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DIVISORS ON GENERIC COMPLETE INTERSECTIONS IN PROJECTIVE SPACE

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Abstract. Let V be a generic complete intersection of hypersurfaces of degree $d_1, d_2, \cdots, d_m$ in n-dimensional projective space. We study the question when a divisor on V is nonrational or of general type, and give an alternative proof of a result of Ein. We also give some improvement of Ein's result in the case $d_1 + d_2 + \cdots + d_m = n + 2$.

0. Introduction

Let V be a generic complete intersection of hypersurfaces of degree $d_1, d_2, \cdots, d_m$ in $\mathbf{P}^n$. A conjecture of Kobayashi (cf. [L]) states that V is hyperbolic if $d = d_1 + d_2 + \cdots + d_m \geq n+2$. In general, S. Lang [L] has conjectured that a variety X is hyperbolic if and only if every subvariety of X is of general type. In this paper, we will prove the following

Theorem 1. *Let V be a complete intersection of m generic hypersurfaces of degree $d_1, d_2, \cdots, d_m$ in $\mathbf{P}^n$, $M \subset V$ a reduced and irreducible divisor, $p_g(M)$ the geometric genus of the desingularization of M. Assume that $1 \leq m \leq n-3$ and $d_i \geq 2$ for all i. Then*
(1) *$p_g(M) \geq n-1$ if $d = d_1 + d_2 + \cdots + d_m \geq n+2$,*
(2) *M is of general type if $d = d_1 + d_2 + \cdots + d_m > n+2$.*

In [E1,E2], Ein has shown that M is nonrational if $d \geq n+2$, and is of general type if $d > n+2$. Here we are going to give an alternative proof of it. Ein also proved that every subvariety of V of dimension l is nonrational if $d \geq 2n-m-l+1$, and is of general type if $d > 2n-m-l+1$. Therefore the improvement we made here is in the case $d = n+2$ and $l = n-m-1$. In particular, we conclude that the divisor M can not be an abelian variety. If a variety X is hyperbolic, then every rational map of an abelian variety or $\mathbf{P}^1$ into X is constant. On the other hand, Lang [L] conjectured that this condition is also sufficient for X to be hyperbolic.

If V is a generic hypersurface in $\mathbf{P}^n$, it was first shown by Clemens [CKM] that V contains no rational curves, if deg $V \geq n-1$. In [X1], we study generic surfaces in $\mathbf{P}^3$, obtain that every curve C on S has geometric genus $g(C) \geq \frac{1}{2}d(d-3)-2$ (d =deg S), and the bound is sharp. We also obtain results about divisors on a generic hypersurface in $\mathbf{P}^n$. In [X2], we generalize these results to some nongeneric cases.

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When V is a generic quintic 3-fold in $\mathbf{P}^4$, a conjecture of Clemens says that V should contain only finitely many rational curves of given degree, which is equivalent to the statement that every divisor on V must have a nonnegative Kodaira dimension. Chang and Ran [CR] has proved that V does not contain a reduced and irreducible divisor which admits a desingularization having a numerically effective anticanonical bundle.

To establish Theorem 1, we need to get control over the singularities of the divisor M on V. The method we use here is deformation of singularity as we did in [X1].

Throughout this paper we work over the complex number field $\mathbb{C}$.

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1. Deformation of Singularities

For simplicity of notations, we will give a proof of Theorem 1 in the case $m = 2$.

First of all, we recall some definitions from [X1].

Let V be an n-dimensional smooth variety, and $M \subset V$ be a reduced and irreducible divisor. According to Hironaka [H], there is a desingularization of M:

$$V_{m+1} \xrightarrow{\pi_{m+1}} V_m \xrightarrow{\pi_m} \cdots \xrightarrow{\pi_2} V_1 \xrightarrow{\pi_1} V_0 = V,$$

so that the proper transform $\tilde{M}$ of M in V_{m+1} is smooth. Here $V_j \xrightarrow{\pi_j} V_{j-1}$ is the blow-up of V_{j-1} along a ν_{j-1}-dimensional submanifold X_{j-1} with $E_{j-1} \subset V_j$ the exceptional divisor. If X_{j-1} is a μ_{j-1}-fold singular submanifold of the proper transform of M in V_{j-1}, we say that M has a *type* $\mu = (\mu_j, X_j, E_j \mid j \in \{0, 1, \dots, m\})$ *singularity.*

If $M \subset V$ has a type $\mu = (\mu_j, X_j, E_j \mid j \in \Gamma)$ singularity, $\Omega \subset V$ is an open set, we localize our definition by saying that M has a type $\mu_\Omega = (\mu_j, X_j, E_j \mid j \in \Gamma_\Omega = \{j \mid \exists q \in E_j,\, q$, is an infinitely near point of some $p \in \Omega\})$ singularity on Ω.

