n)} \chi_{l-1}(\alpha_i) - d_{kl}^{\varepsilon, j} \right| < \varepsilon 2^{\|k\|}, \forall l \leq k; k, l \in \mathbb{Z}_+^d \right\}$$

where, recall, $I_k^\omega(n)$ is the set of $i \in \{1, \dots, n\}$ such that $(m \circ \theta^i)(\omega) = k$. Let $\tilde{\Omega}$ be the set of all ω such that $\lim_{n \rightarrow \infty} q^\omega(n) = q$ and (2.19),(2.35),(2.36) hold true. Then $P(\tilde{\Omega}) = 1$. Fix an $\omega \in \tilde{\Omega}$. Observe that if $q_k = 0$ then $\lim_{n \rightarrow \infty} q_k^\omega(n) = 0$, and so $\lim_{n \rightarrow \infty} d_{kl}^\omega(x, n) = d_{kl} = 0$ for any $x \in \mathbb{T}^d$, $l \leq k$, and $D = (d_{kl}) \in \mathcal{D}_{qr}$. Similarly to [Eg] and Lemma 5.3 of [Bi1,I] (see also related results in [Ca]), in order to obtain an upper bound for $HD(U_r^\omega)$ I need a good upper bound for the number of cylinders in $\mathcal{G}_{\varepsilon,j}^\omega(n)$. Since the number of elements in $I_k^\omega(n)$ for big n is about nq_k , I can estimate the number $\sigma_{\varepsilon,j}^\omega(n, k)$ of choices of indices $i \in I_k^\omega(n)$ with $\alpha_i = l - \mathbf{1}$, $l \leq k$, such that the condition in braces in (2.38) holds true for a fixed k (i.e. $|\{i \in I_k^\omega(n) : \alpha_i = l - \mathbf{1}\}| \approx nd_{kl}^{\varepsilon,j}$, $\forall l \leq k$) via multinomial coefficients (cf. [Eg],[Bi1,I]),

$$(2.39) \quad \sigma_{\varepsilon,j}^\omega(n, k) \leq \frac{[nq_k]!}{\prod_{l \leq k} ([nd_{kl}^{\varepsilon,j}]!)} e^{L_k^\omega \varepsilon n},$$

where $0! = 1$, $[\cdot]$ denotes the integer part, and $L_k^\omega > 0$ depends only on k and ω . Next, pick a big $K \in \mathbb{Z}_+$. I use (2.39) only for $k \leq \mathbf{K} = (K, \dots, K) \in \mathbb{Z}_+^d$, and for all other k I estimate $\sigma_{\varepsilon,j}^\omega(n, k)$ by the total number of choices of $\alpha_i \leq k$ for all $i \in I_k^\omega(n)$, which is $\prod_{a=1}^d k_a^{M_k^\omega(n)}$. Thus by (2.34) and (2.39) together with Stirling's formula I derive that for big n ,

$$(2.40) \quad \begin{aligned} \frac{1}{n} \log |\mathcal{G}_{\varepsilon,j}^\omega(n)| &\leq \frac{1}{n} \sum_{k \in \mathbb{Z}^d} \log \sigma_{\varepsilon,j}^\omega(n, k) \\ &\leq \frac{1}{n} \sum_{k \leq \mathbf{K}} \log \left(\frac{(nq_k e^{-1})^{nq_k}}{\prod_{l \leq k} (nd_{kl}^{\varepsilon,j} e^{-1})^{nd_{kl}^{\varepsilon,j}}} e^{2L_k^\omega \varepsilon n} \right) + \frac{1}{n} \sum_{k \not\leq \mathbf{K}} \log \prod_{i=1}^d k_i^{M_k^\omega(n)} \\ &\leq \sum_{k \leq \mathbf{K}} \left(q_k \log q_k - \sum_{l \leq k} d_{kl}^{\varepsilon,j} \log d_{kl}^{\varepsilon,j} + 2L_k^\omega \varepsilon \right) + \sum_{k \not\leq \mathbf{K}} q_k^\omega(n) \sum_{i=1}^d \log k_i \\ &= - \sum_{k \leq \mathbf{K}} \sum_{l \leq k} q_k a_{kl}^{\varepsilon,j} \log a_{kl}^{\varepsilon,j} + 2\varepsilon \sum_{k \leq \mathbf{K}} L_k^\omega + \sum_{k \not\leq \mathbf{K}} q_k^\omega(n) \sum_{i=1}^d \log k_i \\ &\leq H \int \log m_1 dP + 2\varepsilon \sum_{k \leq \mathbf{K}} L_k^\omega + \sum_{k \not\leq \mathbf{K}} q_k^\omega(n) \sum_{i=1}^d \log k_i, \end{aligned}$$

where H was defined in (2.8). Observe that

$$(2.41) \quad \begin{aligned} \lim_{n \rightarrow \infty} \sum_{k \not\leq \mathbf{K}} q_k^\omega(n) \sum_{i=1}^d \log k_i &= \lim_{n \rightarrow \infty} \sum_{k \in \mathbb{Z}_+^d} q_k^\omega(n) \sum_{i=1}^d \log k_i - \lim_{n \rightarrow \infty} \sum_{k \leq \mathbf{K}} q_k^\omega(n) \sum_{i=1}^d \log k_i \\ &= d \int \log m_1 dP - \sum_{i=1}^d \int_{\{m \leq \mathbf{K}\}} \log m_i dP = \sum_{i=1}^d \int_{\{m \not\leq \mathbf{K}\}} \log m_i dP. \end{aligned}$$

For an arbitrarily small δ pick K so that the right hand side of (2.41) does not exceed $\delta/2$ and then choose first ε and then $n_0 = n_0^{\omega, \varepsilon}$ so that for all $n \geq n_0$,

$$2\varepsilon \sum_{k \leq \mathbf{K}} L_k^\varepsilon + \sum_{k \notin \mathbf{K}} q_k^\omega(n) \sum_{i=1}^d \log k_i + \frac{1}{n} \log J_\varepsilon \leq \delta.$$

This together with (2.40) yields

$$\frac{1}{n} \log |\mathcal{G}_\varepsilon^\omega(n)| \leq \frac{1}{n} \log(J_\varepsilon \max_{1 \leq j \leq J_\varepsilon} |\mathcal{G}_{\varepsilon, j}^\omega(n)|) \leq H \int \log m_1 dP + \delta.$$

Taking into account (2.13) and (2.19), I find that for any $\kappa > H + \delta(\int \log m dP)^{-1}$,

$$\sum_{C \in \mathcal{G}_\varepsilon^\omega(n)} |C|^\kappa \leq \exp \left(n(H \int \log m_1 dP + \delta - \frac{\kappa}{n} \sum_{i=1}^{n-1} \log m_1(\theta^i \omega)) \right)$$

is bounded for all n big enough. Since δ above is arbitrary, this together with (2.37) yields $HD(U_r^\omega) \leq H$, completing the proof of (2.8). It remains to show that \mathcal{A}_{qr} is nonempty if and only if the condition (2.9) holds true. This is a corollary of the following more general result. I am indebted to B. Weiss for useful discussions and references related to it.

2.4. Proposition. *Let \mathcal{P} be a countable partially ordered set and $q = (q_k, k \in \mathcal{P})$, $r = (r_k, k \in \mathcal{P})$ be infinite probability vectors indexed by \mathcal{P} . Let \mathcal{F} be the collection of all filters in \mathcal{P} , i.e. the subsets F of \mathcal{P} such that if $l \in F$ and $k \geq l$ then $k \in F$. Denote by \mathcal{A}_{qr} the set of all infinite probability matrices $A = (a_{kl}; k, l \in \mathcal{P})$ indexed by \mathcal{P} such that $a_{kl} = 0$ unless $k \geq l$ and $\sum_{k \in \mathcal{P}} q_k a_{kl} = r_l$ for all $l \in \mathcal{P}$. Then \mathcal{A}_{qr} is not empty if and only if*

$$(2.42) \quad \sum_{k \in F} q_k \geq \sum_{k \in F} r_k, \quad \forall F \in \mathcal{F}.$$

Proof. Let X and Y be random variables with values in \mathcal{P} such that $P\{X = k\} = q_k$ and $P\{Y = k\} = r_k$ for all $k \in \mathcal{P}$. It is easy to see that $\mathcal{A}_{qr} \neq \emptyset$ if and only if the random variables X and Y can be considered on one probability space (i.e. they can be coupled) so that

$$(2.43) \quad P\{X \geq Y\} = 1.$$

Indeed, if $A = (a_{kl}; k, l \in \mathcal{P}) \in \mathcal{A}_{qr}$, then we can define the joint distribution of X and Y by $P\{X = k, Y = l\} = q_k a_{kl}$, and (2.43) will be satisfied. On the other hand, suppose we found a joint distribution of X and Y satisfying (2.43); then set $a_{kl} = q_k^{-1} P\{X = k, Y = l\}$ if $q_k \neq 0$ and, say, $a_{kl} = 1$ for $l = k$ and $= 0$ otherwise, if $q_k = 0$. After this reduction Proposition 2.4 follows from Theorem B.1 in Chapter 17 of [MO], which, in turn, is a consequence of Theorem 11 from [St]. In case $\mathcal{P} = \mathbb{Z}_+^d$ the problem can be easily reduced to the case when \mathcal{P} is finite, and then the result follows from one of corollaries of the Max-Flow Min-Cut theorem

known as the Supply-Demand theorem (see [LP], Corollary 2.1.5). This completes the proof of Proposition 2.4, as well as Theorem 2.1 itself. \square

2.5. Remark. The student Z. Hellman (who participated in my course) suggested for the deterministic case the following small parameter argument, which yields also in our circumstances the required upper bound for $HD(V_A^\omega)$ when some of the a_{kl} 's are zero. Let n_k be the number of $l \leq k$ for which $a_{kl} = 0$, and $\delta = (\delta_k, k \in \mathbb{Z}_+^d)$ be an infinite vector with positive entries δ_k which are so small that if $n_k \geq 1$ then $a_{kl} - \delta_k n_k^{-1} > 0$ provided $a_{kl} > 0$, and

$$-\log(\delta_k n_k^{-1}) \geq -n_k^{-1} \sum_{l \leq k, a_{kl} > 0} \log(a_{kl} - \delta_k n_k^{-1}).$$

If $n_k = 0$ set $a_{kl}^{(\delta)} = a_{kl}$, otherwise put $a_{kl}^{(\delta)} = a_{kl} - \delta_k n_k^{-1}$ if $a_{kl} > 0$ and $a_{kl}^{(\delta)} = \delta_k n_k^{-1}$ if $a_{kl} = 0$. Then

$$\sum_{l \leq k, a_{kl} > 0} a_{kl} \log a_{kl} \geq \sum_{l \leq k} a_{kl}^{(\delta)} \log a_{kl}^{(\delta)},$$

and so if $A^{(\delta)} = (a_{kl}^{(\delta)})$ and

$$W_{A^{(\delta)}}^\omega = \left\{ x : \limsup_{n \rightarrow \infty} \left(-\frac{1}{n} \log \nu_{p_{A^{(\delta)}}}^\omega (C_n^\omega(x)) \right) \leq -H_{A^{(\delta)}} \int \log m_1 dP \right\},$$

then with probability one,

$$V_A^\omega \subset \left\{ x : \lim_{n \rightarrow \infty} \left(-\frac{1}{n} \log \nu_{p_{A^{(\delta)}}}^\omega (C_n^\omega(x)) \right) = - \sum_{k \in \mathbb{Z}_+^d} q_k \sum_{l \leq k} a_{kl} \log a_{kl}^{(\delta)} \right\} \subset W_{A^{(\delta)}}^\omega.$$

From here, (2.24), and Birkhoff's ergodic theorem it follows as above that with probability one $HD(V_A^\omega) \leq HD(W_{A^{(\delta)}}^\omega) \leq H_{A^{(\delta)}}$. Letting $\delta \rightarrow 0$ (i.e. $\delta_k \rightarrow 0$ for all $k \in \mathbb{Z}_+^d$), one obtains the required upper bound $HD(V_A^\omega) \leq H_A$.

