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RECIPROCITY LAWS IN THE VERLINDE FORMULAE FOR THE CLASSICAL GROUPS

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ABSTRACT. The Verlinde formula is computed for each of the simply-connected classical Lie groups, and it is shown that the resulting formula obeys certain reciprocity laws with respect to the exchange of the rank and the level. Some corresponding dualities between spaces of sections of theta line bundles over moduli spaces of G-bundles on curves are conjectured but not proved.

INTRODUCTION

The purpose of this article is to show that the Verlinde formulae for the classical simple complex Lie groups exhibit curious reciprocity laws analogous to those already observed for the unitary groups [B1],[DT],[Z]. To each such group G and nonnegative integers $l, g \in \mathbf{Z}$ one associates a natural number $N_l(G) = N_l(G, g)$, called the *Verlinde number*. This is defined in §2 below; for its derivation in the formalism of *fusion rings* see [B2]. Then the basic reciprocity law just alluded to is:

$$\frac{N_l(SL_n)}{n^g} = \frac{N_n(SL_l)}{l^g}. \tag{0.1}$$

Given a smooth projective complex curve C of genus g, and any reductive complex algebraic group G, there exists a moduli space $\mathcal{M}(G)$ for algebraic principal G-bundles over C, whose connected components are normal irreducible projective varieties indexed by the fundamental group of G. Associated to any finite dimensional representation $G \to GL(V)$, one can construct a natural determinant line bundle $\Theta(V)$ over $\mathcal{M}(G)$. Moreover, when G is simply-connected there is a nonnegative integer m_V associated to the representation (see [KNR]) such that the Verlinde number $N_{m_V}(G, g)$ is precisely the dimension of the space of sections:

$$\dim H^0(\mathcal{M}(G), \Theta(V)) = N_{m_V}(G, g). \tag{0.2}$$

In particular, Donagi and Tu showed that as a consequence of this and of (0.1) one has

$$\dim H^0(\mathcal{M}(GL_n, 0), \mathcal{L}^l) = \dim H^0(\mathcal{M}(SL_l), \mathcal{L}^n), \tag{0.3}$$

where $\mathcal{L} = \Theta(V)$, V being the standard representation on each side, and where $\mathcal{M}(GL_n, 0)$ denotes the component of moduli space at degree 0; and they conjectured that the respective vector spaces are canonically dual (see [DT] for the full

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story). In the present paper we find some more coincidences of dimension of this kind, though we shall not pursue the subsequent question of finding isomorphisms of the vector spaces.

The first case—and indeed the simplest—is that of the symplectic groups:

$$N_l(\mathrm{Sp}_n) = N_n(\mathrm{Sp}_l) \qquad \text{for all } l, n \geq 1. \tag{0.4}$$

The immediate consequence of this is then:

(0.5) Theorem. *Let $G = \mathrm{Sp}_n$ be the complex symplectic group of rank $n \geq 1$, and $V = \mathbf{C}^{2n}$ be its standard representation; and let $\mathcal{L} = \Theta(V) \in \mathrm{Pic}\big(\mathcal{M}(\mathrm{Sp}_n)\big)$. Then for any $l \geq 1$,*

$$\dim H^0(\mathcal{M}(\mathrm{Sp}_n), \mathcal{L}^l) = \dim H^0(\mathcal{M}(\mathrm{Sp}_l), \mathcal{L}^n).$$

Noting that in the case $l = 1$ the moduli space $\mathcal{M}(\mathrm{Sp}_1) = \mathcal{M}(SL_2) = M_C(2, \mathcal{O})$ is that of semistable rank 2 vector bundles with trivial determinant, and that then $\mathcal{L}$ is the usual ample generator of $\mathrm{Pic}\, M_C(2, \mathcal{O})$, we have:

(0.6) Corollary. *For any $n \geq 1$, $\dim H^0(M_C(2, \mathcal{O}), \mathcal{L}^n) = \dim H^0(\mathcal{M}(\mathrm{Sp}_n), \mathcal{L})$.*

One may perhaps view this as indicating a quaternionic analogue of the well-known duality

$$H^0(J(C), \Theta^{\otimes n}) \cong H^0(\mathcal{M}(SL_n), \mathcal{L})^\vee.$$

(Indeed, while the Jacobian $J(C)$ can be viewed as the space of representations of $\pi_1(C)$ in the circle group, $M_C(2, \mathcal{O})$ is that of its representations in the 3-sphere of unit quaternions.)

The reciprocity laws for the spin groups, on the other hand, are rather more subtle. To explain these, we need to note that the Verlinde number $N_l(G)$ is defined as a certain sum over the finite set P_l of integrable representations of level l of the affine Lie algebra associated to G. We now consider, for each of the classical groups, a finite group Γ of symmetries of the extended Dynkin diagram of this affine Lie algebra, and its natural action on P_l for each $l \in \mathbf{N}$. It turns out that for each G one obtains reciprocity after replacing $N_l(G)$ by a modified Verlinde number $\widetilde{N}_l(G)$ defined as a sum *over the orbit space* P_l/Γ (see §3 below). In particular, for the symplectic case Γ is trivial and one has the straight reciprocity relation (0.4); while for the unitary case $G = SL_n$, Γ is the cyclic group $\mathbf{Z}_n$, and it is precisely this that accounts for the power of n (resp. l) arising in (0.1).

For the odd and even spin groups Γ is $\mathbf{Z}_2$ and $\mathbf{Z}_2 \times \mathbf{Z}_2$ respectively, and the resulting reciprocity law for $\widetilde{N}_l(\mathrm{Spin}_m)$ is stated as theorem (4.13). Unfortunately, however, it is far from clear how to interpret this geometrically, except in one case: that of l, m both odd. For this one needs to consider the two-component moduli space $\mathcal{N}(m) = \mathcal{M}(\mathrm{Spin}_m) \cup \mathcal{M}^-(\mathrm{Spin}_m)$, where $\mathcal{M}^-(\mathrm{Spin}_m)$ denotes the moduli variety of Clifford bundles on C with fixed spinor norm of odd degree. Equivalently $\mathcal{N}(m)$ is an étale cover of the two-component moduli space $\mathcal{M}(SO_m)$. For $m = 3$, $\mathcal{N}(3)$ is the union of $M_C(2, \mathcal{O})$ and $M_C(2, \mathcal{O}(x))$, for $x \in C$.

Now consider the line bundle $\Theta(\mathbf{C}^m) \to \mathcal{N}(m)$ where $\mathbf{C}^m$ is the standard orthogonal representation of the Clifford group. Our results appear to imply the following reciprocity law: if there exists, for each odd $m \in \mathbf{N}$, a line bundle $\mathcal{L} \to \mathcal{N}(m)$ such that $\mathcal{L}^2 = \Theta(\mathbf{C}^m)$, then it satisfies:

$$\dim H^0(\mathcal{N}(m), \mathcal{L}^l) = \dim H^0(\mathcal{N}(l), \mathcal{L}^m) \qquad \text{for } l, m \text{ both odd.} \tag{0.7}$$

This is explained in the final section of the paper; its validity, however, depends on an unproved Verlinde formula on $\mathcal{M}^-(\mathrm{Spin}_m)$ (Conjecture (5.2)), which generalises the 'twisted' Verlinde formula for rank 2 vector bundles of odd degree.

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1. Some trigonometric identities

We begin by proving, in this section, a family of rather surprising trigonometric identities which underpin the reciprocity laws to be described later on.

To begin, let p be a positive integer and let $f(r) = f_p(r) = 4\sin^2(r\pi/p) = (1-\zeta_p^r)(1-\zeta_p^{-r})$, where $\zeta_p = e^{2\pi i/p}$. Given a finite set $U = \{u_1, \dots, u_r\}$ of rational numbers, we shall consider the following products (where an empty product is deemed to be 1):

$$\Pi_p(U) = \prod_{1\le i<j\le r} \Big(f(u_i-u_j)f(u_i+u_j)\Big);$$

$$\Phi_p(U) = \Pi_p(U)\mathcal{N}_p(U) \quad \text{where} \quad \mathcal{N}_p(U) = \prod_{i=1}^r f(u_i);$$

$$\Delta_p(U) = \Pi_p(U)\mathcal{Q}_p(U) \quad \text{where} \quad \mathcal{Q}_p(U) = \prod_{i=1}^r f(2u_i).$$

We shall usually drop the subscript p. Now let $\varepsilon = 0$ or 1; and consider the set $S = S(p,\varepsilon) = (\varepsilon/2 + \mathbf{Z}) \cap (0, p/2)$.

The main result of this section is the following proposition, which can be thought of as a 'failed duality' relation between Π of a set and of its complement in S. We shall subsequently repair this in several ways, using the products Φ and Δ.

(1.1) Proposition. *Let $U \subset S$, and let $U' = S\backslash U$ be its complement. Then*

$$\frac{\Pi(U)}{\Pi(U')} = \frac{p^{|U|-|U'|+1-\delta}2^{\delta(3-\delta)/2}}{\mathcal{N}(U)^{2-\varepsilon}\mathcal{N}(p/2-U)^{2-\eta}}$$

where η is 0 or 1, whichever is congruent to $p+\varepsilon$ modulo 2; and where $\delta = \varepsilon+\eta-1$. (So δ is zero if p is odd and $(-1)^{\varepsilon+1}$ if p is even.)

