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K-THEORETIC CLASSIFICATION FOR CERTAIN INDUCTIVE LIMIT Z_2 ACTIONS ON REAL RANK ZERO C*-ALGEBRAS

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Abstract. In this paper a K-theoretic classification is given of the C*-algebra dynamical systems $(A, \alpha, Z_2) = \varinjlim(A_n, \alpha_n, Z_2)$ where A is of real rank zero, each A_n is a finite direct sum of matrix algebras over finite connected graphs, and each α_n is induced by an action on each component of the spectrum of A_n. Corresponding to the trivial actions is the K-theoretic classification for real rank zero C*-algebras that can be expressed as finite direct sums of matrix algebras over finite graphs obtained in Mem. Amer. Math. Soc. no. 547, vol. 114.

1. Introduction

In this paper a K-theoretic classification is given of certain C*-dynamical systems. Let A be a unital C*-algebra, let G be an abelian compact group and let α be an action of G on A, one can form the C*-dynamical system (A, G, α). The K-theory data for the system will be (i) $K_*(A) = K_0(A) \oplus K_1(A)$ together with the graded dimension range defined in [6]: The pairs $([e], [u]) \in K_*(A)$, where e is a projection in A and u is a partial unitary with support (and range) e. The action α_* on $K_*(A)$ induced by α; (ii) the K-group $K_*(A \times_\alpha G)$ of the crossed product $A \times_\alpha G$ together with the graded dimension range, the action $\hat{\alpha}_*$ on $K_*(A \times_\alpha G)$ induced by the dual action $\hat{\alpha}$ and the special element in $K_0(A \times_\alpha G)^+$ corresponding to the projection obtained by averaging the canonical unitaries of the crossed product; and (iii) the natural map $K_*(A) \to K_*(A \times_\alpha G)$.

The C*-algebras involved in the present classification will be real rank zero C*-algebras (the set of invertible selfadjoint elements is dense in the set of all selfadjoint elements) that can be expressed as inductive limits of finite direct sums of matrix algebras over finite graphs. We will restrict our attention to the group $Z/2Z$. The actions will be certain inductive limit actions induced by the actions on the graphs. We will show that for this class of C*-dynamical systems the K-theory data described above is a complete invariant.

Recently, a number of people constructed various inductive limit actions on this type of C*-algebras and answered some difficult questions ([1] [3] [4] [8]). The present paper was motivated by these works and by the classification results obtained in [6] and [11].

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There were some results along this direction. In the case that the actions are inductive limit inner actions of a compact group on finite-dimensional C*-algebras, or on finite direct sums of matrix algebras over cylinders, the similar classifications, with a slightly different invariant, were obtained in [10] [4] (also see [9]). For general inductive limit Z_2 actions on finite-dimensional C*-algebras, the classification was obtained in [7].

2. Actions

2.1. Let X be a finite connected graph and let σ be an order two action on X. σ induces an order two action α on $C(X)$ or an order two action $1_n \otimes \alpha$ on $M_n \otimes C(X) = M_n(C(X))$. Still denoting this action by α we will call $M_n(C(X))$ a basic building block and $(M_n(C(X)), \alpha, Z_2)$ a basic building block C*-dynamical system.

In this paper, we will consider the following type of C*-dynamical systems $(A, \alpha, Z_2) = \varinjlim(A_n, \alpha_n, Z_2)$: (i) A is of real rank zero, (ii) A_n is a finite direct sum of basic building blocks, and (iii) $\alpha = \varinjlim \alpha_n$ with each α_n an Z_2 action induced by an action on the spectrum of A_n preserving connected components.

For the sake of notation, we will omit Z_2 in a C*-dynamical system (B, β, Z_2) and denote it by (B, β).

2.2. We first describe an order two action on a basic building block. Let X be a finite connected graph and let σ be a Z_2 action on X. Denote by V' the set of the vertices of X. A point on an open edge is not a vertex in the ordinary sense. In our context, we can call this point a vertex without changing the C*-algebra. So we will add some new vertices to our graphs. Let V be the finite subset of X consisting of $V' \cup \sigma(V')$ and all the isolated fixed points. Now σ has the following properties: $\sigma(V) = V$ and if L is an edge of X, $\sigma(L)$ is another edge unless σ acts as an identity on L. In this paper, we will always choose the vertices of X (in connection with σ) this way.

Since V is closed, one may have the quotient space X/V and the quotient map π: $X \to X/V$, sending x to $[x]$. σ induces an order two action $\tilde{\sigma}$ on X/V by $\tilde{\sigma}([x]) = [\sigma(x)]$. Clearly, π intertwines σ and $\tilde{\sigma}$. Now X/V is a graph of several circles joined at a single point. The fixed points of $\tilde{\sigma}$ is this vertex possibly together with some other circles. In the following, we will call a graph of this type a special graph. We will use this construction in section 4.

3. Perturbation

3.1. Let $A = M_n(C(X))$ and $B = M_m(C(Y))$ be two basic building blocks and let α and β be two Z_2 actions on A and B, induced by two actions σ and τ on X and Y, respectively. For a unital equivariant $*$-homomorphism ϕ from (A, α) to (B, β), one has

$$\phi(f \circ \sigma)(t) = \phi(f)(\tau(t))$$

for $t \in Y$ and $f \in A$.

Let $t_0 \in Y$. $\phi(\cdot)(t_0)$ is a representation of A in M_m. Since X is the spectrum of A, there exists $\{\lambda_i\}_{i=1}^k \subset X$ and there exists a unitary $U \in M_m$ such that $\phi(\cdot)(t_0)$

has the following expression:

$$\phi(f)(t_0) = U \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_k) \end{pmatrix} U^*, \quad f \in A,$$

where $k = m/n$. Here $\{\lambda_i\}_{i=1}^k$ and U depend on t_0. The following formula gives a representation of ϕ at $\tau(t_0)$:

$$\phi(f)(\tau(t)) = U \begin{pmatrix} f(\sigma(\lambda_1)) & & \\ & \ddots & \\ & & f(\sigma(\lambda_k)) \end{pmatrix} U^*, \quad f \in A.$$

In particular, $\sigma(\lambda_i) = \lambda_i \quad (i = 1, 2, \ldots, k)$ if $\tau(t_0) = t_0$.

From now on, we will call $\{\lambda_i\}_{i=1}^k$ the points corresponding to the representation of ϕ at t_0.

Theorem. *Let $(A, \alpha), (B, \beta)$ and ϕ be as above. For any $\epsilon > 0$ and any finite subset $F \subset A$, there exists a unital $*$-homomorphism ϕ' from (A, α) to (B, β) such that*

(1) $\|\phi(f) - \phi'(f)\| < \epsilon$ *for* $f \in F$,
(2) *On each edge L of Y, identified with $I = [0, 1]$,*

$$\phi'(f)(t) = W(t) \begin{pmatrix} f(s_1(t)) & & \\ & \ddots & \\ & & f(s_k(t)) \end{pmatrix} W^*(t), \qquad t \in I,\ f \in A,$$

where $W(\cdot)$ is a unitary in $M_n(C(I))$, $\{s_i(\cdot)\}_{i=1}^k \subset C(I, X)$ and $k = m/n$.

Proof. First, we remark that in the case where L is a circle, $W(0)$ may not be equal to $W(1)$ and $s_i(0)$ may not be $s_i(1)$. But the two representations will agree.

The proof basically follows from the proof of Theorem 3.1 in [11], in which case there were no actions. Certain modifications of that proof will provide for ϕ' to be equivariant.

Let $S \subset Y$ be the closure of a fundamental domain of τ in the following sense. S is a closed subset of X that contains all the fixed points of τ together with a collection of some edges of Y. For an edge L of Y in this collection, $\tau(L)$ is not in this collection. Notice that S and X have the same vertices.

First, define ϕ' at the vertices of S as in the proof of Theorem 3.1 of [11] (before Step 1). Let $y_0 \in S$ be a vertex, then ϕ has the representation, say:

$$\phi(f)(y_0) = U \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_k) \end{pmatrix} U^*, \quad f \in A,$$

where $k = m/n$, $\{\lambda_i\}_{i=1}^k \subset X$ and U a unitary in M_m (depending on y_0). In the case that y_0 is a fixed point, the points $\{\lambda_i\}_{i=1}^k$ are in the fixed point set of σ. We may perturb $\{\lambda_i\}_{i=1}^k$ a little so that they are different except for those isolated fixed points of σ. This will be our ϕ' at y_0.

If y_0 is not a fixed point and y_0' is another vertex of S with $\tau(y_0) = y_0'$, then we define ϕ' to satisfy the equation $\phi'(f)(y_0') = \phi'(f \circ \sigma)(y_0)$ for all $f \in A$. Notice that in this case, the evaluation points $\{\lambda_i\}_{i=1}^k$ for ϕ' at y_0 can be chosen to be different since we can move them around a little. Continuing this way, ϕ' is defined on the vertices of S.

Let L be an edge of Y where τ acts as an identity. One can define ϕ' on L to have the desired form in (2) of the present theorem the same way as in the proof of Theorem 3.1 of [11] (Step 1 to Step 4), provided that all $s_i(t)$ are in the fixed point set of σ. We now show that this is possible. The construction in [11], roughly speaking, is as follows. Divide L into small intervals. Let $[t_i, t_{i+1}]$ be such an interval. We have

$$\phi(f)(t_i) = U_i \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_k) \end{pmatrix} U_i^*, \quad f \in A,$$

and

$$\phi(f)(t_{i+1}) = U_{i+1} \begin{pmatrix} f(\lambda_1') & & \\ & \ddots & \\ & & f(\lambda_k') \end{pmatrix} U_{i+1}^*, \quad f \in A.$$

$\{\lambda_i\}_{i=1}^k$ and $\{\lambda_i'\}_{i=1}^k$ are points in X (for us, they are in the fixed point set of σ). U_i and U_{i+1} are two unitaries in M_m. Because of the continuity of ϕ, if t_i and t_{i+1} are close enough, these two groups of points are within $1/2$, up to permutation, one by one. Notice that the construction of ϕ' on $[t_i, t_{i+1}]$ in the proof of Theorem 3.1 of [11] was to connect each pair of the points (Step 1) and the unitaries U_i and U_{i+1} (Step 2 to Step 4). Notice that the fixed point set of σ is a disjoint union of finitely many connected components. Two different components are of distance one. It is clear now that we can connect each pair inside the fixed point set of σ. In this way, ϕ' has been defined on L.

For $L \subset S$ with $\tau(L)$ not being contained in S, we define ϕ' as in [11].

Note that at the ends of L (for any L), these representations and the previously defined ones may differ by some unitaries. However, the map is well-defined. Now ϕ' is defined on S.

To complete the construction of ϕ' on Y, one uses τ to transfer the definition of ϕ' on S to $Y \setminus S$. More precisely, one defines ϕ' by

$$\phi'(f)(t) = \phi'(f \circ \sigma)(\tau^{-1}(t)), \qquad t \in Y \setminus S, \quad f \in A.$$

It is clear that ϕ' is a desired map. ∎

Remark. We need a similar theorem for maps between finite direct sums of basic building block C*-dynamical systems. We can always assume that the second algebra is a single block. Suppose that $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$. Denote by P_i the central projection corresponding to the i^{th} block. Since each P_i is in the fixed point subalgebra, there is a unitary $U \in B^\beta$ such that $Ad(U)(P_i)$ is a diagonal constant projection for all i. Hence, $Ad(U) \circ \phi$ can be approximated by $\psi = \phi_1 \oplus \phi_2 \oplus \cdots \oplus \phi_k$ on a given finite set $F \subset A$ within given ϵ, where each ϕ_i has the form of ϕ' in the theorem. Furthermore, ψ is equivariant. Now, it is clear that ϕ can be approximated by the equivariant $*$-homomorphism $Ad(U)\psi$ on F to within ϵ.

4. Sequences with special graphs

In this section we will show that one can replace arbitrary graphs by special graphs in the sense of 2.2, possibly with multiple vertices (see 4.1), for a given sequence of C*-dynamical systems in the sense of 2.1. On the other hand, the actions involved may have more complicated forms.

4.1. Let X be a finite connected graph. A C*-algebra A will be said to be a basic building block with non-Hausdorff spectrum if A is isomorphic to a sub-C*-algebra of $M_n(C(X))$ of the following form:

$$\{f \in M_n(C(X)) | f \text{ has diagonal block forms at the vertices}\}$$

We will say that A has non-Hausdorff spectrum X (cf. [11]). Sometimes, we will simply denote $A \subset M_n(C(X))$.

Let A be as above. Denote by $L_1, L_2, \ldots, L_k$ the edges of X. There is a natural embedding $\imath$ of A into $\bigoplus_{i=1}^k M_n(C(L_i))$. For a unitary $V \in \bigoplus_{i=1}^k M_n(C(L_i))$, one can associate it with another embedding, i.e., $\imath$ followed by AdV.