2.6. Remark. The supremum in (2.8) of the expression in (2.7) over \mathcal{A}_{qr} can be found, in principle, employing the Lagrange multipliers method, though it does not seem easy to find a nice general analytical formula for H since we also have constraints on $A = (a_{kl}) \in \mathcal{A}_{qr}$ given by inequalities $a_{kl} \geq 0$, and so the maximum may be attained on the boundary of \mathcal{A}_{qr} .

2.7. Remark. The limit in braces in (2.31) gives the relativised entropy of τ with respect to the measure ν_{p_A} , and this is connected with the relativised Shannon-McMillan-Breiman theorem (see [Bo]).

2.8. Remark. In the same way as in [Sm], using general iterated logarithm type theorems from [Ph] one can show that with probability one the Hausdorff measure in dimension H_A of the sets V_A^ω equals infinity except for the case when $H_A = d$.

2.9. Remark. Though [KK] contains a general proof of ergodicity of Gibbs measures for the skew product transformation τ , I give here for the reader's convenience a

direct independent proof of this fact. Observe that if $j_1, j_2, n \in \mathbb{Z}_+$ and $n \geq j_1$, then

$$\begin{aligned} & \nu_p^\omega \left(C_{\alpha_0, \dots, \alpha_{j_1-1}}^\omega \cap (f^n(\omega))^{-1} C_{\beta_0, \dots, \beta_{j_2-1}}^{\theta^n \omega} \right) \\ &= \prod_{i=0}^{j_1-1} p_{\alpha_i}^{\theta^i \omega} \left(\sum_{\gamma_0, \dots, \gamma_{n-j_1}} \prod_{i=0}^{n-j_1} p_{\gamma_i}^{\theta^{k+i} \omega} \right) \prod_{i=0}^{j_2-1} p_{\beta_i}^{\theta^{n+i} \omega} \\ &= \nu_p^\omega \left(C_{\alpha_0, \dots, \alpha_{j_1-1}}^\omega \right) \nu_p^{\theta^n \omega} \left(C_{\beta_0, \dots, \beta_{j_2-1}}^{\theta^n \omega} \right). \end{aligned}$$

Approximating sets by disjoint unions of cylinder sets, I derive that for any measurable collection of measurable sets $\Gamma^\omega \subset \mathbb{T}^d$,

$$(2.43) \quad \nu_p^\omega \left(\Gamma^\omega \cap (f^n(\omega))^{-1} \Gamma^{\theta^n \omega} \right) - \nu_p^\omega(\Gamma^\omega) \nu_p^{\theta^n \omega}(\Gamma^{\theta^n \omega}) \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now suppose that $\Gamma \subset \mathbb{T}^d \times \Omega$ is a measurable set and $\tau^{-1}\Gamma = \Gamma$. Then for $\Gamma^\omega = \{x : (x, \omega) \in \Gamma\}$ it follows that

$$(2.44) \quad (f^n(\omega))^{-1} \Gamma^{\theta^n \omega} = \Gamma^\omega.$$

Since ν_p is τ -invariant, then, by (2.30) and (2.44) for P -a.a. ω ,

$$(2.45) \quad \nu_p^{\theta^n \omega}(\Gamma^{\theta^n \omega}) = (f^n(\omega) \nu_p^\omega)(\Gamma^{\theta^n \omega}) = \nu_p^\omega((f^n(\omega))^{-1} \Gamma^{\theta^n \omega}) = \nu_p^\omega(\Gamma^\omega),$$

and so by (2.43) and (2.44), for P -a.a. ω ,

$$\nu_p^\omega(\Gamma^\omega) = (\nu_p^\omega(\Gamma^\omega))^2.$$

Hence, $\nu_p^\omega(\Gamma^\omega) = 0$ or 1 for P -a.a. ω , which together with (2.45) and ergodicity of P with respect to θ yield, that either $\nu_p^\omega(\Gamma^\omega) = 0$ for P -a.a. ω or $\nu_p^\omega(\Gamma^\omega) = 1$ for P -a.a. ω , and so $\nu_p(\Gamma) = 0$ or 1 , completing the proof of ergodicity of ν_p .

2.10. Example. Let $d = 1$, $q_2 = q_3 = 1/2$, and $r_1 = r_2 = r_3 = 1/3$ with all other q_i and r_i equal zero. Then \mathcal{A}_{qr} consists of all matrices $A = (a_{kl})$ such that $a_{33} = 2/3$, $a_{21} = \gamma$, $a_{22} = 1 - \gamma$, $a_{31} = 2/3 - \gamma$, $a_{32} = \gamma - 1/3$, $a_{ii} = a_{11} = 1$, $\forall i > 3$, where $1/3 \leq \gamma \leq 2/3$ and all other a_{ij} equal zero. Then

$$\begin{aligned} H_A &= -\frac{1}{2}(\gamma \log \gamma + (1 - \gamma) \log(1 - \gamma) + (2/3 - \gamma) \log(2/3 - \gamma) \\ &\quad + (\gamma - 1/3) \log(\gamma - 1/3) + 2/3 \log 2/3). \end{aligned}$$

Maximizing H_A in $\gamma \in [1/3, 2/3]$, I obtain that

$$H = HD(U_r^\omega) = 1 - \frac{\log 2}{3(\log 2 + \log 3)},$$

which is the absolute maximum here since it is attained at $\gamma = 1/2$ inside of our domain.

2.11. Example. Let $d = 1$, $q_2 = 1/3$, $q_3 = 2/3$, and $r_1 = r_3 = 1/2$ with all other q_i and r_i equal to zero, which corresponds to a "random Cantor set". Then \mathcal{A}_{qr} consists of only one matrix $A = (a_{kl})$ such that $a_{31} = 1/4$, $a_{33} = 3/4$, $a_{11} = a_{21} = a_{ii} = 1$, $\forall i > 3$ and all other a_{ij} equal zero. Then

$$H = H_A = \frac{2 \log 2 - \frac{3}{4} \log 3}{\log 2 + 2 \log 3}.$$

3. GENERALIZATIONS AND RELATED RESULTS

Next, similarly to [Bi1] I shall study frequencies of pairs of consecutive “vector-digits” in the expansion (1.2). For any $\bar{k} = (k^{(1)}, k^{(2)})$, $\bar{l} = (l^{(1)}, l^{(2)}) \in \mathbb{Z}_+^{2d}$, where $k^{(i)}, l^{(i)} \in \mathbb{Z}_+^d$, $i = 1, 2$, set

$$(3.1) \quad N_{\bar{k}\bar{l}}^\omega(x, n) = |\{i \geq 0, i < n : m(\theta^i \omega) = k^{(1)}, m(\theta^{i+1} \omega) = k^{(2)}, \\ x_i(\omega) = l^{(1)} - \mathbf{1}, x_{i+1}(\omega) = l^{(2)} - \mathbf{1}\}|$$

and

$$(3.2) \quad N_{\bar{l}}^\omega(x, n) = \sum_{\bar{k} \in \mathbb{Z}_+^{2d}} N_{\bar{k}\bar{l}}^\omega.$$

Let $r = (r_{\bar{k}}, \bar{k} \in \mathbb{Z}_+^{2d})$ be an infinite probability vector and $A = (a_{\bar{k}\bar{l}}, \bar{k}, \bar{l} \in \mathbb{Z}_+^{2d})$ be an infinite probability matrix such that $a_{\bar{k}\bar{l}} = 0$ unless $\bar{l} \leq \bar{k}$ (i.e. $l^{(i)} \leq k^{(i)}$, $i = 1, 2$). Set

$$(3.3) \quad U_r^\omega = \left\{ x \in \mathbb{T}^d : \lim_{n \rightarrow \infty} \frac{1}{n} N_{\bar{l}}^\omega(x, n) = r_{\bar{l}} \text{ for all } \bar{l} \in \mathbb{Z}_+^{2d} \right\}$$

and

$$(3.4) \quad V_A^\omega = \left\{ x \in \mathbb{Z}_+^d : \lim_{n \rightarrow \infty} \frac{1}{n} N_{\bar{k}\bar{l}}^\omega(x, n) = q_{\bar{k}} a_{\bar{k}\bar{l}} \text{ for all } \bar{k}, \bar{l} \in \mathbb{Z}_+^{2d} \right\},$$

where $q_{\bar{k}} = P\{m = k^{(1)}, m \circ \theta = k^{(2)}\}$, $\bar{k} = (k^{(1)}, k^{(2)})$.

3.1. Theorem. *With probability one,*

$$(3.5) \quad HD(V_A^\omega) = \frac{-\sum_{\bar{k} \in \mathbb{Z}_+^{2d}} q_{\bar{k}} \sum_{\bar{l} \leq \bar{k}} a_{\bar{k}\bar{l}} \log(a_{\bar{k}\bar{l}} a_{\bar{k}l^{(1)}}^{-1})}{\int \log m_1 dP} \stackrel{\text{def}}{=} H_A,$$

where $a_{\bar{k}l^{(1)}} = \sum_{l^{(2)} \in \mathbb{Z}_+^d} a_{\bar{k}l}$ if $l = (l^{(1)}, l^{(2)})$ with the convention $0 \log 0 = 0$. Furthermore, for P -a.a. ω ,

$$(3.6) \quad HD(U_r^\omega) = \sup_{A \in \mathcal{A}_{qr}} H_A \stackrel{\text{def}}{=} H,$$

where the supremum in (3.6) is taken over the set \mathcal{A}_{qr} of all infinite probability matrices $A = (a_{\bar{k}\bar{l}})$ such that $a_{\bar{k}\bar{l}} = 0$ unless $\bar{l} \leq \bar{k}$ and $qA = r$. The set \mathcal{A}_{qr} is nonempty if and only if

$$(3.7) \quad \sum_{\bar{l} \in F} q_{\bar{l}} \geq \sum_{\bar{l} \in F} r_{\bar{l}}$$

for any filter F in \mathbb{Z}_+^{2d} . If (3.7) does not hold true for some filter F , then with probability one U_r^ω is empty.

