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SHARP UPPER BOUND FOR THE FIRST NON-ZERO NEUMANN EIGENVALUE FOR BOUNDED DOMAINS IN RANK-1 SYMMETRIC SPACES

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Abstract. In this paper, we prove that for a bounded domain Ω in a rank-1 symmetric space, the first non-zero Neumann eigenvalue $\mu_1(\Omega) \leq \mu_1(B(r_1))$ where $B(r_1)$ denotes the geodesic ball of radius r_1 such that

$$vol(\Omega) = vol(B(r_1))$$

and equality holds iff $\Omega = B(r_1)$. This result generalises the works of Szego, Weinberger and Ashbaugh-Benguria for bounded domains in the spaces of constant curvature.

1. Introduction and Statement of theorems

In this paper we study the Neumann eigenvalue problem

$$\begin{aligned} \Delta u &= \mu u \qquad \text{in} \quad \Omega, \\ \nu . u &\equiv 0 \qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded domain in a rank-1 symmetric space, $\partial\Omega$ is the boundary of Ω, ν is the outward normal to Ω and $\nu.u$ denotes the directional derivative of u in the direction ν.

In 1954, Szego [6] proved that for all simply connected domains of given area in $\mathbb{R}^2$, the maximum of the first non-zero Neumann eigenvalue is attained for a ball. Later, Weinberger [7] extended this result for bounded domains in $\mathbb{R}^n$ for all $n \geq 2$.

Recently Ashbaugh and Benguria [1] have studied the problem (1) for a domain contained in a hemisphere of the Euclidean sphere S^n. For such a domain Ω they have proved that $\mu_1(\Omega) \leq \mu_1(r_1)$ where $B(r_1)$ denotes a geodesic ball of radius r_1 such that $vol(\Omega) = vol(B(r_1))$ and the equality holds iff Ω is a geodesic ball. They also show, using the methods of [7], that a similar result is also true for real hyperbolic space $\mathbb{H}^n$.

In this paper, we consider bounded domains in the remaining rank-1 symmetric spaces. If Ω is a domain in a rank-1 symmetric space of compact type, then we have a restriction on the size of the domain Ω viz., that Ω is contained in a geodesic ball of radius $\frac{i(M)}{4}$, where $i(M)$ denotes the injectivity radius of (M, g). We prove the following theorems.

Theorem 1. *Let Ω be a domain contained in a geodesic ball of radius $\frac{i(M)}{4}$ in a rank-1 symmetric space (M^n, ds^2) of compact type, where ds^2 denotes the canonical*

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Riemannian metric on M^n with sectional curvature $1 \leq K_M \leq 4$. Then

$$\mu_1(\Omega) \leq \mu_1(B(r_1)) := \mu_1(r_1)$$

where $B(r_1)$ is a geodesic ball of radius r_1 having the same volume as that of Ω. Further the equality holds iff Ω is a geodesic ball.

Theorem 2. *Let Ω be a bounded domain in a rank-1 symmetric space (M^n, ds^2) of non-compact type, where ds^2 denotes the canonical Riemannian metric on M^n with sectional curvature $-4 \leq K_M \leq -1$. Then*

$$\mu_1(\Omega) \leq \mu_1(B(r_1)) = \mu(r_1)$$

where $B(r_1)$ denotes a geodesic ball of radius r_1 having the same volume as that of Ω. Further the equality holds iff Ω is a geodesic ball.

As mentioned earlier, for the symmetric spaces of constant sectional curvature, the above results have been established in [7] and [1]. See also the concluding remarks in section 5.

The crucial step in the works of [7] and [1] is what has come to be known as the *centre of mass theorem.* In this paper we formulate this in a more geometric and conceptual way and present a simple proof. After decomposing the Laplacian Δ_M in geodesic polar coordinates and identifying the correct test functions, the analytical arguments developed in [7] and [1] carry through.

We refer to [2] and [5] for the basic Riemannian geometry used in this paper.

2. The centre of mass theorem for domains in complete Riemannian manifolds

Let (M, g) be a complete Riemannian manifold. For a point $p \in M$, let us denote by $r(p)$ the convexity radius of (M, g) at p. Let Ω be a domain in (M, g) such that Ω is contained in $B(p, r(p))$ for some $p \in M$. let us denote by $C\Omega$ the convex hull of Ω. Let $\exp_q : T_qM \to M$ be the exponential map and let $X = (x_1, x_2, \ldots, x_n)$ be a system of normal coordinates centred at q. We identify $C\Omega$ with $\exp_q^{-1}(C\Omega)$ for each $q \in C\Omega$. We denote $g_q(X, X)$ as $\| X \|_q^2$ for $X \in T_qM$. Our centre of mass theorem is the following.