We begin by gathering together some obvious identities.

(1.2)
(i) $f(r) = f(-r)$;
(ii) $f(r) = f(p-r)$;
(iii) $f(p/2) = 4$;
(iv) $f(2r) = f(r)f(p/2-r)$.

We also have

(1.3) $\prod_{r=1}^{p-1} f(r) = p^2$.

Proof. The left-hand side is

$$\prod_{r=1}^{p-1}[(1-\zeta_p^r)(1-\zeta_p^{-r})] = \Big(\prod_{r=1}^{p-1}(1-\zeta_p^r)\Big)^2 = \left.\frac{X^p-1}{X-1}\right|_{X=1}^2 = p^2.$$

Finally, the products Π and Φ are easily seen to satisfy:

(1.4)

(i) *For* $x \notin U$, $\Pi(U \cup \{x\}) = \Pi(U)\mathcal{N}(x-U)\mathcal{N}(x+U)$;

(ii) *in particular, if* $0 \notin U$ *then* $\Pi(U \cup \{0\}) = \Phi(U)\mathcal{N}(U)$;

(iii) $\Pi(U) = \Pi(p/2 - U)$.

The proof of Proposition (1.1) will be by induction on $|U|$; for this the necessary steps are contained in the following:

(1.5) Lemma.

(i) $|S| = (p-1+\delta)/2$.

(ii) $\mathcal{Q}(S) = p(4/p)^\delta$ *and* $\mathcal{N}(S) = p^{1-\varepsilon}2^\delta$.

(iii) *Initial step.*

$$\Pi(S) = p^{|S|-1+\delta}2^{\delta(\delta-3)/2} = \begin{cases} p^{(p-3)/2} & \text{if } p \text{ is odd;} \\ 4p^{p/2-3} & \text{if } p \text{ is even and } \varepsilon = 0; \\ p^{p/2}/2 & \text{if } p \text{ is even and } \varepsilon = 1. \end{cases}$$

(iv) *Inductive step. Let* $U \subset S$ *and choose* $x \in U' = S \setminus U$. *Then*

$$\frac{\Pi(U \cup \{x\})}{\Pi(U' \setminus \{x\})} = \frac{p^2}{f(x)^{2-\varepsilon} f(p/2-x)^{2-\eta}} \times \frac{\Pi(U)}{\Pi(U')}.$$

Proof. (i) Put m and M for the smallest and largest elements of S. Then $m = 1 - \varepsilon/2$ and $M = (p+\eta)/2 - 1$. But $|S| = M - m + 1$ and so the result follows.

(ii) $2S(p,0)$ (resp. $2S(p,1)$) consists of all the even (resp. odd) integers in $(0,p)$. Therefore, by (1.3),

$$\mathcal{Q}(S(p,0))\,\mathcal{Q}(S(p,1)) = p^2.$$

If p is odd (so $\eta = 1-\varepsilon$ and $\delta = 0$) then $x \mapsto p/2 - x$ provides a bijection between $S(p,0)$ and $S(p,1)$ which preserves the value of $f(2x)$. So

$$\mathcal{Q}(S(p,0)) = \mathcal{Q}(S(p,1)) = p.$$

Now the image of $\mathcal{N}(S(p,0))$ under the automorphism $\zeta_p \mapsto \zeta_p^2$ of the field of p-th roots of 1 is $\mathcal{Q}(S(p,0)) = p$. So $\mathcal{N}(S(p,0)) = p$ as required. Moreover, $\mathcal{N}(S(p,1))\mathcal{N}(S(p,0)) = \mathcal{N}(S(p,1))\mathcal{N}(p/2 - S(p,1)) = \mathcal{Q}(S(p,1)) = p$. Thus $\mathcal{N}(S(p,1)) = 1$ as required.

If p is even then $\eta = \varepsilon$. Moreover, $S(p,0) = \{1,2,\dots,p/2-1\}$. So, since $f_p(2r) = f_{p/2}(r)$, we have $\mathcal{Q}_p(S(p,0)) = \mathcal{N}_{p/2}(S(p,0)) = (p/2)^2$ by (1.3). And then $\mathcal{Q}_p(S(p,1)) = \mathcal{N}_{p/2}(S(p,1)) = 4$. But $p/2 - S(p,\varepsilon) = S(p,\varepsilon)$, so

$$\mathcal{N}_p(S(p,\varepsilon))^2 = \mathcal{N}_p(S(p,\varepsilon))\mathcal{N}_p(p/2 - S(p,\varepsilon)) = \mathcal{Q}_p(S(p,\varepsilon))$$

and the result follows.

(iii) If we remove the restriction $i < j$ in the formula for $\Pi(S)$ we get

$$\Pi(S)^2 = \prod_{\substack{i,j \in S \\ i \neq j}} f(i-j)f(i+j) = \prod_{\substack{i,|j| \in S \\ i \neq \pm j}} f(i+j).$$

So

$$\Pi(S)^2\mathcal{Q}(S) = \prod_{\substack{i,|j|\in S\\ i\neq -j}} f(i+j).$$

Squaring by including negative i:

$$\Pi(S)^4\mathcal{Q}(S)^2 = \prod_{\substack{|i|,|j|\in S\\ i\neq -j}} f(i+j) = \prod_{r=1}^{p-1} f(r)^{t_r},$$

where t_r is the number of ways r can be represented, mod p, in the form $i+j$ with $|i|$, $|j| \in S$.

Let Γ be a (multiplicative) cyclic group of order $2p$ with generator $\gamma^{1/2}$, and put $\Sigma = 1+\gamma+\cdots+\gamma^{p-1} \in \mathbf{Z}\Gamma$. We find, after the proverbial moment's thought, that

$$\sigma \stackrel{\text{def}}{=} \sum_{|i|\in S} \gamma^i = \gamma^{\varepsilon/2}\Sigma - (1-\varepsilon) - (1-\eta)\gamma^{p/2}.$$

But t_r is the coefficient of γ^r in the expansion of

$$\begin{aligned}\sigma^2 &= \gamma^{\varepsilon}\Sigma^2 - 2\Sigma\big(\gamma^{\varepsilon/2}(1-\varepsilon) + (1-\eta)\gamma^{(\varepsilon+p)/2}\big)\\ &\qquad + (1-\varepsilon)^2 + 2(1-\varepsilon)(1-\eta)\gamma^{p/2} + (1-\eta)\gamma^p\\ &= p\Sigma - 2\Sigma(2-\varepsilon-\eta) + 2(1-\varepsilon)(1-\eta)\gamma^{p/2} + 1 - \varepsilon + (1-\eta).\end{aligned}$$

So $t_r = p - 2(2-\varepsilon-\eta)$ if $r \not\equiv 0$ or $p/2$ mod p and $p-2(1-\varepsilon)$ if $r = p/2$ (for, if this happens, then p is even and $\eta = \varepsilon$). Thus, by (1.2)(iii) and (1.3),

$$\Pi_p(S)^4\mathcal{Q}_p(S)^2 = p^{2(p-2(2-\varepsilon-\eta))}4^{2(1-\varepsilon)(1-\eta)}$$

and the result follows by part (ii).

(iv) Using (1.3) and writing V for $U' \setminus \{x\}$, the left-hand side is

$$\begin{aligned}&\mathcal{N}(x+U)\,\mathcal{N}(x-U)\,\mathcal{N}(x+V)\,\mathcal{N}(x-V)\\ &\qquad = \mathcal{N}(x+(S\setminus\{x\})) \times \mathcal{N}(x-(S\setminus\{x\}))\\ &\qquad = \left(\prod_{i=x+m}^{x+M} f(i)\right) \times \frac{1}{f(2x)} \times \prod_{i=x-M}^{-1} f(i) \times \prod_{i=1}^{x-m} f(i)\\ &\qquad = \prod_{i=1}^{x-m} f(i) \times \prod_{i=x+m}^{x+M} f(i) \times \prod_{i=x+p-M}^{p-1} f(i) \times \frac{1}{f(2x)}.\end{aligned}$$

Now $(x+m)$ exceeds $(x-m)$ by $2m = 2-\varepsilon$ so these integers are consecutive if $\varepsilon = 1$, whilst if $\varepsilon = 0$ the integer x lies between them. Again, $(x+p-M)$ exceeds $(x+M)$ by $p-2M = 2-\eta$ so these integers are consecutive if $\eta = 1$, and if $\eta = 0$ then the integer $x - p/2$ lies in between. Thus the above expression is

$$\prod_{i=1}^{p-1} f(i) \Big/ \left(f(2x)f(x)^{1-\varepsilon}f(p/2-x)^{1-\eta}\right),$$

and the result follows by (1.3) and (1.2)(iv). ∎

Proof of (1.1). This now follows, as already indicated, from parts (iv) and (iii) of the previous lemma by induction on $|U|$. ∎

The trigonometric identities referred to in the title of this section can now be derived from proposition (1.1) by examining the various possibilities for p and ε.