Proof. The argument remains the same as in Theorem 2.1 except that now I have to use random Markov measures in place of random Bernoulli measures. Namely, set

$p_{\bar{l}}^\omega = p_{l^{(1)}, l^{(2)}}^\omega = a_{\bar{k}\bar{l}} a_{\bar{k}l^{(1)}}^{-1}$, $\bar{k} = (k^{(1)}, k^{(2)})$ and $p_{l^{(1)}}^\omega = a_{\bar{k}l^{(1)}}$ if $m(\omega) = k^{(1)}$, $m(\theta\omega) = k^{(2)}$, where $a_{\bar{k}l^{(1)}}$ is the same as above. By the Kolmogorov extension theorem there exist a family of measures $\nu_{p_A}^\omega, \omega \in \Omega$, such that for any (n, ω) -cylinder,

$$(3.8) \quad \nu_{p_A}^\omega(C_{\alpha_0, \dots, \alpha_{n-1}}^\omega) = p_{\alpha_0+1}^\omega p_{\alpha_0+1, \alpha_1+1}^\omega \cdots p_{\alpha_{n-2}+1, \alpha_{n-1}+1}^\omega,$$

and so

$$\begin{aligned} -\frac{1}{n} \log \nu_{p_A}^\omega(C_n^\omega(x)) &= -\frac{1}{n} \log p_{x_0(\omega)+1}^\omega - \frac{1}{n} \sum_{i=0}^{n-2} \log p_{x_i(\omega)+1, x_{i+1}(\omega)+1}^{\theta^i \omega} \\ &= -\frac{1}{n} \log p_{x_0(\omega)+1}^\omega - \frac{1}{n} \sum_{\bar{k} \in \mathbb{Z}^{2d}} \sum_{\bar{l} \leq \bar{k}} N_{\bar{k}\bar{l}}^\omega(x, n) \log(a_{\bar{k}\bar{l}} a_{\bar{k}l^{(1)}}^{-1}). \end{aligned}$$

The measure ν_{p_A} defined on $\mathbb{T}^d \times \Omega$ by $\nu_{p_A}(W) = \int \nu_{p_A}^\omega(W) dP(\omega)$, $W^\omega = \{x \in \mathbb{T}^d : (x, \omega) \in W\}$, is the unique equilibrium state (Gibbs measure) corresponding to the function $\psi(x, \omega) = \log p_{x_0(\omega), x_1(\omega)}^\omega$ (called, naturally, a random Markov measure), and so it is ergodic (see [KK]). Thus I can proceed verbatim in the same way as in Theorem 2.1 in order to complete the proof of Theorem 3.1. \square

Next, let R be a compact set of infinite probability vectors $r = (r_{\bar{k}}, \bar{k} \in \mathbb{Z}_+^{2d})$ and \mathcal{A} be a compact set of infinite probability matrices $A = (a_{\bar{k}\bar{l}}, \bar{k}, \bar{l} \in \mathbb{Z}_+^{2d})$ such that $a_{\bar{k}\bar{l}} = 0$ unless $\bar{l} \leq \bar{k}$. In the same way as in Section 2 set $r_{\bar{l}}^\omega(x, n) = \frac{1}{n} N_{\bar{l}}^\omega(x, n)$, $d_{\bar{k}\bar{l}}^\omega(x, n) = \frac{1}{n} N_{\bar{k}\bar{l}}^\omega(x, n)$ and let $r^\omega = (r_{\bar{l}}^\omega)$, $D^\omega(x, n) = (d_{\bar{k}\bar{l}}^\omega)$ be the corresponding vector and matrix. Denote also $\mathcal{D} = \{D = (d_{\bar{k}\bar{l}}) : d_{\bar{k}\bar{l}} = q_{\bar{k}} a_{\bar{k}\bar{l}} \text{ and } A = (a_{\bar{k}\bar{l}}) \in \mathcal{A}\}$. Denote by ρ the distances between corresponding vectors and matrices, so that $\rho(r, \tilde{r}) = \sum_{\bar{k} \in \mathbb{Z}_+^{2d}} |r_{\bar{k}} - \tilde{r}_{\bar{k}}|$ and $\rho(B, C) = \sum_{\bar{k}, \bar{l} \in \mathbb{Z}_+^{2d}} 2^{-\|\bar{k}\|} |b_{\bar{k}\bar{l}} - c_{\bar{k}\bar{l}}|$, where $\|k\| = \|k^{(1)}\| + \|k^{(2)}\|$ if $k = (k^{(1)}, k^{(2)})$. Next, define the sets

$$(3.9) \quad U_R^\omega = \{x \in \mathbb{T}^d : \lim_{n \rightarrow \infty} \rho(r^\omega(x, n), R) = 0\}$$

and

$$(3.10) \quad V_A^\omega = \{x \in \mathbb{T}^d : \lim_{n \rightarrow \infty} \rho(D^\omega(x, n), \mathcal{D}) = 0\}.$$

In the same way as I derived (2.8) from (2.7) I obtain

3.2. Theorem. *With probability one,*

$$(3.11) \quad HD(U_R^\omega) = \sup_{\bar{r} \in R} HD(U_{\bar{r}}^\omega) \quad \text{and} \quad HD(V_A^\omega) = \sup_{A \in \mathcal{A}} HD(V_A^\omega)$$

where the right hand side of (3.11) is given by Theorem 3.1.

Next, I shall consider another version of Theorem 2.1 where the condition (1.1) will be relaxed but U_r^ω and V_A^ω will be product sets generalizing Example 6 in [Bi3]. Namely, let

$$(3.12) \quad 0 < \gamma_i = \int \log m_i dP < \infty, \quad i = 1, \dots, d,$$

but the γ_i 's may be different now. For $k, l \in \mathbb{Z}_+$ and $i = 1, \dots, d$ set

$$(3.13) \quad \begin{aligned} N_{kl}^{\omega, i}(x, n) &= |\{j \geq 0, j < n : m_i(\theta^j \omega) = k, x_{ij}(\omega) = l - 1\}|, \\ N_l^{\omega, i}(x, n) &= \sum_{k=1}^{\infty} N_{kl}^{\omega, i}(x, n), \end{aligned}$$

where $m(\omega)$ and $x_j(\omega)$ are the same as in the Introduction. Let $r^{(i)} = (r_k^{(i)}; k \in \mathbb{Z}_+)$ and $A^{(i)} = (a_{kl}^{(i)}; k, l \in \mathbb{Z}_+)$, $i = 1, \dots, d$, be infinite probability vectors and matrices, respectively. Denote by π_i the projection to the i -th coordinate on \mathbb{T}^d . Set

$$(3.14) \quad U_{r^{(i)}}^{\omega} = \{\pi_i x : x \in \mathbb{T}^d, \lim_{n \rightarrow \infty} \frac{1}{n} N_l^{\omega, i}(x, n) = r_l^{(i)}, \forall l \in \mathbb{Z}_+\}$$

and

$$(3.15) \quad V_{A^{(i)}}^{\omega} = \{\pi_i x : x \in \mathbb{T}^d, \lim_{n \rightarrow \infty} N_{kl}^{\omega, i}(x, n) = q_k^{(i)} a_{kl}^{(i)} \forall k, l \in \mathbb{Z}_+\}$$

where $q_k^{(i)} = P\{m_i = k\}$. For $\alpha_0, \dots, \alpha_{n-1} \in \mathbb{Z}_+$ I consider also the one dimensional (ω, n) -cylinders

$$C_{\alpha_0, \dots, \alpha_{n-1}}^{\omega, i} = \{\pi_i x : x_{ij}(\omega) = \alpha_j \forall j = 0, \dots, n-1\}.$$

Let $n = (n_1, \dots, n_d)$, where $n_i = [n_1 \frac{2^i}{\gamma_i}]$. Then in the same way as in Lemma 2.2 it follows that the Hausdorff dimension of any subset of \mathbb{T}^d can be computed considering only covers by the product sets

$$C_{\alpha, n}^{\omega} = \prod_{i=1}^d C_{\alpha_0^{(i)}, \dots, \alpha_{n_i-1}^{(i)}}^{\omega, i}.$$

Let $p_A^{\omega, i} = (p_l^{\omega, i}; l \in \mathbb{Z}_+)$, $i = 1, \dots, d$, be infinite probability vectors such that $\sum_{l=1}^{m_i(\omega)} p_l^{\omega, i} = 1$ and $p_l^{\omega, i} = a_{kl}^{(i)}$ if $\omega \in \{\omega : m_i(\omega) = k\}$. Then by the Kolmogorov theorem there exist probability measures $\nu_{p_A}^{\omega, i}$ on $[0, 1)$ such that

$$(3.16) \quad \nu_{p_A}^{\omega, i}(C_{\alpha_0, \dots, \alpha_{n-1}}^{\omega, i}) = \prod_{j=0}^{n-1} p_{\alpha_j+1}^{\theta^j \omega, i}.$$

Let $\nu_{p_A}^{\omega} = \prod_{i=1}^d \nu_{p_A}^{\omega, i}$ be the product measure. Denote by C_n^{ω} the set $C_{\alpha, n}^{\omega}$ containing a point x . Then

$$(3.17) \quad \begin{aligned} -\frac{1}{n_1} \log \nu_{p_A}^{\omega}(C_n^{\omega}(x)) &= -\frac{1}{n_1} \sum_{i=1}^d \sum_{j=0}^{n_i-1} \log p_{x_{ij}(\omega)+1}^{\theta^j \omega, i} \\ &= -\frac{1}{n_1} \sum_{i=1}^d \sum_{k=1}^{\infty} \sum_{l=1}^k N_{kl}^{\omega, i}(x, n_i) \log a_{kl}^{(i)}. \end{aligned}$$

By the ergodic theorem, with probability one,

$$(3.18) \quad \lim_{n_1 \rightarrow \infty} \frac{1}{n_1} \log \text{Leb}(C_n^\omega(x)) = \lim_{n_1 \rightarrow \infty} \sum_{i=1}^d \sum_{j=0}^{n_i-1} \log \frac{1}{m_i(\theta^j \omega)} = \gamma_1 d$$

and for any $x \in V_{A^{(i)}}^\omega$,

$$(3.19) \quad \lim_{n_1 \rightarrow \infty} \frac{1}{n_1} N_{kl}^{\omega, i}(x, n_i) = \frac{\gamma_1}{\gamma_i} q_k^{(i)} a_{kl}^{(i)}.$$

Thus

$$(3.20) \quad V_{A^{(1)}}^\omega \times V_{A^{(2)}}^\omega \times \cdots \times V_{A^{(d)}}^\omega \subset \left\{ x : \lim_{n \rightarrow \infty} \frac{\log \nu_{p_A}^\omega(C_{\alpha, n}^\omega(x))}{\log \text{Leb}(C_{\alpha, n}^\omega(x))} = \sum_{i=1}^d H_{A^{(i)}} \right\},$$

where

$$H_{A^{(i)}} = -\gamma_i^{-1} \sum_{k=1}^{\infty} q_k^{(i)} \sum_{l=1}^k a_{kl}^{(i)} \log a_{kl}^{(i)}.$$

Now similarly to Section 2 I derive from (3.17)-(3.20) the following result.

3.3. Theorem. *With probability one,*

$$(3.21) \quad HD(V_{A^{(1)}}^\omega \times \cdots \times V_{A^{(d)}}^\omega) = \sum_{i=1}^d H_{A^{(i)}}$$

and

$$(3.22) \quad HD(U_{r^{(1)}}^\omega \times \cdots \times U_{r^{(d)}}^\omega) = \sup_{A^{(i)}, i=1, \dots, d} \sum_{i=1}^d H_{A^{(i)}}$$

where the supremum is taken over the probability matrices $A^{(i)} = (a_{kl}^{(i)}; k, l \in \mathbb{Z}_+)$ such that $a_{kl}^{(i)} = 0$ unless $l \leq k$ and $\sum_{k=1}^{\infty} q_k^{(i)} a_{kl}^{(i)} = r_l^{(i)}, \forall i = 1, \dots, d$.

Finally, I shall discuss large deviations for $\frac{1}{n} N_{kl}^\omega(x, n)$ and $\frac{1}{n} N_l^\omega(x, n)$. Introduce a family of probability measures on $\mathbb{T}^d \times \Omega$ by

$$(3.23) \quad \zeta_{x, \omega}^n = \sum_{i=0}^{n-1} \delta_{\tau^i(x, \omega)},$$

where $\delta_{(x, \omega)}$ denotes the unit mass at (x, ω) and τ was defined by (1.8). Since Leb is a partial case of a random Bernoulli measure (see Section 2), and so it is an equilibrium state for τ , then the thermodynamic formalism for the random expanding transformations $f(\omega), \omega \in \Omega$ exhibited in [Ki2],[KK], and [BG2] yields in view of Theorem D in [Ki2] the following relativized large deviations bounds.