Theorem 3. *Let Ω be a bounded domain in (M, g) contained in $B(q_0, r(q_0))$ for some $q_0 \in M$ and let G be a continuous function on $[0, 2r(q_0)]$ which is positive on $(0, 2r(q_0))$. Then there exists a point $p \in C\Omega$ such that*

$$\int_\Omega G(\| X \|_p) X dV = 0$$

where $X = (x_1, x_2, \ldots, x_n)$ is a normal coordinate system centred at p.

Proof. For $q \in C\Omega$, we define

$$v(q) := \int_\Omega G(\| X \|_q) X dV$$

where $X = (x_1, x_2, \ldots, x_n)$ is a geodesic normal coordinate system centred at q.

Now we shall show that the continuous vector field v points inward along the boundary $\partial C\Omega$ of $C\Omega$. Then the theorem follows from the Brouwer's fixed point theorem.

Since $C\Omega$ is convex, it is contained in the half space $H_q := \{X \in T_qM : g(X, \nu(q)) \leq 0\}$ for every $q \in C\Omega$, where $\nu(q)$ denotes the outward normal to

$\partial C\Omega$ at $q \in \partial C\Omega$. This implies that $g(v(q), \nu(q)) < 0$ for all $q \in \partial C\Omega$. Thus v points inward along the boundary of $C\Omega$.

We can find a $\delta > 0$ such that $\exp_q(\delta v(q)) \in C\Omega$ for every $q \in C\Omega$. Then the continuous map $f_v : C\Omega \to C\Omega$ defined by

$$f_v(q) := \exp_q(\delta v(q))$$

has a fixed point $p \in C\Omega$ by the Brouwer's fixed point theorem. Hence $v(p) = 0$. This completes the proof of the theorem. □

Remark. It is clear from the proof that the centre of mass theorem applies to any bounded domain Ω in (M, g) such that $C\Omega$ is properly contained in M.

3. Properties of the first non-zero Neumann eigenvalue for geodesic balls in rank-1 symmetric spaces

Let (M^n, ds^2) denote any one of the following rank-1 symmetric spaces: Complex projective space $C\mathbb{P}^n$, quarternionic projective space $\mathbb{HP}^n$, the Cayley projective plane $Ca\mathbb{P}^2$ or their non-compact duals. Let $\mathbb{K}$ denote $\mathbb{R}$, C, $\mathbb{H}$ or Ca and $k = \mathrm{d}im_{\mathbb{R}}\mathbb{K}$. Throughout out this paper we will use these notations. Let $\mu_1(r_1)$ denote the first non-zero Neumann eigenvalue for a geodesic ball of radius r_1 in (M^n, ds^2).

We begin with the study of Δ_M in geodesic polar coordinates centred at a point $p \in M$.

$$\Delta_M = -\frac{\partial^2}{\partial r^2} - H(r)\frac{\partial}{\partial r} + \Delta_{S(r)}$$

where $H(r)$ denotes the trace of the second fundamental form of the distance sphere $S(r) := S(p, r)$ and $\Delta_{S(r)}$ denotes the Laplacian of $S(r)$.

Now we will describe $H(r)$ and $\Delta_{S(r)}$. Let $v \in T_pM$ be a unit tangent vector and $\gamma_v(r)$ be the geodesic with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. Let us denote by $J(v, r)$ the Riemannian density function along $\gamma_v(r)$. Since (M^n, ds^2) is a rank-1 symmetric space $J(v, r)$ is independent of v and we write it as $J(r)$. We know that, for (M^n, ds^2) of compact type, for $0 \le r < \frac{\pi}{2}$

$$J(r) = \sin^{kn-1} r \cos^{k-1} r$$

and for (M, ds^2) of non-compact type, for all $r \ge 0$

$$J(r) = \sinh^{kn-1} r \cosh^{k-1} r.$$

The trace of the second fundamental form $H(r)$ of $S(r)$ is equal to $J'(r)J^{-1}(r)$. Hence

$$H(r) = (kn - 1)\cot r - (k - 1)\tan r$$

for (M^n, ds^2) of compact type and

$$H(r) = (kn - 1)\coth r + (k - 1)\tanh r$$

for (M^n, ds^2) of non-compact type.

