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RELATIVELY FREE INVARIANT ALGEBRAS OF FINITE REFLECTION GROUPS

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ABSTRACT. Let G be a finite subgroup of $Gl_n(K)$ (K is a field of characteristic 0 and $n \geq 2$) acting by linear substitution on a relatively free algebra $K\langle x_1, \ldots, x_n\rangle/I$ of a variety of unitary associative algebras. The algebra of invariants is relatively free if and only if G is a pseudo-reflection group and I contains the polynomial $[[x_2, x_1], x_1]$.

1. Introduction

Throughout the paper K is a field of characteristic 0 and $K\langle x_1, \ldots, x_n\rangle$ is the unitary free associative $K-$algebra of rank n. The general linear group $Gl_n = Gl_n(K)$ acts on the free algebra by linear substitution. More explicitly, if $g = (g_{ij}) \in Gl_n$, then

$$g \cdot x_j = \sum_{i=1}^{n} g_{ij} x_i,$$

and for any $f(x_1, \ldots, x_n) \in K\langle x_1, \ldots, x_n\rangle$

$$g \cdot f = f(g \cdot x_1, \ldots, g \cdot x_n).$$

An ideal I of the unitary free associative algebra $K\langle x_1, x_2, \ldots\rangle$ of countable rank is called a *T-ideal* if I is invariant under any $K-$algebra endomorphism of the free algebra, that is, $f(x_1, \ldots, x_n) \in I$ implies $f(u_1, \ldots, u_n) \in I$ for any $u_1, \ldots, u_n \in K\langle x_1, x_2, \ldots\rangle$.

$I_n = I \cap K\langle x_1, \ldots, x_n\rangle$ is invariant under the action of Gl_n, so

$$F_n(I) = K\langle x_1, \ldots, x_n\rangle/I_n$$

inherits the Gl_n-structure. $F_n(I)$ is a relatively free algebra of rank n of the variety of unitary associative algebras satisfying all the identities $f = 0$ with $f \in I$. Denote by y_i the image of x_i under the natural homomorphism

$$K\langle x_1, \ldots, x_n\rangle \to K\langle x_1, \ldots, x_n\rangle/I_n.$$

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Let G be a finite subgroup of Gl_n and denote by

$$F_n(I)^G = \{f \in F_n(I) \mid g \cdot f = f \ \forall g \in G\}$$

the *algebra of invariants of* G. We refer to the article of Formanek [8] for a survey of the results on the algebra of invariants of a finite linear group acting on a relatively free algebra. In the special case of the variety of all unitary commutative algebras, that is, when I is generated by the commutator $[x_1, x_2] = x_1x_2 - x_2x_1$, the relatively free algebra

$$F_n(I) = K[x_1, \dots, x_n]$$

is the n variable commutative polynomial algebra, and the study of $F_n(I)^G$ is the topic of classical invariant theory.

An element $g \in Gl_n(K)$ is called a *pseudo-reflection,* if it fixes an $n-1$ dimensional subspace of K^n. If g is of finite order then g is a pseudo-reflection if and only if g has the eigenvalue 1 with multiplicity $n-1$. A finite subgroup $G < Gl_n$ is called a *pseudo-reflection group* if it is generated by pseudo-reflections. Our starting point is the following famous result (Shephard-Todd [13], Chevalley [2]):

Theorem 1.1. *$K[x_1, \dots, x_n]^G$ is a polynomial algebra if and only if G is a pseudo-reflection group. Moreover, if G is a pseudo-reflection group, then there exist n algebraically independent homogeneous forms which generate $K[x_1, \dots, x_n]^G$. The degrees $d_1 \leq \ldots \leq d_n$ of these forms are uniquely determined by G, and are called the degrees of G. We have the equality $|G| = \prod_{i=1}^n d_i$, and the number of pseudo-reflections in G is $\sum_{i=1}^n (d_i - 1)$.*

We shall extend the notion of algebraic independence to any variety of unitary associative $K-$algebras. We fix a T-ideal I, and denote by $\mathcal{V}(I)$ the variety defined by I. Let R be an algebra in $\mathcal{V}(I)$. We say that the elements $f_1, \dots, f_m \in R$ are *algebraically independent,* if $h(f_1, \dots, f_m) = 0$ for some $h \in K\langle x_1, \dots, x_m\rangle$ implies that $h \in I$. Note that this notion of algebraic independence depends on the T-ideal I and belongs to the variety $\mathcal{V}(I)$.

Guralnick proved in [9] that if $n \geq 2$, $k \geq 2$ and I is the T-ideal of the $k \times k$ matrix algebra over K, then $F_n(I)^G$ is not relatively free for any finite group G. As it was pointed out in [8, p. 105], this result implies easily that $F_n(I)^G$ is not relatively free if I is contained in the T-ideal of the 2×2 matrix algebra. In this paper we give a complete answer to the question of determining when $F_n(I)^G$ is relatively free. The answer was conjectured by Drensky [7].

The case $n = 1$ is trivial, because then $F_1(I) \cong K[x_1] \cong K\langle x_1\rangle$ and G is a cyclic group acting by scalar multiplication, so the algebra of invariants is $K[x^m]$ for some positive integer m.

Theorem 1.2. *Let $G < Gl_n(K)$ $(n \geq 2)$ be a finite group and let I be a T-ideal. Then $F_n(I)^G$ is generated by algebraically independent elements if and only if G is a pseudo-reflection group and I contains the polynomial $[[x_2, x_1], x_1]$.*

2. Preliminaries

The algebra $K\langle x_1, \dots, x_n\rangle^G$ is always free and almost never finitely generated by [4] and [10]. However, working in a proper subvariety of the variety of all unitary associative $K-$algebras the notion of transcendence degree makes sense.

Proposition 2.1. *Fix a non-zero T-ideal I and consider the variety $\mathcal{V}(I)$. If R is an algebra in the variety $\mathcal{V}(I)$ generated by n elements, then it does not contain more than n algebraically independent elements.*

Proof. R is a homomorphic image of $F_n(I)$ and obviously the preimages of algebraically independent elements of R are also algebraically independent, so it suffices to prove the proposition for the relatively free algebra $F_n(I)$. Assume that $f_1, \ldots, f_m \in F_n(I)$ are algebraically independent. Let $J \subseteq K\langle x_1, x_2, \ldots\rangle$ be the radical of I, that is,

$$J = \{f \in K\langle x_1, x_2, \ldots\rangle \mid f^N \in I \text{ for some } N\}.$$

It is well known (see for example [12, Theorems 1.5.32, 2.4.7 and 3.2.6]) that J is the set of identities of the $k \times k$ matrix algebra for some $k \geq 1$. We have the natural onto homomorphism

$$K\langle x_1, \ldots, x_n\rangle / I_n \to K\langle x_1, \ldots, x_n\rangle / J_n.$$

Denote by $h_1, \ldots, h_m$ the images of $f_1, \ldots, f_m$. Suppose that for some $p \in K\langle x_1, \ldots, x_m\rangle$ we have $p(h_1, \ldots, h_m) = 0$. Then there exists an integer N such that $p^N(f_1, \ldots, f_m) = 0$ in $F_n(I)$, and the algebraic independence of the f_is implies that $p^N(x_1, \ldots, x_m) \in I_m$. Hence $p(x_1, \ldots, x_m) \in J_m$ by the definition of J_m, and this shows the algebraic independence of $h_1, \ldots, h_m$ in $F_n(J)$.

