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THE FIXED-POINT PROPERTY FOR SIMPLY CONNECTED PLANE CONTINUA

CHARLES L. HAGOPIAN

Abstract. We answer a question of R. Mańka by proving that every simply-connected plane continuum has the fixed-point property. It follows that an arcwise-connected plane continuum has the fixed-point property if and only if its fundamental group is trivial. Let M be a plane continuum with the property that every simple closed curve in M bounds a disk in M. Then every map of M that sends each arc component into itself has a fixed point. Hence every deformation of M has a fixed point. These results are corollaries to the following general theorem. *If M is a plane continuum, $\mathcal{D}$ is a decomposition of M, and each element of $\mathcal{D}$ is simply connected, then every map of M that sends each element of $\mathcal{D}$ into itself has a fixed point.*

1. Introduction

In 1932, K. Borsuk [Bo1] proved:

Theorem 1.1. *Every locally-connected plane continuum that does not separate the plane has the fixed-point property.*

If a locally-connected plane continuum M separates the plane, then M contains a simple closed curve C that is not bounded by a disk in M [Mo, Theorem 43, p. 193]. It follows that C is a retract of M and there exists a fixed-point-free map of M into C. Thus for locally-connected plane continua, the fixed-point property is equivalent to the property of not separating the plane.

An arcwise-connected subset S of the plane has a trivial fundamental group if and only if every simple closed curve in S bounds a disk in S. Every locally-connected continuum is arcwise connected [Mo, Theorem 13, p. 91]. It follows from [Mo, Theorem 43, p. 193] that a locally-connected plane continuum does not separate the plane if and only if its fundamental group is trivial. Hence Theorem 1.1 asserts that a locally-connected plane continuum has the fixed-point property if and only if its fundamental group is trivial.

In 1971, the author [H1] generalized Theorem 1.1 by proving:

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Theorem 1.2. *Every arcwise-connected nonseparating plane continuum has the fixed-point property.*

For arcwise-connected plane continua, the fixed-point property does not imply the nonseparating property. The $\sin 1/x$ circle [Bi1] is an example of an arcwise-connected plane continuum that has the fixed-point property and separates the plane. Note that since the $\sin 1/x$ circle does not contain a simple closed curve, its fundamental group is trivial.

In 1979, the author [H4] proved:

Theorem 1.3. *Every uniquely-arcwise-connected plane continuum has the fixed-point property.*

A disk [Br] is an example that shows the fixed-point property does not imply unique arcwise connectivity in arcwise-connected plane continua.

In [M, $2^0(a)$, p. 434], Mańka asked if the fixed-point property is equivalent to the property of having a trivial fundamental group in arcwise-connected plane continua. See [Bo2] and [V] for more background on this question. We answer Mańka's question in the affirmative by showing that every simply-connected plane continuum has the fixed-point property. Theorems 1.2 and 1.3 are corollaries to this result, since the fundamental group of every arcwise-connected nonseparating plane continuum and every uniquely-arcwise-connected plane continuum is trivial.

2. Definitions

A space S has the *fixed-point property* if for every map f of S into S there is a point p of S such that $f(p) = p$.

A space is *simply connected* if it is arcwise connected and its fundamental group is trivial.

A map f of a space S is a *deformation* if there exists a map h of $S \times [0,1]$ onto S such that $h(p,0) = p$ and $h(p,1) = f(p)$ for each $p \in S$.

A collection $\mathcal{D}$ of sets is a *decomposition* of a space if $\bigcup \mathcal{D}$ is the space and the elements of $\mathcal{D}$ are pairwise disjoint.

A space is *uniquely-arcwise connected* if it is arcwise connected and does not contain a simple closed curve.

A *continuum* is a nondegenerate compact connected metric space.

3. Methodology

The following generalization of Theorem 1.3 is established in [H7].

Theorem 3.1. *Suppose M is a plane continuum, $\mathcal{D}$ is a decomposition of M, and each element of $\mathcal{D}$ is uniquely-arcwise connected. Then every map of M that sends each element of $\mathcal{D}$ into itself has a fixed point.*

Our primary goal is to show that Theorem 3.1 remains true when the elements of $\mathcal{D}$ are only assumed to be simply connected. Then setting $\mathcal{D} = \{M\}$ will establish the fixed-point property for every simply-connected plane continuum.

