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ON BAIRE-1/4 FUNCTIONS

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Abstract. We give descriptions of the spaces $D(K)$ (i.e. the space of differences of bounded semicontinuous functions on K) and especially of $B_{1/4}(K)$ (defined by Haydon, Odell and Rosenthal) as well as for the norms which are defined on them. For example, it is proved that a bounded function on a metric space K belongs to $B_{1/4}(K)$ if and only if the ω^{th}-oscillation, $\text{osc}_\omega f$, of f is bounded and in this case $\|f\|_{1/4} = \| \, |f| + \widetilde{\text{osc}}_\omega f \|_\infty$. Also, we classify $B_{1/4}(K)$ into a decreasing family $(S_\xi(K))_{1 \leq \xi < \omega_1}$ of Banach spaces whose intersection is equal to $D(K)$ and $S_1(K) = B_{1/4}(K)$. These spaces are characterized by spreading models of order ξ equivalent to the summing basis of c_0, and for every function f in $S_\xi(K)$ it is valid that $\text{osc}_{\omega^\xi} f$ is bounded. Finally, using the notion of null-coefficient of order ξ sequence, we characterize the Baire-1 functions not belonging to $S_\xi(K)$.

Introduction

In recent years the study of the first Baire class, $B_1(K)$, of bounded functions on a metric space K led to the definition of interesting subclasses ([H-O-R], [K-L], [F1]). The study of these subclasses revealed significant properties of their elements ([C-M-R], [R2], [F1], [F2]) and provided applications, such as the c_0-dichotomy theorem of Rosenthal ([R1]).

Here we study some subclasses of $D(K)$, and especially $B_{1/4}(K)$, of $B_1(K)$. By $D(K)$ is denoted the class of all functions on K which are differences of bounded semicontinuous functions. A classical result of Baire yields that $f \in D(K)$ if and only if there exists a sequence (f_n) of continuous functions on K satisfying

$$\sup_{x \in K} \sum_n |f_n(x)| < \infty \text{ and } f = \sum_n f_n. \tag{1}$$

The class $D(K)$ is a Banach algebra with respect to the $\|\cdot\|_D$-norm defined as

$$\|f\|_D = \inf \left\{ \sup_{x \in K} \sum_n |f_n(x)| : (f_n) \subseteq C(K) \text{ satisfying (1)} \right\}.$$

The subclass $B_{1/4}(K)$ was first defined in [H-O-R] as follows:

$$B_{1/4}(K) = \{ f : K \to \mathbf{R} : \text{there exists } (F_n) \subseteq D(K) \text{ such that } \|F_n - f\|_\infty \to 0 \text{ and } \sup_n \|F_n\|_D < \infty \}.$$

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This class is a Banach algebra with respect to the $\|\cdot\|_{1/4}$-norm, given by

$$\|f\|_{1/4} = \inf\left\{\sup_n \|F_n\|_D : (F_n) \subseteq D(K) \text{ and } \|F_n - f\|_\infty \to 0\right\}.$$

In the first section we describe the precise connection between the summing basis (s_n) of c_0 and the normed space $(D(K), \|\cdot\|_D)$; so it is proved in Proposition 1.1 that $f \in D(K)$ if and only if there is a sequence (f_n) of continuous functions on K so that $f_n \to f$ pointwise and there is $C > 0$ such that

$$\left\|\sum_{i=1}^{k} \lambda_i f_{n_i}\right\|_\infty \leq C \left\|\sum_{i=1}^{k} \lambda_i s_i\right\| \tag{2}$$

for every $k, n_1, \ldots, n_k \in \mathbf{N}$ and scalars $\lambda_1, \ldots, \lambda_k$.

If this occurs then

$$\|f\|_D = \inf\left\{\ C > 0 : \text{ there exists } (f_n) \subseteq C(K) \text{ satisfying (2) }\right\}.$$

Since for every sequence of continuous functions defined on a compact metric space K and converging pointwise to a discontinuous function, there exists a subsequence (f_n) and $\mu > 0$ such that

$$\mu \left\|\sum_{i=1}^{k} \lambda_i s_i\right\| \leq \left\|\sum_{i=1}^{k} \lambda_i f_{n_i}\right\|_\infty \tag{3}$$

for $k, n_1, \ldots, n_k \in \mathbf{N}$ and scalars $\lambda_1, \ldots, \lambda_k$ ([H-O-R], [R1]), it follows that the functions in $D(K) \setminus C(K)$ (K compact) are characterized as pointwise limits of sequences of continuous functions equivalent to the summing basis of c_0 (Remark 1.2).

In the case of $B_{1/4}(K)$, where K is a compact metric space, the functions are characterized as pointwise limits of sequences of continuous functions on K with a property weaker than (2), namely one for which the inequality (2) is valid only for $(n_1, \ldots, n_k)$ in the Schreier family $\mathcal{F}_1$ (Theorem 2.1). Moreover, if we set

$$\|f\|_s^1 = \inf\Big\{\ C > 0 : \text{there exists } (f_n) \subseteq C(K) \text{ such that } f_n \to f \text{ pointwise and } \Big\|\textstyle\sum_{i=1}^k \lambda_i f_{n_i}\Big\|_\infty \leq C \Big\|\textstyle\sum_{i=1}^k \lambda_i s_i\Big\| \text{ for every } (n_1, \ldots, n_k) \in \mathcal{F}_1 \text{ and scalars } \lambda_1, \ldots, \lambda_k \Big\},$$

then $\|\cdot\|_s^1$ is a norm on $B_{1/4}(K)$ equivalent to the norm $\|\cdot\|_{1/4}$. This answers in the affirmative a question raised by Haydon, Odell and Rosenthal in [H-O-R]. From this result and (3) we have the characterization of functions in $B_{1/4}(K) \setminus C(K)$ (K compact) as pointwise limits of sequences of continuous functions generating spreading models equivalent to the summing basis of c_0.

More generally, we define analogously the spaces $S_\xi(K)$ and the norms $\|\cdot\|_s^\xi$ on them, employing the higher order Schreier family $\mathcal{F}_\xi$, for $1 \leq \xi < \omega_1$, as defined by Alspach and Argyros ([A-A]). According to Proposition 3.4, $\big(S_\xi(K), \|\cdot\|_s^\xi\big)$ are Banach spaces, which, for separable metric spaces K, constitute a decreasing hierarchy whose intersection is equal to $D(K)$ (Theorem 3.8) and of course $S_1(K) = B_{1/4}(K)$. We further provide alternative descriptions of the spaces $S_\xi(K)$, $1 \leq \xi < \omega_1$, and characterize the Baire-1 functions not belonging to $S_\xi(K)$ (Theorem 3.11), employing the notion of a null-coefficient of order ξ sequence, defined in [F2].

Because of Mazur's theorem, $S_\xi(K)$ is actually a Banach space invariant. That is, if X is a separable Banach space, $x^{**} \in X^{**} \setminus X$, and $K = Ba(X^*, w^*)$, then if

$f = x^{**}|K$, $f \in S_\xi(K)$ if and only if there exists a sequence (x_n) in X such that (x_n) generates a spreading model of order ξ equivalent to (s_n) and converges in the w^*-topology to f. Moreover, then

$$|f|_s^\xi = \inf\{C > 0 : \text{there exists } (x_n) \subset X \text{ and such that } x_n \xrightarrow{w^*} f \text{ and } \|\textstyle\sum_{i=1}^k \lambda_i x_{n_i}\|_\infty \le C\|\sum_{i=1}^k \lambda_i s_i\| \text{ for every } (n_i, \dots, n_k) \in \mathcal{F}_\xi \text{ and scalars } \lambda_1, \dots, \lambda_k\}.$$

A nice relation between the space $(B_{1/4}(K), \|\cdot\|_{1/4})$ and the transfinite oscillations of a function is given in Theorem 2.9. Rosenthal in [R1] defined for every function f the α^{th}-oscillation, $\text{osc}_\alpha f$, of f for every ordinal α (cf. Definition 2.5).In [R2] the author proved the following structural result for $D(K)$: Let f be a real bounded function on an infinite metric space K. Then $f \in D(K)$ if and only if there exist an ordinal α such that $\text{osc}_\alpha f$ is bounded and $\text{osc}_\alpha f = \text{osc}_\beta f$ for all $\beta > \alpha$. Letting τ be the least such α, then

$$\|f\|_D = \|\,|f| + \text{osc}_\tau f\|_\infty \text{ for all } f \in D(K).$$

We prove an analogous structural result for the case of $B_{1/4}(K)$. Precisely, we have the following theorem: Let f be a real bounded function on a metric space K. Then $f \in B_{1/4}(K)$ if and only if $\text{osc}_\omega f$ is bounded. In this case

$$\|f\|_{1/4} = \|\,|f| + \widetilde{\text{osc}}_\omega f\|_\infty \text{ for } f \in B_{1/4}(K).$$

According to the principal result in [F2], $\text{osc}_{\omega^\xi} f$ is bounded for every function f in $S_\xi(K)$ and every ordinal ξ. It is an open problem whether the functions in $S_\xi(K)$ are characterized by this property.

1. Differences of Bounded Semicontinuous Functions

Let K be a metric space. We denote by $C(K)$ the class of continuous functions on K and by $B_1(K)$ the space of bounded first Baire class functions on K (i.e. the pointwise limits of uniformly bounded sequences of continuous functions).