Given any resolution of the singularity of $M \subset V$ as above, if $D \subset V$ is a divisor, such that

$$\pi_j^*(\cdots(\pi_2^*(\pi_1^*(D) - \delta_0 E_0) - \delta_1 E_1) - \cdots) - \delta_{j-1} E_{j-1}$$

is an effective divisor for all $j = 1, 2, \dots, m+1$, then we say that D has a *weak type* $\delta = (\delta_j, X_j, E_j \mid j \in \{0, 1, \dots, m\})$ *singularity.* It is easy to see that a type μ singularity implies a weak type μ singularity.

Assume that $M \subset V$ has a type $\mu = (\mu_j, X_j, E_j \mid j \in \{0, 1, \dots, m\})$ singularity. The following lemma describes the connection between the singularities of M and the canonical bundle of the desingularization $\tilde{M}$ of M.

Lemma 2. *A section of $K_V \otimes M$ with a weak type $\mu - 1 = (\mu_j - 1, X_j, E_j \mid j \in \{0, 1, \dots, m\})$ singularity induces a section of $K_{\tilde{M}}$.*

Proof. Proposition 1.1 in [X1]. q.e.d.

Definition. Let $T \subset \mathbb{C}^N$ be an open neighborhood of the origin $0 \in T$. Assuming that $\sigma\colon M \longrightarrow T$ is a family of reduced equidimensional algebraic varieties, $M_t = \sigma^{-1}(t)$, then we say that the family M_t is μ-equisingular at $t = 0$ in the sense that we can resolve the singularity of M_t simultaneously, that is, there is a proper morphism $\pi\colon \tilde{M} \longrightarrow M$, so that $\sigma \circ \pi\colon \tilde{M} \longrightarrow T$ is a flat map and

$$\sigma \circ \pi\colon \tilde{M}_t = (\sigma \circ \pi)^{-1}(t) \longrightarrow M_t$$

is a resolution of the singularities of M_t. Moreover, if M_t has a type $\mu(t) = (\mu_j(t), X_j(t), E_j(t) \mid j \in \Gamma(t))$ singularity with the above resolution, then $\mu_j(t) = \mu_j$ and $\Gamma(t) = \Gamma$ are independent of t, and the exceptional divisors and the singular loci of the desingularization $\tilde{M}_t \longrightarrow M_t$ have the same configuration for all t.

Now we state a lemma concerning the local deformation theory of singular divisors.

Lemma 3. *If $M_t = \{g_t(z_1, \dots, z_n) = 0\}$ is a μ-equisingular family of varieties defined in an open set $\Omega \subset \mathbb{C}^n$, and M_t has a type $\mu(t)_\Omega = (\mu_j, X_j(t), E_j(t) \mid j \in \{0, \dots, m\})$ singularity on Ω, then the variety $\left\{\frac{dg_t}{dt} \mid_{t=0} = 0\right\}$ has a weak type $\mu(0)_\Omega - 1 = (\mu_j - 1, X_j(0), E_j(0) \mid j \in \{0, \dots, m\})$ singularity on Ω.*

Proof. Lemma 4.4 in [X1]. q.e.d.

Let $\{Z_i\}$ be some homogeneous coordinates of $\mathbf{P}^n$, $F \in H^0(\mathbf{P}^n, \mathcal{O}(r))$ and $G \in H^0(\mathbf{P}^n, \mathcal{O}(l))$ be homogeneous polynomials. We define

$$\frac{\partial(F,G)}{\partial(Z_i, Z_j)} = \det \begin{vmatrix} \frac{\partial F}{\partial Z_i} & \frac{\partial F}{\partial Z_j} \\ \frac{\partial G}{\partial Z_i} & \frac{\partial G}{\partial Z_j} \end{vmatrix}.$$

The next lemma tells us how to use deformation of singularities to produce special homogeneous polynomials.

Lemma 4. *Let $F_{1,t} \in H^0(\mathbf{P}^n, \mathcal{O}(d_1)), F_{2,t} \in H^0(\mathbf{P}^n, \mathcal{O}(d_2)), G_t \in H^0(\mathbf{P}^n, \mathcal{O}(k))$, and $M_t = \{F_{1,t} = 0\} \cap \{F_{2,t} = 0\} \cap \{G_t = 0\}$ be a μ-equisingular family of varieties with a type $\mu(t) = (\mu_j, X_j(t), E_j(t) \mid j \in \Gamma)$ singularity. Setting*

$$\frac{dF_{1,t}}{dt} \mid_{t=0} = F_1', \quad \frac{dF_{2,t}}{dt} \mid_{t=0} = F_2', \quad \frac{dG_t}{dt} \mid_{t=0} = G',$$

and assuming that both the varieties $\{F_{i,t} = 0\}$ $(i = 1, 2)$ and $\{F_{1,t} = 0\} \cap \{F_{2,t} = 0\}$ are smooth for t in a neighborhood of 0. Then the divisor

$$\left\{\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} F_1' - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} F_2' = 0\right\}$$

$(i, j = 0, 1, \dots, n)$ on $V = \{F_{1,0} = 0\} \cap \{F_{2,0} = 0\}$ has a weak type $\mu(0) - 1 = (\mu_j - 1, X_j(0), E_j(0) \mid j \in \Gamma)$ singularity, where $\{Z_0, Z_1, \cdots, Z_n\}$ are homogeneous coordinates of $\mathbf{P}^n$.