3.4. Theorem. *Let Ω be a compact metric space. Then with probability one, for any closed $K \subset \mathcal{P}(\mathbb{T}^d \times \Omega)$,*

$$(3.24) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{Leb}\{x \in \mathbb{T}^d : \zeta_{x,\omega}^n \in K\} \leq - \inf_{\nu \in K} I(\nu),$$

and for any open $G \subset \mathcal{P}(\mathbb{T}^d \times \Omega)$,

$$(3.25) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \log \text{Leb}\{x \in \mathbb{T}^d : \zeta_{x,\omega}^n \in G\} \geq - \inf_{\nu \in G} I(\nu)$$

where $\mathcal{P}(M)$ denotes the space of probability measures on M with the topology of weak convergence, $I(\nu) = -d \int \log m_1 dP - h_\nu^{(\tau)}(\tau)$ if ν is τ -invariant with the marginal on Ω equal to P , and $I(\nu) = \infty$, otherwise. Here $h_\nu^{(\tau)}(\tau)$ denotes the relativized entropy of τ with respect to ν (see [Ki2] and [Bo]).

Applying the contraction principle arguments (see [DZ], Section 4.2) with respect to an appropriate topology, one can derive from Theorem 3.4 that with probability one, for any $-\infty \leq a < b \leq \infty$,

$$(3.26) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{Leb}\{x \in \mathbb{T}^d : \frac{1}{n} N_l^\omega(x, n) \in [a, b]\} \leq - \inf_{c \in [a, b]} J_l^{(1)}(c),$$

$$(3.27) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \log \text{Leb}\{x \in \mathbb{T}^d : \frac{1}{n} N_l^\omega(x, n) \in (a, b)\} \geq - \inf_{c \in (a, b)} J_l^{(1)}(c),$$

$$(3.28) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{Leb}\{x \in \mathbb{T}^d : \frac{1}{n} N_{kl}^\omega(x, n) \in [a, b]\} \leq - \inf_{c \in [a, b]} J_{kl}^{(2)}(c),$$

and

$$(3.29) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \log \text{Leb}\{x \in \mathbb{T}^d : \frac{1}{n} N_{kl}^\omega(x, n) \in (a, b)\} \geq - \inf_{c \in (a, b)} J_{kl}^{(2)}(c)$$

where

$$J_l^{(1)}(c) = \inf\{I(\nu) : \int \chi_{l(1)-1}(x_0(\omega)) \chi_{l(2)-1}(x_1(\omega)) d\nu(x, \omega) = c\}$$

and

$$J_{kl}^{(2)}(c) = \inf\{I(\nu) : \int \chi_{k(1)}(m(\omega)) \times \chi_{k(2)}(m(\theta\omega)) \chi_{l(1)-1}(x_0(\omega)) \chi_{l(2)-1}(x_1(\omega)) d\nu(x, \omega) = c\}.$$

A delicate point in the application of the contraction principle here is that the functions $\varphi_l^\omega(x) = \chi_{l(1)}(x_0(\omega))$ are not continuous in $x \in \mathbb{T}^d$ when ω is fixed with respect to the usual topology of \mathbb{T}^d . One can overcome this taking into account that the points of discontinuity of φ_l^ω in \mathbb{T}^d have zero measure with respect to P -a.a. disintegrations of all equilibrium states of τ corresponding to a large class of functions (see [KK]) and the set of such equilibrium states is dense in $\mathcal{P}(\mathbb{T}^d \times \Omega)$ (see [Ki1]). This is connected also with the symbolic representation of the random dynamical system $\tau = (f(\omega), \theta)$ as a random subshift of finite type (see [BG1,2] and [KK]).

4. DIMENSIONS OF RANDOM REPELLERS

Let K^ω , $\omega \in \Omega$, be a measurable family of compact subsets of \mathbb{T}^d satisfying (1.10) with $f(\omega)$ the same as in the Introduction. Set $K = \{(x, \omega) : x \in K^\omega\}$. Then we have the setup of “random bundle maps” from [BG2], and can speak about the relativized topological entropy $h_{\text{top}}^{(r)}(\tau, K)$ of the skew product transformation τ restricted to the τ -invariant set K .

4.1. Theorem. *With probability one,*

$$(4.1) \quad HD(K^\omega) = \frac{h_{\text{top}}^{(r)}(\tau, K)}{\int \log m_1 dP}.$$

Proof. The relativized variational principle from [Bo] (see also [LW], where Ω was supposed to be compact) can be easily generalized to this setup, yielding that for any $\varepsilon > 0$ there exists a τ -invariant ergodic probability measure ν_ε on K such that $d\nu_\varepsilon(x, \omega) = d\nu_\varepsilon^\omega(x)dP(\omega)$, with probability one $\nu_\varepsilon^\omega(K^\omega) = 1$ and $f(\omega)\nu_\varepsilon^\omega = \nu_\varepsilon^{\theta\omega}$, and

$$(4.2) \quad h_{\nu_\varepsilon}^{(r)}(\tau) \geq h_{\text{top}}^{(r)} - \varepsilon,$$

where $h_\nu^{(r)}(\tau)$ denotes the relativized metric entropy of τ with respect to a τ -invariant measure ν (see [Bo]). Then by the relativized Shannon-McMillan-Breiman theorem (see [Bo]), for ν_ε -a.a. (x, ω) ,

$$(4.3) \quad \lim_{n \rightarrow \infty} \left(-\frac{1}{n} \log \nu_\varepsilon^\omega(C_n^\omega(x))\right) = h_{\nu_\varepsilon}^{(r)}(\tau),$$

and by the Fubini theorem, for P -a.a. ω ,

$$(4.4) \quad \nu_\varepsilon^\omega \left\{ x \in K^\omega : \lim_{n \rightarrow \infty} \left(-\frac{1}{n} \log \nu_\varepsilon^\omega(C_n^\omega(x))\right) = h_{\nu_\varepsilon}^{(r)}(\tau) \right\} = 1,$$

where $C_n^\omega(x)$ was defined before Proposition 2.3. Denote the set in braces in (4.4) by K_0^ω . Then by Proposition 2.3, in the same way as in Section 2, I derive from (4.2)-(4.4),

$$(4.5) \quad HD(K^\omega) \geq HD(K_0^\omega) = \frac{h_{\nu_\varepsilon}^{(r)}(\tau)}{\int \log m_1 dP} \geq \frac{h_{\text{top}}^{(r)} - \varepsilon}{\int \log m_1 dP}.$$

Since ε here is arbitrary, I obtain that the left hand side of (4.1) is not less than the right hand side of (4.1). In order to obtain an upper bound for $HD(K^\omega)$ I shall need the following result, which is contained, essentially, in [BG2] and [KK].

4.2. Lemma. *With probability one,*

$$(4.6) \quad h_{\text{top}}^{(r)}(\tau, K) = \lim_{n \rightarrow \infty} \frac{1}{n} \log r_n^\omega(K)$$

where

$$r_n^\omega(K) = |\{C_{\alpha_0, \dots, \alpha_{n-1}}^\omega : C_{\alpha_0, \dots, \alpha_{n-1}}^\omega \cap K^\omega \neq \emptyset\}|.$$

Proof. For any $x, y \in \mathbb{T}^d$ set $\rho_n^\omega(x, y) = \max_{0 \leq j \leq n-1} \text{dist}(f^j(\omega)x, f^j(\omega)y)$, where $f^j(\omega) = f(\theta^j \omega) \circ \dots \circ f(\theta \omega) \circ f(\omega)$, and denote $B_\delta^\omega(x, n) = \{y \in \mathbb{T}^d : \rho_n^\omega(x, y) \leq \delta\}$. A subset $E \subset K^\omega$ is called (ω, δ, n) -separated if for any $x, y \in E, x \neq y$, one has $\rho_n^\omega(x, y) > \delta$. If E_n^ω is any family of maximal (ω, δ, n) -separated sets, then with probability one (see [Ki1], [Bo], [KK]),

$$(4.7) \quad h_{\text{top}}^{(\tau)}(\tau, K) = \lim_{n \rightarrow \infty} \frac{1}{n} \log |E_n^\omega|.$$

Now let E be a maximal (ω, δ, n) -separated set in K^ω . Then $\bigcup_{x \in E} B_x^\omega(\delta, n) \supset K^\omega$, and $B_x^\omega(\delta/2, n), B_y^\omega(\delta/2, n)$ are disjoint for any $x, y \in K^\omega, x \neq y$. It is clear that any $B_x^\omega(\delta/2, n)$ contains a ball of radius $\frac{1}{2}\delta(\max_{1 \leq i \leq d} \prod_{j=0}^{n-1} m_i(\theta^j \omega))^{-1}$ centered at x . By volume considerations, any cylinder $C_{\alpha_0, \dots, \alpha_{n-1}}^\omega$ cannot intersect more than $10\delta^{-d}k_n(\omega)$ such balls, where $k_n(\omega)$ is the same as in (2.17). Then $|E| \leq 10\delta^{-d}k_n(\omega)r_n^\omega(K)$, and since with probability one $\frac{1}{n} \log k_n(\omega) \rightarrow 0$ as $n \rightarrow \infty$, I derive from (4.7) that $h_{\text{top}}^{(\tau)}(\tau, K) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log r_n^\omega(K)$. In order to obtain the inequality in the other direction, observe that each (ω, n) -cylinder which intersects K^ω intersects also some of the sets $B_x^\omega(\delta, n), x \in E$. Each such set is contained in a cube with side $2\delta(\min_{1 \leq i \leq d} \prod_{j=0}^{n-1} m_i(\theta^j \omega))^{-1}$. By volume considerations such a cube can intersect no more than $(3\delta)^d k_n(\omega)$ (ω, n) -cylinders. Thus $r_n^\omega(K) \leq (3\delta)^d k_n(\omega)|E|$, which by (4.7) yields $h_{\text{top}}^{(\tau)}(\tau, K) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \log r_n^\omega(K)$, completing the proof of (4.6). \square

Now I can complete the proof of Theorem 4.1. By (2.13), (2.18), and Lemma 4.2 for any $\delta, \rho > 0$ and P -a.a. ω there exists $n_0(\omega)$ such that for all $n \geq n_0(\omega)$,

$$(4.8) \quad \lambda_\kappa^\omega(K^\omega, \rho) \leq r_n^\omega(K) |C_{\alpha_0, \dots, \alpha_{n-1}}^\omega| \leq e^{n(h_{\text{top}}^{(\tau)}(\tau, K) + \delta)} e^{-n\kappa(d \int \log m_1 dP - \delta)},$$

where λ_κ^ω was defined by (2.14). Let

$$\kappa > \frac{h_{\text{top}}^{(\tau)}(\tau, K)}{d \int \log m_1 dP} \quad \text{and} \quad \delta < \frac{1}{2}(\kappa d \int \log m_1 dP - h_{\text{top}}^{(\tau)}(\tau, K)).$$

Then the right hand side of (4.8) tends to zero as $n \rightarrow \infty$, and so in view of Lemma 2.2 I derive that

$$HD(K^\omega) \leq \frac{h_{\text{top}}^{(\tau)}(\tau, K)}{\int \log m_1 dP}$$

concluding the proof of Theorem 4.1. \square

4.3. Remark. One obtains the sets $K^\omega, \omega \in \Omega$, satisfying (1.10) by choosing, for instance, a measurable family of finite subsets $\Psi(\omega) \subset \{l-1 : l \in \mathbb{Z}_+^d, l \leq m(\omega)\} = \mathcal{L}(\omega)$ and setting $K^\omega = \{x \in \mathbb{T}^d : x_j(\omega) \in \Psi(\theta^j \omega) \forall j = 0, 1, \dots\}$. If with positive probability $\Psi(\omega) \neq \mathcal{L}(\omega)$, then for P -a.a. ω the K^ω are proper closed subsets of \mathbb{T}^d which can be called random Cantor sets. They are statistically self similar in the sense that if $\alpha_j \in \Psi(\theta^j \omega), j = 0, 1, \dots, n-1$, then looking at the intersection of K^ω with the cylinder $C_{\alpha_0, \dots, \alpha_{n-1}}^\omega$ and rescaling by means of $M(\theta^{n-1} \omega) \dots M(\omega)$, where $M(\omega)$ is the same as in (1.2), we obtain $K^{\theta^n \omega}$, namely, $f^n(\omega)(K^\omega \cap C_{\alpha_0, \dots, \alpha_{n-1}}^\omega) =$

$K^{\theta^n \omega}$, which has the same distribution as K^ω . Observe that $h_{\text{top}}^{(r)}(\tau, K)$ for such sets K^ω is equal to $\int \log |\Psi(\omega)| dP(\omega)$. More general family of sets K^ω satisfying (1.10) can be obtained by taking a measurable family of matrices $B(\omega) = (b_{kl}(\omega))$, $k, l \in \mathbb{Z}_+^d$, $k \leq m(\omega), l \leq m(\theta\omega)$ with 0 and 1 entries and by setting $K^\omega = \{x \in \mathbb{T}^d : b_{x_i(\omega)+1, x_{i+1}(\omega)+1} = 1 \forall i = 0, 1, \dots\}$. In this case, with probability one,

$$(4.9) \quad h_{\text{top}}^{(r)}(\tau, K) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|B(\theta^{n-1}\omega) \cdots B(\theta\omega)B(\omega)\|.$$

This example is connected with random subshifts of finite type considered in [BG2] and [KK]. A somewhat related situation was studied in [Bed].