The center of $F_m(J)$ is a commutative domain and its transcendence degree is $d = (m-1)k^2 + 1$ (see for example [8, p. 105]). Let

$$c_1 = c_1(y_1, \ldots, y_m), \ldots, c_d = c_d(y_1, \ldots, y_m)$$

be a transcendence basis of the center of $F_m(J)$ (so the $c_i(x_1, \ldots, x_m)$ are central polynomials for the $k \times k$ matrix algebra). The algebraic independence of $h_1, \ldots, h_m$ in $F_n(J)$ together with the algebraic independence (in the ordinary sense) of $c_1, \ldots, c_d$ in the center of $F_m(J)$ implies that $c_i(h_1, \ldots, h_m)$ $(i = 1, \ldots, d)$ are algebraically independent in the center of $F_n(J)$. But this center has transcendence degree $(n-1)k^2 + 1$, implying that $m \leq n$. $\square$

We use the following crucial argument from [8] or [9]. Any non-zero T-ideal I is contained in the T-ideal generated by $[x_1, x_2]$. So we have the natural onto homomorphism

$$F_n(I) \to K[x_1, \ldots, x_n]$$

which commutes with the action of G, and by the complete reducibility of this action the homomorphism

$$F_n(I)^G \to K[x_1, \ldots, x_n]^G$$

is also onto. Therefore the image of a generating set of $F_n(I)^G$ is a generating set of $K[x_1, \ldots, x_n]^G$, hence $F_n(I)^G$ can not be generated by less than n elements. Thus if $F_n(I)^G$ is generated by algebraically independent elements, then by Proposition 2.1 it is generated by n elements.

Assume that $f_1, \ldots, f_n$ generate $F_n(I)^G$. Their images in $K[x_1, \ldots, x_n]$ generate $K[x_1, \ldots, x_n]^G$, hence G must be a pseudo-reflection group. We may assume by [9, Lemma 2] that $f_1, \ldots, f_n$ are homogeneous, therefore their degrees are the degrees of G. Moreover, the proof of [9, Lemma 2] shows that if $h_1, \ldots, h_n \in F_n(I)^G$ such that their images generate $K[x_1, \ldots, x_n]^G$, then $F_n(I)^G = K\langle h_1, \ldots, h_n\rangle$.

By the above discussion Theorem 1.2 splits into the next two statements:

Theorem 2.2. *Let G be a pseudo-reflection group and let I be a T-ideal containing $[[x_2, x_1], x_1]$. If $f_1, \dots, f_n$ are homogeneous elements in $F_n(I)^G$ such that their images generate $K[x_1, \dots, x_n]^G$, then $f_1, \dots, f_n$ are algebraically independent in $F_n(I)$ and they generate $F_n(I)^G$. In particular, $F_n(I)^G \cong F_n(I)$.*

Theorem 2.3. *If the T-ideal I does not contain the polynomial $[[x_2, x_1], x_1]$, then $F_n(I)^G$ can not be generated by n elements for any $n \geq 2$ and finite group $G < Gl_n(K)$.*

We recall some basic facts about the T-ideals of the unitary free associative $K-$algebra. Let B denote the subalgebra of $K\langle x_1, x_2, \dots\rangle$ generated by all the commutators $[x_{i_1}, \dots, x_{i_r}]$ with $r \geq 2$, where $[x_1, \dots, x_r]$ is defined inductively by $[x_1, \dots, x_r] = [[x_1, \dots, x_{r-1}], x_r]$ for $r \geq 3$. The elements of B are called *proper* polynomials, and they can be characterised as the polynomials with zero partial derivatives. Clearly, $B_n = B \cap K\langle x_1, \dots, x_n\rangle$ is a Gl_n-submodule of $K\langle x_1, \dots, x_n\rangle$. Drensky showed (see [5, Theorem 2.2]) that any T-ideal I of the free unitary algebra is generated by the proper polynomials that it contains, and as Gl_n-modules

$$F_n(I) \cong K[x_1, \dots, x_n] \otimes (B_n/B_n \cap I). \tag{2.1}$$

Both I_n and B_n are multigraded subalgebras with respect to the usual multigrading on $K\langle x_1, \dots, x_n\rangle$, so $F_n(I)$ and $B_n/B_n \cap I$ inherit the multigrading.

For any $\alpha = (\alpha_1, \dots, \alpha_n)$ and multigraded algebra R we put

$$R^{(\alpha)} = \{f \in R \mid f \text{ is multihomogeneous of multidegree } \alpha\},$$

and for any $\mathbb{N}-$graded algebra R and non-negative integer d

$$R^{(d)} = \{f \in R \mid f \text{ is homogeneous of total degree } d\}.$$

Obviously $B_n^{(0)} = K$ and $B_n^{(1)} = \emptyset$. If $n \geq 2$, then $B_n^{(2)}$ is an irreducible Gl_n-module generated by $[x_1, x_2]$, and $B_n^{(3)}$ is an irreducible Gl_n-module generated by $[x_2, x_1, x_1]$. Therefore there exists a unique maximal T-ideal M among the T-ideals not containing $[x_2, x_1, x_1]$, M is generated by the proper polynomials of degree greater than 3 (in particular, M has no elements of degree less than 4). If I is a T-ideal not containing $[x_2, x_1, x_1]$, then $F_n(M)^G$ is a homomorphic image of $F_n(I)^G$, thus Theorem 2.3 is a consequence of the following more special statement:

Proposition 2.4. *Let M be the T-ideal generated by the proper polynomials of degree greater than 3, and let $G < Gl_n$ $(n \geq 2)$ be a finite pseudo-reflection group. Then $F_n(M)^G$ can not be generated by n elements.*

The Hilbert series of a multigraded algebra R is an n variable formal power series defined by

$$H(R; t_1, \dots, t_n) = \sum_{\alpha = (\alpha_1, \dots, \alpha_n)} dim_K(R^{(\alpha)}) t_1^{\alpha_1} \dots t_n^{\alpha_n},$$

and the Hilbert series of an $\mathbb{N}-$graded algebra R is the formal power series

$$H(R; t) = \sum_{d=0}^{\infty} dim_K(R^{(d)}) t^d.$$

Let G be a finite subgroup of $Gl_n(K)$. For any element $g \in G$ denote by $\omega_1(g), \dots, \omega_n(d)$ the eigenvalues of g in the algebraic closure of K. $F_n(I)^G$ is an $\mathbb{N}-$graded algebra, and we have the noncommutative Molien-Weyl formula (see [8, Theorem 7]):

$$H(F_n(I)^G;t) = \frac{1}{|G|}\sum_{g\in G} H(F_n(I);\omega_1(g)t,\dots,\omega_n(g)t). \tag{2.2}$$