Proof. For any point $P \in M_0$, we can find an open set $\Omega \ni P$ of V, and generic homogeneous coordinates $\{Z_i'\}$ with

$$Z_i' = \sum_{j=0}^{n} l_{ij} Z_j \quad (i = 0, 1, \dots, n),$$

so that

$$\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i', Z_j')} \neq 0$$

on Ω for all $i \neq j$ $(i, j = 0, 1, \cdots, n)$. Assuming M_0 has a type $\mu_\Omega(0) = (\mu_j, X_j(0), E_j(0) \mid j \in \Gamma_\Omega)$ singularity on Ω. Denoting

$$\{z_1, z_2, \cdots, z_n\} = \{\frac{Z_1'}{Z_0'}, \frac{Z_2'}{Z_0'}, \cdots, \frac{Z_n'}{Z_0'}\},$$

if we solve the equation

$$F_{1,t}(1, z_1, z_2, \cdots, z_n) = 0, \quad F_{2,t}(1, z_1, z_2, \cdots, z_n) = 0$$

near the point $P(t)$, where $P(0) = P$, and get

$$z_1 = \varphi_{1,t}(z_3, \cdots, z_n), \quad z_2 = \varphi_{2,t}(z_3, \cdots, z_n),$$

then on some open set of $\mathbb{C}^{n-2}$, M_t is a μ-equisingular family of divisors locally defined by the equation

$$G_t(1, \varphi_{1,t}, \varphi_{2,t}, z_3, \cdots, z_n) = 0.$$

By Lemma 3, the divisor locally defined by the equation

$$\frac{dG_t}{dt}(1, \varphi_{1,t}(z_3, \cdots, z_n), \varphi_{2,t}(z_3, \cdots, z_n), z_3, \cdots, z_n) \mid_{t=0} = 0$$

on Ω has a weak type $\mu_\Omega(0) - 1 = (\mu_j - 1, X_j(0), E_j(0) \mid j \in \Gamma_\Omega)$ singularity.

Now a detailed computation shows that

$$\begin{aligned}\frac{dG_t}{dt}(1, \varphi_{1,t}, \varphi_{2,t}, z_3, \cdots, z_n) \mid_{t=0} &= G' + \frac{\partial G_0}{\partial Z_1'}\frac{d\varphi_{1,t}}{dt} \mid_{t=0} + \frac{\partial G_0}{\partial Z_2'}\frac{d\varphi_{2,t}}{dt} \mid_{t=0} \\ &= \Big\{\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_1', Z_2')}\Big\}^{-1}\Big\{\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_1', Z_2')}G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z_1', Z_2')}F_1' - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_1', Z_2')}F_2'\Big\}.\end{aligned}$$

Then the divisor

$$\Big\{\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_1', Z_2')}G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z_1', Z_2')}F_1' - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_1', Z_2')}F_2' = 0\Big\}$$

has a weak type $\mu_\Omega(0) - 1$ singularity on Ω. Similarly, the divisor

$$\Big\{\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i', Z_j')}G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i', Z_j')}F_1' - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i', Z_j')}F_2' = 0\Big\} \qquad (i, j = 0, 1, \cdots, n)$$

has a weak type $\mu_\Omega(0) - 1$ singularity on Ω. Finally, since the expression

$$\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)}G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)}F_1' - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)}F_2'$$

is a linear combination of expressions

$$\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z'_s,Z'_l)}G' - \frac{\partial(G_0,F_{2,0})}{\partial(Z'_s,Z'_l)}F'_1 - \frac{\partial(F_{1,0},G_0)}{\partial(Z'_s,Z'_l)}F'_2, \quad (s,l=0,1,\cdots,n)$$

and weak type $\mu_\Omega(0)-1$ singularity is additive (cf. section 1 in [X1]), we conclude that the divisor

$$\Big\{\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}G' - \frac{\partial(G_0,F_{2,0})}{\partial(Z_i,Z_j)}F'_1 - \frac{\partial(F_{1,0},G_0)}{\partial(Z_i,Z_j)}F'_2 = 0\Big\}$$

has a weak type $\mu_\Omega(0)-1$ singularity on Ω, hence it has a weak type $\mu(0)-1$ singularity on V. q.e.d.