An important subclass of $B_1(K)$ is the class of differences of bounded semicontinuous functions on K, denoted by $D(K)$. It is easy to see that

$$D(K) = \Big\{ f \in B_1(K) : f = u - v, \text{ where } u, v \ge 0 \text{ are bounded and lower semicontinuous functions} \Big\}.$$

The class $D(K)$ is a Banach algebra with respect to the norm $\|\cdot\|_D$, defined as follows:

$$\|f\|_D = \inf \Big\{ \|u + v\|_\infty : f = u - v \text{ for } u, v \ge 0, \text{ bounded and lower semicontinuous functions} \Big\}.$$

This infimum is attained according to [R1]. A result of Baire gives that

$$D(K) = \Big\{ f \in B_1(K) : \text{there exists } (f_n) \text{ in } C(K) \text{ such that } f = \textstyle\sum_n f_n \text{ pointwise and } \|\sum_n |f_n|\|_\infty < \infty \Big\}$$

and it follows that

$$\|f\|_D = \inf\left\{ \left\|\sum_n |f_n|\right\|_\infty : (f_n) \subseteq C(K) \text{ and } f = \sum_n f_n \text{ pointwise}\right\}$$

for every $f \in D(K)$ (see [R2]). It is easy to see that $\|f\|_\infty \leq \|f\|_D$ for every $f \in D(K)$ but the two norms are not equivalent in general.

In the following proposition we give the fundamental connection between the summing basis (s_n) of c_0 and the functions in $D(K)$, as well as between (s_n) and the norm $\|\cdot\|_D$.

1.1. **Proposition.** *Let K be a metric space. Then*

$$D(K) = \Big\{ f \in B_1(K)\colon \text{there exists } (f_n) \text{ in } C(K) \text{ and } C > 0 \text{ so that } f_n \to f \text{ pointwise and } \|\textstyle\sum_{i=1}^n \lambda_i f_i\|_\infty \leq C\, \|\sum_{i=1}^n \lambda_i s_i\| \text{ for all } n \in \mathbf{N} \text{ and scalars } \lambda_1, \ldots, \lambda_n \Big\},$$

where (s_n) is the summing basis of c_0. Also, for every $f \in D(K)$,

$$\|f\|_D = \|f\|_s = \inf\Big\{ C > 0\colon \text{there exists } (f_n) \subseteq C(K) \text{ such that } f_n \to f \text{ pointwise and } \|\textstyle\sum_{i=1}^n \lambda_i f_i\|_\infty \leq C\, \|\sum_{i=1}^n \lambda_i s_i\| \text{ for all } n \in \mathbf{N} \text{ and scalars } \lambda_1, \ldots, \lambda_n \Big\}.$$

Proof. If $f \in D(K)$ then there exists a sequence $(g_n)_{n=1}^\infty$ in $C(K)$ such that $f = \sum_{n=1}^\infty g_n$ and $C = \|\sum_n |g_n|\|_\infty < \infty$. Set $f_n = \sum_{i=1}^n g_i$ for every $n \in \mathbf{N}$. Of course, $f_n \to f$ pointwise and

$$\left\|\sum_{i=1}^n \lambda_i f_i\right\|_\infty = \left\|\sum_{i=1}^n (\lambda_i + \cdots + \lambda_n) g_i\right\|_\infty \leq \left\|\sum_{i=1}^n |g_i|\right\|_\infty \cdot \left\|\sum_{i=1}^n \lambda_i s_i\right\| \leq C \cdot \left\|\sum_{i=1}^n \lambda_i s_i\right\|.$$

Hence, $\|f\|_s \leq \|f\|_D$ for every $f \in D(K)$.

On the other hand, if there exist (f_n) in $C(K)$ and $C > 0$ such that $f_n \to f$ pointwise and $\|\sum_{i=1}^n \lambda_i f_i\|_\infty \leq C\|\sum_{i=1}^n \lambda_i s_i\|$ for every $n \in \mathbf{N}$ and scalars $\lambda_1, \ldots, \lambda_n$, then if we set $g_0 = 0$ and $g_n = f_n - f_{n-1}$ for every $n \in \mathbf{N}$, we have that $\sum_{n=1}^\infty g_n = f$. Also, for $x \in K$ and $n \in \mathbf{N}$,

$$\begin{aligned} \sum_{i=1}^n |g_i|(x) &= \sum_{i=1}^n |f_i - f_{i-1}|(x) = \sum_{i=1}^n \varepsilon_i (f_i - f_{i-1})(x) \\ &= \left|\sum_{i=1}^n (\varepsilon_i - \varepsilon_{i+1}) f_i\right|(x) \leq C, \end{aligned}$$

where $\varepsilon_i \in \{-1, 1\}$ so that $\varepsilon_i (f_i - f_{i-1})(x) \geq 0$ for every $i = 1, \ldots, n$ and $\varepsilon_{n+1} = 0$. Hence, we have that $\|f\|_D \leq \|f\|_s$ for every $f \in D(K)$. □

1.2. *Remark.* It is known ([H-O-R], [R1]) that, for a compact metric space K, every bounded sequence (f_n) in $C(K)$ converging pointwise to a discontinuous function f has a basic subsequence (g_n) which dominates the summing basis (s_n) of c_0, i.e. there exists $\mu > 0$ such that $\mu\|\sum_{i=1}^n \lambda_i s_i\| \leq \|\sum_{i=1}^n \lambda_i g_i\|_\infty$ for every $n \in \mathbf{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$. Hence, for a compact metric space K,

$$D(K) \setminus C(K) = \Big\{ f : K \to \mathbf{R} : \text{there exists } (f_n) \subseteq C(K) \text{ such that } f_n \to f \text{ pointwise and } (f_n) \text{ is equivalent to } (s_n) \Big\}.$$

This result has been proved in [R1] also. Using Mazur's theorem we have that every uniformly bounded sequence (f_n) converging pointwise to a function f in $D(K) \setminus C(K)$ has a convex block subsequence equivalent to (s_n).

2. Baire-1/4 Functions

As we mentioned before, the supremum norm is not equivalent, in general, to the $\|\cdot\|_D$-norm in $D(K)$. The closure of $D(K)$ in $(B_1(K), \|\cdot\|_\infty)$ has been denoted by $B_{1/2}(K)$ in [H-O-R]. In the same paper the authors defined the subclass $B_{1/4}(K)$ of $B_1(K)$ as follows:

$$B_{1/4}(K) = \Big\{ f \in B_1(K) : \text{there exists } (F_n) \subseteq D(K) \text{ such that } \|F_n - f\|_\infty \to 0 \text{ and } \sup_n \|F_n\|_D < \infty \Big\}.$$

The space $B_{1/4}(K)$ is complete with respect to the norm

$$\|f\|_{1/4} = \inf \left\{ \sup_n \|F_n\|_D : (F_n) \subseteq D(K) \text{ and } \|F_n - f\|_\infty \to 0 \right\}.$$

In the following theorem we will give a characterization of $B_{1/4}(K)$ and we will define the $\|\cdot\|_s^1$-norm on it, in analogy to $D(K)$ (Proposition 1.1). We will prove that this norm is equivalent to the $\|\cdot\|_{1/4}$-norm answering affirmatively the question raised by Haydon, Odell and Rosenthal in [H-O-R]. The techniques of this proof have been employed before in [F1]. The additional work here is to establish the relation between the norms. For completeness we give the proof in detail. We will use the Schreier family $\mathcal{F}_1$ which is:

$$\mathcal{F}_1 = \Big\{ (n_1, \dots, n_k) : k < n_1 < \dots < n_k \in \mathbf{N} \Big\}.$$

2.1. **Theorem.** *Let K be a compact metric space. Then*

$$B_{1/4}(K) = \Big\{ f \in B_1(K) : \textit{there exists } (f_n) \textit{ in } C(K) \textit{ and } C > 0 \textit{ so that } f_n \to f \textit{ pointwise and } \Big\| \sum_{i=1}^k \lambda_i f_{n_i} \Big\|_\infty \le C \Big\| \sum_{i=1}^k \lambda_i s_i \Big\| \textit{ for every } (n_1, \dots, n_k) \in \mathcal{F}_1 \textit{ and scalars } \lambda_1, \dots, \lambda_k \Big\}.$$

Also, defining for $f \in B_{1/4}(K)$

$$\|f\|_s^1 = \inf \Big\{ C > 0\text{: } \textit{there exists } (f_n) \textit{ in } C(K) \textit{ such that } f_n \to f \textit{ pointwise and } \Big\| \sum_{i=1}^k \lambda_i f_{n_i} \Big\|_\infty \le C \Big\| \sum_{i=1}^k \lambda_i s_i \Big\| \textit{ for every } (n_1, \dots, n_k) \in \mathcal{F}_1 \textit{ and scalars } \lambda_1, \dots, \lambda_k \Big\},$$

$\|\cdot\|_s^1$ is a norm on $B_{1/4}(K)$ equivalent to the norm $\|\cdot\|_{1/4}$. Moreover,

$$\|f\|_s^1 \le \|f\|_{1/4} \le 4\|f\|_s^1 \textit{ for every } f \in B_{1/4}(K).$$

Proof. Let $f \in B_{1/4}(K)$. According to the definition of $(B_{1/4}(K), \|\cdot\|_{1/4})$, for every $\delta > 0$ there exists a sequence (F_m) in $D(K)$ so that $\|F_m - f\|_\infty \to 0$ and $\sup_m \|F_m\|_D < \|f\|_{1/4} + \delta$. Let $M = \|f\|_{1/4} + \delta$ and (ϵ_m) a decreasing sequence of positive numbers such that $\epsilon_m < \frac{\delta}{2m}$ and $\sum_{i=m+1}^\infty \epsilon_i < \epsilon_m$ for every $m \in \mathbf{N}$. We can assume that $\|F_{m+1} - F_m\|_\infty < \epsilon_{m+1}$ for every $m \in \mathbf{N}$. Hence, for every $m \in \mathbf{N}$ there exists a sequence $(g_n^m)_{n=1}^\infty \subseteq C(K)$ converging pointwise to $F_{m+1} - F_m$ and $\|g_n^m\|_\infty < \epsilon_{m+1}$ for all $n \in \mathbf{N}$.