3. Proof of Theorem 2.2

Denote by J the T-ideal generated by $[x_2, x_1, x_1]$, and let

$$s_d(x_1,\dots,x_d) = \sum_{\pi\in Sym(d)} (-1)^\pi x_{\pi(1)}\dots x_{\pi(d)}$$

be the *standard polynomial.* It is well known (see [6, 3.2.1. Theorem]) that $B_n^{(d)}/B_n^{(d)} \cap J = 0$, if d is odd or $d > n$, and it is an irreducible Gl_n-module generated by $s_d(y_1,\dots,y_d)$ if d is even and $d \le n$. Let $J(m)$ be the T-ideal generated by $[x_2, x_1, x_1]$ and $s_{2m}(x_1,\dots,x_{2m})$, so the only T-ideals containing $[x_2, x_1, x_1]$ are J and $J(m)$ $(m = 1, 2, \dots)$. We note that

$$K\langle x_1,\dots,x_n\rangle/J(m)_n = K\langle x_1,\dots,x_n\rangle/J_n \quad \text{if } 2m > n.$$

Let $f_1,\dots,f_n \in F_n(J(m))^G$ be homogeneous invariants such that their images in $K[x_1,\dots,x_n]$ generate $K[x_1,\dots,x_n]^G$. We prove by induction on m that $f_1,\dots,f_n$ are algebraically independent in $F_n(J(m))$. In the case $m = 1$ there is nothing to prove. Suppose that the statement is true for $F_n(J(k))$, where $k \le m$. We may assume that $2m \le n$. The algebra $F_n(J(m+1))$ has a linear basis (see [6])

$$\{y_1^{\alpha_1}\dots y_n^{\alpha_n} s_{2k}(y_{i_1},\dots,y_{i_{2k}}) | \alpha_1,\dots,\alpha_n \in \mathbb{N},\ 0 \le k \le m,\ 1 \le i_1 < \dots < i_{2k} \le n\}.$$

Suppose that $h(f_1,\dots,f_n) = 0$ in $F_n(J(m+1))$ for some $h \in K\langle x_1,\dots,x_n\rangle$ with $h \notin J(m+1)_n$. Then we may assume that h is of the form

$$h = \sum_{\alpha=(\alpha_1,\dots,\alpha_n)} \sum_{k=0}^{m} \sum_{1\le i_1<\dots<i_{2k}\le n} a_{\alpha,k,i} x_1^{\alpha_1}\dots x_n^{\alpha_n} s_{2k}(x_{i_1},\dots,x_{i_{2k}}),$$

where at least one of the coefficients $a_{\alpha,k,i}$ is non-zero. Let k_0 be the minimal k such that $a_{\alpha,i,k} \ne 0$ for some α, i. If $k_0 \le m-1$, then by the induction hypothesis

$$\sum_{\alpha,i} a_{\alpha,k_0,i} f_1^{\alpha_1}\dots f_n^{\alpha_n} s_{2k_0}(f_{i_1},\dots,f_{i_{2k_0}}) \ \notin\ J_n(k_0+1)/J_n(m+1).$$

On the other hand

$$\sum_{\alpha,k>k_0,i} a_{\alpha,k,i} f_1^{\alpha_1}\dots f_n^{\alpha_n} s_{2k}(f_{i_1},\dots,f_{i_{2k}}) \equiv 0 \text{ (modulo } J_n(k_0+1)/J_n(m+1)),$$

contradicting that $h(f_1,\dots,f_n)=0$ in $F_n(J(m+1))$. Thus $k_0=m$ and we have

$$\sum_{\alpha,i} a_{\alpha,i} f_1^{\alpha_1}\dots f_n^{\alpha_n} s_{2m}(f_{i_1},\dots,f_{i_{2m}})=0$$

in $F_n(J(m+1))$.

Define the maps

$$\frac{\partial}{\partial y_i}: F_n(J(m+1))\to F_n(J(m+1)) \quad (i=1,\dots,n)$$

in the following way. If $f=f(y_1,\dots,y_n)\in F_n(J(m+1))$ is multihomogeneous of multidegree $\alpha=(\alpha_1,\dots,\alpha_n)$, then let $\frac{\partial}{\partial y_i}f$ be the multihomogeneous component of multidegree $(\alpha_1,\dots,\alpha_i-1,\dots,\alpha_n)$ of $f(y_1+1,\dots,y_n+1)$ ($\frac{\partial}{\partial y_i}f=0$ if $\alpha_i=0$). Now extend $\frac{\partial}{\partial y_i}$ to $F_n(J(m+1))$ by linearity. Clearly, $\frac{\partial}{\partial y_i}$ is a derivation.

Lemma 3.1. *For any $f,g\in F_n(J(m+1))$ we have the equality*

$$[f,g]\equiv \sum_{1\le i,j\le n}\frac{\partial f}{\partial y_i}\frac{\partial g}{\partial y_j}[y_i,y_j] \quad (\textit{modulo } C^2),$$

where

$$C=F_n(J(m+1))[F_n(J(m+1)),F_n(J(m+1))]F_n(J(m+1))$$

is the commutator ideal of $F_n(J(m+1))$.

Proof. By the multilinearity of the derivations and [,] it suffices to prove the lemma when

$$f=y_{i_1}\dots y_{i_k} \text{ and } g=y_{j_1}\dots y_{j_l}$$

are monomials. We have

$$\begin{aligned}(y_{i_1}\dots y_{i_k})(y_{j_1}\dots y_{j_l})\\ &= y_{i_1}\dots y_{i_{k-1}}y_{j_1}y_{i_k}y_{j_2}\dots y_{j_l}+y_{i_1}\dots y_{i_{k-1}}[y_{i_k},y_{j_1}]y_{j_2}\dots y_{j_l}\\ &= (y_{i_1}\dots y_{i_{k-1}})(y_{j_2}\dots y_{j_l})[y_{i_k},y_{j_1}]+y_{i_1}\dots y_{i_{k-1}}y_{j_1}y_{i_k}y_{j_2}\dots y_{j_l}\end{aligned}$$

(the second equality follows from the fact that any commutator lies in the center of $F_n(J(m+1))$). Now exchange $y_{i_{k-1}}$ and y_{j_1} in the second term of the above sum, that is, replace this term by

$$y_{i_1}\dots y_{i_{k-2}}y_{j_1}y_{i_{k-1}}y_{i_k}y_{j_2}\dots y_{j_l}+(y_{i_1}\dots y_{i_{k-2}}y_{j_k})(y_{j_2}\dots y_{j_l})[y_{i_{k-1}},y_{j_1}].$$

Continuing this process we obtain

$$\begin{aligned}(y_{i_1}\dots y_{i_k})(y_{j_1}\dots y_{j_l})\\ &= y_{j_1}(y_{i_1}\dots y_{i_k})(y_{j_2}\dots y_{j_l})+\sum_{r=1}^{k}(y_{i_1}\dots \hat{y}_{i_r}\dots y_{i_k})(y_{j_2}\dots y_{j_l})[y_{i_r},y_{j_1}],\end{aligned}$$

where the sign $\hat{y}_{i_r}$ means that we delete y_{i_r} in the word $y_{i_1}\dots y_{i_k}$. On applying the same process to the word $(y_{i_1}\dots y_{i_k})(y_{j_2}\dots y_{j_l})$ we obtain that