Definition. Let A be as above, let $F \subset A$ be a finite subset and let $\epsilon > 0$. We say that F is approximately constant to within ϵ if there exists a unitary $V = \bigoplus_{i=1}^k V_i \in \bigoplus_{i=1}^k M_n(C(L_i))$ such that for any $f \in F$, $s \in L_j$ and $t \in L_i$

$$\|AdV_j(s)(f(s)) - AdV_i(t)(f(t))\| < \epsilon.$$

4.2. For a basic building block $A \subset M_n(C(X))$, one can associate it with a set of selfadjoint elements called test functions [11]. For $M \geq 1$ and $S \subset X$, a closed subset, define

$$K_{S,M}(t) = \mathrm{Max}\{K_{x,M}(t) | x \in S\},$$

where

$$K_{x,M}(t) = \begin{cases} 1, & t = x, \\ 0, & t \in \{t | d(t,x) > 1/M\}, \\ 1 - Md(t,x), & t \in \{t | d(t,x) \leq 1/M\}. \end{cases}$$

$K_{S,M} \otimes I_n \in A$ will be called a test function. It is in the center of A. For any $\delta > 0$, there is a finite subset of test functions which is δ-dense in all the test functions (for fixed M) (cf. Lemma 2.3 [11]).

Theorem. *Let $A = M_n(C(X))$ and $B = M_m(C(Y))$ be two basic building blocks, let α and β be two order two actions on A and B, induced by σ and τ, acting on X and Y, respectively, let $F \subset A$ be a finite subset and let ϕ be a unital $*$-homomorphism from (A, α) to (B, β). For $0 < \delta < \epsilon < 1$, suppose that*

(1) *$\|f(x) - f(x')\| < \epsilon$ whenever $d(x, x') < \delta$ and $f \in F$.*
(2) *The eigenvalues of $\phi(h)(t)$ and $\phi(h)(s)$ are within $\delta' = \delta/2^{a+5}$ one by one, in increasing order, for all the test functions $h \in A$ associated with $M = 1$ and for any s and t in Y. Here a is the number of the vertices of X.*

Then it follows that

(i) *There is a sub-C*-dynamical system $(B^0, \beta) \subset (B, \beta)$ where B^0 is isomorphic to a basic building block with spectrum a special graph, possibly non-Hausdorff, and there exists a unital $*$-homomorphism ϕ_1 from (A, α) to (B^0, β) such that*

$$\|\phi(f) - \phi_1(f)\| < \epsilon, \quad f \in F.$$

(ii) *On each edge L of the spectrum of B^0, without the ends being identified, there exists a unitary $W \in M_m(C(L))$ and there exists $k = m/n$ maps $\{\xi_i\}_{i=1}^k \subset C(L, X)$ such that for $t \in L$ and $f \in A$,*

$$\phi_1(f)(t) = W(t) \begin{pmatrix} f(\xi_1(t)) & & \\ & \ddots & \\ & & f(\xi_k(t)) \end{pmatrix} W^*(t).$$

Furthermore, the variation of each of $\{\xi\}_{i=1}^k$ over L is less than δ. As a consequence, $\phi_1(F)$ is approximately constant to within ϵ in B^0.

(iii) *(B^0, β) is isomorphic to $(D, \tilde{\beta})$ where D is a basic building block with special graph $\tilde{Y}$ as its spectrum and*

$$[t] \in \tilde{Y}, \quad g \in D,$$

for some unitary $U \in M_m$.

Proof. The proof follows closely the proof of Theorem 5.3 of [11]. The difference is that ϕ_1 should be equivariant. This can be achieved by first defining ϕ_1 on a fundamental domain of τ and then transferring it to the other part of Y, as in the proof of Theorem 3.1. We now sketch the construction.

For any point $y \in Y$, there are k points corresponding to the representation of ϕ at Y. We want to find a group of k points of X which is invariant under σ (as a set) such that these k points are within δ' one by one with the k points corresponding to the i representation of ϕ at any point Y. By (2), two groups of points in X corresponding to the representations of ϕ at two different places are within δ' one by one. This follows from Lemma 2.3 in [11]. Fix $t_0 \in Y$ and let $\{x_i\}_{i=1}^k$ be the corresponding points in the representation of ϕ at t_0. If $\tau(t_0) = t_0$, then $x_i = \sigma(x_i)$ and $\{x_i\}_{i=1}^k = \{\sigma(x_i)\}_{i=1}^k$. If τ has no fixed point, we do not have this. We now select a group of points $\{x_i'\}_{i=1}^k$ to satisfy $\{x_i'\}_{i=1}^k = \{\sigma(x_i')\}_{i=1}^k$. We will require that $\{x_i'\}_{i=1}^k$ be close to $\{x_i\}_{i=1}^k$ one by one. To see this, first we notice that $\{\sigma(x_i)\}_{i=1}^k$ are the points corresponding to the representation of ϕ at $\tau(t_0)$. Hence, there is a pairing between $\{x_i\}_{i=1}^k$ and $\{\sigma(x_i)\}_{i=1}^k$ such that the two points in each pair are within δ'. Let us denote these pairs by $\{(x_i, \sigma(x_{n_i}))\}_{i=1}^k$. We prove our claim by induction on k. For $k = 1$, x_1 and $\sigma(x_1)$ are within δ'. If they are the same point, we take $x_1' = x_1$. So we assume that they are different. Suppose that $x_1 \in L$ is an edge of X. $\sigma(x_1)$ must be on another edge (cf. 2.2). So x_1 is within δ' to an end point of L which must be an fixed point of σ. Otherwise, x_1 and $\sigma(x_1)$ should be at least of distance one. We now define x_1' to be this fixed point. This completes the proof for $k = 1$. Suppose that the claim is true for integers less than or equal to $k - 1$. We now show that this is also true for k. If $i = n_i$ for some i, then x_i must be close to a fixed point of σ as we argued in the case $k = 1$. We then take this fixed point to be x_i' and apply the induction to the rest of the $k - 1$ pairs of points. If there is no such pair, we let $x_1' = x_1$ and $x_{n_1}' = \sigma(x_1)$ and apply the induction to the rest of the $k - 2$ pairs of points. This gives us the desired points $\{x_i'\}_{i=1}^k$. We will denote $\{x_i'\}_{i=1}^k$ by $\{x_i\}_{i=1}^k$ again. We will fix t_0 and $\{x_i\}_{i=1}^k$ in the remainder of the proof.

By Theorem 3.1, we may assume that ϕ takes the form ensured by the theorem. Namely, on each edge L of Y,

$$\phi(f)(t) = W(t)\begin{pmatrix} f(s_1(t)) & & \\ & \ddots & \\ & & f(s_k(t)) \end{pmatrix} W^*(t), \qquad t \in L,\ f \in A.$$

As in the the proof of Theorem 5.3 of [11], ϕ_1 will be defined by changing $s_i(t)$ to $\xi_i(t)$ so that at finitely many points $\{t_j\}_{j=1}^b \subset Y$ which contains all the vertices of Y and many other points we will specify later, $\{\xi_j(t)\}_{i=1}^k = \{x_i\}_{i=1}^k, j = 1, 2, \cdots, b$, and so that $d(\xi_i(t), s_i(t)) < \delta$ for all $t \in L$.

Let $S \subset Y$ be a fundamental domain of τ as described in the beginning of the proof of Theorem 3.1. There are two cases to be considered.

Case (1). $\tau(t_0) = t_0$.

Let $L \subset S$ be an edge which is fixed by τ (if there is such an edge). Divide L into small intervals $\{[a_i, a_{i+1}]\}$. Using the construction in Theorem 5.3 of [11] and Theorem 3.1 above, we may define ϕ_1 on L by:

$$\phi_1(f)(t) = W(t)\begin{pmatrix} f(\xi_1(t)) & & \\ & \ddots & \\ & & f(\xi_k(t)) \end{pmatrix} W^*(t), \quad f \in A.$$

Here $\{\xi_i(t)\}_{i=1}^k = \{x_i\}_{i=1}^k$ for $t \in \{a_i\}$, the partition points, (cf. Theorem 5.3 [11]). The only difference from that of Theorem 5.3 of [11] is that $\{s_i(t)\}_{i=1}^k$ must be in the fixed point set of σ. The construction of $\xi_j(t)$ on $[a_i, a_{i+1}]$ in Theorem 5.3 of [11] was to connect one point in the points corresponding to the representation of ϕ at a_i and another point in the points corresponding to the representation of ϕ at a_{i+1}. These two points were close. In our case, they are all from $\{x_i\}_{i=1}^k$, up to a small perturbation within δ'. Hence, $\xi_i(t)$ stays in the fixed point set of σ. So they must be in the same connected component of the fixed points of σ.

Let $L' \subset S$ be another edge such that $\tau(L')$ is not in S. We define ϕ_1 exactly the same way as in the construction of Theorem 5.3 in [11]. The map has the expression:

$$\phi_1(f)(t) = W'(t)\begin{pmatrix} f(\xi_1'(t)) & & \\ & \ddots & \\ & & f(\xi_k'(t)) \end{pmatrix} W^{'*}(t), \quad f \in A.$$

Again, $\{\xi_i(t)\}_{i=1}^k = \{x_i\}_{i=1}^k$ for finitely many $t \in L'$.

On $\tau(L')$, we define ϕ_1 by using the formula:

$$\phi_1(f)(\tau(t)) = \phi_1(f \circ \sigma)(t), \quad t \in L', \quad f \in A.$$

We remark that if $\{\xi_i'(t)\}_{i=1}^k = \{x_i\}_{i=1}^k$, then $\{\sigma(\xi_i'(t))\}_{i=1}^k = \{x_i\}_{i=1}^k$.

Case (2). τ has no fix point.

We define ϕ_1 the same way as in the Case (1). The difference in the two situations is that we do not have $\sigma(x_i) = x_i$. Rather, we have $\{x_i\}_{i=1}^k = \{\sigma(x_i)\}_{i=1}^k$ as sets.

In summary, there exists $\{t_i\}_{i=1}^b \subset Y$, containing all the vertices of Y and many other points such that

$$\phi_1(f)(t_i) = V_i \begin{pmatrix} f(x_1) & & \\ & \ddots & \\ & & f(x_k) \end{pmatrix} V_i^*, \quad f \in A.$$

We obtained these points by dividing each edge of Y. They were chosen to satisfy the following: ϕ_1 has a representation in each small interval, say $[t_i, t_{i+1}]$,

$$\phi_1(f)(t) = W(t) \begin{pmatrix} f(\xi_1(t)) & & \\ & \ddots & \\ & & f(\xi_k(t)) \end{pmatrix} W^*(t), \quad f \in A,$$

where the variation of each $\xi_i(t)$ is less than δ over $[t_i, t_{i+1}]$ and $d(\xi_i(t), s_i(t)) < \delta$. By refining $\{t_i\}_{i=1}^b$ at the beginning of the construction we may assume that this set is invariant under τ.

Let W be a permutation matrix such that for all $f \in A$,

$$W \begin{pmatrix} f(\sigma(x_1)) & & \\ & \ddots & \\ & & f(\sigma(x_k)) \end{pmatrix} W^* = \begin{pmatrix} f(x_1) & & \\ & \ddots & \\ & & f(x_k) \end{pmatrix}.$$

By relabeling $\{x_i\}_{i=1}^k$, we choose W as follows:

$$W = \begin{cases} I_n \otimes I_k & \text{if } \tau \text{ has a fixed point}, \\ \begin{pmatrix} & & I_n \otimes I_{k_1} \\ & I_n \otimes I_{k_2} & \\ I_n \otimes I_{k_1} & & \end{pmatrix} & \tau \text{ has no fixed point}, \end{cases}$$

where $2k_1 + k_2 = k$. k_2 is the number of fixed points in $\{x_i\}_{i=1}^k$. When σ has no fixed points in $\{x_i\}_{i=1}^k$, $k_2 = 0$. When σ fixes every point of $\{x_i\}_{i=1}^k$, $k_1 = 0$. W depends on the choice of t_0 (and ϕ) at the beginning of the proof. Changing t_0 may change W. But it still has one of the above forms. These are the special cases. Sometimes, we will simply write those identities by 1 with the understanding that they should have suitable sizes. If $\tau(t_i) = t_j$, we can choose $V_j = V_i W$.

B^0 will be a C*-subalgebra of B. The elements in B^0 will be required to satisfy certain conditions on $\{t_j\}_{i=1}^b$. We divide the construction into two cases.

Case (1). τ has a fixed point.

In this case, we will define:

$$B^0 = \{g \in B \mid g(t_i) = g(t_j), \text{ for all } t_i, t_j\}.$$

Case (2). τ has no fixed point.