The difficulty with this extension can be explained in terms of R. H. Bing's dog-chases-rabbit principle [Bi1, p. 123]. When the elements are uniquely-arcwise connected, there is always a unique arc in the continuum between the dog and the rabbit. This arc serves as a constant guide to the dog during the chase. The Borsuk ray [Bo3] is the appropriate tool in that situation.

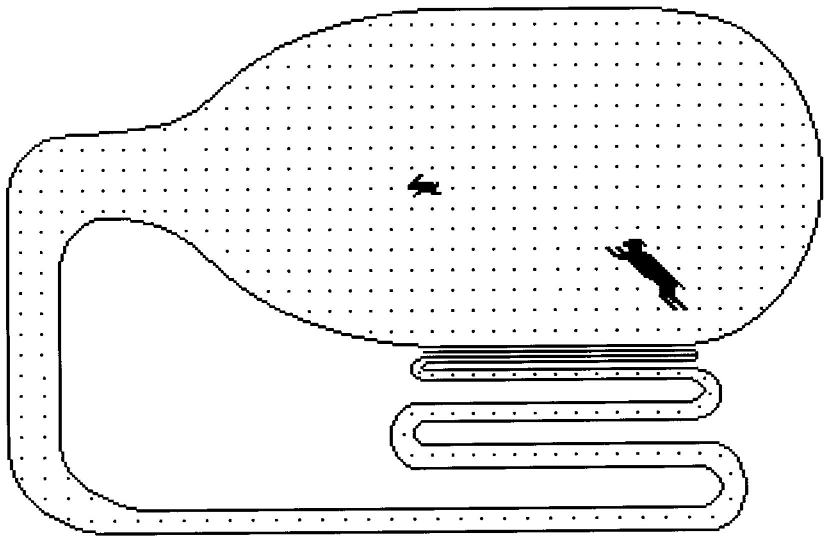

FIGURE 1

When the elements are only simply connected, the dog is forced to hunt without the guiding arc. As in Figure 1, the dog must be able to pursue the rabbit through subsets of the continuum that are open relative to the plane. Thus a ray with a special cut property must be used in place of the Borsuk ray.

In spite of this obstacle, almost all of [H7] applies without modification. In Section 6, we change slightly the notation of [H7] for a subray. All other notation and definitions agree with [H7]. In Sections 6 and 7, we use a lemma of H. Bell [B2, (2.1)] and a construction of K. Sieklucki [S] to establish the existence of rays with the cut property that start at each point of the continuum. In Section 8, we adjust the argument of [H7] to generalize Theorem 3.1.

4. Preliminaries

Let C be a plane continuum. Each component of the complement of C relative to the plane is called a *complementary domain* of C. We denote the complement of the unbounded complementary domain of C by $\mathbb{T}(C)$.

Let $\mathbb{R}^2$ be the Cartesian plane with metric ρ. Let $\mathbb{S}^2$ denote the 2-sphere that is the one-point compactification $\mathbb{R}^2 \cup \{\omega\}$ of $\mathbb{R}^2$. We denote the boundary, closure, and interior of a given set G relative to $\mathbb{S}^2$ by $\operatorname{Bd} G, \operatorname{Cl} G$, and $\operatorname{Int} G$, respectively.

A point p is *accessible* from a set G if there is an arc segment (open arc) in G that has p as an end point. Note that p may belong to G. We denote the set of points in $\mathbb{S}^2$ that are accessible from G by $\operatorname{Ac} G$.

5. Interior domains

Let M be a continuum in $\mathbb{R}^2$. Let $\mathcal{D}$ be a decomposition of M. Let $\mathbb{D}$ be a simply-connected element of $\mathcal{D}$.

A set Θ is an *interior domain* of $\mathbb{D}$ if Θ is a component of $\operatorname{Int} \mathbb{D}$.