Since $F_1 \in D(K)$, by Proposition 1.1, there exists a sequence (f_n^1) in $C(K)$ converging pointwise to F_1 and satisfying

$$\Big\| \sum_{i=1}^k \lambda_i f_i^1 \Big\|_\infty \le M \Big\| \sum_{i=1}^k \lambda_i s_i \Big\|$$

for all $k \in \mathbf{N}$ and scalars $\lambda_1, \dots, \lambda_k$. The sequence $(f_n^1 + g_n^1)$ converges pointwise to F_2. Using Mazur's theorem and the fact that $F_2 \in D(K)$, we can find convex block subsequences $(f_n^{1,2}), (g_n^{1,2})$ of $(f_n^1), (g_n^1)$ respectively such that if $f_n^2 = f_n^{1,2} + g_n^{1,2}$ for every $n \in \mathbf{N}$ then $f_n^2 \to F_2$ pointwise and

$$\left\| \sum_{i=1}^k \lambda_i f_i^2 \right\|_\infty \leq M \left\| \sum_{i=1}^k \lambda_i s_i \right\|$$

for every $k \in \mathbf{N}$ and scalars $\lambda_1, \dots, \lambda_k$. Now, since $f_n^2 + g_n^2$ converges to F_3, there exist convex block subsequences $(f_n^{2,3}), (g_n^{2,3})$ of $(f_n^2), (g_n^2)$ respectively, such that if $f_n^3 = f_n^{2,3} + g_n^{2,3}$ for every $n \in \mathbf{N}$ then $f_n^3 \to F_3$ pointwise and

$$\left\| \sum_{i=1}^k \lambda_i f_i^3 \right\|_\infty \leq M \left\| \sum_{i=1}^k \lambda_i s_i \right\|$$

for every $k \in \mathbf{N}$ and scalars $\lambda_1, \dots, \lambda_k$. Let $(f_n^{1,2,3}), (g_n^{1,2,3})$ be the convex block subsequences of $(f_n^{1,2})$ and $(g_n^{1,2})$ respectively, such that $f_n^{2,3} = f_n^{1,2,3} + g_n^{1,2,3}$ for every $n \in \mathbf{N}$. Hence $f_n^3 = f_n^{1,2,3} + g_n^{1,2,3} + g_n^{2,3}$ for every $n \in \mathbf{N}$. We continue in the obvious way to construct $f_n^{m,\dots,k}$ and $g_n^{m,\dots,k}$ for every $m, k, n \in \mathbf{N}$ with $m \leq k$, so that $(g_n^{m,\dots,k}), (f_n^{m,\dots,k})$ to be convex block subsequences of $(g_n^{m,\dots,l}), (f_n^{m,\dots,l})$ respectively for every $m, l, k \in \mathbf{N}$ with $m \leq l \leq k$ and

$$(*) \qquad f_n^{m,\dots,k} = f_n^{m-1,m,\dots,k} + g_n^{m-1,m,\dots,k}$$

for every $n, k, m \in \mathbf{N}$ with $1 < m \leq k$. Also, for every $m \in \mathbf{N}$, we construct the sequence $(f_n^m)_{n=1}^\infty$ converging pointwise to F_m and

$$(**) \qquad \left\| \sum_{i=1}^k \lambda_i f_i^m \right\|_\infty \leq M \left\| \sum_{i=1}^k \lambda_i s_i \right\|$$

for every $k \in \mathbf{N}$ and scalars $\lambda_1, \dots, \lambda_k$. Finally, we set

$$h_n^m = f_n^{m,\dots,n} \text{ and } d_n^m = g_n^{m\dots,n}$$

for every $m, n \in \mathbf{N}$ with $m \leq n$.

Then, for every $m \in \mathbf{N}$, $(h_n^m)_{n=m}^\infty, (d_n^m)_{n=m}^\infty$ are convex block subsequences of $(f_n^m)_{n=1}^\infty, (g_n^m)_{n=1}^\infty$ respectively, hence $(h_n^m)_{n=m}^\infty$ converges pointwise to F_m, $\|d_n^m\|_\infty < \epsilon_{m+1}$ for every $m, n \in \mathbf{N}$ with $m \leq n$ and $(d_n^m)_{n=m}^\infty$ converges pointwise to $F_{m+1} - F_m$. Also, according to $(*)$, we have that

$$h_n^m = h_n^{m-1} + d_n^{m-1} = h_n^l + d_n^l + \cdots + d_n^{m-1}$$

for every $n, m, l \in \mathbf{N}$ with $l < m \leq n$.

We set $h_n = h_n^n$ for every $n \in \mathbf{N}$. Thus $h_n = h_n^m + d_n^m + \cdots + d_n^{n-1}$ for every $m, n \in \mathbf{N}$ with $m < n$. It is easy to prove that (h_n) converges pointwise to f. If $(n_1, \dots, n_k) \in \mathcal{F}_1$ and $\lambda_1, \lambda_2, \dots, \lambda_k$ are scalars then

$$\left\| \sum_{i=1}^k \lambda_i h_{n_i} \right\|_\infty \leq \left\| \sum_{i=1}^k \lambda_i h_{n_i}^k \right\|_\infty + \left\| \sum_{i=1}^k \lambda_i \left(d_{n_i}^k + \cdots + d_{n_i}^{n_i - 1} \right) \right\|_\infty .$$

First, since $(h_n^k)_{n=k}^\infty$ is a convex block subsequence of $(f_n^k)_{n=1}^\infty$, we have from $(**)$ that

$$\left\| \sum_{i=1}^k \lambda_i h_{n_i}^k \right\|_\infty \leq M \left\| \sum_{i=1}^k \lambda_i s_i \right\| .$$

Secondly,

$$\begin{aligned}\left\|\sum_{i=1}^{k}\lambda_i\left(d_{n_i}^{k}+\cdots+d_{n_i}^{n_i-1}\right)\right\|_\infty &\leq \epsilon_k\cdot\sum_{i=1}^{k}|\lambda_i|\\ &\leq 2k\epsilon_k\left\|\sum_{i=1}^{k}\lambda_i s_i\right\| < \delta\left\|\sum_{i=1}^{k}\lambda_i s_i\right\|.\end{aligned}$$

Hence

$$\left\|\sum_{i=1}^{k}\lambda_i h_{n_i}\right\|_\infty \leq (\|f\|_{1/4}+2\delta)\cdot\left\|\sum_{i=1}^{k}\lambda_i s_i\right\|.$$

This gives

$$\|f\|_s^1 \leq \|f\|_{1/4}+2\delta \text{ for every } \delta>0$$

and finally

$$\|f\|_s^1 \leq \|f\|_{1/4} \text{ for every } f\in B_{1/4}(K).$$

On the other hand, let (f_n) be a sequence in $C(K)$ converging pointwise to f and $C>0$ such that

$$\left\|\sum_{i=1}^{k}\lambda_i f_{n_i}\right\|_\infty \leq C\left\|\sum_{i=1}^{k}\lambda_i s_i\right\|$$

for every $(n_1,\dots,n_k)\in\mathcal{F}_1$ and scalars $\lambda_1,\dots,\lambda_k$. According to a characterization of functions in $B_{1/4}(K)$ given by Haydon, Odell and Rosenthal in [H-O-R], a function f belongs to $B_{1/4}(K)$ if for $\epsilon>0$ there exists a sequence $(g_n)_{n=0}^\infty$ in $C(K)$ with $g_0=0$, converging pointwise to f and such that for every subsequence (g_{n_i}) of (g_n) and $x\in K$ to have

$$\sum_{j\in B((n_i),x)}|g_{n_{j+1}}(x)-g_{n_j}(x)|\leq M,$$

where

$$B((n_i),x)=\left\{j\in\mathbf{N}:|g_{n_{j+1}}(x)-g_{n_j}(x)|\geq\epsilon\right\}.$$

In this case, it is easy to see that $\|f\|_{1/4}\leq 4M$.

For $\epsilon>0$, let m be an integer such that $m>C/\epsilon$. Set $g_n=f_{2m+n}$ for every $n\in\mathbf{N}$. Then, for every strictly increasing sequence (n_i) in $\mathbf{N}$ and $x\in K$ we claim that $\#B((n_i),x)<m$. Indeed, if $j_1,\dots,j_m\in B((n_i),x)$, then

$$m\cdot\epsilon\leq\sum_{i=1}^{m}|g_{n_{j_i+1}}(x)-g_{n_{j_i}}(x)|=\sum_{i=1}^{m}\varepsilon_j(f_{2m+n_{j_i+1}}-f_{2m+n_{j_i}})(x)\leq C,$$

where $\varepsilon_1,\dots,\varepsilon_m\in\{-1,1\}$), so that $\varepsilon_j(f_{2m+n_{j_i+1}}-f_{2m+n_{j_i}})(x)\geq 0$, a contradiction. Hence $\#B((n_i),x)<m$ and thus

$$\sum_{j\in B((n_i),x)}|g_{n_{j+1}}(x)-g_{n_j}(x)|\leq C.$$

Hence $f\in B_{1/4}(K)$ and $\|f\|_{1/4}\leq 4\|f\|_s^1$. □

2.2. *Remark.* It is easy to prove (see [F1]) that a sequence (x_n) in a Banach space X has a subsequence generating a spreading model equivalent to the summing basis (s_n) if and only if it has a subsequence (y_n) with the following property:

there exist $\mu, C > 0$ such that

$$\mu \left\| \sum_{i=1}^{k} \lambda_i s_i \right\| \le \left\| \sum_{i=1}^{k} \lambda_i y_{n_i} \right\| \le C \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|$$

for every $(n_1, \ldots, n_k) \in \mathcal{F}_1$ and scalars $\lambda_1, \ldots, \lambda_k$.