As an illustration, we have for $(C\mathbb{P}^n, ds^2)$, $J(r) = \sin^{2n-1} r \cos r$ and $H(r) = (2n-1)\cot r - \tan r$ and for the quarternionic hyperbolic space $(\mathbb{HH}^n, ds^2)$, $J(r) = \sinh^{4n-1} r \cosh^3 r$ and $H(r) = (4n - 1)\coth r + 3\tanh r$. Note that for $Ca\mathbb{P}^2$ we have $n = 2$. Now we study the first non-zero eigenvalue $\lambda_1(S(r))$ of $\Delta_{S(r)}$.

3.1. (M^n, ds^2) **of compact type.** We have a natural Riemannian submersion

$$\Pi : (S(r), ds^2\,|_{S(r)}) \to (M^{n-1}, \sin^2 r ds^2) \tag{2}$$

with totally geodesic fibres, for the distance sphere $S(r)$ in (M^n, ds^2) with the induced metric $ds^2\,|_{S(r)}$. We always assume that $0 < r < i(M)$. The fibre of Π containing a point $\gamma_v(r) = q \in S(r)$, where $v \in T_pM$ is a unit vector, is $\mathbb{K}.v \cap S(r)$. We can write $\Delta_{S(r)}$ as

$$\Delta_{S(r)} = \frac{1}{\sin^2 r \cos^2 r}\Delta_V + \frac{1}{\sin^2 r}\Delta_H$$

where Δ_V denotes the Laplacian along the fibres of the *canonical fibration* of the unit sphere (S^{kn-1}, ds^2) with totally geodesic fibres S^{k-1} and $\Delta_H := \Delta_{(S^{kn-1},ds^2)} - \Delta_V$. We rewrite $\Delta_{S(r)}$ as

$$\Delta_{S(r)} = \frac{1}{\cos^2 r}\Delta_V + \frac{1}{\sin^2 r}\Delta_{(S^{kn-1},ds^2)}.$$

Then we have

$$\frac{1}{\sin^2 r}\Delta_H\,|_{\Pi^* C^\infty(M^{n-1})} = \Pi^* \Delta_{(M^{n-1},\sin^2 r ds^2)}. \tag{3}$$

By equation (3), all the eigenfunctions of $\Delta_{(M^{n-1},sin^2 r ds^2)}$ are also eigenfunctions of $\Delta_{S(r)}$ with the same eigenvalues. In particular the first non-zero eigenvalue $\frac{2kn}{\sin^2 r}$ of $\Delta_{(M^{n-1},\sin^2 r ds^2)}$ occurs as an eigenvalue of $\Delta_{S(r)}$.

The Euclidean coordinate functions X_i, for $1 \le i \le kn$, are the first non-zero eigenfunctions of $\Delta_{(S^{kn-1},ds^2)}$ corresponding to the first eigenvalue $kn - 1$. Since the fibres are all totally geodesic, these eigenfunctions restricted to the fibres of Π are also eigenfunctions with eigenvalue $k - 1$. Hence we get

$$\Delta_{S(r)} X_i = \left(\frac{kn-1}{\sin^2 r} + \frac{k-1}{\cos^2 r}\right) X_i$$

for $1 \le i \le kn$. Now

$$\left(\frac{kn-1}{\sin^2 r} + \frac{k-1}{\cos^2 r}\right) < \frac{2kn}{\sin^2 r}$$

iff

$$r < \tan^{-1}\left(\sqrt{\frac{kn+1}{k-1}}\right).$$

Hence for $r < \tan^{-1}(\sqrt{\frac{kn+1}{k-1}})$, X_i, for $1 \le i \le kn$ are the first eigenfunctions of $\Delta_{S(r)}$ with eigenvalue $\lambda_1(S(r)) = (\frac{kn-1}{\sin^2 r} + \frac{k-1}{\cos^2 r})$.

We remark that $\lambda_1(S(r))$ is a strictly decreasing function of r for $0 \le \frac{\pi}{4}$. This remark will be used later in section 4.