We define B^0 to be the C*-subalgebra of B that satisfies the following two conditions:

(i) For t_i, $t_{i'} \in S \cap \{t_j\}_{i=1}^b$,

$$V_i^* g(t_i) V_i = V_{i'}^* g(t_{i'}) V_{i'} = \begin{pmatrix} a & & \\ & b & \\ & & \tilde{a} \end{pmatrix},$$

where $a, \tilde{a} \in M_{nk_1}$ and $b \in M_{nk_2}$.

(ii) For $t_j = \tau(t_i)$,

$$W^* V_i^* g(t_j) V_i W = V_i^* g(t_i) V_i.$$

Clearly, by our choice of V_j, the two conditions agree if t_i and $t_j = \tau(t_i)$ are in S. $B^0 \subset B$ is a unital C*-subalgebra and $\phi_1(A) \subset B^0$. Furthermore, for $g \in B^0$ and $t_i, t_{i'} \in S \cap \{t_j\}_{j=1}^b$

$$\begin{aligned} V_i^*(\beta(g))(t_i)V_i &= V_i^* g(\tau(t_i))V_i \\ &= W V_i^* g(t_i) V_i W^* \\ &= W \begin{pmatrix} a & & \\ & b & \\ & & \tilde{a} \end{pmatrix} W^* = \begin{pmatrix} \tilde{a} & & \\ & b & \\ & & a \end{pmatrix} \\ &= V_{i'}^*(\beta(g))(t_i')V_{i'} \end{aligned}$$

and for $t_j = \tau(t_i)$, notice that $W^* = W$, we have

$$W^* V_i^*(\beta(g))(t_j) V_i W = W^* V_i^* g(t_i) V_i W = V_i^*(\beta(g))(t_i) V_i$$

so $\beta(g) \in B^0$. Hence $\beta(B^0) = B^0$.

We now change B^0 into a form so that it is easier to see that its spectrum is a special graph. This identification also carries the action. The new action is no longer of the previous form. Let $V \in B$ be a unitary such that $V(t_i) = V_i$ and $V(t_j) = V_i W$ if $\tau(t_i) = t_j$. In Case (1), $W = 1$. So we may choose $V \in B^0$. In general, V is not in B^0. Define $D = AdV^*(B^0)$. Then we have:

Case (1).

$$AdV^* B^0 = \{g \in B \ \mid \ g(t_i) = g(t_j) \text{ for all } t_i, t_j\}.$$

D is isomorphic to $M_m(C(\tilde{Y}))$ where $\tilde{Y}$ is obtained by identifying all the points of $\{t_i\}_{i=1}^b$. The special graph $\tilde{Y}$ is Hausdorff. The action $\tilde{\beta}$ on D has the form

$$\tilde{\beta}(g)([t]) = g([\tau(t)]).$$

Case (2).

$$AdV^* B^0 = \left\{ g \in B \ \mid \ g(t_i) = g(t_j) = \begin{pmatrix} a & & \\ & b & \\ & & \tilde{a} \end{pmatrix} \text{ for all } t_i, t_j \right\}.$$

$D = AdV^* B^0$ is a basic building block with a special graph coming from identifying all t_i of Y, as its (non-Hausdorff) spectrum. At the only vertex, there are three points.

Notice that $V^*(t_i)V(\tau(t_i)) = W$ for all t_i. We can induce a Z_2 action $\tilde{\beta}$ on D by

$$\tilde{\beta}(g)[t] = V^*(t)V(\tau(t))g([\tau(t)])V^*(\tau(t))V(t),$$

where $[t]$ is in the spectrum of D. Now $(B^0, \beta) \cong (D, \tilde{\beta})$ under AdV^*.

In fact, in the two cases, we can choose V in such a way that $W = V^*(t)V(\tau(t))$. Recall that the unitary W, corresponding to a pair $(t_i, \tau(t_i))$ does not depend on t_i. It only depends on whether τ has fixed point or not and how many fixed points are in $\{x_i\}_{i=1}^k$. Let $S \subset Y$ be a fundamental domain, define V on S so that $V(t_i) = V_i$. Notice that if $t_i, t_j \in S$ such that $\tau(t_i) = t_j$, then $V(t_j) = V_i W$. The construction of V on S is first at the vertices of S and then interpolated in between. On $Y \setminus S$, define $V(t) = V(\tau^{-1}(t))W$. It is readily checked that V is well defined. Now $V^*(t)V(\tau(t)) = W$. So $\tilde{\beta}$ is an order two action on the special graph composing with $Ad(W)$. □

Remark. (a) The theorem can be extended to the case that the first algebra is a finite direct sum of basic building blocks. To see this, notice that the central projections of the first algebra are in the fixed point algebra. Hence, there is a unitary U in the fixed point algebra of the second algebra so that the images of these projections under ϕ composed with AdU are diagonal and constant. Our B^0 will be a direct sum of subalgebras of each block cut down by the corresponding projection. On each block, one can apply the theorem. (B^0, β) will be isomorphic to a direct sum of the type of $(D, \tilde{\beta})$ obtained in the theorem. In particular, restricted to a block, W will still has one of the two forms.

(b) Another point we would like to make is that when t_0 is a fixed point of τ, $\{x_i\}_{i=1}^k$ are fixed points of σ. As a consequence, $\xi_i(t)$ can be chosen to stay inside the fixed point set of σ.

4.3. Let $(A, \alpha) = \varinjlim(A_n, \alpha_n)$ be an inductive limit C*-dynamical system in the sense of 2.1. After passing to subsequences and changing notations, one can have the following approximate intertwining, in the sense of [6],

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & \cdots & \longrightarrow & A \\ & \searrow & \uparrow & \searrow & \uparrow & & & & \\ & & A_2^0 & \longrightarrow & A_3^0 & \longrightarrow & \cdots & \longrightarrow & A^0 \end{array}$$

where each A_i^0 is a finite direct sum of C*-algebras of the form obtained in the Theorem 4.2. By Theorem 2.3 [6], A^0 is isomorphic to A. To show that the isomorphism ϕ from A to A^0 is also equivariant, recall that ϕ could be defined (see [6]) as follows: One first defines a map ϕ_i from A_i to A^0 by going across n steps, down, and across and letting n go to infinity. Clearly, ϕ_i is equivariant. ϕ is the map induced by $\{\phi_i\}$ and hence is equivariant.

In another word, we have the following approximate intertwining (with equivariant maps):

$$\begin{array}{ccccccccc} (A_1, \alpha_1) & \longrightarrow & (A_2, \alpha_2) & \longrightarrow & (A_3, \alpha_3) & \longrightarrow & \cdots & \longrightarrow & (A, \alpha) \\ & \searrow & \uparrow & \searrow & \uparrow & & & & \\ & & (A_2^0, \tilde{\alpha}_2) & \longrightarrow & (A_3^0, \tilde{\alpha}_3) & \longrightarrow & \cdots & \longrightarrow & (A^0, \tilde{\alpha}) \end{array}$$

which induces an isomorphism from (A, α) to $(A^0, \tilde{\alpha})$. So we can replace the original sequence by a new sequence so that the spectrum of each basic building block in the new sequence is of special form. The actions are a little more complicated than those that only act on the graphs (cf. Theorem 4.2 and Remark 4.2 (a)). We will describe them in 5.1 below when we deal with the so-called existence theorem.

4.4. A consequence of the above is the following:

Proposition. *Let $(a, \alpha) = \varinjlim(A_n, \alpha_n)$ be an inductive limit C*-dynamical system. Suppose that A is of real rank zero, suppose that each A_n is a finite direct sum of basic building blocks and suppose that each α_n is an Z_2 action induced by an Z_2 action σ_n on the spectrum of A_n (preserving each connected component). If each σ_n has a fixed point on each component, then α is trivial.*

Proof. By 4.3, we may assume that all the connecting maps have the properties of ϕ_1 in Theorem 4.2. By remark (b) of 4.2, we may only need a quotient of each A_n, say $\tilde{A}_n$ with the spectrum the fixed point subset of σ_n. We have a commuting diagram:

$$\begin{array}{ccccccccc} (A_1,\alpha_1) & \longrightarrow & (A_2,\alpha_2) & \longrightarrow & (A_3,\alpha_3) & \longrightarrow & \cdots & \longrightarrow & (A,\alpha) \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & & & & \\ (\tilde{A}_1,\tilde{\alpha}_1) & \longrightarrow & (\tilde{A}_2,\tilde{\alpha}_2) & \longrightarrow & (\tilde{A}_3,\tilde{\alpha}_3) & \longrightarrow & \cdots & \longrightarrow & (\tilde{A},\tilde{\alpha}) \end{array}$$

where $\tilde{\alpha}_i$ is trivial. Hence, (A,α) is isomorphic to $(\tilde{A},\tilde{\alpha})$ with $\tilde{\alpha}$ trivial. □

Remark. If each σ_n has finitely many fixed points, A is AF.

5. Existence

5.1. We will first describe the crossed product and its K-theory data of a basic building block dynamical system obtained in the Theorem 4.2. Let X be a special graph with the only vertex x_0, let $A \subset M_n(C(X))$ be a basic building block, and let α defined as

$$\alpha(f)(t) = Wf(\sigma(t))W^*, \quad t \in X, \quad f \in A,$$

be an Z_2 action. There are two cases: $W = 1$ and $W = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$ to be considered. Here the 1 in the upper right corner and lower left corner are the same size and the one in the middle may have a different size (and one of them could be zero, cf. 4.2).

Type 1: In this case α is implemented by $W = 1$ and σ. X has $l + 2k$ edges (circles joined at the only vertex x_0): $L_1, L_2, \cdots, L_l, L_{11}, \cdots, L_{1k}, L_{21}, \cdots, L_{2k}$. σ fixes $L_1, \cdots, L_l$ and maps L_{1i} to L_{2i}. (l or k could be zero. They are special cases.) To compute $K_*(A)$, let $I \in A$ be the ideal with those elements that vanish at x_0. One has the following four-term exact sequence coming from the six-term exact sequence:

$$0 \longrightarrow K_0(A) \longrightarrow K_0(A/I) \longrightarrow K_1(I) \longrightarrow K_1(A) \longrightarrow 0.$$

It is then ready to check that $K_0(A) = K_0(A/I) = Z$ and $K_1(A) = K_1(I) = Z^l \oplus Z^k \oplus Z^k$. Here each copy of Z corresponds to a circle of X. α_* acts trivially on $K_0(A)$ and interchanges the last two components of $K_1(A)$.

We now compute the crossed product. Let $S = L_1 \vee \cdots \vee L_l \vee L_{11} \vee \cdots \vee L_{1k}$. S is a fundamental domain of σ. For any $f + gU_\alpha \in A \times_\alpha Z_2$, we map it to $\begin{pmatrix} f & g \\ g \circ \sigma & f \circ \sigma \end{pmatrix}$ in the following algebra:

$$B = \left\{ F \in M_{2n}(C(S)) | F = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \text{on } L_1 \vee \cdots \vee L_l \right\}.$$

Here U_α is the canonical unitary in the crossed product. Let $R = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. Then

$$Ad(R)(B) = \{ F \in M_{2n}(C(S)) | F = \mathrm{diag}(a,b) \text{ on } L_1 \vee \cdots \vee L_l \}.$$

This gives an isomorphism of $A \times_\alpha Z_2$ with $Ad(R)(B)$, isomorphic to a basic building block with spectrum S and with multiple edges. Namely, the elements of

$Ad(R)(B)$ are of block form on $L_1 \vee \cdots \vee L_l$. Now $K_0(A \times_\alpha Z_2)$ can be identified with Z^2 and $K_1(A \times_\alpha Z_2)$ can be identified with $Z^l \oplus Z^l \oplus Z^k$. This comes directly from the six-term exact sequence of the K-groups by taking the ideal of $A \times_\alpha Z_2$ which contains all the elements that vanish at x_0. $\hat{\alpha_*}$ can be described as interchanging the two components of K_0 and the first two components of K_1. Here $\hat{\alpha}(f + gU_\alpha) = f - gU_\alpha$ is the dual action. The special element can be computed, after compressing the identifications, as:

$$\left[\frac{1}{2}(I + U_\alpha)\right] = \left[\frac{1}{2}((I, I) + (I, -I))\right] = (n, 0).$$

The natural map from $K_0(A)$ to $K_0(A \times_\alpha Z_2)$ is by sending x to (x, x). The natural map from $K_1(A)$ to $K_1(A \times_\alpha Z_2)$ is by sending (a, b, c) to $(a, a, b+c)$. These directly come from the embedding of A into $A \times_\alpha Z_2$.

Type 2: α is implemented by $W = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$ and σ. X has $2k$ edges $X = L_{11} \vee \cdots \vee L_{1k} \vee L_{21} \vee \cdots \vee L_{2k}$. σ interchanges L_{1i} with L_{2i}. There are three blocks at x_0. $K_0(A)$ can be identified with Z^3 corresponding to the three blocks. Take three rank one projections going through each block, respectively; they form a set of generators of $K_0(A)$. In particular, we may choose constant ones. $K_1(A)$ can be identified with $Z^k \oplus Z^k$. α_* interchanges the first and the last components of K_0 and the two components of K_1, $[1_A] \in K_0(A)$ is (n_1, n_2, n_1) with $2n_1 + n_2 = n$.