Lemma 5.1. *Suppose H is an arc in $\mathbb{D}$ and Θ is an interior domain of $\mathbb{D}$. Then $H \cap \operatorname{Ac} \Theta$ is connected and closed.*

Proof. For each two points p and q of $H \cap \operatorname{Ac}\Theta$, let $H[p,q]$ be the arc in H from p to q and let $J(p,q)$ be an arc segment in Θ from p to q. It follows from the simple connectivity of $\mathbb{D}$ that $\operatorname{Int}\mathbb{T}(H[p,q] \cup J(p,q)) \subset \Theta$. Thus

$$\operatorname{Ac}(\operatorname{Int}\mathbb{T}(H[p,q] \cup J(p,q))) \subset \operatorname{Ac}\Theta.$$

Note that $\operatorname{Cl} J(p,q)$ contains $H[p,q] \setminus \operatorname{Ac}(\operatorname{Int}\mathbb{T}(H[p,q] \cup J(p,q)))$. Since $\operatorname{Cl} J(p,q) \subset \operatorname{Ac}\Theta$, it follows that $H[p,q] \subset \operatorname{Ac}\Theta$. Hence $H \cap \operatorname{Ac}\Theta$ is connected.

For each two points p and q of $H \cap \operatorname{Ac}\Theta$ and each positive number μ, there exists an arc segment $K(p,q,\mu)$ in $J(p,q) \cup \operatorname{Int}\mathbb{T}(H[p,q] \cup J(p,q))$ from p to q that stays within μ of $H[p,q]$. Note that $K(p,q,\mu) \subset \Theta$.

Let x be a point of $\operatorname{Cl}(H \cap \operatorname{Ac}\Theta)$. Let $x_1, x_2, \ldots$ be a sequence of distinct points of $H \cap \operatorname{Ac}\Theta$ converging to x. Assume without loss of generality that each x_i precedes x_{i+1} with respect to the order of H. The set $\Theta \cap \mathbb{T}(H \cup \bigcup\{K(x_i, x_{i+2}, i^{-1}) : i = 1, 2, \ldots\})$ contains an arc segment that has x as an end point. Thus $x \in H \cap \operatorname{Ac}\Theta$. Hence $H \cap \operatorname{Ac}\Theta$ is closed. □

6. Rays eventually restricted to one interior domain

Suppose there is a fixed-point-free map f of M into M such that $f(\mathbb{D}) \subset \mathbb{D}$.

By the compactness of M and the continuity of f, there is a positive number δ such that

$$\rho(x, f(x)) > \delta \quad \text{for every point } x \text{ of } M. \tag{6.1}$$

For convenience we assume without loss of generality that

$$\delta > 52. \tag{6.2}$$

For a given point x of $\mathbb{D}$, let $\mathbb{P}_x$ be the image in $\mathbb{D}$ of the nonnegative real numbers $[0, +\infty)$ under a one-to-one continuous function φ with the property that $\varphi(0) = x$. The function φ determines a linear ordering $\ll$ of $\mathbb{P}_x$ with x as the first point. We call $\mathbb{P}_x$ a *ray*.

Let z be a point of $\mathbb{P}_x \setminus \{x\}$. Let X and Y be arcs in $\mathbb{D}$ such that $X \cap Y = X \cap \mathbb{P}_x = Y \cap \mathbb{P}_x = \{z\}$. The arc $\mathbb{U}(z) = X \cup Y$ is called a *cut* across $\mathbb{P}_x$ at z if X and Y abut an arc in $\mathbb{P}_x$ from opposite sides [Mo, p. 180] and the end points of $\mathbb{U}(z)$ belong to $\operatorname{Bd}\mathbb{D}$.

The cut $\mathbb{U}(z)$ is called an ε-*cut* if the diameter of $\mathbb{U}(z)$ is less than ε.

The cut $\mathbb{U}(z)$ across $\mathbb{P}_x$ at a point z is *adherent* if every arc in $\mathbb{D}$ from x to $f(z)$ intersects $\mathbb{U}(z)$.

Because the elements of $\mathcal{D}$ are not assumed to be uniquely-arcwise connected, it is necessary to change the notation used in [H7] as follows:

For each point y of $\mathbb{P}_x$, let $\mathbb{P}_x(y)$ denote the ray $\{z \in \mathbb{P}_x : y = z \text{ or } y \ll z\}$.

The ray $\mathbb{P}_x$ has the *cut property* if for each positive number ε there is a point y of $\mathbb{P}_x$ such that (a) for each point z of $\mathbb{P}_x(y) \cap \operatorname{Int}\mathbb{D}$, there exists an adherent ε-cut across $\mathbb{P}_x$ at z, and (b) for each point z of $\mathbb{P}_x(y) \setminus \operatorname{Int}\mathbb{D}$, every arc in $\mathbb{D}$ from x to $f(z)$ contains z and no arc segment in $\mathbb{P}_x$ that contains z lies in a simple closed curve in $\mathbb{D}$.