Hence, it follows from the previous theorem and Remark 1.2, for a compact metric space K that

$$B_{1/4}(K) \setminus C(K) = \Big\{ f \in B_1(K) : \text{there exists } (f_n) \subseteq C(K) \text{ such that } f_n \to f \text{ pointwise and } (f_n) \text{ generates spreading model equivalent to } (s_n) \Big\}.$$

This result has been proved in [F1] also. Furthermore, it has been proved in [H-O-R] that every uniformly bounded sequence (f_n) in $C(K)$ converging pointwise to a function in $B_{1/4}(K) \setminus C(K)$ has a convex block subsequence generating a spreading model equivalent to (s_n).

In the following proposition we will give another description of $B_{1/4}(K)$ and we will prove the equality of the norm $\| \cdot \|_s^1$ with a norm on $B_{1/4}(K)$ analogous to the $\| \cdot \|_D$-norm on $D(K)$.

2.3. **Proposition.** *For every compact metric space K, a function $f : K \to \mathbf{R}$ belongs to $B_{1/4}(K)$ if and only if there exists (f_n) in $C(K)$ such that $f = \sum_{n=1}^{\infty} f_n$ pointwise and for $n_0 = f_0 = 0$,*

$$\sup \left\{ \left\| \sum_{i=1}^{k} |f_{n_{i-1}+1} + \cdots + f_{n_i}| \right\|_\infty : (n_1, \ldots, n_k) \in \mathcal{F}_1 \right\} < \infty.$$

Also, for every $f \in B_{1/4}(K)$ we have

$$\|f\|_s^1 = \|f\|_D^1 = \inf \left\{ \sup \left\{ \left\| \sum_{i=1}^{k} |f_{n_{i-1}+1} + \cdots + f_{n_i}| \right\|_\infty : (n_1, \ldots, n_k) \in \mathcal{F}_1 \right\} : (f_n) \subseteq C(K) \text{ with } f = \sum_{n=1}^{\infty} f_n \right\}.$$

Proof. If $f \in B_{1/4}(K)$ then for every $\epsilon > 0$, from the previous theorem, there exists $(g_n)_{n=0}^{\infty} \subseteq C(K)$, $g_0 = 0$, such that $g_n \to f$ pointwise and

$$\left\| \sum_{i=1}^{k} \lambda_i g_{n_i} \right\|_\infty \le \left(\|f\|_s^1 + \epsilon \right) \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|$$

for every $(n_1, \ldots, n_k) \in \mathcal{F}_1$ and scalars $\lambda_1, \ldots, \lambda_k$. Set $f_n = g_n - g_{n-1}$ for every $n \in \mathbf{N}$. Then $f = \sum_{n=1}^{\infty} f_n$ pointwise. Also, for $(n_1, \ldots, n_k) \in \mathcal{F}_1$ and $x \in K$ we have

$$\begin{aligned} \sum_{i=1}^{k} |f_{n_{i-1}+1} + \cdots + f_{n_i}|(x) &= \sum_{i=1}^{k} \varepsilon_i (f_{n_{i-1}+1} + \cdots + f_{n_i})(x) \\ &= \left| \sum_{i=1}^{k} \varepsilon_i (g_{n_i} - g_{n_{i-1}}) \right|(x) = \left| \sum_{i=1}^{k} (\varepsilon_i - \varepsilon_{i+1}) g_{n_i} \right|(x) \le |f|_s^1 + \epsilon, \end{aligned}$$

where $\varepsilon_i \in \{-1,1\}$ so that $\varepsilon_i(f_{n_{i-1}+1}+\cdots+f_{n_i})(x) \geq 0$ for all $i=1,\ldots,k$ and $\varepsilon_{k+1}=0$. This gives that $\|f\|_D^1 \leq \|f\|_s^1$ for every $f \in B_{1/4}(K)$.

On the other hand, let $(g_n) \subseteq C(K)$ and $C>0$ be such that $f=\sum_{n=1}^\infty g_n$ pointwise and

$$\left\|\sum_{i=1}^k |g_{n_{i-1}+1}+\cdots+g_{n_i}|\right\|_\infty \leq C \;\; (n_0=g_0=0)$$

for every $(n_1,\ldots,n_k) \in \mathcal{F}_1$. Set $f_n=\sum_{i=1}^n g_i$ for every $n \in \mathbf{N}$. Of course $f_n \to f$ pointwise. Also, for $(n_1,\ldots,n_k) \in \mathcal{F}_1$, $x \in K$ and scalars $\lambda_1,\ldots,\lambda_k$ we have

$$\begin{aligned}
\left|\sum_{i=1}^k \lambda_i f_{n_i}\right|(x) &= \left|\sum_{i=1}^k \lambda_i(g_1+\cdots+g_{n_i})\right|(x) \\
&= \left|\sum_{i=1}^k (\lambda_i+\cdots+\lambda_k)\cdot(g_{n_{i-1}+1}+\cdots+g_{n_i})\right|(x) \\
&\leq \sum_{i=1}^k \left|\sum_{j=i}^k \lambda_j\right| \cdot \left|\sum_{j=n_{i-1}+1}^{n_i} g_j\right|(x) \\
&\leq \left\|\sum_{i=1}^k \lambda_i s_i\right\| \cdot \left(\sum_{i=1}^k \left|\sum_{j=n_{i-1}+1}^{n_i} g_j\right|\right)(x) \leq C\cdot\left\|\sum_{i=1}^k \lambda_i s_i\right\|.
\end{aligned}$$

Hence $f \in B_{1/4}(K)$ and $\|f\|_s^1 \leq \|f\|_D^1$. This completes the proof. ☐

2.4. **Corollary.** *For every compact metric space K, a function $f: K \to \mathbf{R}$ belongs to $B_{1/4}(K)$ if and only if there exists (f_n) in $C(K)$ such that $f_n \to f$ pointwise and for $n_0=f_0=0$,*

$$\sup\left\{\left\|\sum_{i=1}^k |f_{n_i}-f_{n_i-1}|\right\|_\infty : (n_1,\ldots,n_k) \in \mathcal{F}_1\right\} < \infty.$$

Also, for every $f \in B_{1/4}(K)$ we have

$$\|f\|_s^1 = \inf\Big\{ \sup\Big\{\Big\|\textstyle\sum_{i=1}^k |f_{n_i}-f_{n_i-1}|\Big\|_\infty : (n_1,\ldots,n_k) \in \mathcal{F}_1\Big\} : \text{ where } (f_n) \subseteq C(K) \text{ and } f_n \to f \text{ pointwise}\Big\}.$$

In the following theorem we will give a characterization of the functions in $B_{1/4}(K)$ and also an identity for $\|f\|_{1/4}$, where f is in $B_{1/4}(K)$, using the transfinite oscillations of f, which have been defined by H. Rosenthal in [R1]. We recall this definition.

2.5. **Definition.** [R1] Let K be a metric space. One defines the upper semicontinuous envelope $\mathcal{U}g$ of an extended real valued function $g: K \to [-\infty,+\infty]$ as follows:

$$\mathcal{U}g = \inf\{h: K \longrightarrow [-\infty,\infty] : h \text{ is continuous and } h \geq g\}.$$

It is easy to see that for $x \in K$

$$\begin{aligned}\mathcal{U}g(x) &= \overline{\lim_{y\to x}} g(y) = \max\{L \in [-\infty, +\infty] : \exists x_n \to x, g(x_n) \to L\} \\ &= \inf\left\{\sup_{y\in U} g(y) : U \text{ is a neighbourhood of } x\right\}.\end{aligned}$$

In [R1] the author associates with each bounded function $f : K \to \mathbf{R}$ a transfinite increasing family $(\mathrm{osc}_\alpha f)_{1\leq\alpha}$ of upper semicontinuous functions which are called α^{th}- oscillations of f. They have been defined by induction as follows:

$$\mathrm{osc}_0 f = 0.$$

If $\mathrm{osc}_\alpha f$ has been defined, then for every $x \in K$

$$\widetilde{\mathrm{osc}}_{\alpha+1} f(x) = \overline{\lim}_{y\to x}\Big(|f(y) - f(x)| + \mathrm{osc}_\alpha f(y)\Big)$$

and consequently

$$\mathrm{osc}_{\alpha+1} f = \mathcal{U}\,\widetilde{\mathrm{osc}}_{\alpha+1} f.$$

If α is a limit ordinal and $\mathrm{osc}_\beta f$ has been defined for all $\beta < \alpha$ then

$$\widetilde{\mathrm{osc}}_\alpha f = \sup_{\beta<\alpha} \mathrm{osc}_\beta f$$

and consequently

$$\mathrm{osc}_\alpha f = \mathcal{U}\,\widetilde{\mathrm{osc}}_\alpha f.$$

According to [R2], a bounded function $f : K \to \mathbf{R}$ is in $D(K)$ if and only if $\mathrm{osc}_\alpha f$ is a bounded function for every ordinal α. In this case there exists an ordinal α so that $\mathrm{osc}_\alpha f$ is bounded and $\mathrm{osc}_\alpha f = \mathrm{osc}_\beta f$ for all $\beta > \alpha$. Moreover, letting τ be the least such α,

$$\|f\|_D = \||f| + \mathrm{osc}_\tau f\|_\infty .$$

We will prove an analogous structural result for $B_{1/4}(K)$. Precisely, we will prove that a bounded function f is in $B_{1/4}(K)$ if and only if $\mathrm{osc}_\omega f$ is bounded and when this occurs then

$$\|f\|_{1/4} = \||f| + \widetilde{\mathrm{osc}}_\omega f\|_\infty .$$

Before the proof of this theorem we will give three lemmas. In the first lemma we list some elementary relations which are used in the sequel.