Remark. In general, if

$$V_t = \{F_{1,t}=0\}\cap\{F_{2,t}=0\}\cap\cdots\cap\{F_{m,t}=0\}$$

is a complete intersection of m hypersurfaces, and $M_t^* = V_t\cap\{G_t=0\}$ is a μ-equisingular family of divisors. Then one can state and prove an analogy of Lemma 4 with the divisor

$$\Big\{\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}G' - \frac{\partial(G_0,F_{2,0})}{\partial(Z_i,Z_j)}F'_1 - \frac{\partial(F_{1,0},G_0)}{\partial(Z_i,Z_j)}F'_2 = 0\Big\}$$

replaced by a divisor of the form

$$\Big\{\frac{\partial(F_{1,0},F_{2,0},F_{3,0},\cdots,F_{m,0})}{\partial(Z_{i_1},Z_{i_2},Z_{i_3},\cdots,Z_{i_m})}G' - \frac{\partial(G_0,F_{2,0},F_{3,0},\cdots,F_{m,0})}{\partial(Z_{i_1},Z_{i_2},Z_{i_3}\cdots,Z_{i_m})}F'_1$$
$$- \frac{\partial(F_{1,0},G_0,F_{3,0},\cdots,F_{m,0})}{\partial(Z_{i_1},Z_{i_2},Z_{i_3},\cdots,Z_{i_m})}F'_2 - \frac{\partial(F_{1,0},F_{2,0},G_0,\cdots,F_{m,0})}{\partial(Z_{i_1},Z_{i_2},Z_{i_3},\cdots,Z_{i_m})}F'_3$$
$$-\cdots - \frac{\partial(F_{1,0},F_{2,0},F_{3,0},\cdots,G_0)}{\partial(Z_{i_1},Z_{i_2},Z_{i_3},\cdots,Z_{i_m})}F'_m = 0\Big\},$$

here $i_1,\cdots,i_m = 0,1,\cdots,n$.

2. Proof of Theorem 1

Let $V=\{F_1=0\}\cap\{F_2=0\}\subset \mathbf{P}^n$ be a complete intersection of generic hypersurfaces $\{F_1=0\}$ and $\{F_2=0\}$ of degree d_1 and d_2. By our assumption $m\le n-3$, that is dim $V\ge 3$, we know that Pic $V=\mathbb{Z}$ and it is generated by $\mathcal{O}_V(1)$, thanks to the Lefschetz theorem. Now if $M\subset V$ is a reduced and irreducible divisor, then it is a complete intersection of V with another hypersurface $\{G=0\}$ of degree k. Here F_1, F_2 and G are homogeneous polynomials.

Proposition 5. *Let V be a complete intersection of m generic hypersurfaces of degree $d_1,d_2,\cdots,d_m$ in $\mathbf{P}^n$, and $M\subset V$ a reduced and irreducible divisor. Assume that $d=d_1+d_2+\cdots+d_m\ge n+2$, $1\le m\le n-3$ and $d_i\ge 2$ for all i. Then there is a desingularization $\sigma:\tilde M\to M$ of M, and we have*

$$dim\ H^0(\tilde M, K_{\tilde M}\otimes\sigma^*\mathcal{O}(-(d-n-2)))\ge n-1.$$

Remark. This is an improvement of an early result of L. Ein [E2] which states that

$$H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^*\mathcal{O}(-(d-n-2))) \neq 0.$$

Assuming Proposition 5, now we can give the

Proof of Theorem 1. (1) If $d \geq n+2$, then $H^0(M, \mathcal{O}(d-n-2)) \neq 0$, by Proposition 5,

$$\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^*\mathcal{O}(-d+n+2)) \geq n-1.$$

Hence we have

$$\begin{aligned} p_g(M) &= \dim H^0(\tilde{M}, K_{\tilde{M}}) \\ &\geq \dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^*\mathcal{O}(-d+n+2)) + \dim H^0(\tilde{M}, \sigma^*\mathcal{O}(d-n-2)) - 1 \\ &\geq n-1, \end{aligned}$$

thanks to Hopf's theorem.

(2) If $d > n+2$, then $d-n-2 \geq 1$. From

$$\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^*\mathcal{O}(-(d-n-2))) \geq n-1 > 0,$$

we conclude that M is of general type. q.e.d.

We now begin the proof of Proposition 5. For simplicity of notation, we will assume that $m = 2$.

Assume the contrary; namely, for any generic complete intersection of 2 hypersurfaces of degree d_1, d_2, there is a reduced and irreducible divisor on it with

$$\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^*\mathcal{O}(-(d-n-2))) < n-1.$$

Set

$$\begin{aligned} B = \{\{F_1, F_2\} \in H^0(\mathbf{P}^n, \mathcal{O}(d_1)) \times H^0(\mathbf{P}^n, \mathcal{O}(d_2)) | & \text{ both varieties} \\ \{F_i = 0\}(i = 1, 2) \text{ and } \{F_1 = 0\} \cap \{F_2 = 0\} & \text{ are smooth}\}, \\ A_k = \{\{F_1, F_2, G\} \in H^0(\mathbf{P}^n, \mathcal{O}(d_1)) \times H^0(\mathbf{P}^n, & \mathcal{O}(d_2)) \times H^0(\mathbf{P}^n, \mathcal{O}(k)) | \\ \{F_1, F_2\} \in B, M = \{G = 0\} \cap V & \text{ is a reduced and irreducible divisor on} \\ V = \{F_1 = 0\} \cap \{F_2 = 0\}, \ \dim H^0(\tilde{M}, & K_{\tilde{M}} \otimes \sigma^*\mathcal{O}(-(d-n-2))) < n-1\}. \end{aligned}$$