$$\begin{aligned}
(y_{i_1}\dots y_{i_k})(y_{j_1}\dots y_{j_l}) &= \sum_{r=1}^{k}(y_{i_1}\dots \hat{y}_{i_r}\dots y_{i_k})(y_{j_2}\dots y_{j_l})[y_{i_r},y_{j_1}]\\
&+\sum_{r=1}^{k} y_{j_1}(y_{i_1}\dots \hat{y}_{i_r}\dots y_{i_k})(y_{j_3}\dots y_{j_l})[y_{i_r},y_{j_2}]\\
&+ y_{j_1}y_{j_2}(y_{i_1}\dots y_{i_k})(y_{j_3}\dots y_{j_l})\\
&\overset{(\mathrm{mod}\ C^2)}{\equiv} \sum_{r=1}^{k}(y_{i_1}\dots \hat{y}_{i_r}\dots y_{i_k})(\hat{y}_{j_1}y_{j_2}\dots y_{j_l})[y_{i_r},y_{j_1}]\\
&+\sum_{r=1}^{k}(y_{i_1}\dots \hat{y}_{i_r}\dots y_{i_k})(y_{j_1}\hat{y}_{j_2}y_{j_3}\dots y_{j_l})[y_{i_r},y_{j_2}]\\
&+ y_{j_1}y_{j_2}(y_{i_1}\dots y_{i_k})(y_{j_3}\dots y_{j_l})\\
&= \sum_{i=1}^{n}(\frac{\partial}{\partial y_i}(y_{i_1}\dots y_{i_k}))(\hat{y}_{j_1}y_{j_2}\dots y_{j_l})[y_i,y_{j_1}]\\
&+\sum_{i=1}^{n}(\frac{\partial}{\partial y_i}(y_{i_1}\dots y_{i_k}))(y_{j_1}\hat{y}_{j_2}\dots y_{j_l})[y_i,y_{j_2}]\\
&+ y_{j_1}y_{j_2}(y_{i_1}\dots y_{i_k})(y_{j_3}\dots y_{j_l}).
\end{aligned}$$

Repeating this algorithm finally we get

$$\begin{aligned}
&(y_{i_1}\dots y_{i_k})(y_{j_1}\dots y_{j_l})\\
&\overset{(\mathrm{mod}\ C^2)}{\equiv} (y_{j_1}\dots y_{j_l})(y_{i_1}\dots y_{i_k})+\sum_{i=1}^{n}\sum_{r=1}^{l}(\frac{\partial}{\partial y_i}(y_{i_1}\dots y_{i_k}))(y_{j_1}\dots \hat{y}_{j_r}\dots y_{j_l})[y_i,y_{j_r}]\\
&= (y_{j_1}\dots y_{j_l})(y_{i_1}\dots y_{i_k})+\sum_{i=1}^{n}\sum_{j=1}^{n}(\frac{\partial}{\partial y_i}(y_{i_1}\dots y_{i_k}))(\frac{\partial}{\partial y_j}(y_{j_1}\dots y_{j_l}))[y_i,y_j],
\end{aligned}$$

which explicitly shows the claim. □

Now we use Lemma 3.1 to rewrite $A = s_{2m}(f_1,\dots,f_{2m})$. By the definition of s_{2m}

$$A=\frac{1}{2^m}\sum_{\pi\in Sym(2m)}(-1)^{\pi}[f_{\pi(1)},f_{\pi(2)}]\dots[f_{\pi(2m-1)},f_{\pi(2m)}].$$

We have $C^{m+1}=0$ in $F_n(J(m+1))$, because the commutators are central elements of $F_n(J(m+1))$ and any proper polynomial of degree greater than $2m+1$ is contained in $J(m+1)$. Therefore in the right hand side of the above equality we may replace $[f_{\pi(2k-1)},f_{\pi(2k)}]$ by something that is congruent with it modulo C^2.

So by Lemma 3.1 we have

$$A = \frac{1}{2^m} \sum_{\pi \in Sym(2m)} (-1)^\pi \sum_{(i_1,\dots,i_{2m})} \frac{\partial f_{\pi(1)}}{\partial y_{i_1}} \frac{\partial f_{\pi(2)}}{\partial y_{i_2}} [y_{i_1}, y_{i_2}] \dots \frac{\partial f_{\pi(2m-1)}}{\partial y_{i_{2m-1}}} \frac{\partial f_{\pi(2m)}}{\partial y_{i_{2m}}} [y_{i_{2m-1}}, y_{i_{2m}}]$$

$$= \frac{1}{2^m} \sum_{\pi \in Sym(2m)} (-1)^\pi \sum_{(i_1,\dots,i_{2m})} \frac{\partial f_{\pi(1)}}{\partial y_{i_1}} \dots \frac{\partial f_{\pi(2m)}}{\partial y_{i_{2m}}} [y_{i_1}, y_{i_2}] \dots [y_{i_{2m-1}}, y_{i_{2m}}].$$

The polynomial $[x_1, x_2][x_1, x_3]$ is contained in J, hence if the $i_1, \dots, i_{2m}$ are not pairwise different, then the corresponding term is zero in the above expression, implying that

$$A = \frac{1}{2^m} \sum_{\pi \in Sym(2m)} (-1)^\pi \sum_{1 \le i_1 < \dots < i_{2m} \le n} \sum_{\rho \in Sym(2m)} \frac{\partial f_{\pi(1)}}{\partial y_{i_{\rho(1)}}} \dots \frac{\partial f_{\pi(2m)}}{\partial y_{i_{\rho(2m)}}} [y_{i_{\rho(1)}}, y_{i_{\rho(2)}}] \dots [y_{i_{\rho(2m-1)}}, y_{i_{\rho(2m)}}]$$

$$= \sum_{1 \le i_1 < \dots < i_{2m} \le n} \frac{1}{2^m} \sum_{\rho \in Sym(2m)} (-1)^\rho [y_{i_{\rho(1)}}, y_{i_{\rho(2)}}] \dots [y_{i_{\rho(2m-1)}}, y_{i_{\rho(2m)}}] \times \sum_{\pi \in Sym(2m)} (-1)^\pi (-1)^\rho \frac{\partial f_{\pi(1)}}{\partial y_{i_{\rho(1)}}} \dots \frac{\partial f_{\pi(2m)}}{\partial y_{i_{\rho(2m)}}}.$$

Again by $C^{m+1} = 0$ we may permute $\frac{\partial f_{\pi(1)}}{\partial y_{i_{\rho(1)}}}, \dots, \frac{\partial f_{\pi(2m)}}{\partial y_{i_{\rho(2m)}}}$ among each other in the above expression, and we get

$$A = \sum_{1 \le i_1 < \dots < i_{2m} \le n} \sum_{\sigma \in Sym(2m)} (-1)^\sigma \frac{\partial f_1}{\partial y_{i_{\sigma(1)}}} \dots \frac{\partial f_{2m}}{\partial y_{i_{\sigma(2m)}}}) s_{2m}(y_{i_1}, \dots, y_{i_{2m}}).$$

For any $1 \le i_1 < \dots < i_{2m} \le n$ and $1 \le j_1 < \dots < j_{2m} \le n$ we put

$$f^{i_1,\dots,i_{2m}}_{j_1,\dots,j_{2m}} = \sum_{\sigma \in Sym(2m)} (-1)^\sigma \frac{\partial f_{i_1}}{\partial y_{j_{\sigma(1)}}} \dots \frac{\partial f_{i_{2m}}}{\partial y_{j_{\sigma(2m)}}}.$$