3.2. (M^n, ds^2) **of non-compact type.** We will denote by $(M^n)^*$ the compact dual of M^n. As in the compact type, here also, we have a natural Riemannian submersion

$$\Pi : (S(r), ds^2\,|_{S(r)}) \to ((M^{n-1})^*, \sinh^2 r ds^2) \tag{4}$$

with totally geodesic fibres, for the distance sphere $S(r) := S(p,r)$ in (M^n, ds^2). For a point $q \in S(r)$, the fibre through the point $q = \gamma_v(r)$, where $v \in T_pM$ is a unit vector, is $\mathbb{K}.v \cap S(r)$. As before we have

$$\Delta_{S(r)} = \frac{-1}{\cosh^2 r}\Delta_V + \frac{1}{\sinh^2 r}\Delta_{(S^{kn-1},ds^2)} \tag{5}$$

and the euclidean coordinate functions X_i's, for $1 \le i \le kn$ are eigen functions of $\Delta_{S(r)}$ with eigenvalue $\lambda_1(S(r)) = (\frac{kn-1}{\sinh^2 r} - \frac{k-1}{\cosh^2 r})$. Now $(\frac{kn-1}{\sinh^2 r} - \frac{k-1}{\cosh^2 r})$ will be the first non-zero eigenvalue of $\Delta_{S(r)}$ so long as

$$(\frac{kn-1}{\sinh^2 r} - \frac{k-1}{\cosh^2 r}) < \frac{2kn}{\sinh^2 r}$$

and this inequality holds for all $r > 0$. Hence $\lambda_1(S(r)) = (\frac{kn-1}{\sinh^2 r} - \frac{k-1}{\cosh^2 r})$ for all $r > 0$. Again we remark that $\lambda_1(S(r))$ is a strictly decreasing function of r for $r > 0$. See also [3] for further study of Laplacians and Riemannian submersions with totally geodesic fibres.

3.3. Now we shall study the first non-zero Neumann eigenvalue $\mu_1(r_1)$. The first non-zero eigenvalue of problem (1) is, by the separation of variables technique, either the second eigenvalue τ_2 of

$$-\frac{1}{J(r)}Q\frac{\partial}{\partial r}(J(r)Q\frac{\partial}{\partial r}f) = \tau f \tag{6}$$

where f is a function defined on $[0, r_1]$ satisfying the boundary conditions $f(0)$ finite and $f'(0) = 0$ or the first eigenvalue μ_1 of

$$-\frac{1}{J(r)}Q\frac{\partial}{\partial r}(J(r)Q\frac{\partial}{\partial r}g) + \lambda_1(S(r))g = \mu g. \tag{7}$$

where g is a function defined on $[0, r_1]$ with boundary conditions $g(0)$ finite and $g'(0) = 0$. We note that $g(0) = 0$ and also that the first eigenvalue of equation (6) is zero. Since g is a first eigenfunction of equation (7) and also that $g(0) = 0$, g does not change sign in $(0, r_1)$. We assume that g is positive in $(0, r_1)$.

Let f and g be the eigenfunctions of equation (6) and equation (7) with eigenvalues τ_2 and μ_1 respectively. Let h be a non-trivial solution of

$$-\frac{1}{J(r)}Q\frac{\partial}{\partial r}(J(r)Q\frac{\partial}{\partial r}h) = \mu_1 h. \tag{8}$$

on $[0, r_1]$. By differentiating the equation (8), we see that h' satisfies equation (7) with the same eigenvalue μ_1. Hence h' and g are proportional. We can assume that $h' = g$. Since f and h satisfy the same equation with eigenvalues τ_2 and μ_1 respectively, we have

$$Q\frac{\partial}{\partial r}(J(r)(h'f - f'h)) = (\tau_2 - \mu_1)fhJ(r). \tag{9}$$

Since f is an eigenfunction corresponding to the second eigenvalue it must change sign in $(0, r_1)$, say at $a \in (0, r_1)$.We may assume that f is positive in $(0, a)$ and $f < 0$ in (a, r_1). Also we have $f'(a) < 0$. Now integrating the equation (9), we get

$$\begin{aligned}(\tau_2 - \mu_1)\int_0^a fhJ(r)dr &= J(r)(h'f - f'h)\mid_0^a \\ &= -J(a)f'(a)h(a)\end{aligned} \tag{10}$$

Since g is positive in $(0, r_1)$ and $\mu_1 h(r_1) = g'(r_1) - H(r_1)g(r_1) < 0$, we get $h(r_1) < 0$. Thus, $h' = g$ and $h(r_1) < 0$ together imply that $h \leq 0$ in $(0, r_1)$. Now from the equation (10), it follows that $\mu_1 < \tau_2$. Thus we have proved that $\mu_1 = \mu_1(r_1)$.