Let $S = L_{11} \vee \cdots \vee L_{1k}$ be a fundamental domain of σ. $A \times_\alpha Z_2$ can be identified with M_{2n} over S with fibers isomorphic to $M_{2n_1} \oplus M_{n_2} \oplus M_{n_2}$ at x_0. To be more precise, the identification is as follows: we send $f + gU_\alpha$ to $\begin{pmatrix} f & g \\ \alpha(g) & \alpha(f) \end{pmatrix}$ to obtain:

$$A \times_\alpha Z_2 = \{F \in M_{2n}(C(S)) \mid F(x_0) = \begin{pmatrix} a & b \\ WbW^* & WaW^* \end{pmatrix}\}.$$

Notice that the fiber of $A \times_\alpha Z_2$ at x_0 is isomorphic to the crossed product of $M_{n_1} \oplus M_{n_2} \oplus M_{n_1}$ by $Ad(W)$. It is $M_{2n_1} \oplus M_{n_2} \oplus M_{n_2}$ (cf. [7]). Combining them together will be our identification. $K_0(A \times_\alpha Z_2)$ is hence Z^3 and $K_1(A \times_\alpha Z_2) = Z^k$. $\hat{\alpha_*}$ acts trivially on K_1 but interchanges the last two components of K_0. The special element will be $(n_1, n_2, 0)$. The map from $K_0(A)$ to $K_0(A \times_\alpha Z_2)$ is to send (x, y, z) to $(x + z, y, y)$ and the map from $K_1(A)$ to $K_1(A \times_\alpha Z_2)$ is to send (x, y) to $(x + y)$. We remark that these computations are similar to the finite dimension situation dealt with in [7].

As we pointed out in 4.2, n_1 and n_2 could be zero.

5.2. The following is the main theorem of this section.

Theorem. *Let (A, α) and (B, β) be two basic building block C^*-dynamical systems of the types in 5.1. Suppose that $\phi = \phi_0 \oplus \phi_1$ is a dimension range preserving group homomorphism from $(K_*(A), \alpha_*)$ to $(K_*(B), \beta_*)$ mapping $[1_A]$ to $[1_B]$ and suppose that $\tilde{\phi} = \tilde{\phi}_0 \oplus \tilde{\phi}_1$ is a dimension range preserving homomorphism from $(K_*(A \times_\alpha Z_2), \hat{\alpha}*)$ to $(K_*(B \times_\beta Z_2), \hat{\beta}_*)$ mapping the special element to the special* *element. Suppose further that the following diagram commutes:*

$$\begin{array}{ccc} K_*(A) & \longrightarrow & K_*(B) \\ \downarrow & & \downarrow \\ K_*(A \times_\alpha Z_2) & \longrightarrow & K_*(B \times_\beta Z_2). \end{array}$$

Then there exists a unital $*$-homomorphism ψ from (A,α) to (B,β) such that ψ induces ϕ and the extension of ψ to $A \times_\alpha Z_2$ induces $\tilde{\phi}$.

Proof. We will use the notation in 5.1 for A. For B, we will use 'dash'. For example, $B \subseteq M_{n'}(C(X'))$, etc. Note that (in general) if we could define a unital ψ from (A,α) to (B,β) to induce ϕ, it will preserve the dimension ranges and the classes of the identity. Let $\tilde{\psi}$ be its extension to the crossed product. $\tilde{\psi}_*$ will intertwine the actions $\tilde{\alpha}_*$ and $\tilde{\beta}_*$. It will send the special element to the special element. It will also induce the commuting diagram:

$$\begin{array}{ccc} K_*(A) & \longrightarrow & K_*(B) \\ \downarrow & & \downarrow \\ K_*(A \times_\alpha Z_2) & \longrightarrow & K_*(B \times_\beta Z_2). \end{array}$$

It is not clear though that $\tilde{\psi}_* = \tilde{\phi}$. We will see that $\tilde{\psi}_*$ does equal $\tilde{\phi}$ in our special cases.

(a) (A,α) is of type 1.

We first consider the case when (B,β) is also of type 1. Write $\phi_1 = (\phi_{ij})_{i,j=1}^3$. Since ϕ_1 intertwines the actions α_* and β_*, one has

$$\phi_1 = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{12} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{21} & \phi_{23} & \phi_{22} \end{pmatrix}.$$

Similarly,

$$\tilde{\phi}_1 = \begin{pmatrix} \tilde{\phi}_{11} & \tilde{\phi}_{12} & \tilde{\phi}_{13} \\ \tilde{\phi}_{12} & \tilde{\phi}_{11} & \tilde{\phi}_{13} \\ \tilde{\phi}_{31} & \tilde{\phi}_{31} & \tilde{\phi}_{33} \end{pmatrix}.$$

By the commuting diagram, one has

$$\begin{cases} \tilde{\phi}_{13} = \phi_{12}, \\ \tilde{\phi}_{31} = \phi_{21}, \\ \tilde{\phi}_{33} = \phi_{22} + \phi_{23}, \\ \tilde{\phi}_{11} + \tilde{\phi}_{12} = \phi_{11}. \end{cases}$$

ϕ_0 is just an integer n'/n. Write $\tilde{\phi}_0 = (a_{ij})_1^2$. Since it intertwines the dual actions, $\tilde{\phi}_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{11} \end{pmatrix}$. Now the commuting diagram gives $a_{11} + a_{12} = n'/n$. The condition on the special elements gives $a_{11}n = n'$ and $a_{12}n = 0$. Hence $\tilde{\phi}_0 = \begin{pmatrix} n'/n & \\ & n'/n \end{pmatrix}$, completely determined by ϕ_0. Take $((1,0),(1,0,0)) \in K_*(A \times_\alpha Z_2)$ to be in the dimension range. Then

$$\Big((n'/n, 0),\ (\tilde{\phi}_{11}, \tilde{\phi}_{12}, \tilde{\phi}_{31})\Big) \in Z^2 \oplus Z^{l'} \oplus Z^{l'} \oplus Z^{k'}.$$

(cf. 5.1). Since the second component of Z^2 corresponds to the second copy of $Z^{l'}$ in the dimension range. Hence, $\tilde{\phi}_{12} = 0$. Now $\tilde{\phi}_1$ is determined by the entries of ϕ_1.

So if we could define a unital $*$-homomorphism ψ to realize ϕ and to intertwine α and β, then ψ will realize all other K-theory data.

Let $S' \subset X'$ be a fundamental domain. Define a map η from S' to X, according to ϕ_1 as follows: On each circle of S', $\eta(t)$ will go around the circles of X. The winding numbers will be the entries of the first two rows of ϕ_1. On $X' \setminus S'$, define

$$\eta(t) = \sigma^{-1}(\eta(\tau(t))), \qquad t \in X' \setminus S'.$$

The map ψ defined by

$$\psi(f)(t) = \begin{pmatrix} f(\eta(t)) & & & \\ & f(x_0) & & \\ & & \ddots & \\ & & & f(x_0) \end{pmatrix}, \quad f \in A,$$

is equivariant. Clearly, ψ induces ϕ_0 and ϕ_1. The extension of ψ to $A \times_\alpha Z_2$ automatically induces $\tilde{\phi}_0$ and $\tilde{\phi}_1$.

Next, we assume that (b, β) is of type 2.

Write $\phi_1 = (\phi_{ij})$ as a two by three matrix. Since it intertwines the actions, one has

$$\phi_1 = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{11} & \phi_{13} & \phi_{12} \end{pmatrix}.$$

Similarly, one has

$$\tilde{\phi}_1 = (\tilde{\phi}_{11}, \tilde{\phi}_{11}, \tilde{\phi}_{13}).$$

By the commuting diagram,

$$\begin{cases} \tilde{\phi}_{11} = \phi_{11}, \\ \tilde{\phi}_{13} = \phi_{12} + \phi_{13}. \end{cases}$$

ϕ_0 can be identified as (a, b, a) and $\tilde{\phi}_0$ can be written as

$$\tilde{\phi}_0 = \begin{pmatrix} a_{11} & a_{11} \\ a_{21} & a_{22} \\ a_{22} & a_{21}. \end{pmatrix}.$$

By the commuting diagram, one has

$$\begin{cases} a_{11} = a, \\ a_{21} + a_{22} = b. \end{cases}$$

The condition on the special element gives us

$$\begin{pmatrix} a & a \\ a_{21} & a_{22} \\ a_{22} & a_{21} \end{pmatrix} \begin{pmatrix} n \\ 0 \end{pmatrix} = \begin{pmatrix} n_1' \\ n_2' \\ 0 \end{pmatrix}.$$

Hence, $a_{22} = 0$. $\tilde{\phi}_0$ and $\tilde{\phi}_1$ are determined by the entries of ϕ and ϕ_1, respectively.

Again, it suffices to define a unital $*$-homomorphism from (A, α) to (B, β) to realize ϕ.

Define a map η from S', a fundamental domain of τ, to X, according to the first row of ϕ_1 in the sense of the previous case. Define ψ as follows:

$$\psi(f)(t) = \begin{pmatrix} f(\eta(t)) & & & \\ & f(x_0) & & \\ & & \ddots & \\ & & & f(x_0) \end{pmatrix}, \quad t \in S', \quad f \in A.$$

On $X' \setminus S'$, define

$$\psi(f)(t) = W^* \begin{pmatrix} f(\sigma \circ \eta(\tau^{-1}(t))) & & & \\ & f(x_0) & & \\ & & \ddots & \\ & & & f(x_0) \end{pmatrix} W, \quad f \in A.$$

ψ is well-defined since $\eta(x_0') = x_0$. Furthermore, a direct computation shows that ψ is equivariant.

(b) (A, α) is of type 2.

First consider the case that (B, β) is of type 1.

Write $\phi_0 = (a, b, a)$ and write

$$\tilde{\phi}_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{13} & a_{12} \end{pmatrix}.$$

We can do this because these two maps respect the corresponding actions. The commuting diagram together with the condition on the special elements give the following:

$$\begin{cases} a_{11} & = & a_{13} & = & a & = & 0, \\ a_{12} & = & n'/n_2 & = & b. \end{cases}$$

So $\phi_0 = (0, b, 0)$ and $\tilde{\phi}_0 = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}$. Take $\eta = ((1, 0, 0), x) \in K_*(A)^+$ for any $x \in K_1(A)$,

$$\phi(\eta) = (0, \phi_1(x)) \in K_*(B)^+.$$

Hence, $\phi_1 = 0$.

To construct ψ, write $x_0 = (x_i)_{i=1}^3$. Define

$$\psi(g)(t) = \begin{pmatrix} g(x_2) & & \\ & \ddots & \\ & & g(x_2) \end{pmatrix}, \quad t \in X', \quad g \in A.$$

Since $\alpha(g)(x_0) = \mathrm{diag}(g(x_3), g(x_2), g(x_1))$, ψ is equivariant.

Next, let (B, β) be of type 2. Write $\phi_0 = (a_{ij})_{i,j=1}^3$ and $\tilde{\phi}_0 = (a_{ij})_{i,j=1}^3$. Since they intertwine the corresponding actions, we have

$$\phi_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{pmatrix}$$

and

$$\tilde{\phi}_0 = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} \\ \tilde{a}_{21} & \tilde{a}_{23} & \tilde{a}_{22} \end{pmatrix}.$$

By the commuting diagram, we have

$$\begin{cases} \tilde{a}_{11} = a_{11} + a_{13}, \\ \tilde{a}_{12} = a_{12}, \\ \tilde{a}_{21} = a_{21}, \\ \tilde{a}_{23} + \tilde{a}_{22} = a_{22}. \end{cases}$$

The condition on the special element gives us

$$\tilde{\phi}_0 \begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} = \begin{pmatrix} n_1' \\ n_2' \\ 0 \end{pmatrix},$$

i.e., $\tilde{a}_{21} = \tilde{a}_{23} = 0$. $\tilde{\phi}_0$ is completely determined.

Write $\phi_1 = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{12} & \phi_{11} \end{pmatrix}$. By the commuting diagram, $\tilde{\phi}_1 = \phi_{11} + \phi_{12}$. Since

$$\phi_0 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ a_{13} & a_{12} & a_{11} \end{pmatrix},$$

and since ϕ_0 sends $[1_A]$ to $[1_B]$, $a_{22} \neq 0$.