It is not difficult to understand that the Hausdorff and Minkowski dimensions (the latter is called also the box dimension, see [Fa]) coincide for the sets K^ω considered in Theorem 4.1. I shall show next that these dimensions will usually be different for such sets if (1.1) is replaced by (3.12) with possibly different γ_i 's, which in my random set yields the results similar to [Mc] and [KP2]. Thus, assume that $m(\omega) = (m_1(\omega), \dots, m_d(\omega)) \in \mathbb{Z}_+^d$ satisfies (3.12) with $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_d$. As in Remark 4.3, choose a measurable family of subsets $\Psi(\omega) \subset \{l - \mathbf{1} : l \in \mathbb{Z}_+^d, l \leq m(\omega)\}$ and put $K^\omega = \{x \in \mathbb{T}^d : x_j(\omega) \in \Psi(\theta^j \omega)\}$. For all $\alpha_j \in \mathbb{Z}_+^d$, $\alpha_j \leq m(\theta^j \omega) - \mathbf{1}$, $j = 0, 1, \dots, n - 1$, set

$$R_{\alpha_0, \dots, \alpha_{n-1}}^\omega = \bigcup_{\beta_0, \dots, \beta_{n-1}} C_{\beta_0, \dots, \beta_{n-1}}^\omega$$

where $C_{\beta_0, \dots, \beta_{n-1}}^\omega$ are (ω, n) -cylinders as above and the union is taken over all $\beta \in \mathbb{Z}_+^d$, $\alpha_j \leq m(\theta^j \omega) - \mathbf{1}$, $j = 0, \dots, n - 1$, such that $\beta_{ij} = \alpha_{ij}$, $\forall j = 0, 1, \dots, [n\gamma_1\gamma_i^{-1}] - 1$ and $\forall i = 1, \dots, d$. Denote also by $R_n^\omega(x)$ the set $R_{\alpha_0, \dots, \alpha_{n-1}}^\omega$ which contains $x \in K^\omega$. Clearly, the sets $R_{\alpha_0, \dots, \alpha_{n-1}}^\omega$ are either disjoint or coincide, and K^ω is the disjoint union of such sets. These sets are random parallelepipeds which are ‘‘approximate cubes’’ in the same sense as (ω, n) -cylinders were under the condition (1.1), namely that with probability one the ratio of the maximal and the minimal sides is subexponential in n as $n \rightarrow \infty$. Via the same arguments as in Lemma 2.2 this enables me to compute the Hausdorff and Minkowski dimensions of K^ω using only covers by the sets $R_{\alpha_0, \dots, \alpha_{n-1}}^\omega$. Following [KP2], introduce functions $Z_k^\omega(l) = Z_k^\omega(l_1, \dots, l_k)$, $l = (l_1, \dots, l_d) \in \Psi(\omega)$, successively for $k = d - 1, \dots, 1$ by

$$(4.10) \quad Z_{k-1}^\omega(l_1, \dots, l_{k-1}) = \sum_{l_k=0}^{m_k(\omega)-1} (Z_k^\omega(l_1, \dots, l_k))^{\gamma_k \gamma_{k+1}^{-1}},$$

where $\gamma_{d+1} = \infty$ and $Z_d^\omega(l) = \chi_{\Psi(\omega)}(l)$ is the indicator of $\Psi(\omega)$.

4.4. Theorem. *With probability one*

$$(4.11) \quad HD(K^\omega) = \gamma_1^{-1} \int \log Z_0^\omega dP(\omega).$$

Proof. For each $\alpha \in \Psi(\omega)$ set

$$(4.12) \quad q_{k,\alpha}^\omega = (Z_k^\omega(\alpha))^{\gamma_k \gamma_{k+1}^{-1}} (Z_{k-1}^\omega(\alpha))^{-1}, \quad p_\alpha^\omega = \prod_{k=1}^d q_{k,\alpha}^\omega,$$

and

$$(4.13) \quad \nu^\omega(C_{\alpha_0, \dots, \alpha_{n-1}}^\omega) = \prod_{j=0}^{n-1} p_{\alpha_j}^{\theta^j \omega}.$$

It follows from (4.10) that $\sum_{\alpha \in \Psi(\omega)} p_\alpha^\omega = 1$, and so Kolmogorov's extension theorem yields that (4.12) determines a measurable in ω family of probability measures ν^ω on \mathbb{T}^d such that $\nu^\omega(K^\omega) = 1$. In the same way as in Section 2 I we conclude that $\nu^\omega, \omega \in \Omega$, satisfy (2.30) and the measure ν given by $d\nu(x, \omega) = d\nu^\omega(x) dP(\omega)$ is τ -invariant and ergodic. It follows from (4.10), (4.12), and (4.13) that for all $\alpha_j \in \Psi(\theta^j \omega), j = 0, \dots, n-1$,

$$(4.14) \quad \nu^\omega(R_{\alpha_0, \dots, \alpha_{n-1}}^\omega) = \prod_{k=1}^d \prod_{j=0}^{N_k(n)} q_{k, \alpha_j}^{\theta^j \omega},$$

where $N_k(n) = [n\gamma_1\gamma_k^{-1}] - 1$, and so for each $x \in K^\omega$,

$$(4.15) \quad \begin{aligned} \log \nu^\omega(R_n^\omega(x)) &= \sum_{k=1}^d \sum_{j=0}^{N_k(n)} \log q_{k, x_j(\omega)}^{\theta^j \omega} \\ &= \sum_{k=1}^d \sum_{j=0}^{N_k(n)} \left(\gamma_k \gamma_{k+1}^{-1} \log Z_k^{\theta^j \omega}(x_j(\omega)) - \log Z_{k-1}^{\theta^j \omega}(x_j(\omega)) \right) \\ &= - \sum_{j=0}^{n-1} \log Z_0^{\theta^j \omega} + \sum_{k=1}^{d-1} \left(\gamma_k \gamma_{k+1}^{-1} g_k^\omega(x, N_k(n)) - g_k^\omega(x, N_{k+1}(n)) \right), \end{aligned}$$

where $g_k^\omega(x, j) = \sum_{i=0}^j \log Z_k^{\theta^i \omega}(x_i(\omega))$. By the ergodic theorem, for ν -a.a. (x, ω) ,

$$(4.16) \quad \lim_{n \rightarrow \infty} \frac{1}{n} g_k^\omega(x, N_j(n)) = \gamma_1 \gamma_j^{-1} \int \log Z_k^\omega(x_0(\omega)) d\nu^\omega(x) dP(\omega),$$

and together with (4.15) these yield that, for ν -a.a. (x, ω) ,

$$(4.17) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \nu^\omega(R_n^\omega(x)) = - \int \log Z_0^\omega dP(\omega).$$

Observe that P -a.s., for all $x \in \mathbb{T}^d$,

$$(4.18) \quad \begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log \text{Leb}(R_n^\omega(x)) &= \lim_{n \rightarrow \infty} -\frac{1}{n} \sum_{k=1}^d \sum_{j=0}^{N_k(n)} \log m_k(\theta^j \omega) \\ &= - \sum_{k=1}^d \gamma_1 \gamma_k^{-1} \int \log m_k dP = -\gamma_1 d. \end{aligned}$$

Denote by K_0^ω the subset of x 's from K^ω for which (4.17) holds true P -a.s. Then arguments similar to Lemma 2.2 enable me to apply Proposition 2.3 to the sets $R_n^\omega(x)$ in place of (ω, n) -cylinders in Section 2, which together with (2.22) yields

$$(4.19) \quad HD(K^\omega) \geq HD(K_0^\omega) \geq \gamma_1^{-1} \int \log Z_0^\omega dP(\omega).$$

In order to derive the inequality in the other direction I shall need first the following result which improves Lemma 6.1 from [KP2] with essentially the same proof, which I give here for the reader's convenience.

4.5 Lemma. (i) Let $\varphi_k(y)$, $k = 1, \dots, d$, be measurable functions on $(0, \infty)$ satisfying

$$(4.20) \quad \max_{1 \leq k \leq d} \sup_{y \geq \delta} y^{-1} |\varphi_k(y)| < \infty$$

for all $\delta > 0$. Then for any numbers $\rho_k > 0$, $k = 1, \dots, d$,

$$(4.21) \quad \limsup_{y \rightarrow \infty} \sum_{k=1}^d y^{-1} (\varphi_k(y) - \rho_k^{-1} \varphi_k(\rho_k y)) \geq 0.$$

(ii) Let $\varphi_k(n)$, $k = 1, \dots, d$, be functions on \mathbb{Z}_+ satisfying (4.20) and

$$(4.22) \quad \max_{1 \leq k \leq d} \limsup_{n \rightarrow \infty} n^{-1} |\varphi_k(n) - \varphi_k(n-1)| = 0;$$

then for any numbers $\rho_k > 0$, $k = 1, \dots, d$,

$$(4.23) \quad \limsup_{n \rightarrow \infty} \sum_{k=1}^d n^{-1} (\varphi_k(n) - \rho_k^{-1} \varphi_k([\rho_k n])) \geq 0.$$

Proof. Set $\psi_k(z) = e^{-z} \varphi_k(e^z)$, $k = 1, \dots, d$. Clearly, for any $u > 0$,

$$(4.24) \quad \left| \int_0^u \sum_{k=1}^d (\psi_k(z) - \psi_k(z + \log \rho_k)) dz \right| \leq \sum_{k=1}^d \left(\left| \int_0^{\log \rho_k} \psi_k(z) dz \right| + \left| \int_u^{u + \log \rho_k} \psi_k(z) dz \right| \right).$$

By (4.20) $\psi_k(z)$ is bounded on any interval $[a, \infty)$, $a > -\infty$, and so the right hand side of (4.24) is a bounded function of u . It follows that

$$(4.25) \quad \limsup_{z \rightarrow \infty} \sum_{k=1}^d (\psi_k(z) - \psi_k(z + \log \rho_k)) \geq 0,$$

since otherwise the left hand side of (4.24) would grow at least linearly in u for u big enough. Set $y = e^z$; then $\psi_k(z) = y^{-1} \varphi_k(y)$, and $\psi_k(z + \log \rho_k) = \rho_k^{-1} y^{-1} \varphi_k(\rho_k y)$, and (4.21) follows from (4.25). If the φ_k 's are defined only on \mathbb{Z}_+ and satisfy (4.20), I extend them linearly inside each interval $[n, n + 1]$, and (4.23) will follow from (4.20)-(4.22). \square