Consider the natural homomorphism

$$\psi : F_n(J(m+1)) \to K[x_1, \dots, x_n].$$

The image of any $f \in F_n(J(m+1))$ has a normal form

$$\psi(f) = \sum_{\alpha = (\alpha_1, \dots, \alpha_n)} b_\alpha x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

Now let ϕ be the map $F_n(J(m+1)) \to F_n(J(m+1))$ defined by

$$\phi(f) = \sum_\alpha b_\alpha y_1^{\alpha_1} \dots y_n^{\alpha_n}.$$

Obviously, we have $\phi(f) \equiv f \pmod{C}$, hence $C^{m+1} = 0$ implies that

$$fs_{2m}(y_{i_1}, \ldots, y_{i_{2m}}) = \phi(f)s_{2m}(y_{i_1}, \ldots, y_{i_{2m}}).$$

Summarizing, we have

$$\begin{aligned} 0 &= h(f_1, \ldots, f_n) \\ &= \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)} \sum_{1 \le i_1 < \ldots < i_{2m} \le n} a_{\alpha,i} f_1^{\alpha_1} \ldots f_n^{\alpha_n} s_{2m}(f_{i_1}, \ldots, f_{i_{2m}}) \\ &= \sum_{1 \le j_1 < \ldots < j_{2m} \le n} \phi(\sum_{\alpha,i} a_{\alpha,i} f_1^{\alpha_1} \ldots f_n^{\alpha_n} f_{j_1,\ldots,j_{2m}}^{i_1,\ldots,i_{2m}}) s_{2m}(y_{j_1}, \ldots, y_{j_{2m}}). \end{aligned}$$

Now since the elements $y_1^{\alpha_1} \ldots y_n^{\alpha_n} s_{2m}(y_{i_1}, \ldots, y_{i_{2m}})$ are linearly independent in $F_n(J(m+1))$,

$$\phi(\sum_{\alpha,i} a_{\alpha,i} f_1^{\alpha_1} \ldots f_n^{\alpha_n} f_{j_1,\ldots,j_{2m}}^{i_1,\ldots,i_{2m}})$$

must be zero for all $j_1, \ldots, j_{2m}$, implying that

$$\psi(\sum_{\alpha,i} a_{\alpha,i} f_1^{\alpha_1} \ldots f_n^{\alpha_n} f_{j_1,\ldots,j_{2m}}^{i_1,\ldots,i_{2m}}) = 0,$$

because the monomials $y_1^{\beta_1} \ldots y_n^{\beta_n}$ are linearly independent in $F_n(J(m+1))$.

Introduce the symbols $dx_1, \ldots, dx_n$, and let

$$E = K\langle dx_1, \ldots, dx_n \mid dx_i dx_j = -dx_j dx_i,\ 1 \le i, j \le n\rangle$$

be the Grassmann algebra of an n dimensional linear space. Consider the map

$$d : K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n] \otimes E$$

defined by

$$df = d(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

The commutative polynomials $h_i = \psi(f_i)$ $(i = 1, \ldots, n)$ form an algebraically independent generating set of $K[x_1, \ldots, x_n]^G$. Direct computation shows that

$$\begin{aligned} &\sum_{\alpha,i} a_{\alpha,i} h_1^{\alpha_1} \ldots h_n^{\alpha_n} dh_{i_1} \ldots dh_{i_{2m}} \\ &\qquad = \sum_{1 \le j_1 < \ldots < j_{2m} \le n} \psi(\sum_{\alpha,i} a_{\alpha,i} f_1^{\alpha_1} \ldots f_n^{\alpha_n} f_{j_1,\ldots,j_{2m}}^{i_1,\ldots,i_{2m}}) dx_{j_1} \ldots dx_{j_n}, \end{aligned}$$

and by our hypothesis the latter is zero. Now [14, Theorem] implies that

$$\{h_1^{\alpha_1} \ldots h_n^{\alpha_n} dh_{i_1} \ldots dh_{i_{2m}} \mid \alpha_1, \ldots, \alpha_n \in \mathbb{N},\ 1 \le i_1 < \ldots < i_{2m} \le n\}$$

is a linearly independent subset of $K[x_1, \ldots, x_n] \otimes E$, hence $a_{\alpha,i} = 0$ for every α, i, contradicting the hypothesis that $f_1, \ldots, f_n \in F_n(J(m+1))$ are algebraically dependent.

Remark. The above argument did not use the fact that $f_1, \dots, f_n$ are invariants. So we have proved for all $m = 1, 2, \dots$ that if $f_1, \dots, f_n \in F_n(J(m))$ are algebraically independent (in the ordinary sense) modulo the commutator ideal, then they are algebraically independent in $F_n(J(m))$.

We have to show that $f_1, \dots, f_n$ generate $F_n(J(m))^G$. They are homogeneous of degree $d_1, \dots, d_n$, and as we pointed out earlier these are the degrees of G. We conclude from the formula (2.1) and the Gl_n-structure of $B_n/B_n \cap J(m)$ that the Hilbert series of $F_n(J(m))$ is

$$H(F_n(J(m)); t_1, \dots, t_n) = \frac{\sum_{i=0}^{m-1} \sigma_{2i}(t_1, \dots, t_n)}{\prod_{j=1}^{n}(1 - t_j)},$$

where σ_k is the kth elementary symmetric function ($\sigma_0 = 1$ and $\sigma_k = 0$ if $k > n$). We have the isomorphism

$$K\langle f_1, \dots, f_n\rangle \cong F_n(J(m)),$$

hence the $\mathbb{N}-$graded Hilbert series of the graded subalgebra $K\langle f_1, \dots, f_n\rangle$ of $F_n(J(m))^G$ is

$$H(F_n(J(m)); t^{d_1}, \dots, t^{d_n}) = \frac{\sum_{i=0}^{m-1} \sigma_{2i}(t^{d_1}, \dots, t^{d_n})}{\prod_{j=1}^{n}(1 - t^{d_j})}.$$

Using the noncommutative Molien-Weyl formula (2.2)

$$H(F_n(J(m))^G; t) = \frac{1}{|G|} \sum_{g \in G} \frac{\sum_{i=0}^{m-1} \sigma_{2i}(\omega_1(g)t, \dots, \omega_n(g)t)}{\prod_{j=1}^{n}(1 - \omega_j(g)t)}.$$

Solomon's formula

$$\frac{1}{|G|} \sum_{g \in G} \frac{\sigma_p(\omega_1(g), \dots, \omega_n(g))}{\prod_{j=1}^{n}(1 - \omega_j(g)t)} = \frac{\sigma_p(t^{d_1-1}, \dots, t^{d_n-1})}{\prod_{j=1}^{n}(1 - t^{d_j})} \tag{3.1}$$

(see [14, formula (5)]) says that $F_n(J(m))^G$ has the same Hilbert series as its subalgebra $K\langle f_1, \dots, f_n\rangle$, therefore the two algebras must coincide. This finishes the proof of Theorem 2.2.

Remark. The polynomial $[x_2, x_1, x_1]$ appears in the theory of PI-algebras as a generator of the T-ideal of identities of the infinite dimensional Grassmann algebra (cf. [11]). It is interesting to note that both parts of the above proof use Solomon's results from [14] on a pseudo-reflection group acting on the tensor product of $K[x_1, \dots, x_n]$ and the Grassmann algebra of an n dimensional vector space.