Now we study the properties of the function g and the function $\mu_1(r_1)$. Let us recall that g satisfies

$$Q\frac{\partial}{\partial r}(J(r)Q\frac{\partial}{\partial r}g) = (\lambda_1(S(r)) - \mu_1(r_1))gJ(r) \tag{11}$$

with boundary conditions $g(0) = 0$ and $g'(r_1) = 0$. Define $\Psi(r) := J(r)g'(r)$. Then $\Psi(0) = 0$ and $\Psi(r_1) = 0$ and $\Psi'(r) > 0$ near 0. This implies that Ψ increases from zero in the beginning and then decreases to zero. In particular $(\lambda_1(S(r)) - \mu_1(r_1))$ must change sign at some point $a \in (0, r_1)$ by the equation (11). Since $\lambda_1(S(r))$ is a strictly decreasing function in $(0, r_1)$, $\Psi'(r) < 0$ in $[a, r_1]$. Hence $\Psi(r) > 0$ and $\mu_1(r_1) > \lambda_1(S(r_1))$. Further, since Ψ is positive in $(0, r_1)$, it follows that $g' > 0$ on $(0, r_1)$. Thus we have proved the following

Lemma 1. *$g'(r) > 0$ in $(0, r_1)$ and $\mu_1(r_1) > \lambda_1(S(r_1))$.*

We note that for M of compact type, we have the restriction $0 < r_1 \leq \frac{\pi}{4}$. Using the lemma we prove the following.

Proposition 1. *$\mu_1(r_1)$ is a decreasing function of r_1.*

Proof. We set up the prüfer variables $\rho(r)$ and $\theta(r)$ for a g satisfying the Sturm-Liouville system

$$Q\frac{\partial}{\partial r}(P(r)Q\frac{\partial}{\partial r}g) + Q(r)g = 0$$

in $(0, r_1)$ with boundary conditions $g(0) = 0$ and $g'(r_1) = 0$, where $P = J(r)$ and $Q(r) = (\lambda_1(S(r)) - \mu_1(r_1))J(r)$. The variables $\rho(r)$ and $\theta(r) = \theta(r, \mu_1(r_1))$ are defined as $\rho(r)\cos\theta(r) = P(r)Q\frac{\partial}{\partial r}g(r)$ and $g(r) = \rho(r)\sin\theta(r)$. By Lemma 1 in section 7 of [4] we know that $\theta(r, \lambda)$ is an increasing function of λ for a fixed $r > 0$. By Lemma 1, $\theta(r, \mu_1(r_1)) \in (0, \frac{\pi}{2})$ for $0 < r < r_1 \leq \frac{\pi}{4}$. Now we claim that for $0 < r_1 < r_2 \leq \frac{\pi}{4}$, $\mu_1(r_1) > \mu_1(r_2)$. If not, then $\mu_1(r_1) \leq \mu_1(r_2)$. Hence

$$\frac{\pi}{2} = \theta(r_1, \mu(r_1)) \leq \theta(r_1, \mu_1(r_2)) \in (0, \frac{\pi}{2})$$

which is a contradiction. This completes the proof of the proposition.

Corollary 1. *For (M^n, ds^2) of compact type, we have $\mu_1(r_1) \geq \mu_1(\frac{\pi}{4}) = \lambda_1(M) = 2k(n+1)$ for $0 < r_1 \leq \frac{\pi}{4}$.*

Proof. The function $g(r) = \sin r \cos r$ satisfies the equation (7) with $\mu = 2k(n+1)$.

4. Proof of Theorem 1

In this section (M^n, ds^2) is of compact type. Let g be the first eigenfunction of the equation (7) on $[0, r_1]$. We define a function B on $[0, r_1]$ by,

$$B(r) = (Q\frac{\partial}{\partial r}g)^2 + \lambda_1(S(r))g^2(r).$$

The following lemma is a main ingredient in the proof of Theorem 1.

Lemma 2. *$B' \leq 0$ on $[0, r_1]$ for $0 < r_1 \leq \frac{\pi}{4}$.*

Proof. Following [1], we define

$$q(r) = \sin 2r \frac{g'}{g}.$$

Then

$$\begin{aligned} B(r) &= \left\{q^2(r) + 4\left[(kn-1)\cos^2 r + (k-1)\sin^2 r\right]\right\} \frac{g^2}{\sin^2 2r} \\ &= \left[q^2 + 4k(n-1)\cos^2 r + 4(k-1)\right] \frac{g^2}{\sin^2 2r} \end{aligned}$$

and

$$\begin{aligned} B'(r) &= 2\left[qq' - 2k(n-1)\sin 2r\right] \frac{g^2}{\sin^2 2r} \\ &\quad + \left(q^2 + 4k(n-1)\cos^2 r + 4(k-1)\right)\left(\frac{q - 2\cos 2r}{\sin 2r}\right)\left(\frac{g^2}{\sin^2 2r}\right) \end{aligned}$$

The lemma follows once we prove that $q' \leq 0$ and $0 \leq q \leq 2\cos 2r$ on $[0, r_1]$. Now we prove the sublemma.