If $a_{11} = a_{13} = 0$, ϕ_1 must be trivial. This is because ϕ preserves the dimension range. More precisely, take $\eta = ((1,0,0),(x,0))$ or $\eta = ((1,0,0),(0,y))$ in $K_*(A)^+$,

$$\phi(\eta) = ((0,0,0), \phi_1(x,0)),$$

$$\phi(\eta) = ((0,0,0), \phi_1(0,y)).$$

So $\phi_1 = 0$. Now ψ can be defined as in the previous case by putting $g(x_0)$ along the diagonal for $g \in A$.

Assume that $a_{11} + a_{13} \neq 0$. Take a fundamental domain of τ, say S'. Define a map η from S' to X, according to the first row of ϕ_1 (to give the winding numbers), such that $\eta(x_0') = x_0$. Namely, the induced map from $C(X)$ to $C(S')$ gives the $K_1 = (\phi_{11}, \phi_{12})$. Let $V \in M_n'$ be a permutation unitary and define for $g \in A$ and $t \in S'$,

$$\psi(g)(t) = V \begin{pmatrix} g(\eta(t)) & & & & & & & & & \\ & g(x_1) & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & g(x_1) & & & & & & \\ & & & & g(x_2) & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & g(x_2) & & & \\ & & & & & & & g(x_3) & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & g(x_3) \end{pmatrix} V^*$$

such that at $t = (x_0')$,

$$\psi(g)(x_0') = \begin{pmatrix} \xi_1(g) & & \\ & \xi_2(g) & \\ & & \xi_3(g) \end{pmatrix}$$

where

$$\xi_1(g) = \begin{pmatrix} g(x_1) & & & & & & & & \\ & \ddots & & & & & & & \\ & & g(x_1) & & & & & & \\ & & & g(x_2) & & & & & \\ & & & & \ddots & & & & \\ & & & & & g(x_2) & & & \\ & & & & & & g(x_3) & & \\ & & & & & & & \ddots & \\ & & & & & & & & g(x_3) \end{pmatrix}$$

with a_{1i} copies of $g(x_i)$.

$$\xi_2(g) = \begin{pmatrix} g(x_2) & & \\ & \ddots & \\ & & g(x_2) \end{pmatrix}$$

with a_{22} copies of $g(x_2)$.

$$\xi_3(g) = \begin{pmatrix} g(x_3) & & & & & & & & \\ & \ddots & & & & & & & \\ & & g(x_3) & & & & & & \\ & & & g(x_2) & & & & & \\ & & & & \ddots & & & & \\ & & & & & g(x_2) & & & \\ & & & & & & g(x_1) & & \\ & & & & & & & \ddots & \\ & & & & & & & & g(x_1) \end{pmatrix}$$

with a_{11} copies of $g(x_3)$, a_{12} copies of $g(x_2)$ and a_{13} copies of $g(x_1)$.

For $t \in X' \setminus S'$, define

$$\psi(g)(t) = W^{'*}\psi(\alpha(g))(\tau^{-1}(t))W'.$$

First, at $t = x_0'$, compute

$$\psi(g)(x_0') = W^{'*}\psi(\alpha(g))(x_0')W'.$$

So ψ is a $*$-homomorphism from A to B. Secondly, for $t \in S'$, we have

$$\beta(\psi(g))(t) = W^{'*}\psi(g)(\tau(t))W' = \psi(\alpha(g))(t)$$

and for $t \in X' \setminus S'$, notice that $W^{'2} = 1$, we also have

$$\beta(\psi(g))(t) = W'\psi(g)(\tau(t))W^{'*} = W^{'2}\psi(\alpha(g))(t)W^{'*2} = \psi(\alpha(g))(t).$$

So ψ is equivariant. Now it is readily checked that ξ is a desired map. $\square$

5.3. We must show that one can generalize the theorem to the case when the first algebra is a finite direct sum of basic building blocks. The proof is to reduce this case to the Theorem 5.2.

Corollary. *Let $(A_1 \oplus A_2, \alpha_1 \oplus \alpha_2)$ and (B, β) be two C^*-dynamical systems and assume that (B, β) is a basic building block C^*-dynamical system and each (A_i, α_i) is a finite direct sum of basic building block C^*-dynamical systems in the sense of 5.1. Suppose that ϕ is a dimension range preserving group homomorphism from $(K_*(A_1 \oplus A_2), (\alpha_1 \oplus \alpha_2)_*)$ to $(K_*(B), \beta_*)$ mapping $[I_{A_1} \oplus I_{A_2}]$ to $[I_B]$ and suppose that $\tilde{\phi} = \tilde{\phi}_0 \oplus \tilde{\phi}_1$ is a dimension range preserving homomorphism from $(K_*(A_1 \oplus A_2) \times_{\alpha_1 \oplus \alpha_2} Z_2, (\hat{\alpha}_1 \oplus \hat{\alpha}_2)_*)$ to $(K_*(B \times_\beta Z_2), \hat{\beta}_*)$ mapping the special element to the special element. Suppose further that the following diagram commutes:*

$$\begin{array}{ccc} K_*(A_1 \oplus A_2) & \longrightarrow & K_*((A_1 \oplus A_2) \times_{\alpha_1 \oplus \alpha_2} Z_2) \\ \downarrow & & \downarrow \\ K_*(B) & \longrightarrow & K_*(B \times_\beta Z_2) \end{array}$$

Then there exists a unital $*$-homomorphism ψ from $(A_1 \oplus A_2, \alpha_1 \oplus \alpha_2)$ to (B, β) such that ψ induces ϕ and $\tilde{\phi}$.

Proof. We will reduce the problem from $A_1 \oplus A_2$ to A_i. The Corollary then follows from Theorem 5.2 together with the induction. The idea is to construct a projection $P_i \in B^\beta$ and then consider (A_i, α_i) and $(P_i B P_i, \beta)$. The proof will be divided into two cases.

Case 1. (B, β) is of type 1.

Let $B = M_m(C(Y))$ and let β be implemented by τ. Recall from 5.1 that $K_0(B) = Z$ and $K_0(B \times_\alpha Z_2) = Z^2$. Let U_{α_1} be the canonical unitary in the crossed product $A \times_{\alpha_1} Z_2$ and let

$$\tilde{\phi}_0 \left(\left[\frac{1 + U_{\alpha_1}}{2} \right], 0 \right) = (x_1, y_1).$$

Since $\tilde{\phi}$ intertwines the dual actions, one has

$$\tilde{\phi}_0 \left(\left[\frac{1 - U_{\alpha_1}}{2} \right], 0 \right) = (y_1, x_1).$$

Combining them together and applying the commuting diagram, one has

$$\tilde{\phi}_0([I_{A_1}], 0) = (j_B)_* \phi_0([I_{A_1}], 0) = (x_1 + y_1, x_1 + y_1)$$

where j_B is the natural map from B to $B \times_\alpha Z_2$. This gives us $\phi_0([I_{A_1}], 0) = x_1 + y_1$.

Similarly, $\tilde{\phi}_0 \left(0, \left[\frac{I_{A_2} + U_{\alpha_2}}{2}\right]\right) = (x_2, y_2)$ with $\phi_0(0, [I_{A_2}]) = x_2 + y_2$. Since $\tilde{\phi}_0$ maps the special element to the special element, one has

$$(x_1 + x_2, y_1 + y_2) = (m, 0).$$

Notice that $x_i, y_i \geq 0$, one has $y_i = 0$ and $x_1 + x_2 = m$.

Let P be a constant diagonal projection in B such that $[P] = x_1$. Notice that P is in the fixed point algebra B^β. Hence $\beta(PBP) = PBP$. Now (PBP, β_1), with $\beta_1 = \beta|_{PBP}$, is a type 1 C*-dynamical system.

The embedding i_P of (PBP, β_1) into (B, β) gives rise to maps from $(K_*(PBP), \beta_{1*})$ to $(K_*(B), \beta_*)$ and from $(K_*(PBP \times_{\beta_1} Z_2), \hat{\beta}_{1*})$ to $(K_*(B \times_\beta Z_2), \hat{\beta}_*)$. They are automatically dimension range preserving isomorphisms. More precisely, they are identities on the direct sums of Z. Hence, we have the following diagram:

$$\begin{array}{ccccc} K_*(PBP) & \cong & K_*(B) & \leftarrow & K_*(A_1 \oplus 0) \\ \downarrow & & \downarrow & & \downarrow \\ K_*(PBP \times_{\beta_1} Z_2) & \cong & K_*(B \times_\beta Z_2) & \leftarrow & K_*(A_1 \oplus 0 \times_{\alpha_1 \oplus \alpha_2} Z_2) \end{array}$$

Furthermore, the two rows in the diagram respect the corresponding actions, respectively.

By the above diagram, we can define two maps from $K_*(A_1 \oplus 0)$ to $K_*(PBP)$ and from $K_*(A_1 \oplus 0 \times_{\alpha_1 \oplus 0} Z_2)$ to $K_*(PBP \times_\beta Z_2)$, respectively. (PBP, β_1) was constructed to satisfy the condition of the the Corollary 5.3. So, applying the Theorem 5.2 together with the induction, we obtain a unital $*$-homomorphism ψ_1 from (A_1, α_1) to (PBP, β_1) to realize the K-theory data associated with these two C*-dynamical systems. Similarly, there exists a unital $*$-homomorphism ψ_2 from (A_2, α_2) to $((1-P)B(1-P), \beta|_{(1-P)B(1-P)})$. Now, $\psi = i_P \circ \psi_1 + i_{(1-P)} \circ \psi_2$ is a desired map.

Case 2. (B, β) is of type 2.

Let $B \in M_m(C(Y))$ such that the fiber at y_0 is $M_l \oplus M_h \oplus M_l$. Let $\tilde{\phi}_0\left(\left[\frac{I_{A_1}+U_{\alpha_1}}{2}\right], 0\right) = (x_1, y_1, z_1)$. Then $\tilde{\phi}_0\left(\left[\frac{I_{A_1}-U_{\alpha_1}}{2}\right], 0\right) = (x_1, z_1, y_1)$. This gives us $\tilde{\phi}_0([I_{A_1}], 0) = (2x_1, y_1+z_1, y_1+z_1)$. Use the notations in Case 1,

$$(j_B)_*\phi_0([I_{A_1}], 0) = (2x_1, y_1+z_1, y_1+z_1).$$

Write $\phi_0([I_{A_1}], 0) = (u, v, w)$. Then

$$\begin{cases} u + w = 2x_1, \\ v = y_1 + z_1. \end{cases}$$

Since $[I_{A_1}]$ is invariant under α_{1*}, we have $u = w$. Hence

$$\phi_0([I_{A_1}], 0) = (x_1, y_1+z_1, x_1).$$

Similarly, we have

$$\tilde{\phi}_0\left(0, \left[\frac{I_{A_2}+U_{\alpha_2}}{2}\right]\right) = (x_2, y_2, z_2),$$

$$\phi_0(0, [I_{A_2}]) = (x_2, y_2+z_2, x_2).$$

The condition on the special elements gives us $z_1 = z_2 = 0$, $x_1 + x_2 = l$ and $y_1 = y_2 = h$. Finally,

$$\phi_0([I_{A_1}], 0) = (x_1, y_1, x_1).$$

Let P be a constant diagonal projection in B^β such that $[P] = (x_1, y_1, x_1)$. Again $(PBP, \beta|_{PBP})$ is a basic building block C*-dynamical system. It is of type 1 if $x_1 = 0$. As in Case 1, we can define i_P and $\tilde{i}_P$. Again, we have the following diagram similar to the one we obtained in Case 1:

$$\begin{array}{ccccc} K_*(PBP) & \rightarrow & K_*(B) & \leftarrow & K_*(A_1 \oplus 0) \\ \downarrow & & \downarrow & & \downarrow \\ K_*(PBP \times_{\beta_1} Z_2) & \rightarrow & K_*(B) & \leftarrow & K_*(A_1 \oplus 0 \times_{\alpha_1 \oplus \alpha_2} Z_2) \end{array}$$

Here the two right pointed maps in the upper row and the down row, respectively, are injective. So one can define the inverse on their images. The two images under the two left pointed maps are in the images of the right ones, respectively. Applying exactly the same argument as in the Case 1, we reduce to the C*-dynamical systems

(A_1, α_1) and (PBP, β_1) to satisfy the condition of the Corollary 5.3. We can do the same thing to $(1-P)B(1-P)$. Putting them together and applying induction and the Theorem 5.2, we have the desired result.

□

6. Uniqueness

In section 5 we were able to lift K-group maps to a $*$-homomorphism between the corresponding C*-dynamical systems. Two such liftings can be very different. In this section we will show that under certain conditions two liftings between finite direct sums of basic building block C*-dynamical systems considered in 5.1 are approximately unitarily equivalent via a unitary in the fixed point algebra, on a given finite subset of approximately constant elements in the sense of the definition 6.3 below.