2.6. **Lemma.** *Let f, g be bounded functions on a metric space K and α an ordinal number.*

(1) If $f \leq g$ then $\mathcal{U}f \leq \mathcal{U}g$.
(2) $\mathcal{U}(f + g) \leq \mathcal{U}f + \mathcal{U}g$.
(3) $\mathcal{U}(f - \mathcal{U}g) = \mathcal{U}(\mathcal{U}f - \mathcal{U}g) \leq \mathcal{U}(f - g)$.
(4) $\mathcal{U}f = f$ if and only if f is upper semicontinuous.
(5) $\mathrm{osc}_\alpha f$ is an upper semicontinuous $[0, +\infty]$-valued function on K.
(6) $\mathrm{osc}_\alpha tf = |t|\mathrm{osc}_\alpha f$ for every $t \in \mathbf{R}$.
(7) $\mathrm{osc}_\alpha(f + g) \leq \mathrm{osc}_\alpha f + \mathrm{osc}_\alpha g$.
(8) $\mathrm{osc}_\alpha(f + g) = \mathrm{osc}_\alpha f$ if g is a continuous function on K.
(9) If $\mathrm{osc}_\alpha f$ is bounded then $\mathcal{U}(\mathrm{osc}_\alpha f \pm f) \leq \widetilde{\mathrm{osc}}_{\alpha+1} f \pm f$.

Proof. The assertions (1)-(8) are easily proved. We will prove (9).

Let $x \in K$. We may choose (y_n) a sequence in K tending to x such that

$$\mathcal{U}(\mathrm{osc}_\alpha f + f)(x) = \lim_{n\to\infty} \mathrm{osc}_\alpha f(y_n) + f(y_n).$$

Since the functions f and $\mathrm{osc}_\alpha f$ are bounded, we may assume without loss of generality that the limits

$$\lim_{n\to\infty} \mathrm{osc}_\alpha f(y_n),\ \lim_{n\to\infty} |f(y_n) - f(x)|,\ \lim_{n\to\infty} f(y_n)$$

all exist. We then have that

$$\begin{aligned} \widetilde{\mathrm{osc}}_{\alpha+1} f(x) &\geq \lim_{n\to\infty} \Big(|f(y_n) - f(x)| + \mathrm{osc}_\alpha f(y_n) \Big) \\ &= \lim_{n\to\infty} |f(y_n) - f(x)| + \lim_{n\to\infty} \mathrm{osc}_\alpha f(y_n) \\ &\geq \lim_{n\to\infty} \Big(\mathrm{osc}_\alpha f(y_n) + f(y_n) \Big) - f(x) \\ &= \mathcal{U}(\mathrm{osc}_\alpha f + f)(x) - f(x). \end{aligned}$$

Thus it is proved that $\mathcal{U}(\mathrm{osc}_\alpha f + f) \leq \widetilde{\mathrm{osc}}_{\alpha+1} f + f$. If instead of f we use $-f$, we have that $\mathcal{U}(\mathrm{osc}_\alpha f - f) \leq \widetilde{\mathrm{osc}}_{\alpha+1} f - f$, since $\widetilde{\mathrm{osc}}_\alpha f = \widetilde{\mathrm{osc}}_\alpha(-f)$. □

2.7. **Lemma.** *Let $f : K \to \mathbf{R}$ be a bounded function. For every $n \in \mathbf{N}$ we have that*

$$\mathcal{U}(\mathrm{osc}_{n+2} f - \mathrm{osc}_{n+1} f) \leq \mathcal{U}(\mathrm{osc}_{n+1} f - \mathrm{osc}_n f).$$

Proof. Using (3) of the previous lemma, we have that

$$\begin{aligned} \mathcal{U}(\mathrm{osc}_{n+2} f - \mathrm{osc}_{n+1} f) &= \mathcal{U}(\widetilde{\mathrm{osc}}_{n+2} f - \mathrm{osc}_{n+1} f) \\ &\leq \mathcal{U}(\widetilde{\mathrm{osc}}_{n+2} f - \widetilde{\mathrm{osc}}_{n+1} f), \quad \text{for every } n \in \mathbf{N}. \end{aligned}$$

Hence it is sufficient to prove that

$$\mathcal{U}(\widetilde{\mathrm{osc}}_{n+2} f - \widetilde{\mathrm{osc}}_{n+1} f) \leq \mathcal{U}(\mathrm{osc}_{n+1} f - \mathrm{osc}_n f) \quad \text{for every } n \in \mathbf{N}.$$

By (1) and (4) of the previous lemma, the proof of this lemma will be complete as soon as we prove that

$$\widetilde{\mathrm{osc}}_{n+2} f - \widetilde{\mathrm{osc}}_{n+1} f \leq \mathcal{U}(\mathrm{osc}_{n+1} f - \mathrm{osc}_n f) \quad \text{for every } n \in \mathbf{N}.$$

Case $n = 0$. We have for $x \in K$,

$$\begin{aligned} \widetilde{\mathrm{osc}}_2 f(x) - \widetilde{\mathrm{osc}}_1 f(x) &= \overline{\lim_{y\to x}} \Big(\mathrm{osc}_1 f(y) + |f(y) - f(x)| \Big) - \overline{\lim_{y\to x}} |f(y) - f(x)| \\ &\leq \overline{\lim_{y\to x}} \mathrm{osc}_1 f(y) = \mathcal{U}(\mathrm{osc}_1 f)(x) = \mathrm{osc}_1 f(x) \end{aligned}$$

(since $\mathrm{osc}_1 f$ is upper semicontinuous).

In general for $n > 0$, $n \in \mathbf{N}$, we have for $x \in K$,

$$\begin{aligned} &\widetilde{\mathrm{osc}}_{n+2} f(x) - \widetilde{\mathrm{osc}}_{n+1} f(x) \\ &\quad = \overline{\lim_{y\to x}} (\mathrm{osc}_{n+1} f(y) + |f(y) - f(x)|) - \overline{\lim_{y\to x}} (\mathrm{osc}_n f(y) + |f(y) - f(x)|) \\ &\quad \leq \overline{\lim_{y\to x}} (\mathrm{osc}_{n+1} f(y) - \mathrm{osc}_n f(y)) = \mathcal{U}(\mathrm{osc}_{n+1} f - \mathrm{osc}_n f)(x). \end{aligned}$$

This completes the proof.

The following lemma was proved by A. Louveau ([F-L]). For completeness we give the proof. □

2.8. **Lemma.** [F-L] *Let* $(g_n)_{n=1}^{\infty}$ *be a sequence of bounded, upper semicontinuous functions on a metric space* K *with* $g_0 = 0$. *If the sequence* $(\mathcal{U}(g_{n+1} - g_n))_{n=0}^{\infty}$ *is decreasing, then* $\mathcal{U}(g_{n+1} - g_n) \leq \frac{1}{n+1} \cdot g_{n+1}$ *for every* $n \in \mathbf{N}$.

Proof. For $n = 0$, it reduces to $\mathcal{U}g_1 \leq g_1$, which is trivial since g_1 is upper semicontinuous. Suppose we know it for n. For the induction step, it suffices, since g_{n+2} is usc, to prove:

$$g_{n+2} - g_{n+1} \leq \frac{g_{n+2}}{n+2}; \quad \text{i.e.,} \ g_{n+2} \leq \frac{g_{n+2}}{n+2} + g_{n+1}.$$

But since $1 = \frac{1}{n+2} + \frac{n+1}{n+2}$, it suffices to show

$$\frac{n+1}{n+2} g_{n+2} \leq g_{n+1}, \quad \text{i.e.,} \ g_{n+2} \leq \frac{n+2}{n+1} g_{n+1} = g_{n+1} + \frac{1}{n+1} g_{n+1}.$$

But this follows immediately from the induction step. □

2.9. **Theorem.** *Let* K *be a metric space. Then*

$$B_{1/4}(K) = \left\{ f : K \to \mathbf{R} \ \textit{bounded} \ : \mathrm{osc}_\omega f \ \textit{is} \ \ \textit{bounded} \right\} \quad \textit{and}$$

$$\|f\|_{1/4} = \left\| \, |f| + \widetilde{\mathrm{osc}}_\omega f \right\|_\infty \ \textit{for all} \ f \in B_{1/4}(K).$$

Proof. Suppose $f \in B_{1/4}(K)$. It follows from the definition of $B_{1/4}(K)$ that for every $\epsilon > 0$ one has a sequence (g_n) in $D(K)$ with $\|g_n - f\|_\infty \to 0$ and $\sup_n \|g_n\|_D < \|f\|_{1/4} + \epsilon$. Set $\epsilon_n = \|g_n - f\|_\infty$. Then by induction on k,

$$\mathrm{osc}_k f \leq \mathrm{osc}_k g_n + 2k\epsilon_n \ \text{for every} \ k, n \in \mathbf{N}.$$

Hence

$$\begin{aligned} |f| + \mathrm{osc}_k f &\leq |g_n| + \epsilon_n + \mathrm{osc}_k g_n + 2k\epsilon_n \\ &\leq |g_n| + \mathrm{osc}_\tau g_n + (2k+1)\epsilon_n \\ &\leq \|g_n\|_D + (2k+1)\epsilon_n \ \text{for every} \ k, n \in \mathbf{N}. \end{aligned}$$

Letting first $n \to \infty$ and then $k \to +\infty$, we get

$$|f| + \widetilde{\mathrm{osc}}_\omega f \leq \sup_n \|g_n\|_D \leq \|f\|_{1/4} + \epsilon.$$

Since ϵ is arbitrary, we have that

$$\||f| + \widetilde{\mathrm{osc}}_\omega f\|_\infty \leq \|f\|_{1/4}$$

and of course that $\widetilde{\mathrm{osc}}_\omega f$ and, consequently, $\mathrm{osc}_\omega f$ are bounded functions.