Sublemma. $0 \leq q \leq 2\cos 2r$ *and* $q' \leq 0$ *on* $[0, r_1]$.

Proof. We have

$$q' = \sin 2r \frac{g''}{g} + 2\cos 2r \frac{g'}{g} - \sin 2r (\frac{g'}{g})^2. \tag{12}$$

Now substituting for

$$g'' = -H(r)g' + (\lambda_1(S(r)) - \mu_1(r_1))g$$

in equation (12), we get

$$q' = -(\mu_1(r_1) - \lambda_1(S(r)))\sin 2r - H(r)q + 2q\cot 2r - \frac{q^2}{\sin 2r}. \tag{13}$$

We rewrite equation (13) as

$$\begin{aligned} q' &= -(\mu_1(r_1) - \lambda_1(M) + 4)\sin 2r \\ &\quad + \frac{(2\cos 2r - q)\left[q + (k(n+1) - 2)\cos 2r + k(n-1)\right]}{\sin 2r}. \end{aligned} \tag{14}$$

From the definition, we have $q(0) = 2$ and by an easy computation using the equation (14) we see that $q'(0) = 0$. By differentiating the equation (14) and evaluating at $t = 0$, using Lemma 1, we get that $q''(0) \leq -8$. Further $q(r_1) = 0$ and using Lemma 1 and the equation (14) we see that $q'(r_1) < 0$. □

Now we prove that $q \leq 2\cos 2r$ on $[0, r_1]$ using a comparison theorem (see Theorem 7, p. 267 of [4]). Let $F(r, q)$ denote the right hand side of the equation (14). From the initial values q, q' and q'' at $t = 0$, it follows that $q(r) \leq 2\cos 2r$ for small values of r, say for $r \in [0, a]$, for some $a < r_1$. Now if $q \geq 2\cos 2r$ on $[a, a+\epsilon)$ for some $\epsilon > 0$, we would have, by the equation (14), for $r \in [a, a+\epsilon)$

$$\begin{aligned} q'(r) &\leq -(\mu_1(r_1) - \lambda_1(M) + 4)\sin 2r \\ &< -4\sin 2r \\ &= F(r, \cos 2r). \end{aligned}$$

The inequality in the second step above follows from Lemma 1. Now by the comparison theorem cited above, we conclude that $q \le 2\cos 2r$ in $[a, a+\epsilon)$. Thus we have proved that $q \le 2\cos 2r$ on $[0, r_1]$.

To prove that $q' \le 0$ on $[0, r_1]$, we rewrite the equation (14) as

$$\begin{aligned} q' = &-\mu_1 \sin 2r + \frac{1}{\sin 2r}\left[2k(n+1) - 4 - q^2 - k(n-1)q\right] \\ &+ \cot 2r\left[2k(n-1) - (k(n+1)-4)q\right] \end{aligned}$$

i.e.,

$$\begin{aligned} q' = &-\mu_1 \sin 2r + \frac{1}{\sin 2r}\left[2k(n-1) + 4 - q^2 - k(n-1)q\right] \\ &+ \cot 2r\left[2k(n+1) - 8 - (k(n+1)-4)q\right] + 4(k-2)\frac{(1-\cos 2r)}{\sin 2r}. \end{aligned}$$

Since $q \le 2\cos 2r$ and $k \ge 2$, $2k(n-1) + 4 - q^2 - k(n-1)q \ge 0$ and $2k(n+1) - 8 - (k(n+1)-4)q \ge 0$. Hence the right hand side $F(r, q)$ of the above equation is convex in the variable r, as $-\sin 2r$, $\frac{1}{\sin 2r}$, $\cot 2r$ and $\tan r$ are convex functions for $0 < r \le \frac{\pi}{4}$. As in [1], we conclude that $q' \le 0$ on $[0, r_1]$ for $0 < r_1 \le \frac{\pi}{4}$.