First we consider the case of $p = 2a$ even—in fact we shall only be interested in the case $a \geq 3$. If $\varepsilon = 0$ then $S = \{1, \dots, a-1\}$, $\eta = 0$ and $\delta = -1$. This case gives us our most basic identity:

(1.6) Corollary. *For any subset $U \subset \{1, \dots, a-1\}$, with complement U', we have*

$$\frac{(2a)^{|U|}}{\Delta_{2a}(U)} = \frac{(2a)^{|U'|}}{\Delta_{2a}(U')}.$$

Proof. From (1.1) and (1.2)(iv) we have

$$\frac{\prod_{2a}(U)}{\prod_{2a}(U')} = \frac{(2a)^{|Y|-|U'|}a^2}{Q_{2a}(U)^2} = (2a)^{|U|-|U'|}\mathcal{Q}_{2a}(U')/\mathcal{Q}_{2a}(U),$$

where by (1.5)(ii) we have used $\mathcal{Q}_{2a}(U')\mathcal{Q}_{2a}(U) = \mathcal{Q}_{2a}(S) = a^2$; and the required relation now follows. ∎

(1.7) Corollary. *Let U, U' be complementary subsets of $\{1, \dots, a-1\}$ as before. Then:*

(i) $$\frac{(2a)^{|U|}}{\Phi_{2a}(U)} = \frac{2(2a)^{|U' \cup \{a\}|}}{\Phi_{2a}(U' \cup \{a\})};$$

(ii) $$\frac{(2a)^{|U|}}{\Pi_{2a}(U)} = \frac{4(2a)^{|U' \cup \{0,a\}|}}{\Pi_{2a}(U' \cup \{0,a\})};$$

(iii) $$\frac{(2a)^{|U \cup \{0\}|}}{\Pi_{2a}(U \cup \{0\})} = \frac{(2a)^{|U' \cup \{a\}|}}{\Pi_{2a}(U' \cup \{a\})}.$$

Proof. We have

$$\begin{aligned}
\frac{\Phi_{2a}(U)}{\Phi_{2a}(U' \cup \{a\})} &= \frac{\Pi_{2a}(U)}{\Pi_{2a}(U')} \times \frac{\mathcal{N}_{2a}(U)}{\mathcal{N}_{2a}(U')f(a)\mathcal{N}_{2a}(a-U')\mathcal{N}_{2a}(a+U')} \\
&= \frac{(2a)^{|U|-|U'|)}}{4} \times \frac{\mathcal{Q}_{2a}(U')}{\mathcal{Q}_{2a}(U)} \times \frac{\mathcal{N}_{2a}(U)}{\mathcal{N}_{2a}(U')\mathcal{N}_{2a}(a-U')^2} \\
&= \frac{(2a)^{|U|-|U'|}}{4} \times \frac{\mathcal{N}_{2a}(a-U')}{\mathcal{N}_{2a}(a-U)} \times \frac{1}{\mathcal{N}_{2a}(a-U')^2} \\
&= \frac{(2a)^{|U|-|U'|}}{4\mathcal{N}_{2a}(S)} = \frac{(2a)^{|U|}}{2(2a)^{|U' \cup \{a\}|}}
\end{aligned}$$

—so that (i) follows. Furthermore,

$$\begin{aligned}\frac{\Pi_{2a}(U)}{\Pi_{2a}(U'\cup\{0,a\})} &= \frac{\Phi_{2a}(U)}{\Phi_{2a}(U'\cup\{a\})}\times\frac{1}{\mathcal{N}_{2a}(U)\mathcal{N}_{2a}(U'\cup\{a\})}\\ &= \frac{(2a)^{|U|}}{2(2a)^{|U'\cup\{a\}|}\mathcal{N}_{2a}(S)f(a)}\\ &= \frac{(2a)^{|U|}}{4(2a)^{|U'\cup\{0,a\}|}}.\end{aligned}$$

This gives (ii). Finally,

$$\begin{aligned}\frac{\Pi_{2a}(U\cup\{0\})}{\Pi_{2a}(U'\cup\{a\})} &= \frac{\Pi_{2a}(U)}{\Pi_{2a}(U'\cup\{0,a\})}\times\mathcal{N}_{2a}(U)^2\times\mathcal{N}_{2a}(U'\cup\{a\})^2\\ &= \frac{(2a)^{|U|}}{4(2a)^{|U'\cup\{0,a\}|}}\times\mathcal{N}_{2a}(S)\times f(a)\\ &= \frac{(2a)^{|U\cup\{0\}|}}{(2a)^{|U'\cup\{a\}|}},\end{aligned}$$

and (iii) follows. ∎

Keeping $p=2a$ we now consider the case $\varepsilon=\eta=\delta=1$. Proposition (1.1) says in this case

$$\frac{\Pi_{2a}(U)}{\Pi_{2a}(U')} = 2\frac{(2a)^{|U|-|U'|}}{\mathcal{N}_{2a}(U)\mathcal{N}_{2a}(a-U)}.$$

Now, applying (1.4)(iii) and (1.5)(ii),

$$\frac{\Phi_{2a}(U)}{\Phi_{2a}(a-U')} = 2\frac{(2a)^{|U|-|U'|}}{\mathcal{N}_{2a}(a-U')\mathcal{N}_{2a}(a-U)} = 2\frac{(2a)^{|U|-|U'|}}{\mathcal{N}_{2a}(a-S)} = (2a)^{|U|-|U'|}.$$

Thus we arrive at:

(1.8) Corollary. *This time take $U\subset S=\{\frac{1}{2},\dots,a-\frac{1}{2}\}$, with complement U'; then we have:*

$$\frac{(2a)^{|U|}}{\Phi_{2a}(U)} = \frac{(2a)^{|U'|}}{\Phi_{2a}(a-U')}.$$

Finally, let us take p to be odd. We shall be mainly interested in $\varepsilon=1$; then $\eta=\delta=0$ and $S=\{\frac{1}{2},\dots,\frac{p}{2}-1\}$.

(1.9) Corollary. *For p odd and any complementary subsets $U,U'\subset\{\frac{1}{2},\dots,\frac{p}{2}-1\}$ we have:*

(i)
$$\frac{p^{|U\cup\{p/2\}|}}{\Pi_p(U\cup\{p/2\})} = \frac{p^{|U'|}}{\Phi_p(U')},$$

(ii)
$$\frac{4p^{|U\cup\{p/2\}|}}{\Phi_p(U\cup\{p/2\})} = \frac{p^{|U'|}}{\Pi_p(U')}.$$

Proof. From (1.1) we get

$$\frac{\Pi(U)}{\Pi(U')} = \frac{p^{|U|-|U'|+1}}{\mathcal{N}(U)\mathcal{N}(p/2-U)^2}.$$

Hence

$$\frac{\Pi(U\cup\{p/2\})}{\Phi(U')} = \frac{p^{|U|-|U'|+1}\mathcal{N}(p/2-U)^2}{\mathcal{N}(U')\mathcal{N}(U)\mathcal{N}(p/2-U)^2} = \frac{p^{|U|-|U'|+1}}{\mathcal{N}(S)} = p^{|U|-|U'|+1},$$

and so part (i) follows. Next,

$$\frac{\Phi(U\cup\{p/2\})}{\Pi(U')} = \frac{p^{|U|-|U'|+1}\mathcal{N}(U)f(p/2)\mathcal{N}(p/2-U)^2}{\mathcal{N}(U)\mathcal{N}(p/2-U)^2} = 4p^{|U|-|U'|+1},$$

from which we get part (ii). ∎

(1.10) *Remark.* For the case of p odd and $\varepsilon = 0$, note that there is a bijection $V \mapsto p/2 - V$ between the subsets of

$$S(p,0)\cup\{0\} = \{0,1,\dots,(p-1)/2\}$$

and those of

$$S(p,1)\cup\{p/2\} = \{1/2,\dots,p/2\}$$

which, by (1.4)(iii), preserves Π_p. Therefore one could, if one wished, reinterpret (1.9) in this case also.

We shall conclude this section with a somewhat simpler identity, due to Zagier [Z], which is necessary to complete the picture.

(1.11) Proposition. *Given complementary subsets $U, U' \subset \{0,1,\dots,p-1\}$, for any $p \geq 2$, set (where $U = \{u_i\}$) $\Psi_p(U) = \prod_{i<j} f(u_i - u_j)$. Then*

$$\frac{p^{|U|}}{\Psi_p(U)} = \frac{p^{|U'|}}{\Psi_p(U')}.$$

Proof. The square of the left-hand side is

$$\prod_{u\in U}\Big(\frac{p^2}{\prod_{v\in U\setminus\{u\}} f(u-v)}\Big).$$

Then $p^2 = \prod_{r=1}^{p-1} f(r)$ by (1.3), so the above expression becomes

$$\prod_{u\in U}\Big(\prod_{\substack{v=0\\ v\neq u}}^{p-1} f(u-v) \Big/ \prod_{v\in U\setminus\{u\}} f(u-v)\Big) = \prod_{\substack{u\in U\\ v\in U'}} f(u-v),$$

which is symmetric in U and U', and the proposition follows.