Let $\mathbb{L}_x$ denote the set $\cap\{\operatorname{Cl}\mathbb{P}_x(y) : y \in \mathbb{P}_x\}$. We call $\mathbb{L}_x$ the *limit set* of the ray $\mathbb{P}_x$.

A point p is a *departure point* of an interior domain Θ of $\mathbb{D}$ if p belongs to $\operatorname{Ac}\Theta$ and there exists an arc segment in $\mathbb{D} \setminus \operatorname{Ac}\Theta$ from p to $f(p)$.

A set is ε-*lanky* if it does not contain a circular disk of diameter ε.

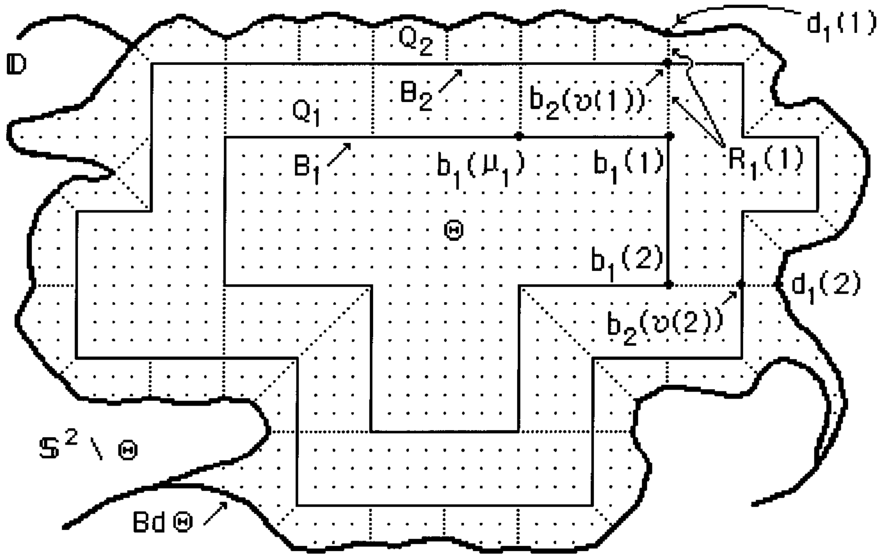

FIGURE 2

A *dendrite* is a locally-connected continuum that does not contain a simple closed curve. A dendrite that is the union of finitely many arcs is a *finite dendrite.*

Lemma 6.3. *Suppose an interior domain Θ of $\mathbb{D}$ does not have a departure point. For each point x of $\mathbb{D}$, there is a ray $\mathbb{P}_x$ with the cut property such that $\mathbb{L}_x$ is not degenerate and $\mathbb{P}_x(t) \subset \Theta$ for a point t of $\mathbb{P}_x$.*

Proof. Since Θ is simply connected, $\mathbb{S}^2 \setminus \Theta$ is a nonseparating plane continuum in $\mathbb{S}^2$.

Sieklucki [S, Lemma 5.5] proved there is a sequence $Q_1, Q_2, \dots$ of disks in $\mathbb{S}^2$ such that $\mathbb{S}^2 \setminus \Theta = \bigcap\{Q_n : n = 1, 2, \dots\}$ and for each n,

$$Q_{n+1} \subset \operatorname{Int} Q_n, \tag{6.4}$$

$$Q_n \cap \Theta \text{ is } 2^{-n}\text{-lanky}, \tag{6.5}$$

(6.6) the boundary B_n of Q_n is a polygonal simple closed curve with consecutive vertices $b_n(1), b_n(2), \dots, b_n(\mu_n)$, $b_n(\mu_{n+1}) = b_n(1)$ ordered clockwise with respect to Θ, and

(6.7) for $j = 1, 2, \dots, \mu_n$, the interval in B_n from $b_n(j)$ to $b_n(j+1)$ has diameter less than 2^{-n} (see Figure 2).