Now I can complete the proof of Theorem 4.4. Observe that for each $k = 1, \dots, d$ and $\alpha \in \Psi(\omega)$,

$$(4.26) \quad Z_k^\omega(\alpha) \leq \prod_{i=1}^d m_i(\omega).$$

Since

$$\lim_{n \rightarrow \infty} n^{-1} (N_{k+1}(n) - N_k([\gamma_k \gamma_{k+1}^{-1} n])) = 0,$$

then by (3.12), (4.26), the definition of $q_k^\omega(x, j)$, and by the ergodic theorem, for P -a.a. ω and for each $x \in K^\omega$,

$$(4.27) \quad \lim_{n \rightarrow \infty} n^{-1} (g^\omega(x, N_k([\gamma_k \gamma_{k+1}^{-1} n])) - g^\omega(x, N_{k+1}(n))) = 0.$$

Set $\varphi_k^\omega(x, n) = \gamma_k \gamma_{k+1}^{-1} g^\omega(x, N_k(n))$; then

$$g^\omega(x, N_k([\gamma_k \gamma_{k+1}^{-1} n])) = \gamma_{k+1} \gamma_k^{-1} \varphi_k^\omega([\gamma_k \gamma_{k+1}^{-1} n]).$$

By (4.26) and the ergodic theorem, for P -a.a. ω ,

$$(4.28) \quad \begin{aligned} \limsup_{n \rightarrow \infty} n^{-1} \sup_{x \in K^\omega} |\varphi_k^\omega(x, n)| &\leq \gamma_k \gamma_{k+1}^{-1} \limsup_{n \rightarrow \infty} n^{-1} \sum_{j=0}^{N_k(n)} \sum_{i=1}^d \log m_i(\theta^j \omega) \\ &= \gamma_1 \gamma_{k+1}^{-1} \sum_{i=1}^d \gamma_i < \infty \end{aligned}$$

and

$$(4.29) \quad \begin{aligned} \limsup_{n \rightarrow \infty} n^{-1} \sup_{x \in K^\omega} |\varphi_k^\omega(x, n) - \varphi_k^\omega(x, n-1)| \\ \leq \gamma_k \gamma_{k+1}^{-1} \limsup_{n \rightarrow \infty} n^{-1} \sum_{i=1}^d \log m_i(\theta^{N_k(n)} \omega) = 0, \end{aligned}$$

which implies both (4.20) and (4.22). Thus I can apply Lemma 4.5 with $\varphi_k(n) = \varphi_k^\omega(x, n)$ and $\rho_k = \gamma_k \gamma_{k+1}^{-1}$, which in view of (4.15), (4.27)–(4.29), and the ergodic theorem yields that, for P -a.a. ω and any $x \in K^\omega$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \nu^\omega(R_n^\omega(x)) \geq - \int \log Z_0^\omega dP(\omega).$$

This together with (4.18) implies that, for P -a.a. ω and each $x \in K^\omega$,

$$(4.30) \quad \liminf_{n \rightarrow \infty} \frac{\log \nu^\omega(R_n^\omega(x))}{\log(\text{Leb} R_n^\omega(x))} \leq \gamma_1^{-1} d^{-1} \int \log Z_0^\omega dP(\omega).$$

Finally, by (2.23) and (2.26) I conclude from (4.30) that with probability one

$$HD(K^\omega) \leq \gamma_1^{-1} \int \log Z_0^\omega dP(\omega),$$

which together with (4.19) completes the proof of Theorem 4.4. □

Concerning the Minkowski dimension of the K^ω 's, one has the following.

4.6. Theorem. (i) *With probability one*

$$(4.31) \quad MD(K^\omega) = \sum_{k=1}^d (\gamma_k^{-1} - \gamma_{k+1}^{-1}) \int \log |\pi_k(\Psi(\omega))| dP(\omega),$$

where π_k is the projection to the first k coordinates and $\gamma_{d+1} = \infty$.

(ii) The equality $MD(K^\omega) = HD(K^\omega)$ holds true with probability one if and only if for each $k = 1, \dots, d - 1$ the cardinalities of the preimages under π_k are the same for all elements of $\pi_k(\Psi(\omega))$ and P -a.a. ω .

Proof. It is not difficult to understand via arguments similar to Lemmas 2.2 and 4.2 that with probability one

$$(4.32) \quad MD(K^\omega) = \gamma_1^{-1} \lim_{n \rightarrow \infty} \frac{1}{n} \log r_n^\omega(K)$$

where $r_n^\omega(K)$ is the number of different sets $R_{\alpha_0, \dots, \alpha_{n-1}}^\omega$ which intersect K^ω . From the definition of K^ω it follows that

$$(4.33) \quad r_n^\omega(K) = \prod_{k=1}^d \prod_{j=N_{k+1}(n)+1}^{N_k(n)} |\pi_k(\Psi(\theta^j \omega))|,$$

where, by convention, $N_{d+1}(n) = 0$. Now (4.32) and (4.33) together with the ergodic theorem yield (4.31). The assertion (ii) is obtained similarly to [KP2] by comparing (4.11) and (4.31). \square

4.7. Remark. Similarly to [KP2] I can obtain in the case $d = 2$ also the Hausdorff dimension of sets $K^\omega, \omega \in \Omega$, determined by random subshifts of finite type in place of the restriction on each $x_j(\omega)$ separately (see Remark 4.3). Also it is not difficult to obtain random (relativized) versions of the Ledrappier-Young formula from [KP2]. Combining methods of [KP1] and of the present paper, one can obtain also the Hausdorff dimension of the sets $(K^\omega + t) \cap K^\omega$, where $t \in \mathbb{T}^d$ with $K^\omega \subset \mathbb{T}^d$ satisfying (1.10). In particular, similarly to Theorem 1.2 in [KP1] I derive that if $d = 1$ and $K_\Psi^\omega = \{x \in [0, 1) : x_j(\omega) \in \Psi(\theta^j \omega) \forall j = 0, 1, \dots\}$ are random Cantor sets, then with probability one

$$HD((K_{\Psi_1}^\omega + t) \cap K_{\Psi_2}^\omega) = \frac{\lambda}{\int \log mdP}$$

for Leb a.a. $t \in [0, 1)$, where $\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|A_{\phi \circ \tau^n(t, \omega)}^{\theta^n \omega} \cdots A_{\phi \circ \tau(t, \omega)}^{\theta \omega} A_{\phi(t, \omega)}^\omega\|$ with $\phi(t, \omega) = t_0(\omega)$ and $A_0^\omega, A_1^\omega, \dots, A_{m(\omega)-1}^\omega$ are two-by-two matrices given by $A_l^\omega(i, j) = |(\Psi_1(\omega) + i + l) \cap (\Psi_2(\omega) + jm(\omega))|$ for $i, j = 0, 1$. The above limit exists and is constant with probability one for Leb a.a. $t \in [0, 1)$, since $\text{Leb} \times P$ is an ergodic τ -invariant probability measure on $[0, 1) \times \Omega$. Similarly to Theorem 3.1 in [KP1], under additional assumptions one can obtain the above Hausdorff dimension in a more explicit form. Suppose, for instance, that the difference set $\Psi(\omega) = \Psi_2(\omega) - \Psi_1(\omega) \subset \mathbb{Z}$ is contained in an arithmetic progression of length $m(\omega)$ and $|\Psi(\omega)| = m(\omega)$. Then with probability one

$$HD((K_{\Psi_1}^\omega + t) \cap K_{\Psi_2}^\omega) = \frac{1}{\int \log mdP} \int \frac{1}{m(\omega)} \sum_{i \in \Psi(\omega)} \log M_i^\omega dP(\omega)$$

for Leb a.a. t from the interval $K_\Psi^\omega = K_{\Psi_2}^\omega - K_{\Psi_1}^\omega$, where $M_i^\omega = |(\Psi_1(\omega) + i) \cap \Psi_2(\omega)|$.

Next, I shall exhibit a version of the Bowen-Ruelle formula for the Hausdorff dimensions of repellers of random conformal maps. Let (Ω, P) and θ be as before,

M be a C^2 locally compact Riemannian manifold, and $f(\omega), \omega \in \Omega$, be a measurable family of C^2 maps of M such that there exist a compact set $M_0 \subset M$ with nonempty interior $\text{int}M_0$ and a function $\lambda_\omega(x)$ satisfying:

$$(i) \quad D_x f(\omega) = \lambda_\omega(x) I_x^\omega, \quad \forall x \in M_0, \omega \in \Omega,$$

where $D_x f(\omega)$ is the differential of $f(\omega)$ at x and I_x^ω is an isometry of the tangent space $T_x M$ onto $T_{f(\omega)} M$.

(ii) With probability one, $\inf_{x \in M_0} \lambda_\omega(x) > 1$.

(iii) If $\gamma_0(\omega) = \sup_{x \in M_0} \log \lambda_\omega(x)$ and $\gamma_1(\omega) = \sup_{x \in M_0} \left| \log \left\| \frac{d\lambda_\omega(x)}{dx} \right\| \right|$ then $\int (\gamma_0 + \gamma_1) dP < \infty$.

I assume also the mixing condition saying that for any open set $U \subset M_0$ there exists $N = N_\omega$ such that $f^N(\omega)U \supset M_0$, where $f^N(\omega) = f(\theta^N \omega) \circ \dots \circ f(\theta \omega) f(\omega)$. As an example of maps satisfying the above conditions one may keep in mind algebraic endomorphisms of the torus \mathbb{T}^d given by integer valued matrices $L = (l_{ij})$ such that $\sum_{j=1}^d l_{ij} l_{kj} = \delta_{ik} \lambda_L^2$ for some $\lambda_L > 1$ independent of i , where $\delta_{ik} = 1$ if $i = k$ and $= 0$, otherwise. Then $\lambda_L^{-1} L$ is an orthogonal matrix, and the conditions above are satisfied for $f(\omega)$ taken out of this family of endomorphisms with $\lambda_\omega(x)$ depending on ω but not on x .

Now let $K^\omega, \omega \in \Omega$, be a measurable family of compact sets satisfying (1.10) and $K^\omega \subset M_1, \forall \omega \in \Omega$ for some compact set $M_1 \subset \text{int}M_0$. I shall use again the metrics $\rho_n^\omega(x, y)$, the sets $B_\delta^\omega(x, n)$, and the notion of (ω, δ, n) -separated sets defined in the proof of Lemma 4.2 with \mathbb{T}^d replaced by M . Let $\psi_\omega(x), \omega \in \Omega$, be a family of continuous functions on M_1 such that $\int \sup_{x \in M_1} |\psi_\omega(x)| dP(\omega) < \infty$. Let $E_n^\omega \subset K^\omega$ be any family of maximal (ω, δ, n) -separated sets in K^ω ; then the relativized topological pressure $Q_K^{(r)}(\psi)$ of $\psi = \psi_\omega(x)$ on $K = \{(x, \omega) : x \in K^\omega\}$ can be obtained for P -a.a. ω by

$$(4.34) \quad Q_K^{(r)}(\psi) = \limsup_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{x \in E_n^\omega} \exp(S_n \psi(x, \omega)),$$

where $S_n \psi(x, \omega) = \sum_{j=0}^{n-1} \psi_{\theta^j \omega}(f^j(\omega)x)$. Note that if ψ is independent of x on K , then

$$(4.35) \quad Q_K^{(r)}(\psi) = Q_K^{(r)}(0) + \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi_{\theta^i \omega} = h_{\text{top}}^{(r)}(\tau, K) + \int \psi_\omega dP(\omega),$$

where, again, τ is the skew product transformation and $h_{\text{top}}^{(r)}(\tau, K)$ is the relativized topological entropy of τ on K . One has also the following relativized variational principle (see [Bo]):

$$(4.36) \quad Q_K^{(r)}(\psi) = \sup_{\mu} \left(\int \psi d\mu + h_{\mu}^{(r)}(\tau) \right),$$

where, again, $h_{\mu}^{(r)}(\tau)$ denotes the relativized entropy of τ with respect to μ , and the supremum is taken over the set of all τ -invariant probability measures on K whose marginal on Ω is P .