2. The Verlinde formula

In this section we shall describe the 'Verlinde number' $N_l(G)$, which depends on the group, on a nonnegative integer g—interpreted as the genus of an algebraic curve—and on an integer l called the 'level'.

Our notation below is as in [B2]:

- G is assumed simple and simply-connected;
- $\mathbf{h}$ is a fixed Cartan subalgebra, identified with $\mathbf{h}^*$ via the Killing form $\langle\ ,\ \rangle$;
- $P \subset \mathbf{h}^*$ is the weight lattice, $Q \subset P$ the root lattice and $Q_{\text{long}} \subset Q$ the sublattice generated by the long roots;
- $\Delta \subset Q$ is the root system, $\mathcal{S} \subset \Delta$ a set of simple roots, with respect to which
- $\Delta_+ \subset \Delta$ is the subset of positive roots, and θ the highest root, $\theta^\vee = 2\theta/\langle\theta,\theta\rangle$ the highest coroot;
- $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ and $h = 1 + \langle\rho,\theta^\vee\rangle$ is the dual Coxeter number;
- for $l \in \mathbf{N}$ we put $P_l = \{\lambda \in P | \lambda \geq 0, \langle\lambda,\theta^\vee\rangle \leq l\}$ when $\langle\ ,\ \rangle$ is normalised so that $\theta = \theta^\vee$; and
- $T_l = P/(l+h)Q_{\text{long}}$.

With these conventions, the *Verlinde number for group G, genus g and level l* is defined to be

$$N_l(G) = |T_l|^{g-1} \sum_{\lambda\in P_l} \Big(\prod_{\alpha\in\Delta_+} 2\sin\frac{\pi}{l+h}\langle\alpha,\rho+\lambda\rangle \Big)^{-2g+2}. \tag{2.1}$$

Remark. Note, from the inclusions $(l+h)Q_{\text{long}} \subset Q_{\text{long}} \subset P$, that

$$\begin{aligned} |T_l| &= (l+h)^{\operatorname{rank} G}|P/Q_{\text{long}}| \\ &= (l+h)^{\operatorname{rank} G}|Z(G_{\text{long}})| \end{aligned}$$

where $G_{\text{long}} \subset G$ is the canonical *simply-laced subgroup*—i.e. has the same rank as G and root system $\Delta \cap Q_{\text{long}}$; and $Z(G_{\text{long}})$ denotes its centre (see [Sz]).

Examples: $\mathbf{SL_{n+1}}$.

As usual we take the Cartan subalgebra $\mathbf{h}$ to consist of diagonal tracefree matrices, and in the dual space $\mathbf{h}^*$ we take $L_1, \dots, L_{n+1}$ to be the restriction of linear forms dual to the standard basis of diagonal matrices; thus the tracefree condition is $L_1 + \cdots + L_{n+1} = 0$. We then have:

$$\begin{aligned} \mathcal{S} &= \{L_1 - L_2, \dots, L_n - L_{n+1}\}, \\ \Delta_+ &= \{L_i - L_j\}_{i<j}, \\ \theta &= \theta^\vee = L_1 - L_{n+1} \end{aligned}$$

with respect to normalised Killing form

$$\langle L_i, L_j\rangle = \begin{cases} n/(n+1) & \text{if } i = j, \\ -1/(n+1) & \text{if } i \neq j. \end{cases}$$

The dual Coxeter number is $h = n+1$. Finally, the fundamental weights (dual with respect to $\langle\ ,\ \rangle$ to the simple coroots) are

$$L_1, L_1 + L_2, \dots, L_1 + \cdots + L_n.$$

Taking these as integral basis for P we shall write $\lambda = (s_1, \dots, s_n) \in P$, $s_1, \dots, s_n \in \mathbf{Z}$; then $\langle \lambda, \theta^\vee \rangle = s_1 + \cdots + s_n$ and we see

$$\lambda \in P_l \qquad \Leftrightarrow \qquad \begin{array}{c} s_1, \dots, s_n \geq 0 \\ s_1 + \cdots + s_n \leq l. \end{array}$$

Also $\rho = (1, \dots, 1)$ and for $\alpha = L_i - L_j \in \Delta_+$ we find

$$\begin{aligned} \langle \alpha, \rho + \lambda \rangle &= s_i + \cdots + s_{j-1} + j - i \\ &= t_i + \cdots + t_{j-1}, \end{aligned}$$

where we have written $t_i = s_i + 1$. Finally, $|T_l| = (n+1)(l+n+1)^n$ by the preceding remark, since $G_{\text{long}} = G$ and $|Z(SL_{n+1})| = n+1$. We therefore arrive at the Verlinde formula in this case:

(2.2)
$$\begin{aligned} N_l(SL_{n+1}) =& \big((n+1)(l+n+1)^n\big)^{g-1} \\ &\times \sum_{\substack{t_1, \dots, t_n \geq 1 \\ t_1 + \cdots + t_n \leq l+n}} \Big(\prod_{1 \leq i < j \leq n+1} \big(2 \sin \frac{\pi}{l+n+1} (t_i + \cdots + t_{j-1})\big) \Big)^{-2g+2}. \end{aligned}$$

$\mathbf{Spin_{2n}}$, $\mathbf{n \geq 4}$.

This time we can take $L_1, , \dots, L_n$ to be an *orthonormal* basis of $\mathbf{h}^*$ with respect to the normalised Killing form (see for example [FH]), with:

$$\begin{aligned} \mathcal{S} &= \{L_1 - L_2, \dots, L_{n-1} - L_n, L_{n-1} + L_n\}, \\ \Delta_+ &= \{L_i - L_j\}_{i<j} \cup \{L_i + L_j\}_{i<j}, \\ \theta &= \theta^\vee = L_1 + L_2. \end{aligned}$$

The dual Coxeter number is $h = 2n - 2$, and the fundamental weights are

$$\begin{aligned} &L_1, \\ &L_1 + L_2, \\ &\dots \\ &L_1 + \cdots + L_{n-2}, \\ &\frac{1}{2}(L_1 + \cdots + L_n), \\ &\frac{1}{2}(L_1 + \cdots + L_{n-1} - L_n). \end{aligned}$$

Thus, if as before we write $\lambda = (s_1, \dots, s_n) \in P$ in terms of this basis, $\langle \lambda, \theta^\vee \rangle = s_1 + 2s_2 + \cdots + 2s_{n-2} + s_{n-1} + s_n$. On the other hand, $|T_l| = 4(l + 2n - 2)^n$ since Spin_{2n} is simply-laced and has centre of order 4. Using the same notation $t_i = s_i + 1$ as in the previous case we obtain the Verlinde formula:

(2.3)

$$N_l(\mathrm{Spin}_{2n}) = \big(4(l+2n-2)^n\big)^{g-1} \times \sum_{\substack{t_1,\dots,t_n\geq 1\\ t_1+2t_2+\cdots 2t_{n-2}+t_{n-1}+t_n\leq l+2n-3}} \Big(\prod_{i=1}^{n-1} \times \prod_{i<j}\Big)^{-2g+2},$$

where

$$\prod_{i=1}^{n-1} = \prod_{i=1}^{n-1} 4\sin\frac{\pi}{l+2n-2}(t_i+\cdots+t_{n-1})\sin\frac{\pi}{l+2n-2}(t_i+\cdots+t_{n-2}+t_n),$$

and

$$\prod_{i<j} = \prod_{1\leq i<j\leq n-1} 2\sin\frac{\pi}{l+2n-2}(t_i+\cdots+t_{j-1}) \times 2\sin\frac{\pi}{l+2n-2}(t_i+\cdots+t_{j-1}+2t_j+\cdots+2t_{n-2}+t_{n-1}+t_n).$$

Remark. This formula is also valid for $n=3$, in fact; one sees that the resulting expression for $N_l(\mathrm{Spin}_6)$ agrees with that for $N_l(SL_4)$ from (2.2), as it should.

$\mathbf{Spin_{2n+1}}$, $\mathbf{n\geq 2}$.