For every $b_n(j)$ ($n = 1, 2, \dots$ and $j = 1, 2, \dots, \mu_n$) there exists a vertex $b_{n+1}(v(j))$ such that

(6.8) the interval $N_n(j)$ in $\mathbb{S}^2$ from $b_n(j)$ to $b_{n+1}(v(j))$ has diameter less than 2^{-n},

$$N_n(j) \setminus \{b_n(j), b_{n+1}(v(j))\} \subset \operatorname{Int} Q_n \setminus Q_{n+1}, \text{ and} \tag{6.9}$$

$$(6.10) \qquad N_n(j) \cap N_n(k) = \varnothing \quad \text{for each integer } k \neq j \ (1 \leq k \leq \mu_n).$$

In Sieklucki's construction [S, proof of Lemma 5.1], each edge of each polygonal simple closed curve B_n is either a horizontal or a vertical interval. This feature is not accurately represented in [S, Figure 3] or [H7, Figure 10].

For each n, let $\Sigma_n = \bigcup\{N_n(j) : j = 1, 2, \ldots, \mu_n\}$.

Let $N = \bigcup\{\Sigma_n : n = 1, 2, \ldots\}$. In Lemma 5.1(iii) of [S], we can assume that $x(b)$ is a nearest boundary point of $Fr\, X$ to b. This condition is established in the proof of Lemma 5.1 but not stated in (iii). When this stronger version of Lemma 5.1(iii) is applied in Sieklucki's construction [S, p. 269, line 11], each component of N is the union of a sequence of collinear intervals. Hence each component of N is a half-open interval in Θ that goes from a vertex of some B_n to a nearest point of $\mathrm{Bd}\,\Theta$.

Define N_1 to be the union of all components of N that intersect B_1. For each integer i $(1 \leq i \leq \mu_1)$, let $R_1(i)$ be the $b_1(i)$-component of N_1. Let $d_1(i)$ be the end point of $R_1(i)$ in $\mathrm{Bd}\,\Theta$. Since $\mathrm{Bd}\,\Theta$ is not degenerate, it follows from Sieklucki's construction that there are two integers i and j such that $d_1(i) \neq d_1(j)$.

We assume without loss of generality that $d_1(1) \neq d_1(\mu_1)$.

For each three integers i, j, k $(1 \leq i < j < k \leq \mu_1)$,

$$(6.11) \qquad \text{if } d_1(i) = d_1(k), \text{ then } d_1(i) = d_1(j).$$

To verify this, assume $d_1(i) \neq d_1(j)$. Then there is a simple closed curve C in $B_1 \cup R_1(i) \cup R_1(k) \cup \{d_1(i)\}$ such that $d_1(j) \in \mathrm{Int}\,\mathbb{T}(C)$. Since $C \subset \mathbb{D}$ and $d_1(j) \in \mathrm{Bd}\,\Theta$, this contradicts the assumption that $\mathbb{D}$ is simply connected. Hence (6.11) is true.

Let O_1 be a subset of N_1 such that each component of O_1 is a component of N_1 and each point of $\{d_1(1), d_1(2), \ldots, d_1(\mu_1)\}$ belongs to the closure of only one component of O_1.

Let $c_1(1), c_1(2), \ldots, c_1(\xi_1), c_1(\xi_1+1) = c_1(1)$ be a list of the vertices of B_1 ordered clockwise that belong to O_1. Since $\mathrm{Bd}\,\Theta$ is a (nondegenerate) continuum, we can assume without loss of generality that B_1 is sufficiently close to $\mathrm{Bd}\,\Theta$ so that $\xi_1 > 2$.

Let E be a component of $(\Theta \cap \mathrm{Int}\, Q_1) \setminus O_1$. Using the terminology of [H7], E is the interior of a section. The two components of $O_1 \cap \mathrm{Bd}\, E$ are called the *sides* of E. The polygonal arc $B_1 \cap \mathrm{Bd}\, E$ is called the *bottom* of E.

Let $\mathbb{U}(E)$ denote the arc $\mathrm{Cl}(\Theta \cap \mathrm{Bd}\, E)$.

Since we are assuming in Lemma 5.1(iii) of [S] that $x(b)$ is a nearest boundary point to b, each side of E is in an interval (that Sieklucki calls $I(b_1^i)$) of diameter less than 1/2 [S, p. 269, line 11]. Hence each side of E is a half-open interval of diameter less than 1/2. Since $\mathbb{U}(E)$ is the closure of the union of the bottom and sides of E, it follows from (6.7) and (6.11) that the diameter of $\mathbb{U}(E)$ is less than 3.