Now to the proof of the theorem. We extend g to a function G on $[0, \frac{\pi}{4}]$ by

$$G(r) = \begin{cases} g(r) & \text{for } 0 \le r \le r_1, \\ g(r_1) & \text{for } r_1 \le r \le \frac{\pi}{4}. \end{cases}$$

Let Ω be a domain in M contained in a ball of radius $\frac{\pi}{8}$. Now we apply the centre of mass theorem with weight function $\frac{G(r)}{r}$ to the domain Ω. Let $p \in C\Omega$ be the centre of mass of Ω. Then, for normal coordinates $(X_1, X_2, \ldots, X_{kn})$ centred at p,

$$\int_\Omega \frac{G(r)}{r} X_i dV = 0$$

for $1 \le i \le kn$. Now from the Rayleigh-Ritz inequality, we have

$$\mu_1(\Omega) \le \frac{\int_\Omega \mid \nabla(\frac{GX_i}{r}) \mid^2 dV}{\int_\Omega (\frac{GX_i}{r})^2 dV}$$

i.e.,

$$\mu_1(\Omega) \int_\Omega (\frac{GX_i}{r})^2 dV \le \int_\Omega \mid \nabla(\frac{GX_i}{r}) \mid^2 dV.$$

By summing over $i = 1, 2, \ldots, kn$ we get

$$\mu_1(\Omega) \le \frac{\sum_{i=1}^{kn} \int_\Omega \mid \nabla(\frac{GX_i}{r}) \mid^2 dV}{\int_\Omega G^2 dV}. \tag{15}$$

By applying the divergence theorem to the terms in the numerator of the right hand side of the equation (15), we get

$$\mu_1(\Omega) \le \frac{\int_\Omega ({G'}^2 + \lambda_1(S(r))G^2) dV}{\int_\Omega G^2 dV}.$$

We denote the function ${G'}^2 + \lambda_1(S(r))G^2$ also by B on $[0, \frac{\pi}{4}]$. By Lemma 2, B is a decreasing function on $[0, r_1]$ and since $\lambda_1(S(r))$ is a decreasing function on $[r_1, \frac{\pi}{4}]$, we see that B is a decreasing function on $[0, \frac{\pi}{4}]$. Also G is an increasing function on $[0, \frac{\pi}{4}]$. Following [7], we have

$$\begin{aligned}\int_\Omega BdV &= \int_{\Omega\cap B(r_1)} BdV + \int_{\Omega\setminus\Omega\cap B(r_1)} BdV \\ &\le \int_{\Omega\cap B(r_1)} BdV + B(r_1)\int_{\Omega\setminus\Omega\cap B(r_1)} dV\end{aligned}$$

and

$$\int_{B(r_1)} BdV = \int_{\Omega\cap B(r_1)} BdV + \int_{B(r_1)\setminus\Omega\cap B(r_1)} BdV$$

i.e.,

$$\int_{\Omega\cap B(r_1)} BdV = \int_{B(r_1)} BdV - \int_{B(r_1)\setminus\Omega\cap B(r_1)} BdV.$$

This implies that

$$\int_\Omega BdV \le \int_{B(r_1)} BdV - \int_{B(r_1)\setminus\Omega\cap B(r_1)} BdV + B(r_1)\int_{B(r_1)\setminus\Omega\cap B(r_1)} dV.$$

Since $vol(B(r_1)\setminus\Omega\cap B(r_1)) = vol(\Omega\setminus\Omega\cap B(r_1))$ and B is decreasing,

$$\int_\Omega BdV \le \int_{B(r_1)} BdV$$

By similar arguments we can prove that

$$\int_\Omega G^2 dV \ge \int_{B(r_1)} G^2 dV.$$

Hence $\mu_1(\Omega) \le \mu_1(r_1)$ and equality holds iff $\Omega = B(p, r_1)$.

5. Proof of Theorem 2

In this section (M^n, ds^2) is of non-compact type. Let $\mu_1(r_1)$ denote the first non-zero Neumann eigenvalue for the geodesic ball of radius r_1 for $r_1 > 0$. Let g be the eigenfunction satisfying equation (7) on $[0, r_1]$ with eigenvalue $\mu_1(r_1)$. i.e.,

$$-g'' - ((kn-1)\coth r + (k-1)\tanh r)\, g' + \left(\frac{kn-1}{\sinh^2 r} - \frac{k-1}{\cosh^2 r}\right) g = \mu_1(r_1) g$$

with the boundary conditions $g(0) = 0$ and $g'(r_1) = 0$. We define a function

$$B(r) = (Q\frac{\partial}{\partial r} g)^2 + \lambda_1(S(r)) g^2(r).$$

Now we verify that B is a decreasing function on $[0, r_1]$.