On the other hand, let $f : K \to \mathbf{R}$ be a bounded function with $\mathrm{osc}_\omega f$ also bounded. Set

$$g_n = \frac{\lambda_n - \mathcal{U}(\mathrm{osc}_n f - f)}{2} - \frac{\lambda_n - \mathcal{U}(\mathrm{osc}_n f + f)}{2},$$

where $\lambda_n = \| \, |f| + \mathrm{osc}_n f\|_\infty$ for every $n \in \mathbf{N}$. Then $g_n \in D(K)$ and

$$\begin{aligned} \|g_n\|_D &\leq \left\|\lambda_n - \tfrac{1}{2}\mathcal{U}(\mathrm{osc}_n f - f) - \tfrac{1}{2}(\mathrm{osc}_n f + f)\right\|_\infty \leq \\ &\leq \lambda_n \leq \| \, |f| + \widetilde{\mathrm{osc}}_\omega f\|_\infty \ \text{for every} \ n \in \mathbf{N}. \end{aligned}$$

The first inequality holds for every $n \in \mathbf{N}$, since from (1), (2) and (4) of Lemma 2.6 we have

$$\mathcal{U}(\mathrm{osc}_n f - f) + \mathcal{U}(\mathrm{osc}_n f + f) \geq 2\mathcal{U}(\mathrm{osc}_n f) = 2\mathrm{osc}_n f \geq 0 \ \text{and}$$

$$\begin{aligned}&\lambda_n - \tfrac{1}{2}\mathcal{U}(\mathrm{osc}_n f - f) - \tfrac{1}{2}\mathcal{U}(\mathrm{osc}_n f + f)\\ &\quad\geq \lambda_n - \mathcal{U}(\mathrm{osc}_n f + |f|) \geq \lambda_n - \|\mathcal{U}(\mathrm{osc}_n f + |f|)\|_\infty\\ &\quad= \lambda_n - \|\mathrm{osc}_n f + |f|\,\|_\infty = 0.\end{aligned}$$

If we could prove that $\|g_n - f\|_\infty \to 0$, then we would have that $f \in B_{1/4}(K)$ and $\|f\|_{1/4} \leq \|\,|f| + \widetilde{\mathrm{osc}}_\omega f\|_\infty$. Now, according to (9) of Lemma 2.6,

$$\begin{aligned}g_n - f &= \tfrac{1}{2}\mathcal{U}(\mathrm{osc}_n f + f) - \tfrac{1}{2}\mathcal{U}(\mathrm{osc}_n f - f) - f\\ &\leq \tfrac{1}{2}\,(\widetilde{\mathrm{osc}}_{n+1} f + f) - \tfrac{1}{2}(\mathrm{osc}_n f - f) - f\\ &= \tfrac{1}{2}\,(\widetilde{\mathrm{osc}}_{n+1} f - \mathrm{osc}_n f) \leq \tfrac{1}{2}(\mathrm{osc}_{n+1} f - \mathrm{osc}_n f) \text{ for every } n \in \mathbf{N}.\end{aligned}$$

On the other direction,

$$\begin{aligned}g_n - f &= \tfrac{1}{2}\mathcal{U}(\mathrm{osc}_n f + f) - \tfrac{1}{2}\mathcal{U}(\mathrm{osc}_n f - f) - f\\ &\geq \tfrac{1}{2}\,(\mathrm{osc}_n f + f) - \tfrac{1}{2}(\widetilde{\mathrm{osc}}_{n+1} f - f) - f\\ &= -\tfrac{1}{2}\,(\widetilde{\mathrm{osc}}_{n+1} f - \mathrm{osc}_n f) \geq -\tfrac{1}{2}(\mathrm{osc}_{n+1} f - \mathrm{osc}_n f) \text{ for every } n \in \mathbf{N}.\end{aligned}$$

Hence

$$|g_n - f| \leq \frac{1}{2}(\mathrm{osc}_{n+1} f - \mathrm{osc}_n f) \text{ for every } n \in \mathbf{N}.$$

According to Lemma 2.7, the sequence $(\mathcal{U}(\mathrm{osc}_{n+1} f - \mathrm{osc}_n f))_{n=0}^\infty$ is decreasing. Hence, using Lemma 2.8, we have that

$$\begin{aligned}|g_n - f| &\leq \frac{1}{2}(\mathrm{osc}_{n+1} f - \mathrm{osc}_n f) \leq \frac{1}{2}\mathcal{U}(\mathrm{osc}_{n+1} f - \mathrm{osc}_n f)\\ &\leq \frac{1}{n+1}\mathrm{osc}_{n+1} f \leq \frac{1}{n+1}\mathrm{osc}_\omega f \leq \frac{1}{n+1}\|\mathrm{osc}_\omega f\|_\infty.\end{aligned}$$

Thus $\|g_n - f\|_\infty \leq \frac{1}{n+1} \cdot \|\mathrm{osc}_\omega f\|_\infty$ and, finally, $\|g_n - f\|_\infty \to 0$. This finishes the proof of the theorem. □

2.10. *Remark.* Using the invariants $(f_\alpha)_{1\leq\alpha}$ which have been introduced by Kechris and Louveau in [K-L] and which are similar to the α^{th}- oscillations of the function f, we proved with Louveau ([F-L]) that a bounded function f is in $B_{1/4}(K)$ if and only if f_ω is bounded and in this case

$$\frac{1}{3}\|f_\omega\|_\infty \leq \|f\|_{1/4} \leq 4\|f_\omega\|_\infty + 5\|f\|_\infty.$$

But the previous theorem shows that the transfinite oscillations appear to be more appropriate than the f_α's.

After proving this theorem, I learned that H. Rosenthal ([R2]) had an analogous result. Precisely, he proved in [R2] that f belongs to $B_{1/4}(K)$ (case $f : K \to \mathbf{C}$) if and only if $\mathrm{osc}_\omega f$ is bounded and when this occurs and f is real valued,

$$\frac{1}{2}(\|f\|_\infty + \|\mathrm{osc}_\omega f\|_\infty) \leq \|f\|_{1/4} \leq \|f\|_\infty + 3\,\|\mathrm{osc}_\omega f\|_\infty.$$

3. A Classification of $B_{1/4}(K)$

We will define a classification of $B_{1/4}(K)$, where K is a separable metric space, into a decreasing hierarchy $(S_\xi(K))_{1\leq\xi<\omega_1}$ of Banach spaces whose intersection is equal to $D(K)$. The functions in $S_\xi(K)$ have a property stronger than the one of the functions in $B_{1/4}(K)$ which is described in Proposition 2.3. Precisely, the families $\mathcal{F}_\xi$, which have been defined by D. Alspach and S. Argyros in [A-A], are used instead of the Schreier family $\mathcal{F}_1$. We quote the definition of the $\mathcal{F}_\xi$'s.

3.1. **Definition** ([A-A])**.** For every limit ordinal ξ, let (ξ_n) be a sequence of ordinal numbers strictly increasing to ξ. Then $\mathcal{F}_0 = \{\{n\} : n \in \mathbf{N}\}$.

Suppose that $\mathcal{F}_\xi$ is defined, then

$$\mathcal{F}_{\xi+1} = \{F \subseteq \mathbf{N} : F \subseteq F_1 \cup \cdots \cup F_n \text{ with } \{n\} < F_1 < \cdots < F_n \text{ and } F_i \in \mathcal{F}_\xi \text{ for all } i = 1, \ldots, n\} .$$

If ξ is a limit ordinal, $\mathcal{F}_\xi = \{F \subseteq \mathbf{N} : F \in \mathcal{F}_{\xi_n} \text{ and } \{n\} \leq F\}$.

Using the families $\mathcal{F}_\xi$, for every ordinal ξ, we extended the notion of spreading model in [F2] as follows:

3.2. **Definition** ([F2])**.** Let X be a Banach space, ξ an ordinal number and (x_n) a sequence in X. We say that (x_n) generates spreading model of order ξ equivalent to a basic sequence (e_n) if there exist $\mu > 0$ and $C > 0$ such that:

$$\mu \left\| \sum_{i=1}^{k} \lambda_i e_{n_i} \right\| \leq \left\| \sum_{i=1}^{k} \lambda_i x_{n_i} \right\| \leq C \left\| \sum_{i=1}^{k} \lambda_i e_{n_i} \right\|$$

for every $(n_1, \ldots, n_k) \in \mathcal{F}_\xi$ and scalars $\lambda_1, \ldots, \lambda_k$.