For $i = 1, 2, \ldots, \xi_1$, let $E_1(i)$ be the component of $(\Theta \cap \mathrm{Int}\, Q_1) \setminus O_1$ whose sides are the components of O_1 that intersect $\{c_1(i), c_1(i+1)\}$. For $i = 1, 2, \ldots, \xi_1$, let $S_1(i)$ denote the side of $E_1(i)$ that contains $c_1(i)$ and let $e_1(i)$ denote the end point of $S_1(i)$ that belongs to $\mathrm{Bd}\,\Theta$.

By (6.1) and (6.2),

$$(6.12) \qquad \mathbb{U}(E_1(i)) \cap f[\mathbb{U}(E_1(i))] = \varnothing \quad \text{for } i = 1, 2, \ldots, \xi_1.$$

Let $\mathbb{V}(E)$ denote $\mathrm{Ac}\, E \cup \{p \in \mathbb{D} \setminus \mathrm{Ac}\,\Theta$: an arc segment in $\mathbb{D} \setminus \mathrm{Ac}\,\Theta$ goes from p to a point of $\mathrm{Ac}\, E\}$.

Since $\mathbb{D} \cap Q_1$ is arcwise connected,

$$\mathbb{D} \cap Q_1 = \bigcup \{\mathbb{V}(E_1(i)) : i = 1, 2, \ldots, \xi_1\}. \tag{6.13}$$

Define $R = \mathbb{T}(B_1) \cup \bigcup \{\operatorname{Cl} S_1(i) : i = 1, 2, \ldots, \xi_1\}$.

Note that

(6.14) for each arcwise-connected set K in $\mathbb{D} \setminus R$, there is an integer i such that $K \subset \mathbb{V}(E_1(i))$.

To see this, assume the contrary. Then by (6.13), there exists a simple closed curve C in $\mathbb{D}$ such that $\operatorname{Int} \mathbb{T}(C)$ contains a point of $\{e_1(1), e_1(2), \ldots, e_1(\xi_1)\}$. Since $\mathbb{D}$ is simply connected, $\operatorname{Int} \mathbb{T}(C) \subset \mathbb{D}$, and this contradicts the assumption that $\{e_1(1), e_1(2), \ldots, e_1(\xi_1)\} \subset \operatorname{Bd} \Theta$. Hence (6.14) is true.

For each positive integer n, given two points p and q of the simple closed curve B_n, we denote the arc, half-open arc, and arc segment in B_n from p to q whose orders agree with a clockwise order on B_n by $[p, q]$, $[p, q)$, and (p, q), respectively.

Next we prove that

(6.15) there is an integer i $(1 \leq i \leq \xi_1)$ such that $f[\mathbb{U}(E_1(i))] \subset \mathbb{V}(E_1(i))$.

Assume (6.15) is false. Define $T = \{p \in B_1 : f(p) \in R\}$. Note that

(6.16) T is not empty.

To see this, assume the contrary. Then by (6.14), there is an integer i such that $f[B_1] \subset \mathbb{V}(E_1(i))$. Thus $f[[c_1(i), c_1(i+1)]] \subset \mathbb{V}(E_1(i))$. Since $f[\mathbb{U}(E_1(i))]$ is an arcwise-connected continuum in $\mathbb{D}$, it follows from (6.12) that $f[\mathbb{U}(E_1(i))] \subset \mathbb{V}(E_1(i))$, and this contradicts the assumption that (6.15) is false. Hence (6.16) is true.

For each $i = 1, 2, \ldots, \xi_1$, define $T_i = \{p \in B_1 : p$ is either the first point or the last point of $[c_1(i), c_1(i+1)]$ in $T\}$. It follows from (6.16) that $\bigcup\{T_i : i = 1, 2, \ldots, \xi_1\}$ is not empty. Let $p_1, p_2, \ldots, p_k, p_{k+1} = p_1$ be a list of the points of $\bigcup\{T_i : i = 1, 2, \ldots, \xi_1\}$ ordered clockwise on B_1.

For each $j = 1, 2, \ldots, k$, define $H_j = [p_j, p_{j+1}] \cup \bigcup\{\operatorname{Cl} S_1(i) : S_1(i) \cap [p_j, p_{j+1}] \neq \varnothing\}$.