4. Proof of Proposition 2.4

Let G be a pseudo-reflection group, and suppose that $F_n(M)^G$ is generated by n elements (where M is the T-ideal generated by all the proper polynomials of degree greater than 3). As we mentioned in Section 2, then $F_n(M)^G$ is generated by its homogeneous elements $f_1, \dots, f_n$ if and only if their images in $K[x_1, \dots, x_n]$

generate $K[x_1,\dots,x_n]^G$. In the sequel we assume that $f_1,\dots,f_n \in F_n(M)^G$ have this property. By (2.1) the Hilbert series of $F_n(M)$ is

$$H(F_n(M);t_1,\dots,t_n) = \frac{1+S_{(1,1)}(t_1,\dots,t_n)+S_{(2,1)}(t_1,\dots,t_n)}{\prod_{i=1}^n(1-t_i)},$$

where $S_\lambda(t_1,\dots,t_n)$ is the *Schur function* corresponding to the partition $\lambda = (\lambda_1,\dots,\lambda_n)$. On expressing the Schur functions $S_{(1,1)}$ and $S_{(2,1)}$ by the elementary symmetric polynomials we obtain

$$H(F_n(M);t_1,\dots,t_n) = \frac{1+\sigma_2(t_1,\dots,t_n)-\sigma_3(t_1,\dots,t_n)+\sigma_1\sigma_2(t_1,\dots,t_n)}{\prod_{i=1}^n(1-t^{d_i})}.$$

Since $F_n(M)^G$ is generated by $f_1,\dots,f_n$, it is an $\mathbb{N}-$graded homomorphic image of $F_n(M)$, where we give the degrees $d_1,\dots,d_n$ to the generators of $F_n(M)$. Hence in the formal power series

$$D(t) = H(F_n(M);t^{d_1},\dots,t^{d_n}) - H(F_n(M)^G;t)$$

each coefficient is a non-negative integer. By (2.2) and (3.1) we have

$$D(t) = \frac{F(t)}{\prod_{i=1}^n(1-t^{d_i})} - H_1(t),$$

where

$$F(t) = (t^{d_1}+\dots+t^{d_n})\sum_{1\le i<j\le n} t^{d_i+d_j}$$

and

$$H_1(t) = \frac{1}{|G|}\sum_{g\in G}\frac{(\omega_1(g)t+\dots+\omega_n(g)t)(\sum_{1\le i<j\le n}\omega_i(g)\omega_j(g)t^2)}{\prod_{i=1}^n(1-\omega_i(g)t)}. \tag{4.1}$$

Steinberg proved (see [1, p. 127]) that

$$\sum_{1\le i<j\le n}\omega_i\omega_j : G\to\mathbb{C}$$

is an irreducible character of G. (Actually, $\sum\omega_i\omega_j$ is the character of $G < Gl(V)$ acting on the second exterior power of V, and we do not need here the irreducibility.) Therefore

$$(\bar\omega_1+\dots+\bar\omega_n)\sum_{1\le i<j\le n}\bar\omega_i\bar\omega_j$$

is a character of degree $n\binom{n}{2}$ of G, and we may decompose it as $\sum_{i=1}^r m_i\chi_i$, where $\chi_1,\dots,\chi_r$ are pairwise different irreducible characters of G and $m_1,\dots,m_r$ are positive integers. Let I be the ideal of $K[x_1,\dots,x_n]$ generated by the invariants of G of strictly positive degree. Chevalley showed in [2] that the representation of

G on $K[x_1,\dots,x_n]/I$ is equivalent to the regular representation. Following [15] we associate with any irreducible character χ of G the polynomial

$$p_\chi(t)=\sum_{i=0}^{\infty}a_i(\chi)t^i,$$

where $a_i(\chi)$ is the multiplicity of χ in the ith homogeneous component of $K[x_1,\dots,x_n]/I$. (Note that what we call $p_\chi(t)$ is $p_{\bar\chi}(t)$ in [15], because there G acts on $K[x_1,\dots,x_n]$ via the adjoint representation.) It turns out that $a_i(\chi)=0$ if $i>\sum_{j=1}^n(d_j-1)$ [15, Lemma 2.9]. The coefficient of t^d in the formal power series

$$\frac{1}{|G|}\sum_{g\in G}\frac{(\omega_1(g)+\ldots+\omega_n(g))\sum_{1\le i<j\le n}\omega_i(g)\omega_j(g)}{\prod_{i=1}^n(1-\omega_i(g)t)}$$

is the scalar product of the character $\sum_{i=1}^r m_i\chi_i$ and the character of G acting on the dth homogeneous component of $K[x_1,\dots,x_n]$. So by [15, 2.6 Proposition and 2.9 Lemma] we have

$$H_1(t)=\frac{t^3\sum_{i=1}^r m_ip_{\chi_i}(t)}{\prod_{i=1}^n(1-t^{d_i})}=\frac{G(t)}{\prod_{i=1}^n(1-t^{d_i})}, \tag{4.2}$$

where $G(t)$ is of the form

$$G(t)=\sum_{i=1}^{n\binom{n}{2}}t^{e_i}$$

with $0\le e_1\le\ldots\le e_{n\binom{n}{2}}\le\sum_{i=1}^n(d_i-1)+3$.

Now we reduce to the case of irreducible groups as it was done by Guralnick in [9]. Assume that $n\ge 2$ is minimal such that $F_n(M)^G$ is generated by n elements for some pseudo-reflection group G.

1. The same argument as in [9] shows that G is not Abelian.

[9, Lemma 3] remains valid with the same proof for any relatively free algebra instead of the generic matrix algebra. Therefore the minimality of n implies that G is irreducible, and we may assume that G is a complex unitary pseudo-reflection group. We shall use the classification of these groups given in [13].

2. $n>2$.

Suppose that $n=2$, and let $\chi=\omega_1+\omega_2$ denote the character of the given representation of G as a subgroup of $Gl_2(\mathbb{C})$. We showed above that the degree of $G(t)$ (see (4.2)) is at most d_1+d_2+1. Though this bound would suffice for our purpose, for sake of completeness we derive some more precise information on the polynomial $G(t)$:

$$\begin{aligned}H_1(t)&=\frac{1}{|G|}\sum_{g\in G}\frac{\chi(g)\omega_1(g)\omega_2(g)t^3}{(1-\omega_1(g)t)(1-\omega_2(g)t)}\\&=\frac{1}{|G|}\frac{t^3}{t^2}\sum_{g\in G}\frac{\chi(g)}{(1-\omega_1(g^{-1})t^{-1})(1-\omega_2(g^{-1})t^{-1})}\\&=\frac{t}{|G|}\sum_{g\in G}\frac{\bar\chi(g)}{(1-\omega_1(g)t^{-1})(1-\omega_2(g)t^{-1})}\\&=t\frac{p_\chi(t^{-1})}{(1-t^{-d_1})(1-t^{-d_2})}\end{aligned}$$

by a similar argument we used before. The character of the representation of G on the 1st homogeneous component of $K[x_1,\dots,x_n]/I$ is χ, so $p_\chi(t) = t + \sum_{i=2}^{d_1+d_2-2} a_i(\chi)t^i$, hence

$$H_1(t) = t\frac{t^{-1} + \sum_{i=2}^{d_1+d_2-2} a_i(\chi)t^{-i}}{(1-t^{-d_1})(1-t^{-d_2})} = \frac{t^{d_1+d_2} + \sum_{j=3}^{d_1+d_2-1} a_{d_1+d_2+1-j}(\chi)t^j}{(1-t^{d_1})(1-t^{d_2})}.$$

On the other hand, the minimal degree term of $\frac{F(t)}{\prod_{i=1}^n(1-t^{d_i})}$ is $t^{2d_1+d_2}$, thus the coefficient of $t^{d_1+d_2}$ in $D(t) = \frac{F(t)-G(t)}{(1-t^{d_1})(1-t^{d_2})}$ is -1. This contradicts the assumption that $D(t)$ has non-negative coefficients.