Again we take $L_1,\dots,L_n$ to be an orthonormal basis of $\mathbf{h}^*$ with respect to the normalised Killing form; for which this time

$$\mathcal{S} = \{L_1-L_2,\dots,L_{n-1}-L_n,L_n\},$$
$$\Delta_+ = \{L_i-L_j\}_{i<j}\cup\{L_i+L_j\}_{i<j}\cup\{L_i\}_i,$$
$$\theta=\theta^\vee = L_1+L_2;$$

whilst $h=2n-1$ and the fundamental weights are

$$L_1, L_1+L_2,\dots,L_1+\cdots+L_{n-1},\frac{1}{2}(L_1+\cdots+L_n).$$

The simply-laced subgroup is now $G_{\mathrm{long}}=\mathrm{Spin}_{2n}\subset\mathrm{Spin}_{2n+1}$, with centre of order 4, and so $|T_l|=4(l+2n-1)^n$. Again writing $\lambda=(s_1,\dots,s_n), \lambda+\rho=(t_1,\dots,t_n)\in P$ in terms of the fundamental weights, we have $\langle\lambda,\theta^\vee\rangle = s_1+2s_2+\cdots+2s_{n-1}+s_n$, and the Verlinde formula is:

(2.4)

$$N_l(\mathrm{Spin}_{2n+1}) = \big(4(l+2n-1)^n\big)^{g-1} \times \sum_{\substack{t_1,\dots,t_n\geq 1\\ t_1+2t_2+\cdots 2t_{n-1}+t_n\leq l+2n-2}} \Big(\prod_{i=1}^{n} \times \prod_{i<j}\Big)^{-2g+2},$$

where

$$\prod_{i=1}^{n} = 2\sin\frac{\pi}{(l+2n-1)}\frac{t_n}{2} \times \prod_{i=1}^{n-1} 2\sin\frac{\pi}{l+2n-1}(t_i+\cdots+t_{n-1}+\frac{t_n}{2}),$$

and

$$\prod_{i<j} = \prod_{1\leq i<j\leq n} 4\sin\frac{\pi}{l+2n-1}(t_i+\cdots+t_{j-1}) \times \sin\frac{\pi}{l+2n-1}(t_i+\cdots+t_{j-1}+2t_j+\cdots+2t_{n-1}+t_n).$$

Remark. Of course, this formula doesn't make sense for $n=1$, and we take $N_l(\mathrm{Spin}_3)=N_l(SL_2)$ by definition.

$\mathbf{Sp_n}$, $\mathbf{n \geq 2}$.

Here again $L_1, \dots, L_n$ denotes an orthonormal basis of $\mathbf{h}^*$; and

$$\begin{aligned} \mathcal{S} &= \{L_1 - L_2, \dots, L_{n-1} - L_n, 2L_n\}, \\ \Delta_+ &= \{L_i - L_j\}_{i<j} \cup \{L_i + L_j\}_{i \leq j}, \\ \theta &= \theta^\vee = 2L_1; \end{aligned}$$

the dual Coxeter number is $h = n + 1$ and the fundamental weights are

$$L_1, L_1 + L_2, \dots, L_1 + \cdots + L_n.$$

The simply-laced subgroup is the (diagonal) product $G_{\text{long}} = \mathrm{Sp}_1 \times \cdots \times \mathrm{Sp}_1$ (n factors), whose centre is of order 2^n; so $|T_l| = 2^n(l + n + 1)^n$. With the usual notation $\lambda = (s_1, \dots, s_n), \lambda + \rho = (t_1, \dots, t_n) \in P$ relative to the fundamental weights, we have $\langle \lambda, \theta^\vee \rangle = s_1 + \cdots + s_n$. Thus one obtains the Verlinde formula:

$$N_l(\mathrm{Sp}_n) = \left(2^n(l + n + 1)^n\right)^{g-1} \times \sum_{\substack{t_1, \dots, t_n \geq 1 \\ t_1 + \cdots + t_n \leq l+n}} \left(\prod_{i=1}^{n} \times \prod_{i<j}\right)^{-2g+2}, \tag{2.5}$$

where

$$\prod_{i=1}^{n} = \prod_{i=1}^{n} 2 \sin \frac{\pi}{l + n + 1}(t_i + \cdots + t_n),$$

and

$$\begin{aligned} \prod_{i<j} = \prod_{1 \leq i < j \leq n} & 2 \sin \frac{\pi}{2(l + n + 1)}(t_i + \cdots + t_{j-1}) \\ & \times 2 \sin \frac{\pi}{2(l + n + 1)}(t_i + \cdots + t_{j-1} + 2t_j + \cdots + 2t_n). \end{aligned}$$

Remark. For $n = 2$ one sees that the formula for $N_l(\mathrm{Sp}_2)$ agrees with (2.4) for $N_l(\mathrm{Spin}_5)$, as it should. (2.5) is also valid for $n = 1$, where it gives $N_l(\mathrm{Sp}_1) = N_l(SL_2)$ agreeing with (2.2).

3. Symplectic and unitary reciprocity

Let us begin by proving Theorem (0.5) stated in the introduction. We first need to recall, as mentioned in the introduction, that to each representation V of a simple group G there is an integer invariant m_V with the property that for any integer l,

$$\dim H^0(\mathcal{M}(G), \Theta(V)^{\otimes l}) = N_{m_V l}(G).$$

For the definition of m_V we refer to [KNR]; it is easy to compute that it takes the value 1 when $G = \mathrm{Sp}_n$ and V is the standard $2n$-dimensional representation. Thus in this case

$$\dim H^0(\mathcal{M}(\mathrm{Sp}_n), \Theta(\mathbf{C}^{2n})^{\otimes l}) = N_l(\mathrm{Sp}_n)$$

so that Theorem (0.5) is equivalent to (0.4).

To prove the latter, set

$$\begin{aligned} u_1 &= t_1 + \cdots + t_n, \\ &\cdots \\ u_{n-1} &= t_{n-1} + t_n, \\ u_n &= t_n. \end{aligned}$$

Then we can rewrite (2.5) in the form

$$N_l(\mathrm{Sp}_n) = \sum_{U \subset \{1,\dots,k-1\}} \Big(\frac{(2k)^n}{\Delta_{2k}(U)} \Big)^{g-1}$$

where $k = l + n + 1$, where the sum is taken over subsets $U = \{u_1, \dots, u_n\} \subset \{1, \dots, k-1\}$, and where $\Delta_{2k}(U)$ is defined as in §1. It follows at once from (1.6), therefore, that

$$N_l(\mathrm{Sp}_n) = \sum_{\substack{U' \subset \{1,\dots,k-1\} \\ |U'|=l}} \Big(\frac{(2k)^l}{\Delta_{2k}(U')} \Big)^{g-1} = N_n(\mathrm{Sp}_l).$$

We now want to address the problem of determining the appropriate analogue of (0.4) for the spin groups: a little experimentation reveals that there is certainly no such simple relation in these cases.

Recall that the *extended Dynkin diagram* associated to a simple Lie algebra is obtained by adjoining to the usual Dynkin diagram, whose nodes label the simple roots in some root system, an extra node corresponding to $-\theta$ where θ is the highest root. The values of the Killing form on this enlarged set of vectors is then encoded as a graph structure on the nodes in the usual way. (See [K].)

Now for a given level $l \in \mathbf{N}$ one represents a weight $\lambda \in P_l$ by attaching to the original nodes the components $a_i \in \mathbf{Z}$ of λ along the simple roots; and to the new node the translate by l of its component along $-\theta$, i.e. the number $a_0 = l - \langle \lambda, \theta^\vee \rangle$. In this way one identifies

$$P_l = \{(a_0, \dots, a_n) |\ a_i \in \mathbf{Z}_{\geq 0}, \ \ \sum m_i a_i = l\ \},$$

where $m_i = \langle \psi_i, \theta^\vee \rangle$, $\psi_1, \dots, \psi_n$ being the simple roots.

It follows that by permuting the a_i, the symmetry group of the extended Dynkin diagram acts in a natural way on P_l. In what follows we shall fix attention on a certain subgroup Γ of this symmetry group, for each of the classical simple algebras, as described in Figure 1 (the white node in each case is that corresponding to $-\theta$):

In other words, the subgroup Γ is in each case obtained by forgetting the 'end-to-end' symmetry, when this is present. We do not at present offer any geometrical reason for this choice of group: it merely happens to work in the computations below!

For each of the classical groups the Verlinde number has the form

$$N_l(G) = \sum_{\lambda \in P_l} \Big(\frac{|T_l|}{A(\lambda)} \Big)^{g-1}$$

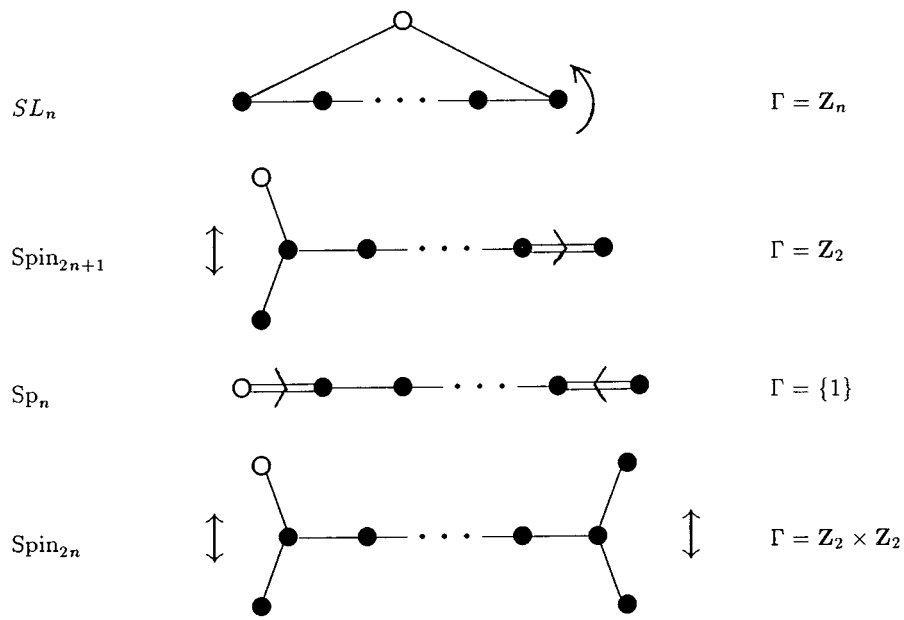

FIGURE 1

where the constant $|T_l|$ was defined in §2, and *where* $A(\lambda)$ *is a function invariant under the action of* Γ. We shall therefore consider the modified expression

$$\widetilde{N}_l(G) = \sum_{[\lambda]\in P_l/\Gamma} \Big(\frac{1}{|\Gamma(\lambda)|} \times \frac{|T_l|}{A(\lambda)}\Big)^{g-1}, \tag{3.1}$$

where $\Gamma(\lambda)$ is the Γ-orbit of $\lambda \in P_l$; and we claim that for each G there is a reciprocity law analogous to (0.4) that holds for $\widetilde{N}_l(G)$. Indeed, (0.4) is itself a special case of this as Γ is trivial for $G = \mathrm{Sp}_n$. In the next section we shall work out the reciprocity properties of (3.1) in detail for the spin groups; first we shall show how the reciprocity (0.1) for the unitary groups fits into this framework.