6.1. First, a definition.

Definition. Let A and B be two basic building blocks with special graph spectra X and X', and with generic fibers M_n and $M_{n'}$, respectively. We will say that a unital $*$-homomorphism ϕ from A to B has standard form, if

(i) On each edge L of X', identified with $I = [0,1]$, without the end points being identified, ϕ has the following expression:

$$\phi(f)(t) = U(t) \begin{pmatrix} f(s_1(t)) & & \\ & \ddots & \\ & & f(s_a(t)) \end{pmatrix} U^*(t)$$

for $t \in [0,1]$ and $f \in A$. Here $U \in M_{n'}(C[0,1])$ and $\{s_i\}_{i=1}^a \subset C(I, X)$.

(ii) After identifying each edge of X with $[0,1]$, each of $\{s_i(t)\}_{i=1}^a$ has one of the following forms:

$$\{t, \quad 1-t, \quad \text{a vertex point}\}.$$

For a $*$-homomorphism ϕ with standard form, we will call $\{s_i(t)\}_{i=1}^a$ the eigenvalue structure of ϕ (on L).

6.2. The following lemma serves as a tool to reduce the so called uniqueness theorem from finite direct sums of basic building block C*-dynamical systems to single ones.

Lemma. *Let $(A_1 \oplus A_2, \alpha_1 \oplus \alpha_2)$ and (B, β) be two C*-dynamical systems where (B, β) is a basic building block C*-dynamical system and where each (A_i, α_i) is a finite direct sum of basic building block C*-dynamical systems in the sense of 5.1. Suppose that ϕ is a unital $*$-homomorphism from $(A_1 \oplus A_2, \alpha_1 \oplus \alpha_2)$ to (B, β) with the property ensured by Theorem 3.1 (cf. Remark 3.1). Then there exists a unitary $U \in B^\beta$ such that $U\phi(1,0)U^*$ and $U\phi(0,1)U^*$ are two constant diagonal projections.*

Proof. It is enough to consider $\phi(1,0)$ since the result for $\phi(0,1)$ will follow automatically. Recall that on each edge L (always a circle) of Y, the spectrum of B, identified with $[0,1]$, $\phi(f,0)$ has the expression:

$$\phi(f,0)(t) = V(t)\begin{pmatrix} f(\xi_1(t)) & & & & & \\ & \ddots & & & & \\ & & f(\xi_l(t)) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} V^*(t)$$

for $t \in L$ and $f \in A$ (cf. Theorem 4.2 (ii)). In particular,

$$\phi(1,0)(t) = V(t)\begin{pmatrix} 1 & \\ & 0 \end{pmatrix} V^*(t), \quad t \in L.$$

We consider two cases separately.

Case 1. (B,β) is of type 1 with $B = M_m(C(Y))$.

Let τ be the action on Y that implements β, let $S \subset Y$ be a fundamental domain of τ and let $L \subset S$. Since

$$V(0)\begin{pmatrix} 1 & \\ & 0 \end{pmatrix} V^*(0) = V(1)\begin{pmatrix} 1 & \\ & 0 \end{pmatrix} V^*(1),$$

$V^*(1)V(0)$ must be of diagonal block form

$$V^*(1)V(0) = \begin{pmatrix} V_1 & \\ & V_2 \end{pmatrix}.$$

Let $X \in M_m(C(I))$ such that X commutes with $\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$ and such that $X(0) = I$ and $X(1) = V^*(1)V(0)$. Denote $V(t)X(t)$ by $U(t)$. Then

$$\phi(1,0)(t) = U(t)\begin{pmatrix} 1 & \\ & 0 \end{pmatrix} U^*(t), \quad t \in L,$$

and $U(0) = U(1) = V(0)$.

Let $L' \subset \overline{Y \setminus S}$ be an edge with $\tau(L) = L'$. We have for $t \in L$

$$\phi(1,0)(\tau(t)) = (\beta\phi(1,0))(t) = \phi(\alpha(1),0)(t) = \phi(1,0)(t).$$

So we may use the same U on L' to diagonalize $\phi(1,0)$. This computation also includes the case that $L = L'$. Let L_1 and L_2 be two different edges that meet at the vertex $y_0 \in Y$. Suppose that the unitary to diagonalize $\phi(1,0)$ is U_1 and U_2 on L_1 and L_2, respectively. Since $U_2^*(0)U_1(0) = U_2^*(1)U_1(1)$ commutes with $\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$, we may change U_2 so that $U_2(0) = U_2(1) = U_1(0)$. Now it is clear that there exists $U \in B$ such that

$$\phi(1,0) = U\begin{pmatrix} 1 & \\ & 0 \end{pmatrix} U^*.$$

Since we use $U(\tau(t)) = U(t)$, $U \in B^\beta$.

Case 2. (B,β) is of type 2.

Suppose that at y_0, the only vertex of Y, B has fiber $M_l \oplus M_h \oplus M_l$, β is implemented by τ and $W = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$ (see 5.1). Notice that y_0 is the only fixed point of τ. Let $S \subset Y$ be a fundamental domain and let $L \subset S$ be an edge identified with $[0,1]$. Changing V by a permutation, we may write

$$\phi(1,0)(t) = V(t)PV^*(t), \quad t \in L,$$

where $P = \operatorname{diag}(P_1, P_2, P_1) \in B^\beta$ is a constant diagonal projection. The construction is similar to the construction in Case 1. Next, we will change V so that $V(0) = V(1)$. We will also change them once more so that these V on different edges agree at y_0. Compute:

$$\phi(1,0)(0) = \phi(\alpha_1(1),0)(0) = \beta(\phi(1,0))(0) = W\phi(1,0)(0)W^*.$$

Since $\phi(1,0)(0) \in M_l \oplus M_h \oplus M_l$ and since $\phi(1,0)(0)$ commutes with W,

$$\phi(1,0)(0) = \begin{pmatrix} a & & \\ & b & \\ & & a \end{pmatrix}.$$

Hence there exists a unitary $Z \in M_l \oplus M_h \oplus M_l$, commuting with W such that

$$\phi(1,0)(0) = Z \begin{pmatrix} P_1 & & \\ & P_2 & \\ & & P_1 \end{pmatrix} Z^*.$$

Here Z and $P = \operatorname{diag}(P_1, P_2, P_1)$ are all in B^β. Now

$$V(0)PV^*(0) = ZPZ^*$$

$V^*(0)Z$ commutes with P. Notice that for $t \in L$,

$$\phi(1,0)(t) = V(t)(V^*(0)Z)P(Z^*V(0))V^*(t).$$

So we may replace $V(t)$ by $V(t)V^*(0)Z$. $V(0)V^*(0)Z = Z$ commutes with W. We remark that Z does not depend on the edges.

First, let $L \subset S'$. Define $U(t)$ on L by $V(t)V^*(0)Z$. On $L' = \tau(L)$, define

$$U(\tau(t)) = W^*U(t)W$$

Since $U(0) = V(0)$ (see the proof of Case 1). $U \in B^\beta$ is well-defined. ☐

The following Proposition will be used in the proof of the uniqueness theorem.

Proposition. *Let (A,α) and (B,β) be two finite direct sums of basic building block C*-dynamical systems in the sense of 5.1 and let ϕ and ψ be two unital $*$-homomorphisms from (A,α) to (B,β). Suppose that ϕ and ψ are of the standard forms. For any $\epsilon > 0$ and for any finite subset $F \subset A$, if*

(i) *$K_0(\phi) = K_0(\psi)$ and they intertwine α_* and β_*,*
(ii) *ϕ and ψ have the same eigenvalue structure on each edge of the spectrum of B.*

Then there exists a unitary $U \in B^\beta$ such that

$$\|U\phi(f)U^* - \psi(f)\| < \epsilon.$$

Proof. If all the actions are trivial, this is the Proposition 7.3 of [11]. We will modify that proof so that $U \in B^\beta$. By the Proposition 6.2, we can reduce to the case where both (A,α) and (B,β) are single basic building block C$*$-dynamical systems. We divide the proof into two cases.

Case 1. (B, β) is of type 1.

Let $B = M_m(C(Y))$ and let τ be an action on Y that induces β. Define $U \in M_m(C(S))$, where S is a fundamental domain of τ, as in the proof of Proposition 7.3 in [11]. On $Y \setminus S$, define $U(t)$ by $U(\tau(t))$. Since the definitions agree at y_0, the vertex of Y, $U \in B$. Furthermore, U is defined to be in B^β.

Case (3). (B, β) is of type 2.

Use the notation in Case 1. Assume that the fiber B_0 of B at y_0, the vertex of Y, is $M_l \oplus M_h \oplus M_l$. Let $I \subset A$ and $J \subset B$ be the ideals that consist of elements which vanish at the vertices of X and Y, the spectrum of A and B, respectively. ϕ and ψ induce maps from A/I to B/J, α and β induce actions, still denoted by α and β, on A/I and B/J too. This is because ϕ and ψ have standard forms. Furthermore, $(K_0(B), \beta_*) \cong (K_0(B/J), \beta_*)$ and $(K_0(A), \alpha_*) \cong (K_0(A/I), \alpha_*)$, under the quotient map. This follows from fact that the connecting maps in the six-term exact sequences are zero.

Restricted to the quotients, we are in a finite dimension C*-dynamical system situation. By Theorem 4.1 (Case 4 in [7]), there exists a unitary U_0 in B_0 that commutes with W such that

$$U_0 \phi(f)(y_0) U_0^* = \psi(f)(y_0)$$

for all $f \in A$. Here W is the unitary part of β (see 5.1).

Now the rest follows from the proof of Case 1 together with the proof of Proposition 7.3 in [11]. Namely, first define U on S, a fundamental domain of τ, with $U(y_0) = U_0$, then transfer it to $Y \setminus S$ by β. Since $U(y_0) = U_0$ and since it commutes with W, U is well-defined and $U \in B^\beta$. □

6.3. Before we state and proof the uniqueness theorem, we need to refine the notation of approximately constant elements. Let $B \subset M_m(C(Y))$ be a basic building block and let (B, β) be a basic building block dynamical system considered in 5.1, where β is implemented by an order two action τ on Y and an order two unitary W in M_m (could be identity). Denote the edges of Y by $L_1, L_2, \dots, L_l$.

Definition. *Let (B, β) be as above, let $F \subset B$ be a finite subset and let $\epsilon > 0$. We say F is approximately constant to within ϵ associated with β if there exists a unitary $V = \bigoplus_{i=1}^l V_i \in \bigoplus_{i=1}^l M_n(C(L_i))$ such that for $f \in F$, $s \in L_i$ and $t \in L_j$,*

$$\|AdV_i(s)f(s) - AdV_j(t)f(t)\| < \epsilon$$

and such that $V_j(\tau(t)) = V_i(t)W$ if $\tau(L_i) = L_j$.

We remark that the approximately constant elements we obtained in Theorem 4.3 (in B^0) were indeed approximately constant to within ϵ associated with $\beta|_{B^0}$.

Theorem. *Let (A, α) and (B, β) be two finite direct sums of basic building block C^*-dynamical systems in the sense of 5.1 and let ϕ and ψ be two unital $*$-homomorphisms from (A, α) to (B, β). Let $\epsilon > 0$ and let $F \subset A$ be a finite subset of approximately constant elements to within ϵ associated with α. Suppose that*

(i) *$K_*\phi = K_*\psi$, and*
(ii) *ϕ and ψ have the form insured by the Perturbation Theorem 3.1 such that the variation of each generalized eigenvalue map is within $1/4$ on each edge of the spectrum of B.*

It follows that there exists a unitary U in B^β such that

$$\|U\phi(f)U^* - \psi(f)\| < 21\epsilon$$

for all $f \in F$.

Proof. By the Proposition 6.2, we may assume that both (A, α) and (B, β) are single basic building block C*-dynamical systems. We divide the proof into two cases. We will use the notations in the beginning of this section for (B, β).

Case 1. (B, β) is of type 1.

Let L be an edge of Y, identified with $[0, 1]$, write

$$\phi(g)(t) = V(t) \begin{pmatrix} g(\lambda_1(t)) & & \\ & \ddots & \\ & & g(\lambda_k(t)) \end{pmatrix} V^*(t)$$

for $t \in L$ and $g \in A$. Let σ and $\tilde{W}$ implement α. For $\tau(t) \in L' = \tau(L)$,

$$\phi(g)(\tau(t)) = V(t) \begin{pmatrix} \alpha(g)(\lambda_1(t)) & & \\ & \ddots & \\ & & \alpha(g)(\lambda_k(t)) \end{pmatrix} V^*(t).$$

This is the condition (for all the pairs L and $\tau(L)$) that ϕ is equivariant. We are going to deform and perturb ϕ into standard form. The perturbation will be small and it will not change the induced maps, obtained in the middle stages, on K-groups. The operations will be similar to those in the proof of Theorem 7.4 in [11]. The difference is that the maps should be always equivariant.

(a) Deformation inside an edge of Y.