For each $j = 1, 2, \ldots, k$,

$$H_j \cap f[[p_j, p_{j+1}]] = \varnothing. \tag{6.17}$$

To see this, let α be the integer such that $p_j \in [c_1(\alpha), c_1(\alpha+1))$. If $p_{j+1} \in [c_1(\alpha), c_1(\alpha+1)]$, then it follows from (6.12) that (6.17) is true. Therefore we assume that p_{j+1} is not in $[c_1(\alpha), c_1(\alpha+1)]$.

It follows from the definition of each T_i that $\{f(p_j), f(p_{j+1})\} \subset R$ and $f[(p_j, p_{j+1})] \subset \mathbb{D} \setminus R$. There is an integer β $(1 \leq \beta \leq \xi_1)$ such that

$$\{f(p_j), f(p_{j+1})\} \subset \mathbb{U}(E_1(\beta));$$

for otherwise, $R \cup f[[p_j, p_{j+1}]]$ contains a simple closed curve C such that a point of $\{e_1(1), e_1(2), \ldots, e_1(\xi_1)\}$ belongs to $\operatorname{Int} \mathbb{T}(C)$, contradicting the simple connectivity of $\mathbb{D}$. Applying (6.14), we assume without loss of generality that $f[(p_j, p_{j+1})] \subset \mathbb{V}(E_1(\beta))$. Thus $f[[p_j, p_{j+1}]] \subset \mathbb{V}(E_1(\beta))$.

Now assume (6.17) is false. Since $H_j \cap f[(p_j, p_{j+1})] = \varnothing$, it follows that $H_j \cap \{f(p_j), f(p_{j+1})\} \neq \varnothing$. If $[c_1(\beta), c_1(\beta+1)] \subset [p_j, p_{j+1}]$, then $f[[c_1(\beta), c_1(\beta+1)]] \subset \mathbb{V}(E_1(\beta))$. Thus by (6.12) and (6.14), $f[\mathbb{U}(E_1(\beta))] \subset \mathbb{V}(E_1(\beta))$, and this contradicts the assumption that (6.15) is false. Therefore, $\{p_j, p_{j+1}\} \cap [c_1(\beta), c_1(\beta+1)] \neq \varnothing$. Since $\{f(p_j), f(p_{j+1})\} \subset \mathbb{U}(E_1(\beta))$, this contradicts (6.12). Hence (6.17) is true.

By (6.17), for each $j = 1, 2, \ldots, k$, there is a unique arc A_j in $\operatorname{Bd} R \setminus H_j$ with end points $f(p_j)$ and $f(p_{j+1})$. For $j = 1, 2, \ldots, k$, let $K_j = A_j \cup f[[p_j, p_{j+1}]]$. By (6.17), $H_j \cap K_j = \varnothing$. Furthermore, there is an arc in $(\operatorname{Bd} R) \setminus K_j$ from a point of $\{p_j, p_{j+1}\}$ to a point of $\{e_1(1), e_2(2), \ldots, e_1(\xi_1)\}$. It follows from the simple connectivity of $\mathbb{D}$ that $\mathbb{T}(K_j) \cap [p_j, p_{j+1}] = \varnothing$ for $j = 1, 2, \ldots, k$. Let $\mathcal{S} = \{\operatorname{Cl} S_1(i) : i = 1, 2, \ldots, \xi_1\}$. Let g be the quotient map of $\mathbb{R}^2$ onto the plane $\mathbb{R}^2/\mathcal{S}$ [Mo, Theorem 2, p. 311]. Let h be the fixed-point-free map of the disk $g[\mathbb{T}(B_1)]$ into $\mathbb{R}^2/\mathcal{S}$ such that $h \circ g(p) = g \circ f(p)$ for each point p of $\mathbb{T}(B_1)$. For $j = 1, 2, \ldots, k$, the arc $g[A_j]$ joins $h \circ g(p_j)$ to $h \circ g(p_{j+1})$ in the simple closed curve $g[B_1]$, and $g[[p_j, p_{j+1}]] \cap \mathbb{T}(g[A_j] \cup h \circ g[[p_j, p_{j+1}]]) = \varnothing$. By [B2, (1.2)], every extension of $h|g[B_1]$ to a map of $g[\mathbb{T}(B_1)]$ into $\mathbb{R}^2/\mathcal{S}$ has a fixed point, and this contradicts the fact that h moves each point of $g[\mathbb{T}(B_1)]$. Hence (6.15) is true.