To begin, let us rewrite the Verlinde formula (2.2) for SL_n. We have $n-1$ indices $t_1, \dots, t_{n-1}$ in the sum, and analogously to the symplectic case we shall make the change of variables

$$\begin{aligned} u_1 &= t_1 + \cdots + t_{n-1}, \\ &\cdots \\ u_{n-1} &= t_{n-1}, \\ u_n &= 0. \end{aligned}$$

(These numbers are the row lengths for a Young diagram representing the weight $\lambda + \rho$.) Thus $P_l(SL_n)$ can be viewed as the collection of sets $U = \{k > u_1 > \cdots > u_{n-1} > u_n = 0\}$, where $k = l + n$. In other words:

$$P_l(SL_n) = \{U \subset \mathbf{Z}_k |\ 0 \in U,\ |U| = n\ \}.$$

On the other hand $\Gamma = \mathbf{Z}_n$, acting on P_l by cyclic permutation of

$$t_1, \dots, t_{n-1} \quad \text{and} \quad t_0 \stackrel{\text{def}}{=} k - t_1 - \cdots - t_{n-1}.$$

It is easy to check that $\mathbf{Z}_n$ acts on the corresponding subset $U \subset \mathbf{Z}_k$ *by translation in* $\mathbf{Z}_k$ *carrying successive members of* U *to* 0.

In particular we observe, first, that there is a canonical bijection

$$P_l(SL_n)/\mathbf{Z}_n \cong P_n(SL_l)/\mathbf{Z}_l \tag{3.2}$$

by taking a subset $U \subset \mathbf{Z}_k$ to its complement, modulo translation; and second, that we can identify the stabiliser subgroups

$$\mathrm{Stab}_{\mathbf{Z}_n}(U) \cong \mathrm{Stab}_{\mathbf{Z}_l}(U') \tag{3.3}$$

(where $U' = \mathbf{Z}_k \backslash U$) with each other and with the subgroup of $\mathbf{Z}_k$ which preserves, via translation, the partition $\mathbf{Z}_k = U \cup U'$.

The Verlinde formula (2.2) can now be written

$$N_l(SL_n) = \sum_{U \in P_l} \Big(\frac{nk^{n-1}}{\Psi_k(U)} \Big)^{g-1},$$

where $\Psi_k(U)$ is defined as in Proposition (1.11), and hence

$$\frac{N_l(SL_n)}{n^g} = \frac{1}{n} \sum_{U \in P_l} \Big(\frac{k^{n-1}}{\Psi_k(U)} \Big)^{g-1}. \tag{3.4}$$

On the other hand, the "strategy" (3.1) tells us to put

$$\widetilde{N}_l(SL_n) = \sum_{[U] \in P_l/\mathbf{Z}_n} \Big(|\mathrm{Stab}_{\mathbf{Z}_n}(U)| \times \frac{k^{n-1}}{\Psi_k(U)} \Big)^{g-1}. \tag{3.5}$$

Then it follows at once from (3.2), (3.3) and (1.6) that

$$\widetilde{N}_l(SL_n) = \widetilde{N}_n(SL_l). \tag{3.6}$$

Note that if l and n are coprime then $\mathbf{Z}_n$ acts freely on $P_l(SL_n)$, so that

$$\widetilde{N}_l(SL_n) = \frac{N_l(SL_n)}{n^g}; \tag{3.7}$$

and we have therefore proved (0.1) in this case. In general, though, the action is not free and equality does not hold in (3.7); however, the difference is

$$\widetilde{N}_l(SL_n) - \frac{N_l(SL_n)}{n^g} = \sum_{[U] \in P_l/\mathbf{Z}_n} \frac{k^{n-1}}{\Psi_k(U)} \Big(|\mathrm{Stab}_{\mathbf{Z}_n}(U)|^{g-1} - \frac{1}{|\mathrm{Stab}_{\mathbf{Z}_n}(U)|} \Big)$$

which again is symmetric in n and l. Hence by (3.6) the reciprocity law (0.1) follows.

Remark. The above proof is essentially equivalent to Zagier's in [Z, §3]. He works with the set S of arbitrary subsets of $\mathbf{Z}_k$ of size n; and sums modulo the full translation action of $\mathbf{Z}_k$. This is equivalent to our description via the obvious bijection $S/\mathbf{Z}_k \cong P_l/\mathbf{Z}_n$.

4. Spin reciprocity

We can now formulate the reciprocity laws for the Verlinde numbers of the spin groups (Theorem (4.13) below).

Spin(2n + 1).

Put

$$\begin{aligned} u_1 &= t_1 + \cdots + t_{n-1} + \frac{t_n}{2}, \\ &\cdots \\ u_{n-1} &= t_{n-1} + \frac{t_n}{2}, \\ u_n &= \frac{t_n}{2}, \\ u_{n+1} &= 0. \end{aligned}$$

Thus $P_l(\mathrm{Spin}_{2n+1})$ becomes identified with the collection of sets $\{u_1 > \cdots > u_n > 0\}$ such that:

- $u_i \in \frac{1}{2}\mathbf{Z}$;
- $u_i - u_{i+1} \in \mathbf{Z}$ for $i = 1, \ldots n-1$;
- $u_1 + u_2 < k \stackrel{\text{def}}{=} l + 2n - 1$.

The group $\Gamma = \mathbf{Z}_2$ acts by exchanging t_1 with $t_0 \stackrel{\text{def}}{=} k - t_1 - 2t_2 - \cdots - 2t_{n-1} - t_n$; in other words $u_1 \leftrightarrow k - u_1$.

It follows that we can identify

$$P_l(\mathrm{Spin}_{2n+1})/\Gamma = \{U = (u_1, \ldots, u_n) |\ \frac{k}{2} \geq u_1 > \cdots > u_n > 0, \quad u_i \in \frac{1}{2}\mathbf{Z},\ u_i - u_{i+1} \in \mathbf{Z}\ \} \tag{4.1}$$

where $k = l + 2n - 1$; and that

$$|\Gamma(U)| = \begin{cases} 2 & \text{if } u_1 < \frac{k}{2}, \\ 1 & \text{if } u_1 = \frac{k}{2}. \end{cases} \tag{4.2}$$

By (2.4) the Verlinde number is

$$N_l(\mathrm{Spin}_{2n+1}) = \sum_{U \in P_l} \Big(\frac{4k^n}{\Phi_k(U)} \Big)^{g-1} \tag{4.3}$$

where

$$\Phi_k(U) = \Big(\prod_{1 \leq i < j \leq n+1} 2 \sin \frac{\pi}{k}(u_i - u_j) \prod_{1 \leq i < j \leq n} 2 \sin \frac{\pi}{k}(u_i + u_j) \Big)^2 \tag{4.4}$$

as defined in §1. Now, by (3.1) we put (using (4.1), (4.2) and (4.4))

$$\widetilde{N}_l(\mathrm{Spin}_{2n+1}) = \sum_{U \in P_l/\Gamma} \Big(\frac{1}{|\Gamma(U)|} \times \frac{4k^n}{\Phi_k(U)} \Big)^{g-1}. \tag{4.5}$$

Spin(2n).

This time set

$$\begin{aligned} u_1 &= t_1 + \cdots + t_{n-2} + \frac{t_{n-1}+t_n}{2}, \\ &\cdots \\ u_{n-2} &= t_{n-2} + \frac{t_{n-1}+t_n}{2}, \\ u_{n-1} &= \frac{t_{n-1}+t_n}{2}, \\ u_n &= \frac{-t_{n-1}+t_n}{2}. \end{aligned}$$

Then $P_l(\mathrm{Spin}_{2n})$ consists of sets $U = \{u_i > \cdots > u_n\}$ where

- $u_i \in \frac{1}{2}\mathbf{Z}$;
- $u_i - u_{i+1} \in \mathbf{Z}$ for $i = 1, \ldots n-1$;
- $u_1 + u_2 < k \stackrel{\text{def}}{=} l + 2n - 2$; and
- $u_{n-1} + u_n > 0$.