The first step is to deform ϕ to $\phi^{(1)}$ such that the eigenvalue maps, still denoted by $\{\lambda_i\}_{i=1}^k$, have the property that $\lambda_i(t)$ achieves the vertex x_0 of X only at some of the following three points $t = 0, 1/2$ and 1 unless λ_i is a constant map x_0. The deformation is the same as that in the proof of Theorem 7.4 of [11], Step 1 and Step 2. Let $[c, d] \subset [0, 1] = L$ such that $\lambda_1([c, d])$ is inside an edge $\tilde{L}$ of X, Let $s_1(t)$ be defined on $[c, d]$ with $s_1([c, d]) \subset \tilde{L}$. Let $H(s, t)$ be a deformation of λ_1 to s_1 such that $H(s, 0) = \lambda_1(c)$ and $H(s, 1) = \lambda_1(d)$. We will define a deformation ϕ_s such that $\phi_0 = \phi$ and such that the eigenvalue maps of ϕ_1 are $\lambda'_1, \lambda'_2, \cdots, \lambda'_k$ and $\lambda'_1 = s_1$ on $[c, d]$. Furthermore, ϕ_s is equivariant.

On L, for $t \in [c, d]$ and $g \in A$, define

$$\phi_s(g)(t) = V(t) \begin{pmatrix} Z^*(\lambda_1(t))(ZgZ^*)(H(s,t))Z(\lambda_1(t)) & & \\ & \ddots & \\ & & g(\lambda_k(t)) \end{pmatrix} V^*(t)$$

where $Z(t) \in M_n(C(L))$ such that ZfZ^* is almost constant to within ϵ over L. On $\tau(L)$, define

$$\phi_s(g)(t) = Ad\left(V(t)\begin{pmatrix} \tilde{W}\left(\tilde{W}^* Z^*(\lambda_1(t))\right) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\right)$$

$$\begin{pmatrix} \left(Z(\sigma^{-1})\tilde{W}g\tilde{W}^* Z^*(\sigma^{-1})\right)\sigma(H(s,t)) & & \\ & \ddots & \\ & & \alpha(g)(\lambda_k(t)) \end{pmatrix}$$

$$= Ad\left(V(t)\begin{pmatrix} Z^*(\lambda_1(t))Z(H(s,t)) & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\right)$$

$$\begin{pmatrix} \alpha(g)(H(s,t)) & & \\ & \ddots & \\ & & \alpha(g)(\lambda_k(t)) \end{pmatrix}.$$

For the time being define ϕ_s to be ϕ at other places of Y. Then ϕ_s is equivariant. Notice that this construction can be carried out for $\lambda_2 \dots \lambda_k$ and on other pairs of edges of Y at the same time.

We need two more deformations to obtain $\phi^{(1)}$. Let us summarize as follows. Suppose that $\lambda_1(t_1)$ and $\lambda_1(t_2)$ are x_0 and $\lambda_1(t)$ stay in an edge of X over $[t_1, t_2]$. Then this type of deformation will deform $\lambda_1(t)$ into x_0 over $[t_1, t_2]$. The second deformation will be shrinking x_0 over an interval in L. More clearly suppose that $\lambda_1(t) = x_0$ on $[a, b] \subset (0, 1)$, we may deform it to a new one $\tilde{\lambda}_1$ with only $\tilde{\lambda}_1(a) = x_0$. The third deformation will be moving a to $1/2$. Together we need three deformations and hence 3ϵ. The deformations are the same as those in the proof Theorem 7.4 of [11], Step 1 and Step 2. The final map, say $\phi^{(1)}$, has the desired property. We will denote $\phi^{(1)}$ by ϕ.

(b) Deform at y_0.

The purpose of this is to deform $\lambda_i(y_0)$ to x_0 so that the new map has standard form. This kind of deformations can be found in the proof of Theorem 5.4 of [11], Step 4. Recall that at y_0 , ϕ has representations:

$$\phi(g)(y_0) = V(y_0)\begin{pmatrix} g(\lambda_1(y_0)) & & \\ & \ddots & \\ & & g(\lambda_k(y_0)) \end{pmatrix} V^*(y_0),$$

$$\phi(g)(\tau(y_0)) = V(y_0)\begin{pmatrix} \alpha(g)(\lambda_1(y_0)) & & \\ & \ddots & \\ & & \alpha(g)(\lambda_k(y_0)) \end{pmatrix} V^*(g_0).$$

This gives us

$$\begin{pmatrix} g(\lambda_1) & & \\ & \ddots & \\ & & g(\lambda_k) \end{pmatrix} = \begin{pmatrix} \tilde{W}g(\sigma(\lambda_1))\tilde{W}^* & & \\ & \ddots & \\ & & \tilde{W}g(\sigma(\lambda_k))\tilde{W}^* \end{pmatrix}$$

for all $g \in A$. Here $\lambda_i = \lambda_i(y_0)$. If x_0 is a multiple point, then $x_0 = (x_{01}, x_{02}, x_{03})$ or $x_0 = (x_{01}, x_{02}))$. If λ_i is one or two vertex points but not a whole vertex x_0, $\tilde{W}g(\sigma(\lambda_i))\tilde{W}^*$ just means $\alpha(g)(\lambda_i)$. Notice that we did not change these points in (a). We will not change them in the following operations.

The above equation gives $\{\lambda_i\}_{i=1}^k = \{\sigma(\lambda_i)\}_{i=1}^k$ as sets. We would like to see what is required to move those non-vertex points. Suppose that $\lambda_1 = \sigma(\lambda_i)$. Then $\sigma(\lambda_1) = \lambda_i$. If we deform λ_1 to $\lambda_1' \in X$, we must deform λ_i to $\sigma(\lambda_1')$ at the same time to ensure that the maps are equivalent during the deformation. For the sake of notation, let us assume that $i = k = 2$. We view this as a two by two block of the original one. $\tilde{W}$ is then a permutation matrix.

Let $\lambda_1(0) \in \tilde{L} \subset X$ and assume that $\lambda_1 \neq x_0$. Now $\sigma(\lambda_1(0)) = \lambda_2(0)$. Let $H(s,t)$ be a deformation of $\lambda_1(t)$ to $s_1(t)$ with $t \in [0, 1/2]$ and with $s_1(t) \in \tilde{L}$. Here $H(s, 1/2) = \lambda_1(1/2) = s_1(1/2)$. Using the notations in (a) on L, define

$$\phi_s(g)(t) = Ad\left(V(t)\begin{pmatrix} Z^*(\lambda_1(t)) & \\ & Z_1^*(\lambda_2(t)) \end{pmatrix}\right)$$
$$\begin{pmatrix} (ZgZ^*)(H(s,t)) & \\ & (Z_1gZ_1^*)(\sigma(H(s,t))) \end{pmatrix},$$

where ZgZ^* is almost constant on $\tilde{L}$ and $Z_1gZ_1^*$ is almost constant on $\sigma(\tilde{L})$.

For $\tau(t) \in L'$, define

$$\phi_s(g)(\tau(t)) =$$
$$Ad\left(V_1(t)\begin{pmatrix} Z^*(\lambda_1(t)) & 0 \\ 0 & Z_1^*(\lambda_2(t)) \end{pmatrix}\right)$$
$$\begin{pmatrix} Z(\sigma^{-1})\tilde{W}g\tilde{W}^*Z^*(\sigma^{-1})(\sigma(H(s,t))) & 0 \\ 0 & Z_1(\sigma^{-1})\tilde{W}g\tilde{W}^*Z_1^*(\sigma^{-1})(H(s,t)) \end{pmatrix}.$$

Notice that $Z_1(\sigma(t)) = Z(t)\tilde{W}$, by the definition of approximate constant associated with β. Here, $V_1(t)$ is the unitary corresponding to the representation of ϕ on $\tau(\tilde{L})$. Since ϕ intertwines α and β, $V_1(t) = Ad\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}V(t)$.

At $t = y_0$, one has on the one hand,

$$\phi_s(g)(y_0) = Ad\left(V(y_0)\begin{pmatrix} Z^*(\lambda_1) & \\ & Z_1^*(\lambda_2) \end{pmatrix}\right)$$
$$\begin{pmatrix} (ZgZ^*)(H(s,y_0)) & \\ & (Z_1gZ_1^*)(\sigma(H(s,y_0))) \end{pmatrix}$$

and on the other hand,

$$\phi_s(g)(\tau(y_0)) =$$
$$Ad\left(V_1(y_0)\begin{pmatrix} Z^*(\lambda_1) & \\ & Z_1^*(\lambda_2) \end{pmatrix}\right)$$
$$\begin{pmatrix} Ad(Z(H(s,y_0))\tilde{W})g(\sigma(H(s,y_0))) & \\ & Ad(\tilde{W}^*Z_1^*(\sigma^{-1}(H(s,y_0))))g(H(s,y_0)) \end{pmatrix}.$$

So they do agree. If we do the same at $t = 1$, then ϕ_s is well-defined. After doing this, ϕ_s becomes a map from (A, α) to (B, β), changing $\lambda_1(0)$ to $s_1(0) \in \tilde{L}$.

Let ϕ be obtained in (a). For those eigenvalue maps λ_i with $\lambda_i(1/2) = x_0$, using the deformation described above, deform $\lambda_i(0)$ to x_0 along the shortest route. Deform $\lambda_i(1)$ to x_0 along the same direction. Since the variation of each λ_i, after the deformation in (a), is within $1/4$, each new eigenvalue map stays in one edge of X. Apply the deformation in (a) again; we may put them into standard form. The new map and ϕ differ by $2\epsilon + 3\epsilon = 5\epsilon$ on F. We will denote the new map by ϕ again.

(c) Perturbation inside each edge of Y.

Let $\tilde{L}$ be a circle of X. Suppose that on $L \subset Y$, ϕ has the representation in (a). If $\lambda_1(L) = \tilde{L}$ and $\lambda_2(L) = \tilde{L}$ but they go in different directions, we would like to cancel them in the sense of [11] (Theorem 7.4, Step 5). More precisely, since $\lambda_1(1/2) = \lambda_2(1/2)$, one can perturb λ_1 and λ_2 to two new maps so that they do not cross over. This is a technique introduced in [5] and used in [6] and [11]. After this perturbation, λ_1 and λ_2 become $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ such that they do not complete a circle. We may require the resulting map and ϕ differ within ϵ on F as well as on the generators of K_*. Finally, we put each eigenvalue map into standard form by using the operation (a).

After these three steps, ϕ becomes $\phi^{(1)}$. ϕ and $\phi^{(1)}$ differ 7ϵ on F. Furthermore, $K_*(\phi) = K_*(\phi^{(1)})$. Similarly, we may turn ψ into $\psi^{(1)}$ such that ψ and $\psi^{(1)}$ differ 7ϵ on F and such that $K_*(\psi) = K_*(\psi^{(1)})$. Now $\psi^{(1)}$ and $\phi^{(1)}$ are of the standard forms. Since $K_1(\psi^{(1)}) = K_1(\phi^{(1)})$, $\psi^{(1)}$ and $\phi^{(1)}$ have the same eigenvalue structure. As a consequence of Proposition 6.3, there exists a unitary $U \in B^\beta$ such that

$$\|U\phi^{(1)}(f)U^* - \psi^{(1)}(f)\| < 7\epsilon, \quad f \in F.$$

Finally,

$$\|U\phi(f)U^* - \psi(f)\| < 21\epsilon, \qquad f \in F.$$

Case 2. (B, β) is of type 2.

Suppose that β is implemented by τ and W (cf. 5.1). Let L be an edge of Y and let $L' = \tau(L)$. Write

$$\phi(g)(t) = V(t) \begin{pmatrix} g(\lambda_1(t)) & & \\ & \ddots & \\ & & g(\lambda_k(t)) \end{pmatrix} V^*(t)$$

and

$$\phi(g)(\tau(t)) = WV(t) \begin{pmatrix} \alpha(g)(\lambda_1(t)) & & \\ & \ddots & \\ & & \alpha(g)(\lambda_k(t)) \end{pmatrix} V^*(t)W^*$$

for $t \in L$ and $g \in A$. It is clear that the operation (a) of Case 1 can be carried out here exactly the same way.

To examine operation (b), let us look at the only fixed point y_0 of τ. One has

$$\phi(g)(y_0) = V(0) \begin{pmatrix} g(\lambda_1(0)) & & \\ & \ddots & \\ & & g(\lambda_k(0)) \end{pmatrix} V^*(0)$$

and

$$\phi(g)(y_0) = W^*V(0)\begin{pmatrix} \alpha(g)(\lambda_1(0)) & & \\ & \ddots & \\ & & \alpha(g)(\lambda_k(0)) \end{pmatrix} V^*(0)W$$

$$= Ad\left(WV(0)\begin{pmatrix} \tilde{W} & & \\ & \ddots & \\ & & \tilde{W} \end{pmatrix}\right)\begin{pmatrix} g(\sigma(\lambda_1(0))) & & \\ & \ddots & \\ & & g(\sigma(\lambda_k(0))) \end{pmatrix}.$$

Same as in Case 1, if $\lambda_1(0) = \sigma(\lambda_i(0))$ then we should deform $\lambda_1(0)$ and $\lambda_i(0)$ at the same time. It will not destroy the above equality.