Now we will define the spaces $S_\xi(K)$ for every ordinal ξ, which are characterized by spreading models of order ξ equivalent to the summing basis (s_n) of c_0.

3.3. **Definition.** Let K be a metric space and ξ an ordinal number. We define the space

$$S_\xi(K) = \Big\{ f : K \to \mathbf{R} : \text{there exists } (f_n) \subseteq C(K) \text{ and } C > 0 \text{ such that } f_n \to f \text{ pointwise and } \|\textstyle\sum_{i=1}^{k} \lambda_i f_{n_i}\|_\infty \leq C \,\|\sum_{i=1}^{k} \lambda_i s_i\| \text{ for every } (n_1, \ldots, n_k) \in \mathcal{F}_\xi \text{ and scalars } \lambda_1, \ldots, \lambda_k \Big\}$$

and the norm $\|\cdot\|_s^\xi$ on it as follows:

$$\|f\|_s^\xi = \inf \Big\{ C > 0 : \text{there exists } (f_n) \subseteq C(K) \text{ such that } f_n \to f \text{ pointwise and } \|\textstyle\sum_{i=1}^{k} \lambda_i f_{n_i}\|_\infty \leq C \,\|\sum_{i=1}^{k} \lambda_i s_i\| \text{ for every } (n_1, \ldots, n_k) \text{ in } \mathcal{F}_\xi \text{ and scalars } \lambda_1, \ldots, \lambda_k \Big\}$$

If K is a compact metric space, it is easy to prove (see Remark 1.2) that

$$S_\xi(K) \setminus C(K) = \Big\{ f : K \to \mathbf{R} : \text{there exists } (f_n) \text{ in } C(K) \text{ such that } f_n \to f \text{ pointwise and } (f_n) \text{ generates spreading model of order } \xi \text{ equivalent to } (s_n) \Big\}.$$

Of course, $S_1(K) = B_{1/4}(K)$ for a compact metric space K. Also, for every ordinal number ξ, $S_\xi(K)$ is a linear subspace of $B_1(K)$. Although the family $(\mathcal{F}_\xi)_{1\leq\xi}$ is not increasing, it has the property: for every $1 \leq \beta < \xi$, there exists $n_0 = n_0(\beta, \xi)$ in $\mathbf{N}$ such that if $A \in \mathcal{F}_\beta$ and $\{n_0\} \leq A$ then $A \in \mathcal{F}_\xi$. Hence, it is easy to prove that the family $(S_\xi(K))_{1\leq\xi}$ is decreasing and, also, $\|f\|_s^\beta \leq \|f\|_s^\xi$ for every $1 \leq \beta < \xi$ and f in $S_\xi(K)$.

3.4. **Proposition.** *For every ordinal number ξ, $\big(S_\xi(K), \|\cdot\|_s^\xi\big)$ is a Banach space.*

Proof. Let ξ be an ordinal number and (F_n) a Cauchy sequence in $(S_\xi(K), \|\cdot\|_s^\xi)$. We can assume that $\|F_{n+1} - F_n\|_s^\xi < \frac{1}{2^n}$ for every $n \in \mathbf{N}$. So, for every $n \in \mathbf{N}$ we can find a sequence $(\phi_m^n)_{m=1}^\infty \subseteq C(K)$ converging pointwise to $F_{n+1} - F_n$ and satisfying

$$\left\|\sum_{i=1}^k \lambda_i \phi_{m_i}^n\right\|_\infty \leq \frac{1}{2^n}\left\|\sum_{i=1}^k \lambda_i s_i\right\|$$

for every $(m_1, \dots, m_k) \in \mathcal{F}_\xi$ and scalars $\lambda_1, \dots, \lambda_k$. Since $\|f\|_\infty \leq \|f\|_s^\xi$ for every $f \in S_\xi(K)$, there exists $F \in B_1(K)$ such that $\|F_n - F\|_\infty \to 0$.

Let $n_0 \in \mathbf{N}$. Set $\Phi_n = F_{n+1} - F_n$ for every $n \in \mathbf{N}$, and $f_n = \phi_n^{n_0} + \cdots + \phi_n^n$ for every $n \geq n_0$. Then $F - F_{n_0} = \sum_{n=n_0}^\infty \Phi_n$. Also, $f_n \to F - F_{n_0}$ pointwise. Indeed,

$$\left\|f_n - (\phi_n^{n_0} + \cdots + \phi_n^l)\right\|_\infty = \left\|\phi_n^{l+1} + \cdots + \phi_n^n\right\|_\infty \leq \sum_{i=l+1}^n \frac{1}{2^i} = \frac{1}{2^l}$$

for every $n_0 \leq l < n \in \mathbf{N}$. Hence, letting $n \to \infty$, we have for every $x \in K$ and $l \geq n_0$,

$$\Phi_{n_0}(x) + \cdots + \Phi_l(x) - \frac{1}{2^l} \leq \underline{\lim}_n f_n(x) \leq \overline{\lim}_n f_n(x) \leq \Phi_{n_0}(x) + \cdots + \Phi_l(x) + \frac{1}{2^l}.$$

Letting $l \to \infty$, this gives that $f_n \to F - F_{n_0}$ pointwise.

On the other hand, for every $(n_1, \dots, n_k) \in \mathcal{F}_\xi$ and scalars $\lambda_1, \dots, \lambda_k$ we have that

$$\begin{aligned}
\left\|\sum_{i=1}^k \lambda_i f_{n_i}\right\|_\infty &= \left\|\sum_{i=1}^k \left(\lambda_i \phi_{n_i}^{n_0} + \cdots + \lambda_i \phi_{n_i}^{n_i}\right)\right\|_\infty \\
&\leq \sum_{j=n_0}^{n_1} \left\|\sum_{i=1}^k \lambda_i \phi_{n_i}^j\right\|_\infty + \sum_{j=n_1+1}^{n_2} \left\|\sum_{i=2}^k \lambda_i \phi_{n_i}^j\right\|_\infty + \cdots + \sum_{j=n_{k-1}+1}^{n_k} |\lambda_k| \left\|\phi_{n_k}^j\right\|_\infty \\
&\leq \sum_{j=n_0}^{n_1} \frac{1}{2^j}\left\|\sum_{i=1}^k \lambda_i s_i\right\| + \sum_{j=n_1+1}^{n_2} \frac{1}{2^j}\left\|\sum_{i=2}^k \lambda_i s_i\right\| + \cdots + \sum_{j=n_{k-1}+1}^{n_k} \frac{1}{2^j}\|\lambda_k s_k\| \\
&\leq \left(\sum_{j=n_0}^\infty \frac{1}{2^j}\right) \cdot 2 \cdot \left\|\sum_{i=1}^k \lambda_i s_i\right\| = \frac{1}{2^{n_0}} \cdot \left\|\sum_{i=1}^k \lambda_i s_i\right\|.
\end{aligned}$$

Hence $F - F_{n_0} \in S_\xi(K)$, whence $F \in S_\xi(K)$. Also, we have that

$$\|F - F_{n_0}\|_s^\xi \leq \frac{1}{2^{n_0}} \text{ for every } n_0 \in \mathbf{N},$$

which gives that (F_n) converges to F with respect to the $\|\cdot\|_s^\xi$-norm. This completes the proof. ☐

We will give more descriptions of the spaces $S_\xi(K)$ in analogy to $B_{1/4}(K)$ (see Proposition 2.3 and Corollary 2.4).

3.5. **Proposition.** *For every metric space K and ordinal number ξ, a function $f : K \to \mathbf{R}$ belongs to $S_\xi(K)$ if and only if there exists (f_n) in $C(K)$ such that $f = \sum_{n=1}^\infty f_n$ pointwise and for $n_0 = f_0 = 0$,*

$$\sup\left\{\left\|\sum_{i=1}^k |f_{n_{i-1}+1} + \cdots + f_{n_i}|\right\|_\infty : (n_1, \dots, n_k) \in \mathcal{F}_\xi\right\} < \infty.$$

Also, for every $f \in S_\xi(K)$,

$$\|f\|_s^\xi = \inf\Big\{ \sup\Big\{\Big\|\textstyle\sum_{i=1}^k |f_{n_{i-1}+1}+\cdots+f_{n_i}|\Big\|_\infty : (n_1,\dots,n_k)\in\mathcal{F}_\xi\Big\} : \text{for every } (f_n) \text{ in } C(K) \text{ with } f=\textstyle\sum_n f_n \text{ pointwise}\Big\}.$$

Proof. The proof is analogous to the proof of Proposition 2.3. □

3.6. **Corollary.** *For every metric space* K *and ordinal number* ξ, *a function* $f : K \to \mathbf{R}$ *belongs to* $S_\xi(K)$ *if and only if there exists* (f_n) *in* $C(K)$ *such that* $f_n \to f$ *pointwise and for* $n_0 = f_0 = 0$,

$$\sup\left\{\left\|\sum_{i=1}^k |f_{n_i}-f_{n_i-1}|\right\|_\infty : (n_1,\dots,n_k)\in\mathcal{F}_\xi\right\} < \infty.$$

Also, for every $f \in S_\xi(K)$,

$$\|f\|_s^\xi = \inf\Big\{ \sup\Big\{\Big\|\textstyle\sum_{i=1}^k |f_{n_i}-f_{n_{i-1}}|\Big\|_\infty : (n_1,\dots,n_k)\in\mathcal{F}_\xi\Big\} : \text{for every } (f_n)\subseteq C(K) \text{ with } f_n\to f \text{ pointwise}\Big\}.$$

From a result in [F2], we have the following connection between the functions in $S_\xi(K)$ and the transfinite oscillations.