Then the map

$$\bigcup_{k=1}^{\infty} A_k \to B$$

is dominant by assumption. Hence the map $A_k \to B$ is dominant for some k. Therefore at some regular point $\{F_1, F_2\}$ of B, we can find a smooth section $B \to A_k$, that is, there is a triple

$$\{F_1, F_2, G\} \in H^0(\mathbf{P}^n, \mathcal{O}(d_1)) \times H^0(\mathbf{P}^n, \mathcal{O}(d_2)) \times H^0(\mathbf{P}^n, \mathcal{O}(k)),$$

which has the following property: both varieties $\{F_i = 0\}$ $(i = 1, 2)$ and $V = \{F_1 = 0\} \cap \{F_2 = 0\}$ are smooth, the divisor $M = V \cap \{G = 0\}$ is reduced and irreducible, and for any deformation $F_{1,t}$ of $F_1 = F_{1,0}$ and $F_{2,t}$ of $F_2 = F_{2,0}$, there is a unique deformation G_t of $G = G_0$, so that the divisor

$$M_t = \{F_{1,t} = 0\} \cap \{F_{2,t} = 0\} \cap \{G_t = 0\}$$

on $\{F_{1,t} = 0\} \cap \{F_{2,t} = 0\}$ has

$$\dim H^0(\tilde{M}_t, K_{\tilde{M}_t} \otimes \sigma_t^* \mathcal{O}(-(d-n-2))) < n-1.$$

Here $\sigma_t^* : \tilde{M}_t \to M_t$ is a desingularization of M_t. Moreover, we can assume that the family M_t is μ-equisingular, and M_t has a type $\mu(t) = (\mu_j, X_j(t), E_j(t) \mid j \in \Gamma)$ singularity.

Let $\{Z_i\}$ be fixed homogeneous coordinates of $\mathbf{P}^n$. By Lemma 4, for any deformation $F_1' \in H^0(\mathbf{P}^n, \mathcal{O}(d_1))$ of F_1 and $F_2' \in H^0(\mathbf{P}^n, \mathcal{O}(d_2))$ of F_2, there is a unique deformation $G' \in H^0(\mathbf{P}^n, \mathcal{O}(k))$ of G, so that the divisor

$$\Big\{\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} G' - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} F_1' - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} F_2' = 0\Big\} \quad (i, j = 0, 1, \dots, n)$$

on $V = \{F_{1,0} = 0\} \cap \{F_{2,0} = 0\} = \{F_1 = 0\} \cap \{F_2 = 0\}$ has a weak type $\mu(0) - 1 = (\mu_j - 1, X_j(0), E_j(0) \mid j \in \Gamma)$ singularity. Denote $G' = \Phi(F_1', F_2')$. Then we have a map

$$\Phi : \quad H^0(\mathbf{P}^n, \mathcal{O}(d_1)) \times H^0(\mathbf{P}^n, \mathcal{O}(d_2)) \longrightarrow H^0(\mathbf{P}^n, \mathcal{O}(k))/(F_1, F_2, G),$$

here (F_1, F_2, G) is the ideal generated by F_1, F_2, G.

Lemma 6. *Φ is linear in F_1, F_2* mod (F_1, F_2, G).

Proof. Otherwise, since

$$\begin{aligned}
&\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} (\Phi(aF_1' + b\tilde{F}_1', F_2') - a\Phi(F_1', 0) - b\Phi(\tilde{F}_1', 0) - \Phi(0, F_2')) \\
&= \Big\{\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \Phi(aF_1' + b\tilde{F}_1', F_2') - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} (aF_1' + b\tilde{F}_1') - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} F_2'\Big\} \\
&\quad - a\Big\{\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \Phi(F_1', 0) - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} F_1'\Big\} \\
&\quad - b\Big\{\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \Phi(\tilde{F}_1', 0) - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} \tilde{F}_1'\Big\} \\
&\qquad - \Big\{\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)} \Phi(0, F_2') - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} F_2'\Big\},
\end{aligned}$$

and for any point $P \in V$,

$$\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)}(P) \neq 0$$

for some i, j. By Lemma 4 and the additivity of weak type $\mu(0) - 1$ singularity, the divisor

$$\{\Phi(aF_1' + b\tilde{F}_1', F_2') - a\Phi(F_1', 0) - b\Phi(\tilde{F}_1', 0) - \Phi(0, F_2')) = 0\}$$

will have a weak type $\mu(0)-1=(\mu_j-1, X_j(0), E_j(0)|j\in\Gamma)$ singularity on V. On the other hand, by the adjunction formula, we have

$$K_V\otimes M=\mathcal{O}(d+k-n-1).$$

If Φ is not linear mod (F_1,F_2,G), then

$$\Phi(aF_1'+b\tilde{F}_1', F_2')-a\Phi(F_1',0)-b\Phi(\tilde{F}_1',0)-\Phi(0,F_2')$$

will generate a section of $K_{\tilde{M}}\otimes\mathcal{O}(-(d-n-2)-1)$ by Lemma 2, which will imply that

$$\dim\, H^0(\tilde{M}, K_{\tilde{M}}\otimes\sigma^*\mathcal{O}(-(d-n-2)))\geq n-1$$

because $\dim\, H^0(M,\mathcal{O}(1))\geq n-1$. Here we use the fact that $\deg\, F_i=d_i\geq 2$. q.e.d.