3. G is not one of the groups of no. 23, 28, 30, 35, 36, 37 in the table [13, Table VII]. (The argument in 5 below applies also for these groups. However, first we eliminate them by exhibiting some invariants. This argument shows the role of the polynomial $[x_2, x_1, x_1]$ explicitly.)

Suppose that G is one of these groups. It is well known (see for example [16, 4.2.15. Lemma]) that G can be defined over the reals if and only if G has an invariant of degree 2. So we can suppose that $G < Gl_n(\mathbb{R})$ is an orthogonal group, and then a straightforward calculation shows that

$$x_1^2 + \dots + x_n^2 \in K\langle x_1,\dots,x_n\rangle^G.$$

Thus we can choose $f_1,\dots,f_n$ such that

$$f_1 = y_1^2 + \dots + y_n^2.$$

Consider the invariant

$$f = \sum_{g\in G} g\cdot(y_1(y_1^2+\dots+y_n^2)y_1).$$

Comparing the condition $K\langle f_1,\dots,f_n\rangle = F_n(M)^G$ and the degrees of the groups under consideration (cf. [13, Table VII]) we conclude that the only invariants of G of degree 4 are the scalar multiples of f_1^2, therefore $f = af_1^2$ in $F_n(M)$ for some $a \in \mathbb{R}$. Now we have

$$f = \sum_{g\in G}(g_{11}y_1+\dots+g_{n1}y_n)(y_1^2+\dots+y_n^2)(g_{11}y_1+\dots+g_{n1}y_n),$$

and the polynomial

$$h = \sum_{g\in G}(g_{11}x_1+\dots+g_{n1}x_n)(x_1^2+\dots+x_n^2)(g_{11}x_1+\dots+g_{n1}x_n) - a(x_1^2+\dots+x_n^2)^2$$

is contained in M. The homogeneous component of h of degree 4 in x_i is

$$(\sum_{g\in G} g_{i1}^2 - a)x_i^4, \qquad (i = 1,\dots,n),$$

and it is contained in M, implying that $a = \sum_{g\in G} g_{i1}^2$, $i = 1,\dots,n$. Since for any $g \in G$ there exists an i with $g_{i1} \neq 0$, we have $a \neq 0$. The multihomogeneous component of h of multidegree $(2,2)$

$$\sum_{g\in G} g_{11}^2 x_1x_2^2x_1 + \sum_{g\in G} g_{21}^2 x_2x_1^2x_2 - ax_1^2x_2^2 - ax_2^2x_1^2 = a(x_1x_2^2x_1 + x_2x_1^2x_2 - x_1^2x_2^2 - x_2^2x_1^2)$$

is contained in M. Make the substitution $x_1 \to x_1 + 1$, $x_2 \to x_2 + 1$ in the above polynomial, then we get

$$2ax_1x_2x_1 + \text{ other monomials.}$$

Hence M contains a polynomial of degree 3, which contradicts the definition of M.

4. G is not $G(m,p,n)$ (the groups no. 2 in [13, Table VII]).

Recall the definition of $G(m,p,n)$. Let $n \geq 3$ (the case $n = 2$ was handled in 2.), $m \geq 2$, p be positive integers such that p divides m. Let $A(m,p,n) < Gl_n(\mathbb{C})$ be the group of diagonal matrices whose diagonal entries are mth roots of unity and the determinant is an $\frac{m}{p}$th root of unity. Consider $Sym(n)$ as the group of permutation matrices in $Gl_n(\mathbb{C})$. Clearly, $Sym(n)$ normalizes $A(m,p,n)$, and $G(m,p,n)$ is defined as the semidirect product

$$G(m,p,n) = A(m,p,n) \rtimes Sym(n).$$

Consider the polynomials

$$h_i = \sum_{\pi\in Sym(n)} \pi \cdot \sigma_i(x_1^m,\dots,x_n^m), \qquad i = 1,\dots,n-1;$$

$$h_n = \sum_{\pi\in Sym(n)} \pi \cdot (x_1 \dots x_n)^{m/p}.$$

The above semidirect decomposition of $G(m,p,n)$ shows that

$$h_1,\dots,h_n \in K\langle x_1,\dots,x_n\rangle^G,$$

and it is well known that their images in $K[x_1,\dots,x_n]$ form a basic set of invariants, with degrees $m, 2m, 3m, \dots, (n-1)m, \frac{m}{p}n$. So we may suppose that $f_1,\dots,f_n$ are the images of $h_1,\dots,h_n$ under the natural homomorphism $K\langle x_1,\dots,x_n\rangle \to F_n(M)$. Consider the invariant

$$f = \sum_{g\in G} g \cdot (y_1^{m-1} f_1 y_1).$$

Any $g \in G$ can be written in the form $g = a\pi$ for some $a \in A(m,p,n)$ and $\pi \in Sym(n)$. So we have $g(y_1) = \theta y_{\pi(1)}$ for some permutation π and mth root of unity θ, and

$$f = (n-1)!m^n/p \sum_{i=1}^n y_i^{m-1}(y_1^m + \dots + y_n^m)y_i.$$

By our hypothesis f is contained in $K\langle f_1,\dots,f_n\rangle$, and since $deg(f_3),\dots,deg(f_n) > deg(f) = 2m$, f can be expressed with f_1, f_2 and f_n. More precisely, there exists a polynomial

$$h = \sum_{i=1}^{n} x_i^{m-1}(x_1^m + \ldots + x_n^m)x_i - (c_1 h_1^2 + c_2 h_2 + \sum_j a_j h_n b_j) \in M,$$

where $c_1, c_2 \in \mathbb{C}$, and $a_j, b_j \in K\langle x_1,\dots,x_n\rangle$. Consider the multihomogeneous components of h. The coefficient of x_1^{2m} is $1 - c_1$, implying that $c_1 = 1$. Since $n \geq 3$, any monomial of h_n contains the variable x_3, therefore the sum of the coefficients of the monomials of multidegree (m, m) is $2 - 2c_1 - kc_2$ for some positive integer k, showing that $c_2 = 0$. Thus the polynomial

$$x_1^{m-1}x_2^m x_1 + x_2^{m-1}x_1^m x_2 - x_1^m x_2^m - x_2^m x_1^m$$

is contained in M. After the substitution $x_1 \to x_1 + 1$, $x_2 \to x_2 + 1$ we obtain a polynomial in which the coefficient of $x_1x_2x_1$ is $(m-1)m$. Again this contradicts the fact that M has no elements of degree 3.