3.7. **Theorem** ([F2])**.** *Let* K *be a metric space and* ξ *an ordinal number. Then*

$$S_\xi(K) \subseteq \left\{ f : K\to\mathbf{R} : \operatorname{osc}_{\omega^\xi} f \text{ is bounded} \right\}.$$

Proof. It follows from the proof of Theorem 9 in [F2] that, for every function f in $S_\xi(K)$, the function $u_{\omega^\xi}(f)$ is bounded (the functions $u_\alpha(f)$, were introduced in [R1] and are similar to the α^{th}- oscillations of f). But, as it is proved in [R1],

$$\operatorname{osc}_\alpha f \le u_\alpha(f) + u_\alpha(-f)$$

for every ordinal number α. Hence, $\operatorname{osc}_{\omega^\xi} f$ is bounded.

This theorem yields immediately the following result. □

3.8. **Theorem.** *Let* K *be a separable metric space. The intersection of all the classes* $S_\xi(K)$, $1 \le \xi < \omega_1$, *is equal to* $D(K)$.

Proof. It follows from the previous theorem and the fact that f belongs to $D(K)$ if and only if $\operatorname{osc}_\alpha f$ is bounded for every countable ordinal α ([R1]). □

In [F2] we defined for every ordinal ξ the notion of a null-coefficient of order ξ (ξ-n.c.) sequence in a Banach space and we proved that every bounded, Baire-1 function f with $\operatorname{osc}_{\omega^\xi} f$ unbounded has the property that every bounded sequence of continuous functions converging pointwise to f is null-coefficient of order ξ. We will prove in the sequel that this property characterizes the functions in $B_1(K)\setminus S_\xi(K)$.

3.9. **Definition** ([F2])**.** A sequence (x_n) in a Banach space is called null-coefficient of order ξ (ξ-n.c), where ξ is an ordinal number, if whenever the scalars (c_n) satisfy:

$$\sup\left\{\left\|\sum_{i=1}^k c_{n_{2i}}(x_{n_{2i}}-x_{n_{2i-1}})\right\| : (n_1,\dots,n_{2k})\in\mathcal{F}_\xi\right\} < \infty$$

the sequence (c_n) converges to 0.

3.10. **Proposition.** *Let ξ be an ordinal number, and (x_n) a weak-Cauchy and non-weakly convergent sequence in a Banach space. Then (x_n) is not null-coefficient of order ξ if and only if it has a subsequence with spreading model of order ξ equivalent to the summing basis of c_0.*

Proof. If (x_n) is not null-coefficient of order ξ then there exists a bounded sequence of scalars (c_n) such that (c_n) is not null-converging and

$$(*) \qquad \left\| \sum_{i=1}^{k} c_{n_{2i}} (x_{n_{2i}} - x_{n_{2i-1}}) \right\| \leq 1$$

for every $(n_1, \ldots, n_{2k}) \in \mathcal{F}_\xi$.

So we can find $\epsilon > 0$ and a subsequence (c_{n_t}) of (c_n) such that $c_{n_t} > \epsilon$ for every $t \in \mathbf{N}$ (otherwise replace c_n by $-c_n$).

Consider x_n, $n \in \mathbf{N}$, as elements of $C(K)$, where K is the unit ball of the dual of $X = [x_n]$, the closed subspace generated by (x_n), with respect to the weak*-topology. Since (x_n) converges with respect to the w^*-topology to a function $x^{**} \in X^{**} \setminus X$ (Remark 1.2) there exists a subsequence $(x_{n_{t_s}})$ of (x_{n_t}) and $\mu > 0$ such that

$$\mu \left\| \sum_{i=1}^{k} \lambda_i s_i \right\| \leq \left\| \sum_{i=1}^{k} \lambda_i x_{n_{t_i}} \right\|$$

for every $k \in \mathbf{N}$ and scalars $\lambda_1, \ldots, \lambda_k$. Set $y_s = x_{n_{t_s}}$ and $c_{n_{t_s}} = a_s$ for every $s \in \mathbf{N}$.

We will prove that the subsequence (y_s) of (x_n) has spreading model of order ξ equivalent to the summing basis (s_n) of c_0. Indeed, for every $(s_1, \ldots, s_k) \in \mathcal{F}_\xi$ and $x \in K$ we have $y_{s_0} = y_0 = 0$ and

$$\begin{aligned}
\sum_{i=1}^{k} |y_{s_i} - y_{s_{i-1}}|(x) &\leq \frac{1}{\epsilon} \sum_{i=1}^{k} a_{s_i} |y_{s_i} - y_{s_{i-1}}|(x) \\
&= \left| \frac{1}{\epsilon} \sum_{i=1}^{k} a_{s_i} \cdot \varepsilon_{s_i} (y_{s_i} - y_{s_{i-1}}) \right| (x) \quad (\text{where } \varepsilon_{s_i} \in \{-1, 1\}) \\
&\leq \frac{1}{\epsilon} a_{s_1} \|y_{s_1}\| + \frac{1}{\epsilon} \left| \sum_{\substack{i=2 \\ i \text{ odd} \\ \varepsilon_{s_i}=1}}^{k} a_{s_i} (y_{s_i} - y_{s_{i-1}}) \right| (x) \\
&\quad + \frac{1}{\epsilon} \left| \sum_{\substack{i=2 \\ i \text{ odd} \\ \varepsilon_{s_i}=-1}}^{k} a_{s_i} (y_{s_i} - y_{s_{i-1}}) \right| (x) + \frac{1}{\epsilon} \left| \sum_{\substack{i=2 \\ i \text{ even} \\ \varepsilon_{s_i}=1}}^{\infty} a_{s_i} (y_{s_i} - y_{s_{i-1}}) \right| (x) \\
&\quad + \frac{1}{\epsilon} \left| \sum_{\substack{i=2 \\ i \text{ even} \\ \varepsilon_{s_i}=-1}}^{\infty} a_{s_i} (y_{s_i} - y_{s_{i-1}}) \right| (x) \leq \frac{4}{\epsilon} + \frac{1}{\epsilon} \cdot \|(c_n)\|_\infty \cdot \|(\|x_n\|)\|_\infty = C.
\end{aligned}$$

In the last inequality we used $(*)$ and the fact that every subset H of a set F belonging to $\mathcal{F}_\xi$ is in $\mathcal{F}_\xi$ as well and that $(n_{t_{s_1}}, \ldots, n_{t_{s_k}}) \in \mathcal{F}_\xi$ for every $(s_1, \ldots, s_k)$ in $\mathcal{F}_\xi$.

Finally, for every $(s_1, \dots, s_k) \in \mathcal{F}_\xi$ and scalars $\lambda_1, \dots, \lambda_k$ we have

$$\left\| \sum_{i=1}^{k} \lambda_i y_{s_i} \right\| = \left\| \sum_{i=1}^{k} (\lambda_i + \cdots + \lambda_k)(y_{s_i} - y_{s_{i-1}}) \right\| \leq C \left\| \sum_{i=1}^{k} \lambda_i s_i \right\|,$$

which completes the proof. □

3.11. **Theorem.** *Let K be a metric space and ξ an ordinal number. Then*

$$B_1(K) \setminus S_\xi(K) = \Big\{ f \in B_1(K) : \textit{every bounded sequence } (f_n) \textit{ in } C(K) \textit{ converging pointwise to } f \textit{ is null-coefficient of order } \xi \Big\}.$$

Proof. Let $f \in B_1(K) \setminus S_\xi(K)$ and a bounded sequence (f_n) in $C(K)$ converging pointwise to f. Then (f_n) is null-coefficient of order ξ. Indeed, if (f_n) is not ξ-n.c., then according to the proof of the previous proposition, we can find a subsequence (g_n) of (f_n) and $C > 0$ such that

$$\left\| \sum_{i=1}^{k} |f_{n_i} - f_{n_{i-1}}| \right\|_\infty \leq C$$

for all $(n_1, \dots, n_k) \in \mathcal{F}_\xi$. Hence, it follows from Corollary 3.6 that $f \in S_\xi(K)$, a contradiction.

On the other hand, if $f \in S_\xi(K)$ then there exists a sequence $(f_n) \subseteq C(K)$ converging pointwise to f and $C > 0$ such that

$$\left\| \sum_{i=1}^{k} |f_{n_i} - f_{n_{i-1}}| \right\|_\infty \leq C$$

for every $(n_1, \dots, n_k) \in \mathcal{F}_\xi$, according to Corollary 3.6. Thus, if $c_n = 1$ for every $n \in \mathbf{N}$, we have

$$\begin{aligned} \left\| \sum_{i=1}^{k} (f_{n_{2i}} - f_{n_{2i-1}}) \right\|_\infty &\leq \left\| \sum_{i=1}^{k} |f_{n_{2i}} - f_{n_{2i-1}}| \right\|_\infty \\ &\leq \left\| \sum_{i=1}^{2k} |f_{n_i} - f_{n_{i-1}}| \right\|_\infty \leq C \end{aligned}$$

for every $(n_1, \dots, n_{2k}) \in \mathcal{F}_\xi$. Hence (f_n) is not null-coefficient of order ξ.

This completes the proof. □

3.12. **Corollary.** *Let K be a compact metric space. Then*

$$B_1(K) \setminus B_{1/4}(K) = \Big\{ f \in B_1(K) : \textit{every bounded sequence } (f_n) \textit{ in } C(K) \textit{ converging pointwise to } f \textit{ is null-coefficient of order } 1 \Big\}.$$

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