$\Gamma = \mathbf{Z}_2 \times \mathbf{Z}_2$ acts by exchanging $t_{n-1} \leftrightarrow t_n$, and $t_1 \leftrightarrow t_0 \stackrel{\text{def}}{=} k - t_1 - 2t_2 - \cdots - 2t_{n-2} - t_{n-1} - t_n$; or equivalently $u_n \leftrightarrow -u_n$ and $u_1 \leftrightarrow k - u_1$ respectively. So we can identify

$$\begin{aligned} P_l(\mathrm{Spin}_{2n})/\Gamma = \{U = (u_1, \ldots, u_n) \mid\ & \frac{k}{2} \geq u_1 > \cdots > u_n \geq 0, \\ & u_i \in \frac{1}{2}\mathbf{Z},\ u_i - u_{i+1} \in \mathbf{Z}\ \} \end{aligned} \tag{4.6}$$

where $k = l + 2n - 2$. The Verlinde formula (2.3) now reads

$$N_l(\mathrm{Spin}_{2n}) = \sum_{U \in P_l} \Big(\frac{4k^n}{\Pi_k(U)}\Big)^{g-1}, \tag{4.7}$$

where, again as in §1,

$$\Pi_k(U) = \Big(\prod_{1 \leq i < j \leq n} 2\sin\frac{\pi}{k}(u_i - u_j) \times 2\sin\frac{\pi}{k}(u_i + u_j)\Big)^2; \tag{4.8}$$

and (3.1) tells us to consider

$$\widetilde{N}_l(\mathrm{Spin}_{2n}) = \sum_{U \in P_l/\Gamma} \Big(\frac{1}{|\Gamma(U)|} \times \frac{4k^n}{\Pi_k(U)}\Big)^{g-1}, \tag{4.9}$$

where

$$|\Gamma(U)| = \begin{cases} 4 & \text{if } \frac{k}{2} > u_1 > u_n > 0; \\ 2 & \text{if one of } u_1 = \frac{k}{2},\ u_n = 0 \text{ holds}; \\ 1 & \text{if } u_1 = \frac{k}{2} \text{ and } u_n = 0. \end{cases} \tag{4.10}$$

Finally, we shall refine (4.5) and (4.9) still further by considering (for m even or odd) the decomposition

$$\widetilde{N}_l(\mathrm{Spin}_m) = \widetilde{N}_l^+ + \widetilde{N}_l^- \tag{4.11}$$

where $\widetilde{N}_l^+$ denotes the sum taken over *integral* U, i.e. those sets with all $u_i \in \mathbf{Z}$, and $\widetilde{N}_l^-$ denotes the sum taken over *half-integral* U. In other words, $\widetilde{N}_l^+$ is the sum over classes in P_l/Γ of highest weights of representations of SO_m: what the physicists call 'tensor representations'; while $\widetilde{N}_l^-$ is the sum over the 'spinor representations'.

(4.12) *Remark.* $\widetilde{N}_l^+(\mathrm{Spin}_{2n}) = \widetilde{N}_l^-(\mathrm{Spin}_{2n})$ when l is odd.

This follows easily from Temark (1.10), where $\widetilde{N}_l^+(\mathrm{Spin}_{2n})$ is summed over subsets of $\{0, 1, \dots, (k-1)/2\}$ (with $k = l + 2n - 2$), and $\widetilde{N}_l^-(\mathrm{Spin}_{2n})$ over those of $\{1/2, \dots, k/2\}$.

We can now state the main result of this section.

(4.13) Theorem. *Suppose that both of*

$$l, m \geq \begin{cases} 5 & \textit{if odd,} \\ 8 & \textit{if even.} \end{cases}$$

(i) *If* $l \equiv m \mod 2$ *then*

$$\widetilde{N}_l^+(\mathrm{Spin}_m) = \widetilde{N}_m^+(\mathrm{Spin}_l).$$

(ii) *If at least one of* l, m *is odd then*

$$\widetilde{N}_l^-(\mathrm{Spin}_m) = \widetilde{N}_m^-(\mathrm{Spin}_l).$$

Proof. (i) Let us first suppose that $l = 2n+1$, $m = 2r+1$ are both odd; and put $a = k/2 = n + r$. The left-hand side $\widetilde{N}_l^+(\mathrm{Spin}_m)$ is the sum over the integral part of $P_l(\mathrm{Spin}_m)/\mathbf{Z}_2$, which by (4.1) is the set

$$S_{l,m} = \{U \subset \{1, \dots, a\} \mid |U| = r\ \},$$

and this is bijective to the summation set $S_{m,l}$ of the right-hand side by taking complements. By (4.5), therefore, it suffices to show that terms corresponding under this bijection agree, i.e. that

$$|\mathrm{Stab}(U)| \times \frac{k^r}{\Phi_k(U)} = |\mathrm{Stab}(U')| \times \frac{k^n}{\Phi_k(U')}$$

where U, U' are complementary subsets of $\{1, \dots, a\}$. We can assume without loss of generality that $a \in U$. Then $|\mathrm{Stab}(U)| = 2$ and $|\mathrm{Stab}(U')| = 1$, so we have to show:

$$\frac{2(2a)^r}{\Phi_k(U)} = \frac{(2a)^n}{\Phi_k(U')}.$$

But this is precisely (1.7)(i).

The argument for the case of $l = 2n$, $m = 2r$ both even is similar after putting $a = \frac{k}{2} = n + r - 1$, except that now we sum (for the left-hand side) over

$$S_{l,m} = \{U \subset \{0, 1, \dots, a\} \mid |U| = r\ \}.$$

Again this is bijective to $S_{m,l}$ by taking complements; this time however, one has two cases to examine: either both $0, a \in U$ so $|\mathrm{Stab}(U)| = 4$ and $|\mathrm{Stab}(U')| = 1$, or

$0 \in U$, $K \in U'$ (say) and $|\mathrm{Stab}(U)| = |\mathrm{Stab}(U')| = 2$. The first case is (1.7)(ii) and the second is (1.7)(iii).

(ii) First suppose that $l = 2n+1$ and $m = 2r+1$ are both odd. Again put $a = k/2 = n + r$. This time $\widetilde{N}_l^-(\mathrm{Spin}_m)$ is the sum over the half-integral part of $P_l(\mathrm{Spin}_m)/\mathbf{Z}_2$, which is

$$S_{l,m} = \{U \subset \{\frac{1}{2}, \dots, a - \frac{1}{2}\} | |U| = r\}.$$

We shall take for a bijection $S_{l,m} \xrightarrow{\sim} S_{m,l}$ that given by *end-to-end reflection* of the complement U', i.e. $U \mapsto a - U'$. Noting that the $\mathbf{Z}_2$-action is in this case free so that $|\Gamma(U)| = 2$ for all U, we just need to note from (1.8) that

$$\frac{(2a)^r}{\Phi_k(U)} = \frac{(2a)^n}{\Phi_k(a - U')},$$

so the result follows.

Finally, suppose that $l = 2n+1$ is odd and $m = 2r$ is even; so $k = 2n + 2r - 1$ is odd. By definition

$$\widetilde{N}_l^-(\mathrm{Spin}_m) = \sum_{\substack{U \subset \{\frac{1}{2}, \dots, \frac{k}{2}\} \\ |U| = r}} \Big(\frac{1}{|(\mathbf{Z}_2 \times \mathbf{Z}_2)(U)|} \times \frac{4k^r}{\Pi_k(U)} \Big)^{g-1},$$

$$\widetilde{N}_m^-(\mathrm{Spin}_l) = \sum_{\substack{U' \subset \{\frac{1}{2}, \dots, \frac{k}{2}\} \\ |U| = n}} \Big(\frac{1}{|\mathbf{Z}_2(U')|} \times \frac{4k^n}{\Phi_k(U')} \Big)^{g-1}.$$

Taking U, U' to be complementary gives a bijection between the terms of the two sums, and we then just need to check that

$$|\mathrm{Stab}_{\mathbf{Z}_2 \times \mathbf{Z}_2}(U)| \times \frac{k^r}{\Pi_k(U)} = 2|\mathrm{Stab}_{\mathbf{Z}_2}(U')| \times \frac{k^n}{\Phi_k(U')}.$$

Now either $k/2 \in U$, in which case $|\mathrm{Stab}_{\mathbf{Z}_2 \times \mathbf{Z}_2}(U)| = 2$ and $|\mathrm{Stab}_{\mathbf{Z}_2}(U')| = 1$—and the relation reduces to (1.9)(i) (with $p = k$)—or $k/2 \in U'$ so that $|\mathrm{Stab}_{\mathbf{Z}_2 \times \mathbf{Z}_2}(U)| = 1$ and $|\mathrm{Stab}_{\mathbf{Z}_2}(U')| = 2$ and it reduces to (1.9)(ii); either way we're done. ∎

At least in one case, Theorem (4.13) implies a reciprocity relation for the unmodified Verlinde numbers $N_l(\mathrm{Spin}_m)$: this is when l and m are both odd. Here again it is necessary to decompose N_l into its 'tensor' and 'spinor' parts N_l^+ and N_l^-; that is, we decompose the sums (4.3) and (4.7) into the sums over integral and half-integral sets U respectively.