4.8. Theorem. *Let $\varphi_\omega(x) = -\log \lambda_\omega(x)$. Then, under the assumptions above, for P -a.a. ω ,*

$$(4.37) \quad HD(K^\omega) = t_0 \quad \text{if } Q_K^{(r)}(t_0\varphi) = 0.$$

If $\lambda_\omega(x)$, and so $\varphi_\omega(x)$, are independent of x on K , then

$$(4.38) \quad HD(K^\omega) = \frac{h_{\text{top}}^{(r)}(\tau, K)}{\int \log \lambda_\omega dP(\omega)}.$$

Proof. First, observe that (4.38) follows immediately from (4.34)-(4.37) if $\lambda_\omega(x)$ does not depend on x . Now let E_n^ω be a family of maximal (ω, δ, n) -separated sets in $K^\omega, \omega \in \Omega$. In the same way as in Section 3 of [KK] via Proposition 2.1 there, it follows from (i)-(iii) that for any $\varepsilon > 0$ there exists a measurable set $\Omega_\varepsilon \subset \Omega$ with $P(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ and random variables $R_{\varepsilon, \delta}(\omega) < \infty$ such that for any $\delta \leq \delta_0(\omega), x \in K^\omega, P$ -a.a. ω , and n satisfying $\theta^n \omega \in \Omega_\varepsilon$,

$$(4.39) \quad B_{R_{\varepsilon, \delta}(\omega)\Lambda_n^\omega(x)}(x) \supset B_\delta^\omega(x, n) \supset B_{R_{\varepsilon, \delta}^{-1}(\omega)\Lambda_n^\omega(x)}(x),$$

where $B_r(x)$ denotes the ball centered at x with radius r and

$$\Lambda_n^\omega(x) = \left(\prod_{j=0}^{n-1} \lambda_{\theta^j \omega}(f^j(\omega)x) \right)^{-1}.$$

Then

$$(4.40) \quad \sum_{x \in E_n^\omega} \exp(tS_n \varphi(x, \omega)) = \sum_{x \in E_n^\omega} (\Lambda_n^\omega(x))^t.$$

For $t = t_0$ the expression in the left hand side of (4.40) is subexponential in n , and for $t > t_0$ it tends to zero as $n \rightarrow \infty$ in view of (ii). Since

$$\bigcup_{x \in E_n^\omega} B_{R_{\varepsilon, \delta}(\omega)\Lambda_n^\omega(x)}(x) \supset \bigcup_{x \in E_n^\omega} B_\delta^\omega(x, n) \supset K^\omega,$$

I derive from here that

$$(4.41) \quad HD(K^\omega) \leq t_0.$$

In order to prove the inequality in the other direction one can employ a relativized version of the Brin-Katok local entropy formula (see [BK]) saying that for any τ -invariant ergodic probability measure μ on K such that $d\mu(x, \omega) = d\mu^\omega(x)dP(\omega)$ and μ -a.a. (x, ω) ,

$$(4.42) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \mu^\omega(B_\delta^\omega(x, n)) = -h_\mu^{(r)}(\tau).$$

Let μ be the equilibrium state corresponding to the function $t_0\varphi$ (see [Ki1], [KK]). Then by (4.36),

$$(4.43) \quad h_\mu^{(r)}(\tau) = -t_0 \int \varphi d\mu.$$

Since μ is ergodic (see [KK]), then for μ -a.a. (x, ω) ,

$$(4.44) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log \Lambda_n^\omega(x) = - \int \log \lambda_\omega(x) d\mu(x, \omega) < 0$$

and, in particular,

$$(4.45) \quad \lim_{n \rightarrow \infty} \frac{\log \Lambda_{n+1}^\omega(x)}{\log \Lambda_n^\omega(x)} = 1.$$

Observe also that if $n_j^{(\varepsilon)}(\omega)$, $i = 1, 2, \dots$, are successive integers for which $\theta^{n_j^{(\varepsilon)}}(\omega)\omega \in \Omega_\varepsilon$, $\varepsilon \in (0, 1)$, then by the ergodic theorem, for P -a.a. ω ,

$$(4.46) \quad \lim_{j \rightarrow \infty} \frac{n_{j+1}^{(\varepsilon)}(\omega)}{n_j^{(\varepsilon)}(\omega)} = 1.$$

This together with (4.39), (4.42)-(4.45), and the Fubini theorem yields that, with probability one for μ^ω -a.a. x ,

$$(4.47) \quad \lim_{\delta \rightarrow 0} \frac{\log \mu^\omega(B_\delta(x))}{\log \delta} = t_0.$$

Let $K_0^\omega \subset K^\omega$ be the set of x for which (4.47) holds true. Then from Proposition 2.1 in [Yo] (which is one of the general versions of Proposition 2.3 in Section 2) I obtain that $HD(K^\omega) \geq HD(K_0^\omega) = t_0$, which together with (4.41) completes the proof of Theorem 4.8. \square

4.9. Remark. Modifying the proof in [Ru] one can obtain that for P -a.s. ω the disintegrations μ^ω of μ are equivalent to t_0 -Hausdorff measures on K^ω , $\omega \in \Omega$.

4.10. Remark. It is also possible to generalize to the case of random diffeomorphisms Young's formula [Yo] which expresses dimensions of invariant measures of diffeomorphisms of surfaces via entropies and Lyapunov exponents. Other dimensions considered in [Yo] such as Rényi and Lyapunov dimension, as well as the correlation dimension, can be also defined and studied in the case of random diffeomorphisms.

4.11. Remark. One can obtain the lower bound in the proof of Theorem 4.8 by constructing a random Markov partition (see [BG1]) on a small neighborhood of the set M_1 and applying the relativized the Shannon-McMillan-Breiman theorem (see [Bo]) in place of the relativized Brin-Katok local entropy formula. The application of Shannon-McMillan-Breiman theorem is possible since similarly to the deterministic case the boundary of a Markov partition has zero measure with respect to any equilibrium state corresponding to a "good" function (see [KK]), the

argument being completed by some general version (see [Bi3]) of Proposition 2.3 above. Constructing Markov partitions and passing to symbolic representations (random subshifts of finite type, see [BG2]) also enables one to obtain a lot of non-trivial sets K^ω satisfying (1.10) taking sequences which do not contain prescribed symbols (cf. Remark 4.3).

4.12. *Remark.* One can study an expanding map by means of its inverse branches, which form an iterated function system consisting of contractions. Similarly, random expanding maps can be studied via random iterated function systems defined as follows. Let Ω, P, θ be as before, (X, ρ) be a complete metric space, $\mathcal{S} = (f_1, f_2, \dots)$ be a countable set of contractions of X , and $S(\omega)$ be a measurable family of finite subsets of \mathbb{Z}_+ . Set $s(\omega) = \max\{s_{f_i} : i \in S(\omega)\}$, where s_f is the contraction factor of f and $m(\omega) = |S(\omega)|$. Put $A^\omega = \{a = (a_0, a_1, \dots) : a_i \in S(\theta^i \omega)\}$ and consider iterations $f_a^n = f_{a_0} \circ f_{a_1} \circ \dots \circ f_{a_{n-1}}$. Suppose that

$$(4.48) \quad \int \log s(\omega) dP(\omega) < 0 \quad \text{and} \quad \int \log m(\omega) dP(\omega) < \infty.$$

The first condition in (4.48) implies that with probability one for any $x \in X$ and $a \in A^\omega$ the limit $\lim_{n \rightarrow \infty} f_a^n x = x_a(\omega)$ exists and the set $K^\omega = \{x_a(\omega) : a \in A^\omega\}$ satisfies $F(\omega)K^{\theta\omega} = K^\omega$, where $F(\omega)K = \bigcup_{\alpha \in S(\omega)} f_\alpha K$. This enables me to construct random repellers for random expanding (in particular, conformal) maps. Namely, given random expanding maps $f(\omega)$, $\omega \in \Omega$, I choose the set of indices $S(\omega)$ so that it corresponds to inverse branches of $f(\omega)$, which are contracting maps. For instance, if $f(\omega)$'s are taken out of a family of rational maps of the Riemannian sphere with hyperbolic Julia sets close to each other, then all inverse branches will be contractions in some fixed compact set. Another important feature of random iterated function systems consisting of contractions is the possibility to employ the Ruelle-Perron-Frobenius operator, which yields some of the results from [Ki2] and [KK] for this more general setup. As an example, let us see how the random Cantor sets from Remark 4.3 can be obtained via random iterated function systems. Let $d=1$ and $\Psi(\omega) \subset \{0, 1, \dots, m(\omega) - 1\}$. Set

$$\mathcal{S} = \{f : fx = \frac{x}{m} + \frac{k-1}{m}; m, k \in \mathbb{Z}_+, k \leq m\}$$

and

$$S(\omega) = \{k \in \Psi(\omega) : f_k x = \frac{x}{m(\omega)} + \frac{k}{m(\omega)}\}.$$

Then $K^\omega = \{x \in [0, 1] : x_j(\omega) \in \Psi(\theta^j \omega)\}$ can be obtained also as the set of limits $\lim_{n \rightarrow \infty} f_a^n x$ for $a \in A^\omega$. One can construct in this way some random von Koch type curves and other random fractals.

4.13. *Remark.* Let $f(\omega)$, $\omega \in \Omega$, be piecewise linear maps of $[0, 1]$ onto itself such that there exist points $0 = z_0(\omega) < z_1(\omega) < \dots < z_{m(\omega)-1} < z_{m(\omega)} = 1$ with $f(\omega)[z_l(\omega), z_{l+1}(\omega)] = [0, 1] \forall l = 0, \dots, m(\omega) - 1$. If $|z_{l+1}(\omega) - z_l(\omega)| = \frac{1}{m(\omega)} \forall l = 0, 1, \dots, m(\omega) - 1$, then we come back to the previous setup with $d = 1$. Suppose that

$$\begin{aligned} -\infty &< \int \min_{0 \leq l < m(\omega)} \log |z_{l+1}(\omega) - z_l(\omega)| dP(\omega) \\ &\leq \int \max_{0 \leq l < m(\omega)} \log |z_{l+1}(\omega) - z_l(\omega)| dP(\omega) < 0. \end{aligned}$$

The maps $f(\omega)$ are conformal (as any smooth one dimensional map) but, in general, the right and the left derivatives of $f(\omega)x$ in x do not coincide at $z_l(\omega), l = 1, \dots, m(\omega) - 1$. Still one can adapt Theorem 4.8 to this situation (adjusting the method from [Wa] to random transformations similarly to [Ki2] and [KK]) and obtain (4.37) for Hausdorff dimensions of sets K^ω satisfying (1.10), but I do not know an explicit formula such as (4.1) for the Hausdorff dimensions of the corresponding random Cantor sets (for some related results see Section 15.1 in [Fa]). Next, I can assign to each $x \in [0, 1)$ a sequence $x_0(\omega), x_1(\omega), \dots$ (where $x_j(\omega) = l - 1, l = 1, \dots, m(\theta^j \omega)$), if $f^j(\omega)x \in [z_{l-1}(\theta^j \omega), z_l(\theta^j \omega))$, where, as before, $f^j(\omega) = f(\theta^{j-1} \omega) \circ \dots \circ f(\omega)$. I can define again $N_{kl}^\omega(x, n), N_l^\omega(x, n), U_r^\omega$, and V_A^ω as in the Introduction and in Section 2. The cylinder sets $C_n^\omega(x)$ and the random Bernoulli measures $\nu_{p_A}^\omega$ can be defined as in Section 2. It follows that for any $x \in V_A^\omega$,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \nu_{p_A}^\omega(C_n^\omega(x)) = -\sum_k q_k \sum_{l \leq k} a_{kl} \log a_{kl}.$$