Let $\{Y_i\}$ be another homogeneous coordinate of $\mathbf{P}^n$. Now we take a special deformation $F_1'=Y_pU$ $(p=0,1,\cdots,n)$ of F_1 with $U\in H^0(\mathbf{P}^n,\mathcal{O}(d_1-1))$. Since

$$\begin{aligned}\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}&(Y_p\Phi(Y_qU,0)-Y_q\Phi(Y_pU,0))\\ &=Y_p\Big(\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}\Phi(Y_qU,0)-\frac{\partial(G_0,F_{2,0})}{\partial(Z_i,Z_j)}Y_qU\Big)\\ &-Y_q\Big(\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}\Phi(Y_pU,0)-\frac{\partial(G_0,F_{2,0})}{\partial(Z_i,Z_j)}Y_pU\Big),\end{aligned}$$

by Lemma 4 we conclude that the divisor $\{Y_p\Phi(Y_qU,0)-Y_q\Phi(Y_pU,0)=0\}$ on V has a weak type $\mu(0)-1$ singularity.

Lemma 7. *If* $\dim H^0(\tilde{M}, K_{\tilde{M}}\otimes\sigma^*\mathcal{O}(-(d-n-2)))<n-1$, *then there is a linear map*

$$\Phi_1: H^0(\mathbf{P}^n,\mathcal{O}(d_1-1))\times H^0(\mathbf{P}^n,\mathcal{O}(d_2-1))\to H^0(\mathbf{P}^n,\mathcal{O}(k-1))/(F_1,F_2,G),$$

so that for any $U\in H^0(\mathbf{P}^n,\mathcal{O}(d_1-1))$, *and* $W\in H^0(\mathbf{P}^n,\mathcal{O}(d_2-1))$, *the divisor*

$$\Big\{\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}\Phi_1(U,W)-\frac{\partial(G_0,F_{2,0})}{\partial(Z_i,Z_j)}U-\frac{\partial(F_{1,0},G_0)}{\partial(Z_i,Z_j)}W=0\Big\}$$

$(i,j=0,1,\cdots,n)$ *on* V *has a weak type* $\mu(0)-1$ *singularity.*

Proof. Let $Y,H\in H^0(\mathbf{P}^n,\mathcal{O}(1))$ be 2 hyperplanes, and $U\in H^0(\mathbf{P}^n,\mathcal{O}(d_1-1))$ be a fixed polynomial. By the argument before Lemma 7 (choose $Y_p=Y, Y_q=H$), we know that the divisor $\{Y\Phi(HU,0)-H\Phi(YU,0)=0\}$ on V has a weak type $\mu(0)-1$ singularity. Since we have

$$K_V\otimes M=\mathcal{O}(d+k-n-1),$$

and $Y\Phi(HU,0)-H\Phi(YU,0)\in H^0(\mathbf{P}^n,\mathcal{O}(k+1))$, if

$$Y\Phi(HU,0)-H\Phi(YU,0)\not\equiv 0$$

on M, that is $Y\Phi(HU,0) - H\Phi(YU,0) \notin (F_1, F_2, G)$, then it will induce a section of $K_{\tilde{M}} \otimes \sigma^*\mathcal{O}(-(d-n-2))$ by Lemma 2. Denote

$$\Lambda_H = \{Y | Y\Phi(HU,0) - H\Phi(YU,0) \in (F_1, F_2, G)\} \subset H^0(\mathbf{P}^n, \mathcal{O}(1)).$$

The linearity of Φ implies that Λ_H is a linear subspace of $H^0(\mathbf{P}^n, \mathcal{O}(1))$. We conclude that dim $\Lambda_H \geq 2$ by our assumption that

$$\dim\, H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^*\mathcal{O}(-(d-n-2))) \leq n-2.$$

Hence there is a nontrivial hyperplane $Y_H \in \Lambda_H$ such that

$$Y_H \notin (H, F_1, F_2, G),$$

thanks to the fact that deg$F_i \geq 2$.

Let $\sigma : \tilde{M} \to M$ be a desingularization of M. Then the linear system $|\sigma^*\mathcal{O}(1)|$ on $\tilde{M}$ is base point free. Since dim M =dim $V - 1 \geq 2$, and M is reduced and irreducible, Bertini's theorem implies that the generic hyperplane section of $\tilde{M}$ is irreducible. Therefore we can choose a generic hyperplane H, so that $H \cap M$ is irreducible and reduced. By our construction of Y_H, we have

$$Y_H\Phi(HU,0) - H\Phi(Y_H U,0) \in (F_1, F_2, G),$$

that is $Y_H\Phi(HU,0) \in (H, F_1, F_2, G)$. The fact that $Y_H \notin (H, F_1, F_2, G)$ and that $H \cap M$ is irreducible now gives us $\Phi(HU,0) \in (H, F_1, F_2, G)$. Therefore,