Now it is just in the case of the sum $N_l^-(\mathrm{Spin}_m)$ *with l, m both odd* that Γ, in this case $\mathbf{Z}_2$, acts freely on the summation set; so that $|\Gamma(U)| = 2$ for all U in (4.5). At the same time this sum has exactly half as many terms as the sum (4.3), and hence

$$\widetilde{N}_l^-(\mathrm{Spin}_m) = 2^{-g} N_l^-(\mathrm{Spin}_m) \qquad \text{when } l, m \text{ are both odd.} \tag{4.14}$$

From this and (4.13)(ii) we deduce:

(4.15) Corollary. *For $l, m \geq 5$ both odd, $N_l^-(\mathrm{Spin}_m) = N_m^-(\mathrm{Spin}_l)$.*

We shall make some comments on the geometrical meaning of this relation in the next section; first let us complete it by adding in the case of Spin_3.

(4.16) Theorem. *For all odd $l \geq 5$ we have $N_{2l}^-(\mathrm{Spin}_3) = N_3^-(\mathrm{Spin}_l)$.*

Before proving this we need to make more precise the meaning of the left-hand side. By definition the Verlinde numbers of Spin_3 are those of SL_2; and from (2.2) one finds the well-known formula

$$N_l(\mathrm{Spin}_3) = \sum_{r=1}^{k-1} \Big(\frac{2k}{f_k(r)}\Big)^{g-1},$$

where $k = l+2$ and f_k is defined as in §1. The "tensor representations" in $P_l(\mathrm{Spin}_3)$ correspond to the even values of r in this sum, so for N_l^- we just sum over the odd values. Making use of (1.2)(ii) and (iii) we have in particular

$$N_{2l}(\mathrm{Spin}_3) = (l+1)^{g-1} + 2\sum_{r=1}^{l} \Big(\frac{2k}{f_k(r)}\Big)^{g-1},$$

where now $k = 2l+2$; and if in addition we assume that l is odd, then the power of $l+1$ will not enter the sum for N_{2l}^-, and we have:

$$N_{2l}^-(\mathrm{Spin}_3) = 2 \sum_{\substack{r=1 \\ r \text{ odd}}}^{l} \Big(\frac{2k}{f_k(r)}\Big)^{g-1}.$$

Proof of (4.16). Let $l = 2a-1$, so $k = 4a$. We can rewrite the preceding formula as

$$\begin{aligned} N_{2l}^-(\mathrm{Spin}_3) &= 2 \sum_{\substack{U \subset \{\frac{1}{2}, \dots, a-\frac{1}{2}\} \\ |U|=1}} \Big(\frac{8a}{\Phi_{2a}(U)}\Big)^{g-1} \\ &= 2^{2g-1} \sum_{\substack{U \subset \{\frac{1}{2}, \dots, a-\frac{1}{2}\} \\ |U|=a-1}} \Big(\frac{(2a)^{a-1}}{\Phi_{2a}(U)}\Big)^{g-1}, \end{aligned}$$

using Corollary (1.8). On the other hand by (4.5)

$$\widetilde{N}_3^-(\mathrm{Spin}_{2n+1}) = \sum_{\substack{U \subset \{\frac{1}{2}, \dots, n+\frac{1}{2}\} \\ |U|=n}} \Big(\frac{2(2n+2)^n}{\Phi_{2n+2}(U)}\Big)^{g-1}$$

where we have used the fact that $|\Gamma(U)| = 2$ for all U in the sum. Putting $n = a-1$ these expressions combine to give

$$N_{2l}^-(\mathrm{Spin}_3) = 2^g \widetilde{N}_3^-(\mathrm{Spin}_l) = N_3^-(\mathrm{Spin}_l)$$

by (4.14). ∎

5. Final remarks

One would like to interpret the reciprocity results (4.13) geometrically, in terms of the moduli of spin bundles on an algebraic curve, along the lines of (0.3) and (0.5). Unfortunately it is not clear to us how to do this, and it seems an intriguing question. On the other hand, the special relations (4.15) and (4.16) for the odd spin groups are a little more transparent, as we shall now explain.

The moduli space $\mathcal{M}(\mathrm{Spin}_m)$ of (semistable) holomorphic Spin_m-bundles on our curve C has a sister moduli space $\mathcal{M}^-(\mathrm{Spin}_m)$, defined as follows. Let G_m denote the special Clifford group for a nondegenerate quadratic form on $\mathbf{C}^m$; and let $\mathrm{Nm} : G_m \to \mathbf{C}^*$ be the spinor norm. Then by definition $\mathrm{Spin}_m = \ker \mathrm{Nm}$. Moreover, there is a projective moduli space $\mathcal{M}(G_m)$ of semistable G_m-bundles on C, and the spinor norm induces a fibration

$$\delta : \mathcal{M}(G_m) \to \mathrm{Pic}(C).$$

The isomorphism class of the fibre over $L \in \mathrm{Pic}(C)$ depends only on $c_1(L)$ modulo 2; thus

$$\mathcal{M}(\mathrm{Spin}_m) = \delta^{-1}(\mathcal{O}_C) \qquad \text{and} \qquad \mathcal{M}^-(\mathrm{Spin}_m) \stackrel{\text{def}}{=} \delta^{-1}(\mathcal{O}_C(x)) \quad \text{for } x \in C.$$

Equivalently $\mathcal{M}^-(\mathrm{Spin}_m)$ parametrises Clifford bundles on C, with fixed norm, which lift SO_m-bundles with nonzero Stiefel-Whitney class.

The case $m = 3$ is standard: here $G_3 = GL_2$ and Nm is just the determinant homomorphism. Thus $\mathcal{M}(G_3)$ is the moduli space of semistable rank 2 vector bundles; while $\mathcal{M}(\mathrm{Spin}_3) \cong M_C(2, \mathcal{O})$ and $\mathcal{M}^-(\mathrm{Spin}_3) \cong M_C(2, \mathcal{O}(x))$.

Now let $\mathcal{N}(m)$ be the union of $\mathcal{M}(\mathrm{Spin}_m)$ and $\mathcal{M}^-(\mathrm{Spin}_m)$, and consider the line bundle $\Theta(\mathbf{C}^m) \to \mathcal{N}(m)$ where $\mathbf{C}^m$ is the standard orthogonal representation of G_m (see [KNR]). *We shall suppose in what follows that there exists a line bundle $\mathcal{L} \to \mathcal{N}(m)$ such that $\mathcal{L}^2 = \Theta(\mathbf{C}^m)$.*

It is not known to us whether this is the case except when $m = 3$; though from the results of [KNR] it seems plausible (see also [O]). Note that in the case $m = 3$, $\mathcal{L}$ restricts to the ample generator of the Picard group on $M_C(2, \mathcal{O}(x))$ and to *twice* the ample generator on $M_C(2, \mathcal{O})$. Granted this assumption, though, the following facts are known:

(5.1)
(i) $\dim H^0(\mathcal{M}(\mathrm{Spin}_3), \mathcal{L}^l) = N_{2l}(\mathrm{Spin}_3) = N_{2l}^- + N_{2l}^+$;
(ii) $\dim H^0(\mathcal{M}^-(\mathrm{Spin}_3), \mathcal{L}^l) = N_{2l}^-(\mathrm{Spin}_3) - N_{2l}^+(\mathrm{Spin}_3)$;
(iii) $\dim H^0(\mathcal{M}(\mathrm{Spin}_m), \mathcal{L}^l) = N_l(\mathrm{Spin}_m) = N_l^- + N_l^+$ for $m \geq 5$.

Statement (ii) here is the 'twisted' rank 2 Verlinde formula, due originally to Thaddeus.

(5.2) Conjecture. *For any odd $m \geq 5$,*

$$\dim H^0(\mathcal{M}^-(\mathrm{Spin}_m), \mathcal{L}^l) = N_l^-(\mathrm{Spin}_m) - N_l^+(\mathrm{Spin}_m).$$

Note that this makes sense for even values of l *even if our assumption on the existence of $\mathcal{L}$ is false.* Somewhat stronger evidence for (5.2) is given in [O], where

the vector spaces $H^0(\mathcal{N}(m), \Theta(\mathbf{C}^m))$ are shown to have natural interpretations as spaces of theta functions on the Prym varieties associated to the curve.

Finally, (5.1) and (5.2) together imply that for odd m,

$$\dim H^0(\mathcal{N}(m), \mathcal{L}^l) = \begin{cases} 2N_l^-(\mathrm{Spin}_m) & \text{if } m \geq 5, \\ 2N_{2l}^-(\mathrm{Spin}_3) & \text{if } m = 3. \end{cases}$$

Consequently (4.15) and (4.16) together imply the reciprocity law (0.7):

$$\textbf{(5.3)} \qquad \dim H^0(\mathcal{N}(m), \mathcal{L}^l) = \dim H^0(\mathcal{N}(l), \mathcal{L}^m) \qquad \text{for } l, m \text{ both odd.}$$

And curiously, this relation remains true if either of l, m is 1, if we interpret $\mathcal{N}(1)$ to be the set $J_2(C)$ of 2-torsion points in the Jacobian! (See [O].)

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