On the other hand,

$$\text{Leb}(C_n^\omega(x)) = \prod_{j=0}^{n-1} \lambda \circ \tau^j(x, \omega),$$

where $\lambda(x, \omega) = |z_l(\omega) - z_{l-1}(\omega)|$ if $x \in [z_{l-1}(\omega), z_l(\omega))$, which is more difficult to estimate. Since ν_{p_A} is ergodic, I obtain that with probability one, for $\nu_{p_A}^\omega$ -a.a. $x \in V_A^\omega$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log \text{Leb}(C_n^\omega(x)) &= \int \log \lambda d\nu_{p_A} \\ &= \sum_{k,l;l \leq k} \int_{\Omega_k} a_{kl} \log |z_l(\omega) - z_{l-1}(\omega)| dP(\omega) \stackrel{\text{def}}{=} \gamma \end{aligned}$$

where $\Omega_k = \{\omega : m(\omega) = k\}$. This gives only the lower bound

$$\gamma^{-1} \sum_k q_k \sum_{l \leq k} a_{kl} \log a_{kl}$$

for $HD(V_A^\omega)$. It is not clear what is the appropriate upper bound for $HD(V_A^\omega)$ and whether one can obtain an explicit formula for this Hausdorff dimension in our circumstances. Observe that in the nonrandom case it is easy, since then $\text{Leb}(C_n(x)) = \prod_{l=1}^m \lambda_l^{N_l(x)}$, where $\lambda_l = |z_l - z_{l-1}|$, and so for any $x \in V_A$,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \text{Leb}(C_n(x)) = -\sum_{l=1}^m r_l \log \lambda_l,$$

which yields

$$HD(U_r) = \frac{\sum_l r_l \log r_l}{\sum_l r_l \log \lambda_l}.$$

Still, there is a natural generalization which can be dealt with. Suppose that $\tilde{m}(\omega)$ is another \mathbb{Z}_+ -valued random variable satisfying $\int \log \tilde{m} dP < \infty$. Assume that

$\lambda(x, \omega) = \lambda_{\tilde{k}l}$ if $x \in [z_{l-1}(\omega), z_l(\omega))$ and $\tilde{m}(\omega) = \tilde{k}$, i.e. the lengths $|z_l(\omega) - z_{l-1}(\omega)|$ depend only on l and $\tilde{m}(\omega)$. Denote by $N_{\tilde{k}kl}^\omega(x, n)$ the number of j 's between 0 and $n - 1$ such that $m(\theta^j \omega) = k$, $\tilde{m}(\theta^j \omega) = \tilde{k}$, and $x_j(\omega) = l - 1$. Let $B = (b_{\tilde{k}kl}, \tilde{k}, k, l \in \mathbb{Z}_+, l \leq k)$, where $b_{\tilde{k}kl} \geq 0$, $\sum_{l \leq k} b_{\tilde{k}kl} = 1$, and set

$$W_B^\omega = \{x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{1}{n} N_{\tilde{k}kl}^\omega(x, n) = u_{\tilde{k}k} b_{\tilde{k}kl} \forall \tilde{k}, k, l \in \mathbb{Z}_+\},$$

where $u_{\tilde{k},k} = P\{m = k, \tilde{m} = \tilde{k}\}$. Set $p_l^\omega = b_{\tilde{k}kl}$ if $m(\omega) = k$ and $\tilde{m}(\omega) = \tilde{k}$, and denote by $\nu_{p_B}^\omega$ the unique measure satisfying (2.29). Then for any $x \in W_B^\omega$, with probability one,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \nu_{p_B}^\omega(C_n^\omega(x)) = -\sum_{\tilde{k},k} u_{\tilde{k},k} \sum_{l \leq k} b_{\tilde{k}kl} \log b_{\tilde{k}kl}$$

and

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \text{Leb}(C_n^\omega(x)) = -\sum_{\tilde{k},k} u_{\tilde{k},k} \sum_{l \leq k} b_{\tilde{k}kl} \log \lambda_{\tilde{k}l}.$$

Applying a slight modification of Lemma 2.2 (cf. Theorem 1 from [Bi3]), I derive from here that

$$HD(W_B^\omega) = \frac{\sum_{\tilde{k},k} u_{\tilde{k},k} \sum_{l \leq k} b_{\tilde{k}kl} \log b_{\tilde{k}kl}}{\sum_{\tilde{k},k} u_{\tilde{k},k} \sum_{l \leq k} b_{\tilde{k}kl} \log \lambda_{\tilde{k}l}}.$$

Let $\mathcal{B}_{u,q,A}$ be the set of $B = (b_{\tilde{k},k,l})$ as above such that $\sum_{\tilde{k}} u_{\tilde{k}k} b_{\tilde{k}kl} = q_k a_{kl}$ with q_k and $A = (a_{kl})$ as before. Then

$$HD(V_A^\omega) = \sup_{B \in \mathcal{B}_{u,q,A}} HD(W_B^\omega) \quad \text{and} \quad HD(U_r) = \sup_{A \in \mathcal{A}_{qr}} HD(V_A^\omega).$$

4.14. *Remark.* Random subshifts considered in [BG2], and [KK] together with the corresponding relativized thermodynamic formalism are the natural framework for general models of random graph directed constructions (see [Ol] and references there) and random geometric (Moran) constructions (see [PW]) of a different type than considered before. Let σ be the shift on the space $\mathcal{X} = \bar{\mathbb{Z}}_+^{\mathbb{Z}_+}$ of sequences $x = (x_i; i = 1, 2, \dots; x_i \in \bar{\mathbb{Z}}_+)$ considered with the product topology, where $\bar{\mathbb{Z}}_+$ is the one point compactification of \mathbb{Z}_+ . A random subshift of (\mathcal{X}, σ) is a collection of a probability space (Ω, \mathcal{F}, P) together with a \mathbb{Z}_+ -valued random variable $m(\omega)$ satisfying $\int \log m dP < \infty$, a P -preserving transformation θ of Ω , and a measurable family of compact subsets $X^\omega \subset \mathcal{X}$ such that $\sigma X^\omega = X^{\theta\omega}$ and if $x = (x_i) \in X^\omega$ then $x_i \in \{1, 2, \dots, m(\omega)\}$. If there exists a measurable family of $m(\omega) \times m(\theta\omega)$ -matrices $B(\omega)$ with entries $b_{kl}(\omega)$, $k = 1, \dots, m(\omega)$, $l = 1, \dots, m(\theta\omega)$, equal to 0 or 1 and such that $X^\omega = \{x = (x_i) : b_{x_i x_{i+1}}(\theta^i \omega) = 1\}$, then the random subshift is said to be of finite type. For any $\alpha = (\alpha_i) \in X^\omega$ set $\alpha^{(n)} = (\alpha_1, \dots, \alpha_n)$ and $C_{\alpha^{(n)}}^\omega = \{x = (x_i) \in X^\omega : x_i = \alpha_i \forall i = 1, \dots, n\}$. Similarly to [PW] I consider a symbolic construction which is determined by a measurable family of closed (usually, compact) subsets $\Delta_{\alpha^{(n)}}^\omega$ of \mathbb{R}^d (or of another good complete metric space) defined for all $\alpha \in X^\omega$ and such that: $\Delta_{\alpha^{(n)}}^\omega \cap \Delta_{\beta^{(n)}}^\omega = \emptyset$ if $\alpha^{(n)} \neq \beta^{(n)}$, $\Delta_{\alpha^{(n+1)}}^\omega \subset \Delta_{\alpha^{(n)}}^\omega$,

and $\lim_{n \rightarrow \infty} \max_{\alpha^{(n)}} \text{diam}(\Delta_{\alpha^{(n)}}^\omega) = 0$. Now one is interested in dimension parameters of the limiting sets $K^\omega = \bigcap_{n=1}^\infty \bigcup_{\alpha^{(n)}: C_{\alpha^{(n)}}^\omega \neq \emptyset} \Delta_{\alpha^{(n)}}^\omega$. Suppose that there exist random variables $K_n^\omega > 0$ and $0 < \lambda_i^\omega < 1, i = 1, \dots, m(\omega)$, such that each $\Delta_{\alpha^{(n)}}^\omega$ contains a ball of radius $(K_n^\omega)^{-1} r_{\alpha^{(n)}}^\omega$ and is contained in a ball of radius $K_n^\omega r_{\alpha^{(n)}}^\omega$, where $r_{\alpha^{(n)}}^\omega = \lambda_{\alpha_n}^{\theta^{n-1}\omega} \dots \lambda_{\alpha_2}^{\theta\omega} \lambda_{\alpha_1}^\omega, \int \min_i \log \lambda_i^\omega dP(\omega) > -\infty$, and with probability one $\lim_{n \rightarrow \infty} \frac{1}{n} \log K_n^\omega = 0$. Then similarly to Theorem 4.8 one can show that with probability one $HD(K^\omega) = t_0$ provided $Q^{(r)}(t_0 \log \lambda) = 0$, where $Q^{(r)}$ is the relativized topological pressure of the random subshift defined above and the function λ on $X = \{(x, \omega) : x \in X^\omega\}$ is given by $\lambda^\omega(x) = \lambda_{x_1}^\omega, x = (x_i) \in X^\omega$. If $\Delta_{\alpha^{(n)}}^\omega = f_{\alpha_n}(\theta^{n-1}\omega) \circ \dots \circ f_{\alpha_2}(\theta\omega) \circ f_{\alpha_1}(\omega)(\Delta)$, where $f_1(\omega), \dots, f_{m(\omega)}(\omega) : \Delta \rightarrow \Delta$ are conformal contractions and Δ is a fixed compact set, then we arrive at a partial case of the above construction. If, in addition, the random subshift is of finite type, we obtain a certain random graph directed (or Markov) construction. It is possible to study also constructions governed by random sofic systems. Furthermore, similarly to Remark 4.13, under some additional assumptions I can estimate the Hausdorff dimension of subsets of points $z \in K^\omega$ such that their symbolic representations $\alpha = \alpha(z) = (\alpha_1, \alpha_2, \dots) \in X^\omega, z = \bigcap_{\alpha^{(n)}} \Delta_{\alpha^{(n)}}^\omega$, have prescribed frequencies of digits, i.e. the limits $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \chi_l(x_i)$ are given.

4.15. *Remark.* Employing [ABD] (see also [Pe]), one can extend the notion of the correlation dimension to random transformations. Let $f(\omega) : X \rightarrow X$ be continuous random transformations, $\omega \in \Omega$, and let $\theta : \Omega \rightarrow \Omega$ be a P -preserving ergodic transformation on Ω . Let $\mu \in \mathcal{P}(X \times \Omega)$ be an ergodic invariant measure of the skew product transformation τ given by (1.8) with the marginal P on Ω , i.e. $d\mu(x, \omega) = d\mu^\omega(x)dP(\omega)$. Set

$$C^\omega(x, n, r) = \frac{2}{n(n-1)} \text{card} \left\{ (i, j) : \text{dist}(f^i(\omega)x, f^j(\omega)x) \leq r \text{ for } 0 \leq i < j < n \right\}.$$

Then, with probability one, for μ^ω -a.a. x ,

$$\lim_{n \rightarrow \infty} C^\omega(x, n, r) = \int_{X \times X} \int_{\Omega \times \Omega} \chi_{\{\text{dist}(y, z) \leq r\}} d\mu^\omega(x) d\mu^{\tilde{\omega}}(y) dP(\omega) dP(\tilde{\omega}),$$

provided that, for $P \times P$ -a.a. $(\omega, \tilde{\omega})$,

$$\mu^\omega \times \mu^{\tilde{\omega}} \left\{ (y, z) : \text{dist}(y, z) = r \right\} = 0.$$

One can further describe sets of r 's for which this condition holds true similarly to [Pe].

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INSTITUTE OF MATHEMATICS, HEBREW UNIVERSITY OF JERUSALEM, GIVAT RAM, JERUSALEM
91904, ISRAEL

E-mail address: `kifer@math.huji.ac.il`