As for operation (c), the same deformation as in the Case 1 works. We complete the proof of the theorem by applying exactly the same deformations and perturbations as in the Case 1 to ϕ. □

7. The classification

7.1. Given an inductive limit C^*-dynamical system $(A, \alpha) = \varinjlim(A_n, \alpha_n)$ considered in 2.1. By 4.3, one may replace it by another sequence such that the spectrum of each basic building block is a special graph and each connecting map has the property ensured by the Perturbation Theorem. Furthermore, the variation of each eigenvalue map of each connecting map is small over an edge of the spectrum of a basic building block. Using the operation (b) in 6.3, one may deform each of the eigenvalue maps so that it will not take a vertex value in the interior of an edge unless it is a constant map. Now each of these maps stays inside an edge and the variation of each map is at most three times the previous one. The purpose of doing this is to ensure that the composition of eigenvalue maps still has the form ensured by the Perturbation theorem. We remark that we can do these without changing A. This follows from the standard intertwining argument of [6]. In the case that there were no actions, this was done in [6] and [11]. In the present case, the maps in the middle stages of the perturbations and deformations should be equivariant. We showed in Theorem 3.1 and 6.4 that this was possible.

We summarize the results in the following proposition.

Proposition. *Let $(A, \alpha) = \varinjlim(A_n, \alpha_n)$ be an inductive limit C^*-dynamical system considered in 2.1. Then there exists a sequence $(B_1, \beta_1) \to (B_2, \beta_2) \to \cdots$ of finite direct sums of basic building blocks of C^*-dynamical systems, with the same limit (A, α), having the following properties.*

1. *Each (B_n, β_n) is a finite direct sum of basic building blocks in the sense of 5.1.*
2. *The connecting maps have the form obtained in Remark 3.1. More precisely, let $\phi_{n,n+1}$ be the map from B_n, a finite direct sum of k copies of basic building blocks, to B_{n+1} and let ϕ be this map composed with the quotient map from B_{n+1} to a summand. Then for $f_1 \oplus \cdots \oplus f_k \in B_n$ and t in an edge L of the*

spectrum of that summand, ϕ can be written as:

$$\phi(f_1 \oplus \cdots \oplus f_k)(t) = AdU(t) \begin{pmatrix} \begin{pmatrix} f_1(s_1(t)) & & \\ & \ddots & \\ & & f_1(s_p(t)) \end{pmatrix} & & \\ & \ddots & \\ & & \begin{pmatrix} f_k(\eta_1(t)) & & \\ & \ddots & \\ & & f_k(\eta_q(t)) \end{pmatrix} \end{pmatrix}$$

where $U(t)$ and $\{s_i(t)\}_{i=1}^{p} \cup \cdots \cup \{\eta_j(t)\}_{j=1}^{q}$ are continuous over L and the variation of $\{s_i(t)\}_{i=1}^{p} \cup \cdots \cup \{\eta_j(t)\}_{j=1}^{q}$ are so small that when ϕ is composed with any of $\{\phi_{1,n}, \phi_{2,n}, \cdots, \phi_{n-1,n}\}$, the corresponding eigenvalue maps have variations less than $\frac{1}{2^{n+1}}$. As a consequence, for any $\epsilon > 0$ and any finite subset $F \subset A_n$, there exists $m_0 > n$ such that the image of F in each summand of B_m is approximately constant to within ϵ associated with the action on that summand, for all $m \geq m_0$.

Theorem. *Let $(A, \alpha) = \varinjlim(A_n, \alpha_n)$ and $(B, \beta) = \varinjlim(B_n, \beta_n)$ be two inductive limit C^*-dynamical systems considered in 2.1. Suppose that $\phi = \phi_0 \oplus \phi_1$ is a dimension range preserving group isomorphism from $(K_*(A), \alpha_*)$ to $(K_*(B), \beta_*)$ mapping $[I_A]$ to $[I_B]$ and suppose that $\tilde{\phi} = \tilde{\phi}_0 \oplus \tilde{\phi}_1$ is a dimension range preserving isomorphism from $(K_*(A \times_\alpha Z_2), \hat{\alpha}_*)$ to $(K_*(B \times_\beta Z_2), \hat{\beta}_*)$ mapping the special element to the special element. Suppose further that the following diagram commutes:*

$$\begin{array}{ccc} K_*(A) & \longrightarrow & K_*(B) \\ \downarrow & & \downarrow \\ K_*(A \times_\alpha Z_2) & \longrightarrow & K_*(B \times_\beta Z_2) \end{array}$$

Then there exists an isomorphism ψ from (A, α) to (B, β) that induces ϕ and $\tilde{\phi}$.

Proof. The proof is standard. We divide it into three steps.

Step 1. The isomorphism $(K_*(A), \alpha_*) \rightleftharpoons (K_*(B), \beta_*)$ and $(K_*(A \times_\alpha Z_2), \hat{\alpha}_*) \rightleftharpoons (K_*(B \times_\beta Z_2), \hat{\beta}_*)$ can be lifted to an intertwining of subsequences of the sequences $\{(K_*(A_i), \alpha_{i*})\}, \{(K_*(B_i), \beta_{i*})\}, \{(K_*(B_i \times_{\alpha_i} Z_2), \hat{\alpha}_{i*})\}$ and $\{(K_*(B \times_{\beta_i} Z_2), \hat{\beta}_{i*})\}$ which preserve the dimension ranges, the special elements and the commuting diagram.

It follows from [6] [11] that by passing to subsequences and by changing notation, one has

$$\begin{array}{ccccccc} K_*(A_1) & \longrightarrow & K_*(A_2) & \longrightarrow & \cdots & \longrightarrow & K_*(A) \\ \downarrow & \nearrow & \downarrow & \nearrow & & & \Updownarrow \\ K_*(B_1) & \longrightarrow & K_*(B_2) & \longrightarrow & \cdots & \longrightarrow & K_*(B) \end{array}$$

which preserves the dimension range as well as the classes of units.

For the crossed products, the basic building blocks are not exactly the same as in [11], due to the multiple edge. However, the groups are finite direct sums of Z and the following property holds: If $([e], h)$ is in the dimension range, so is $([e], nh)$

for $n \in Z$ (cf. [6] [11]). Hence, by passing to subsequences and changing notation, one also has:

$$\begin{array}{ccccccc} K_*(A_1 \times_{\alpha_1} Z_2) & \longrightarrow & K_*(A_2 \times_{\alpha_2} Z_2) & \longrightarrow & \cdots & \longrightarrow & K_*(A \times_\alpha Z_2) \\ \downarrow & \nearrow & \downarrow & \nearrow & & & \Updownarrow \\ K_*(B_1 \times_{\beta_1} Z_2) & \longrightarrow & K_*(B_2 \times_{\beta_2} Z_2) & \longrightarrow & \cdots & \longrightarrow & K_*(B \times_\beta Z_2) \end{array}$$

which preserves the dimension range and the special elements.

We now show that by passing to subsequences the liftings will also respect the actions. Let us consider the following:

$$\begin{array}{ccc} (K_*(A_1), \alpha_{1*}) & \longrightarrow & (K_*(A), \alpha_*) \\ & \searrow & \Updownarrow \\ & & (K_*(B), \beta_*). \end{array}$$

Let $a \in K_*(A_1)$ be a generator. The image of $\alpha_{1*}(a)$ in $K_*(B)$ is the image of a acted by β_* in $K_*(B)$. So there exists $j \geq 1$ such that

$$\begin{array}{ccc} (K_*(A_1), \alpha_{1*}) & \longrightarrow & (K_*(A), \alpha_*) \\ \downarrow & & \Updownarrow \\ (K_*(B_j), \beta_{j*}) & \longrightarrow & (K_*(B), \beta_*) \end{array}$$

commuting on $(a, \alpha_{1*}(a))$. Since $K_*(A)$ is finitely generated, enlarge j if necessary, we can get the above diagram commuting on $K_*(A_1)$. Now it becomes clear that, continuing this way, after passing to subsequences and changing notation, we have

$$\begin{array}{ccccccc} (K_*(A_1), \alpha_{1*}) & \longrightarrow & (K_*(A_2), \alpha_{2*}) & \longrightarrow & \cdots & \longrightarrow & (K_*(A), \alpha_*) \\ \downarrow & \nearrow & \downarrow & \nearrow & & & \Updownarrow \\ (K_*(B_1), \beta_{1*}) & \longrightarrow & (K_*(B_2), \beta_{2*}) & \longrightarrow & \cdots & \longrightarrow & (K_*(B), \beta_*) \end{array}$$

preserving the dimension ranges and the classes of the identity.

Similarly, we have

$$\begin{array}{ccccc} (K_*(A_1 \times_{\alpha_1} Z_2), \hat{\alpha}_{1*}) & \longrightarrow & \cdots & \longrightarrow & (K_*(A \times_\alpha Z_2), \hat{\alpha}_*) \\ \downarrow & \nearrow & & \nearrow & \Updownarrow \\ (K_*(B_1 \times_{\beta_1} Z_2), \hat{\beta}_{1*}) & \longrightarrow & \cdots & \longrightarrow & (K_*(B \times_\beta Z_2), \hat{\beta}_*). \end{array}$$

preserving the dimension range, the special elements and the classes of identity.

Finally, we will show that these two commuting diagrams respect the map from K-groups of an algebra to the K-groups of the crossed product. First, we have

$$\begin{array}{ccccccc} K_*(A_1 \times_{\alpha_1} Z_2) & \longrightarrow & K_*(A_2 \times_{\alpha_2} Z_2) & \longrightarrow & \cdots & \longrightarrow & K_*(A \times_\alpha Z_2) \\ \uparrow & & \uparrow & & & & \uparrow \\ K_*(A_1) & \longrightarrow & K_*(A_2) & \longrightarrow & \cdots & \longrightarrow & K_*(A) \\ K_*(B_1) & \longrightarrow & K_*(B_2) & \longrightarrow & \cdots & \longrightarrow & K_*(B) \\ \downarrow & & \downarrow & & & & \downarrow \\ K_*(B_1 \times_{\beta_1} Z_2) & \longrightarrow & K_*(B_2 \times_{\beta_2} Z_2) & \longrightarrow & \cdots & \longrightarrow & K_*(B \times_\beta Z_2) \end{array}$$

Let us start with $K(A_1) \to K(A_1 \times_{\alpha_1} Z_2)$. Write

$$\begin{array}{ccccccc} K_*(A_1) & \to & K_*(A) & \cong & K_*(B) & \leftarrow & K_*(B_1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_*(A_1 \times_{\alpha_1} Z_2) & \to & K_*(A \times_\alpha Z_2) & \cong & K_*(B \times_\beta Z_2) & \leftarrow & K_*(B_1 \times_{\beta_1} Z_2). \end{array}$$

It is clear that there exists n such that the following diagram commutes:

$$\begin{array}{ccc} K_*(A_1) & \longrightarrow & K(B_n) \\ \downarrow & & \downarrow \\ K_*(A_1 \times_{\alpha_1} Z_2) & \longrightarrow & K_*(B_n \times_{\beta_n} Z_2). \end{array}$$

Continuing this way, we have proved the claim.

Step 2. By Corollary 5.3, one has the following not necessarily commutative diagram:

$$\begin{array}{ccccccccc} (A_1, \alpha_1) & \longrightarrow & (A_2, \alpha_2) & \longrightarrow & \cdots & \longrightarrow & (A, \alpha) \\ \downarrow & \nearrow & \downarrow & \nearrow & & & \\ (B_1, \beta_1) & \longrightarrow & (B_2, \beta_2) & \longrightarrow & \cdots & \longrightarrow & (B, \beta) \end{array}$$

Step 3. After passing again to suitable subsequences of $\{(A_i, \alpha_i)\}$ and $\{(B_i, \beta_i)\}$ (and changing notation), it is possible to perturb each of the homomorphisms $(A_i, \alpha_i) \to (B_i, \beta_i)$ and $(B_i, \beta_i) \to (A_{i+1}, \alpha_{i+1})$ obtained in Step 2 by an unitary in the fixed point algebra in such a way that the diagram becomes an approximate intertwining, in the sense of [6]. Hence A is isomorphic to B. Furthermore, this isomorphism intertwines α and β (cf. 4.3).

The proof is exactly the same as Step 3 in [11]. One uses Proposition 7.1, Lemma 6.2 and Theorem 6.4. □

Remark. An immediate consequence is that $(A \times_\alpha Z_2, \hat{\alpha})$ is isomorphic to $(B \times_\beta Z_2, \hat{\beta})$ where $\hat{\alpha}$ and $\hat{\beta}$ are dual actions.

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