$$\Phi(HU,0) = HU^* \mod (F_1, F_2, G)$$

for some $U^* \in H^0(\mathbf{P}^n, \mathcal{O}(k-1))$, and U^* is unique mod (F_1, F_2, G) because M is reduced and irreducible. Similarly, for any $W \in H^0(\mathbf{P}^n, \mathcal{O}(d_2-1))$, there is a $W^* \in H^0(\mathbf{P}^n, \mathcal{O}(k-1))$, such that

$$\Phi(0,HW) = HW^* \mod (F_1, F_2, G).$$

Now we define

$$\Phi_1(U,W) = U^* + W^* \in H^0(\mathbf{P}^n, \mathcal{O}(k-1))/(F_1, F_2, G),$$

then Φ_1 is independent of the choice of the generic hyperplane H.

From Lemma 4, we know that the divisor

$$\Big\{\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)}\Phi(HU,HW) - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)}HU - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)}HW = 0\Big\}$$

on V has a weak type $\mu(0)-1$ singularity. Using the fact that

$$\Phi(HU,HW) = \Phi(HU,0) + \Phi(0,HW) = H\Phi_1(U,W) \text{ mod } (F_1, F_2, G),$$

we find that the divisor

$$\Big\{H(\frac{\partial(F_{1,0}, F_{2,0})}{\partial(Z_i, Z_j)}\Phi_1(U,W) - \frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)}U - \frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)}W) = 0\Big\}$$

on V has a weak type $\mu(0)-1$ singularity. Therefore we know that the divisor

$$\Big\{\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}\Phi_1(U,W)-\frac{\partial(G_0,F_{2,0})}{\partial(Z_i,Z_j)}U-\frac{\partial(F_{1,0},G_0)}{\partial(Z_i,Z_j)}W=0\Big\}$$

on V has a weak type $\mu(0)-1$ singularity if we choose the generic hyperplane H such that it is in general position with respect to the singular locus of M. Again, we may assume that Φ_1 to be linear mod (F_1,F_2,G) as we did for Φ. q.e.d.

We continue the proof of Theorem 1. If

$$\dim\, H^0(\tilde{M},K_{\tilde{M}}\otimes\sigma^*\mathcal{O}(-(d-n-2)))<n-1,$$

we can repeat the argument in the proof of Lemma 7 again on the triple

$$(U,W,\Phi_1(U,W))\in H^0(\mathbf{P}^n,\mathcal{O}(d_1-1))\times H^0(\mathbf{P}^n,\mathcal{O}(d_2-1))\times H^0(\mathbf{P}^n,\mathcal{O}(k-1))$$

instead of the triple

$$(F_1',F_2',\Phi(F_1',F_2'))\in H^0(\mathbf{P}^n,\mathcal{O}(d_1))\times H^0(\mathbf{P}^n,\mathcal{O}(d_2))\times H^0(\mathbf{P}^n,\mathcal{O}(k)),$$

and using Lemma 7 instead of Lemma 4. After repeating this process for several times, eventually we arrive at the following situation.

Case (1). $d_1\le k$ and $d_2\le k$. There are

$$R_{ij}\in H^0(\mathbf{P}^n,\mathcal{O}(k-d_1))\text{ and }S_{ij}\in H^0(\mathbf{P}^n,\mathcal{O}(k-d_2)),$$

so that both the divisor

$$\Big\{\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}R_{ij}-\frac{\partial(G_0,F_{2,0})}{\partial(Z_i,Z_j)}\cdot 1=0\Big\}$$

and the divisor

$$\Big\{\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}S_{ij}-\frac{\partial(F_{1,0},G_0)}{\partial(Z_i,Z_j)}\cdot 1=0\Big\}$$

on V have weak type $\mu(0)-1$ singularities. Moreover,

$$R_{ij}\equiv R,S_{ij}\equiv S\text{ mod }(F_1,F_2,G)$$

are independent of i,j, because we assume that the deformation $G'=\Phi(F_1',F_2')$ is unique for given F_1',F_2' (the reason is the same as we assume that Φ is linear).

Consider the following linear equation

$$\alpha\frac{\partial F_{1,0}}{\partial Z_i}+\beta\frac{\partial F_{2,0}}{\partial Z_i}=\frac{\partial G_0}{\partial Z_i}-\frac{\partial F_{1,0}}{\partial Z_i}R-\frac{\partial F_{2,0}}{\partial Z_i}S,$$
$$\alpha\frac{\partial F_{1,0}}{\partial Z_j}+\beta\frac{\partial F_{2,0}}{\partial Z_j}=\frac{\partial G_0}{\partial Z_j}-\frac{\partial F_{1,0}}{\partial Z_j}R-\frac{\partial F_{2,0}}{\partial Z_j}S.$$

When we solve this equation, we get

$$\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}\alpha=\frac{\partial(G_0,F_{2,0})}{\partial(Z_i,Z_j)}\cdot 1-\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}R,$$
$$\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}\beta=\frac{\partial(F_{1,0},G_0)}{\partial(Z_i,Z_j)}\cdot 1-\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}S.$$