$$\begin{aligned}B'(r) &= 2g'g'' + 2gg'\lambda_1(S(r)) \\ &\quad - 2g^2\left[(kn-1)\frac{\cosh r}{\sinh^3 r} - (k-1)\frac{\sinh r}{\cosh^3 r}\right].\end{aligned} \tag{16}$$

Now, by substituting for g'' in equation (16) we get,

$$\begin{aligned}\frac{1}{2}B'(r) = &-((kn-1)\coth r + (k-1)\tanh r)\,(g')^2\\ &-\left[(kn-1)\frac{\cosh r}{\sinh^3 r} - (k-1)\frac{\sinh r}{\cosh^3 r}\right] g^2\\ &+2gg'\left[\frac{kn-1}{\sinh^2 r} - \frac{k-1}{\cosh^2 r}\right] - \mu_1(r_1)gg'.\end{aligned}$$

Now by an easy computation, we get

$$\begin{aligned}\frac{1}{2}B'(r) = &-\frac{k(n-1)}{\sinh^3 r}\left[(g'\sinh r - g)^2\cosh r + 2gg'(\cosh r - 1)\sinh r\right]\\ &-\frac{2(k-1)}{\sinh^3 2r}\left[(g'\sinh 2r - 2g)^2\cosh 2r + 4gg'(\cosh 2r - 1)\sinh 2r\right]\\ &-\mu_1(r_1)gg'\\ \le\,& 0 \qquad \text{by Lemma 1.}\end{aligned}$$

Let Ω be a bounded domain in (M^n, ds^2). Let $B(r_1)$ be a geodesic ball of radius r_1 in M such that $vol(\Omega) = vol(B(r_1))$. We extend the function g to a function G on $[0,\infty)$ by

$$G(r) = \begin{cases} g(r) & \text{for } 0 \le r \le r_1,\\ g(r_1) & \text{for } r_1 \le r < \infty.\end{cases}$$

Now we apply the centre of mass theorem with the weight function $\frac{G(r)}{r}$ to the domain Ω. Let $p \in C\Omega$ be the centre of mass of Ω. Then, for normal coordinates $(X_1, X_2, \ldots, X_{kn})$ centred at p,

$$\int_\Omega \frac{G(r)}{r} X_i dV = 0$$

for $1 \le i \le kn$. By the Rayleigh-Ritz inequality, we have

$$\mu_1(\Omega) \le \frac{\int_\Omega \mid \nabla(\frac{GX_i}{r}) \mid^2 dV}{\int_\Omega (\frac{GX_i}{r})^2 dV}$$

i.e.,

$$\mu_1(\Omega)\int_\Omega (\frac{GX_i}{r})^2 dV \le \int_\Omega \mid \nabla(\frac{GX_i}{r}) \mid^2 dV.$$

By summing over $i = 1, 2, 3, \ldots, kn$, we get

$$\mu_1(\Omega) \le \frac{\sum_{i=1}^{kn}\int_\Omega \mid \nabla(\frac{GX_i}{r}) \mid^2 dV}{\int_\Omega G^2 dV}. \tag{17}$$

By applying the divergence theorem to the terms in the numerator of the right hand side of the equation (17), we get

$$\mu_1(\Omega) \le \frac{\int_\Omega \left((G')^2 + \lambda_1(S(r))G^2\right) dV}{\int_\Omega G^2 dV}.$$

We denote the function $(G')^2 + \lambda_1(S(r))G^2$ also by B on $[0,\infty)$. Since $B' \le 0$, B is a decreasing function for all $r > 0$. Also G is an increasing function for all $r > 0$. As in section 4, we see that $\mu_1(\Omega) \le \mu_1(r_1)$ and equality holds iff $\Omega = B(p, r_1)$.

Concluding Remarks.

1. Our proof of Theorem 1 when applied to (S^n, ds^2) gives the result for a domain contained in a geodesic ball of radius $\frac{\pi}{4}$. A *reflection argument* developed in [1], then shows that the theorem is true for a domain contained in a hemisphere of S^n. But this reflection argument can not be applied to the other symmetric spaces of compact type.
2. The improvement of the size of the domain Ω in Theorem 1 depends on the location of the centre of mass of Ω.
3. In their proof of Theorem 1 for the case of (S^n, ds^2), Ashbaugh and Benguria [1] have used *rearrangement* of the functions B and G. As we have shown, this is not needed.

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