5. G is not one of the groups no. 24, 26, 27, 29, 31, 32, 33, 34 or no. 1 with $n \geq 4$ in [13, Table VII].

Let a, b be functions $\mathbb{N} \to \mathbb{C}$, and let c be a function $\mathbb{N} \to \mathbb{N}$. We say that $a(n) = b(n) + O(c(n))$, if $|a(n) - b(n)| \leq Lc(n)$ for some constant L. We need the following lemma (cf. [16, 2.5.9. Lemma]):

Lemma 4.1. *Let $H(t) \in \mathbb{C}[[t]]$ be a formal power series of the form*

$$H(t) = \frac{A(t)}{\prod_{i=1}^{n}(1 - t^{d_i})} = \sum_{d=0}^{\infty} c_d t^d,$$

where $A(t) \in \mathbb{C}[t]$.

(i) $$(1-t)^n H(t)\Big|_{t=1} = \frac{A(1)}{\prod_{i=1}^{n} d_i}.$$

(ii) $$(1-t)^{n-1}H(t) - \frac{A(1)}{\prod_{i=1}^{n} d_i}\frac{1}{1-t}\Big|_{t=1} = \frac{\frac{1}{2}A(1)\sum_{i=1}^{n}(d_i - 1) - A'(1)}{\prod_{i=1}^{n} d_i}.$$

(iii) *Suppose that $n \geq 3$ and all sets of $n-1$ of the d_i have greatest common divisor* 1. *Then we have*

$$c_d = \frac{A(1)}{(n-1)!\prod_{i=1}^{n} d_i}d^{n-1} + \frac{\frac{1}{2}A(1)\sum_{i=1}^{n} d_i - A'(1)}{(n-2)!\prod_{i=1}^{n} d_i}d^{n-2} + O(d^{n-3}).$$

(*In* [16] $n \geq 4$ *is required, but the proof works also for* $n = 3$.)

Now we investigate the power series $D(t) = \frac{F(t)-G(t)}{\prod_{i=1}^{n}(1-t^{d_i})}$. Since $F(1) = G(1) = n\binom{n}{2}$, by Lemma 4.1 (i) $D(t)$ has no pole $\frac{1}{(1-t)^n}$. By Lemma 4.1 (ii) the coefficient of the pole $\frac{1}{(1-t)^{n-1}}$ in the Laurent series of $\frac{F(t)}{\prod_{i=1}^{n}(1-t^{d_i})}$ is

$$\frac{\frac{1}{2}n\binom{n}{2}\sum_{i=1}^{n}(d_i - 1) - \frac{3}{2}n(n-1)\sum_{i=1}^{n} d_i}{\prod_{i=1}^{n} d_i} = \frac{n(n-1)}{4|G|}((n-6)\sum_{i=1}^{n}(d_i - 1) - 6n).$$

By (4.1) the coefficient of the pole $\frac{1}{(1-t)^{n-1}}$ in the Laurent series of $H_1(t)$ is

$$\frac{1}{|G|}\sum_{g\in R}\frac{(1+\ldots+1+\omega(g))(\binom{n-1}{2}+(n-1)\omega(g))}{1-\omega(g)},$$

where $R=\{g\in G\mid g \text{ is a pseudo-reflection}\}$ and $\omega(g)$ is the eigenvalue of g different from 1. It is well known (and one can derive from Lemma 4.1 (i)) that

$$\sum_{g\in R}\frac{1}{1-\omega(g)}=\frac{1}{2}\sum_{i=1}^{n}(d_i-1).$$

Using this formula we get the equalities

$$\sum_{g\in R}\frac{\omega(g)}{1-\omega(g)}=-\sum_{g\in R}\frac{1}{1-\bar{\omega}(g)}=-\frac{1}{2}\sum_{i=1}^{n}(d_i-1)$$

and

$$\sum_{g\in R}\frac{\omega(g)^2}{1-\omega(g)}=\sum_{g\in R}(-1-\omega(g)+\frac{1}{1-\omega(g)})=N-\frac{1}{2}\sum_{i=1}^{n}(d_i-1),$$

where N is the number of the reflecting hyperplanes. Thus we have

$$\frac{1}{|G|}\sum_{g\in R}\frac{((n-1)+\omega(g))(\binom{n-1}{2}+(n-1)\omega(g))}{1-\omega(g)}$$
$$=\frac{n-1}{4|G|}((n^2-6n+4)\sum_{i=1}^{n}(d_i-1)+4N),$$

and the coefficient of the pole $\frac{1}{(1-t)^{n-1}}$ in $D(t)$ is

$$\frac{n-1}{2|G|}(-3n^2-2\sum_{i=1}^{n}(d_i-1)-2N)=\frac{G'(1)-F'(1)}{|G|}.$$

(The last equality holds by Lemma 4.1 (ii).)

Denote by δ the greatest common divisor of $d_1,\ldots,d_n$. The assumption $F_n(M)^G = K\langle f_1,\ldots,f_n\rangle$ implies that δ divides each of the $e_1,\ldots,e_{n\binom{n}{2}}$, that is, we may write

$$D(t)=\frac{A(s)}{\prod_{i=1}^{n}(1-s^{d_i/\delta})}=\sum_{d=0}^{\infty}c_d s^d,$$

where $s=t^\delta$ and $A(s)\in\mathbb{C}[s]$. One can check in [13, Table VII] that for the groups under consideration the numbers $d_1/\delta,\ldots,d_n/\delta$ satisfy the condition of Lemma 4.1 (iii), so applying this lemma and the equalities $A(1)=0$ and $\frac{\partial}{\partial s}A(s)\Big|_{s=1} = \delta(G'(1)-F'(1))$ we obtain

$$c_d=\frac{\delta(G'(1)-F'(1))}{(n-2)!\prod_{i=1}^{n}d_i}d^{n-2}+O(d^{n-3})$$
$$=\frac{-\delta(n-1)}{2(n-2)!|G|}(3n^2+2\sum_{i=1}^{n}(d_i-1)+2N)d^{n-2}+O(d^{n-3}).$$

This immediately implies that for sufficiently large d the coefficient of t^d in the power series $D(t)$ is strictly negative, which is a contradiction.

6. G is not the group no. 25 in [13, Table VII].

This group has 24 pseudo-reflections, all of them are of order 3 (see [3, p. 412, Table]), so the number of reflecting hyperplanes is 12. By the calculations in 5 we have

$$F'(1) - G'(1) = 3 \cdot 3^2 + 2 \cdot 24 + 2 \cdot 12.$$

On the other hand

$$F'(1) - G'(1) = 9(d_1 + d_2 + d_3) - (e_1 + \ldots + e_9);$$

therefore

$$\frac{e_1 + \ldots + e_9}{9} = 16,$$

implying that $e_1 \leq 16$. But we have

$$F(t) = t^{2d_1+d_2} + \text{ higher degree terms},$$

and since $2d_1 + d_2 = 21$, the coefficient of t^{e_1} in $D(t) = \frac{F(t)-G(t)}{\prod_{i=1}^{3}(1-t^{d_i})}$ is strictly negative. This is a contradiction.

7. G is not $Sym(4)$, that is, the group no. 1 with $n = 3$ in [13, Table VII].

One can calculate the power series $D(t)$ directly and conclude that it has a negative coefficient.

We have eliminated all the finite irreducible complex unitary reflection groups, so the proof of Proposition 2.4 is complete.

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