Hence the divisor

$$\Big\{\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}(\frac{\partial G_0}{\partial Z_i}-\frac{\partial F_{1,0}}{\partial Z_i}R-\frac{\partial F_{2,0}}{\partial Z_i}S)=0\Big\}$$

on V has a weak type $\mu(0)-1$ singularity. For any point $P\in V$, we can choose generic homogeneous coordinates so that

$$\frac{\partial(F_{1,0},F_{2,0})}{\partial(Z_i,Z_j)}\neq 0$$

near P for all $i\neq j$. Then the divisor

$$\Big\{\frac{\partial G}{\partial Z_i}-\frac{\partial F_1}{\partial Z_i}R-\frac{\partial F_2}{\partial Z_i}S=0\Big\}=\Big\{\frac{\partial G_0}{\partial Z_i}-\frac{\partial F_{1,0}}{\partial Z_i}R-\frac{\partial F_{2,0}}{\partial Z_i}S=0\Big\}$$

has a weak type $\mu(0)-1$ singularity in a neighborhood of P. Now let $\{Y_i\}$ be another homogeneous coordinate of $\mathbf{P}^n$. Since

$$\frac{\partial G}{\partial Y_j}-\frac{\partial F_1}{\partial Y_j}R-\frac{\partial F_2}{\partial Y_j}S$$

is a linear combination of expressions

$$\frac{\partial G}{\partial Z_i}-\frac{\partial F_1}{\partial Z_i}R-\frac{\partial F_2}{\partial Z_i}S\quad (i=0,1,\cdots,n)$$

and weak type $\mu(0)-1$ singularity is additive, we conclude that the divisor

$$\{\frac{\partial G}{\partial Y_j}-\frac{\partial F_1}{\partial Y_j}R-\frac{\partial F_2}{\partial Y_j}S=0\}$$

on V has a weak type $\mu(0)-1$ singularity. If

$$\frac{\partial G}{\partial Y_j}-\frac{\partial F_1}{\partial Y_j}R-\frac{\partial F_2}{\partial Y_j}S=0 \quad \mod (F_1,F_2,G)$$

for all j, then

$$\frac{\partial G}{\partial Y_j}-\frac{\partial F_1}{\partial Y_j}R-\frac{\partial F_2}{\partial Y_j}S=0 \quad \mod (F_1,F_2)$$

because it's degree $k-1<k$, and the Euler equation will imply that $G\in(F_1,F_2)$, which is impossible.

Therefore

$$\frac{\partial G}{\partial Y_j}-\frac{\partial F_1}{\partial Y_j}R-\frac{\partial F_2}{\partial Y_j}S\notin(F_1,F_2,G)$$

for some j, that is,

$$\frac{\partial G}{\partial Y_j}-\frac{\partial F_1}{\partial Y_j}R-\frac{\partial F_2}{\partial Y_j}S\not\equiv 0$$

on M. Now we can choose

$$H_1,\cdots,H_{n-1}\in H^0(\mathbf{P}^n,\mathcal{O}(1)),$$

so that H_i generates a linear subspace of dimension $n-1$ and G is not there (in case deg $G = 1$). Then

$$\Big(\frac{\partial G}{\partial Y_j} - \frac{\partial F_1}{\partial Y_j}R - \frac{\partial F_2}{\partial Y_j}S\Big)H_1 H_i \quad (i = 1, 2, \cdots, n-1)$$

will induce $n-1$ linear independent sections of $K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-(d-n-2))$ by Lemma 2. A contradiction.

Case (2). $d_1 \leq k < d_2$. Then Lemma 7 implies that the divisor

$$\Big\{\frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} = 0\Big\}$$

on V has a weak type $\mu(0) - 1$ singularity. The argument in case (1) (take $S = 0$) shows that for some j, the divisor

$$\{\frac{\partial G}{\partial Y_j} - \frac{\partial F_1}{\partial Y_j}R = 0\}$$

on V has a weak type $\mu(0) - 1$ singularity, and it is nontrivial on M. Again we get

$$\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-(d-n-2))) \geq n-1.$$

Case (3). $d_1 > k$ and $d_2 > k$. This time, we conclude that both divisors

$$\Big\{\frac{\partial(F_{1,0}, G_0)}{\partial(Z_i, Z_j)} = 0\Big\} \quad \text{and} \quad \Big\{\frac{\partial(G_0, F_{2,0})}{\partial(Z_i, Z_j)} = 0\Big\}$$

on V have weak type $\mu(0) - 1$ singularities. The argument in case (1) (take $R = S = 0$) shows that for some j, the divisor $\{\frac{\partial G}{\partial Y_j} = 0\}$ on V has a weak type $\mu(0) - 1$ singularity, and it is nontrivial on M. We conclude again that

$$\dim H^0(\tilde{M}, K_{\tilde{M}} \otimes \sigma^* \mathcal{O}(-(d-n-2))) \geq n-1.$$

This completes the proof of Proposition 7.

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