Assume without loss of generality that

$$f[\mathbb{U}(E_1(1))] \subset \mathbb{V}(E_1(1)). \tag{6.18}$$

Assume without loss of generality that $c_1(1) = b_1(1), b_1(2), \ldots, b_1(\alpha) = c_1(2)$ are the consecutive vertices of $B_1 \cap \operatorname{Bd} E_1(1)$ ordered clockwise on B_1. Let $b_2(1), b_2(2), \ldots, b_2(\beta)$ be the consecutive vertices of $B_2 \cap \operatorname{Cl} E_1(1)$ ordered clockwise on B_2.

Define N_2 to be the union of all components of N that intersect $B_2 \cap \operatorname{Cl} E_1(1)$. For each integer i $(1 \le i \le \beta)$, let $N_2(i)$ be the $b_2(i)$-component of N_2. Let $d_2(i)$ be the end point of $N_2(i)$ in $\operatorname{Bd}\Theta$.

Let O_2 be a subset of N_2 such that each component of O_2 is a component of N_2, both $N_2(1)$ and $N_2(\beta)$ are components of O_2, and each point of $\{d_2(1), d_2(2), \ldots, d_2(\beta)\}$ belongs to the closure of only one component of O_2.

Let $b_2(1) = c_2(1), c_2(2), \ldots, c_2(\gamma) = b_2(\beta)$ denote the vertices of B_2 ordered clockwise that belong to O_2. Since $\operatorname{Bd}\Theta \cap \operatorname{Cl} E_1(1)$ is a (nondegenerate) continuum, we can assume without loss of generality that $\gamma > 2$.

For $i = 1, 2, \ldots, \gamma$, let $E_2(i)$ be the component of $(\Theta \cap \operatorname{Int} Q_2) \setminus O_2$ whose sides are contained in the components of O_2 that intersect $\{c_2(i), c_2(i+1)\}$. For $i = 1, 2, \ldots, \gamma$, let $S_2(i)$ denote the side of $E_2(i)$ that contains $c_2(i)$ and let $e_2(i)$ denote the end point of $S_2(i)$ that belongs to $\operatorname{Bd}\Theta$.

Define $E_2(\gamma+1) = \Theta \setminus \operatorname{Cl} E_1(1)$, and note that

$$\mathbb{U}(E_2(\gamma+1)) = \mathbb{U}(E_1(1)). \tag{6.19}$$

By (6.1), (6.2), and (6.19), $\mathbb{U}(E_2(i)) \cap f[\mathbb{U}(E_2(i))] = \varnothing$ for $i = 1, 2, \ldots, \gamma, \gamma+1$.

It follows from the argument for (6.15) that

(6.20) there is an integer σ $(1 \le \sigma \le \gamma+1)$ such that $f[\mathbb{U}(E_2(\sigma))] \subset \mathbb{V}(E_2(\sigma))$.

Note that

$$\sigma \neq \gamma + 1. \tag{6.21}$$

To see this, assume the contrary. Then by (6.19), $f[\mathbb{U}(E_1(1))] \subset \mathbb{V}(E_2(\gamma+1))$. It follows from (6.12), (6.18), and the definition of the set function $\mathbb{V}$ that $f[\mathbb{U}(E_1(1))]$ misses $\operatorname{Ac} E_1(1) \cup \operatorname{Ac} E_2(\gamma+1)$. Let A be an arc in $\mathbb{D}$ from $f[\mathbb{U}(E_1(1))]$ to a point of Θ. Since $\mathbb{D}$ is simply connected and Θ is an interior domain of $\mathbb{D}$, every arc segment in $\mathbb{D} \setminus \operatorname{Ac}\Theta$ from $f[\mathbb{U}(E_1(1))]$ to $\operatorname{Ac}\Theta$ has the same end point in $\operatorname{Ac}\Theta$. Thus the first point of A in $\operatorname{Ac}\Theta$ is either $e_1(1)$ or $e_1(2)$, contradicting the assumption that Θ does not have a departure point. Hence (6.21) is true.

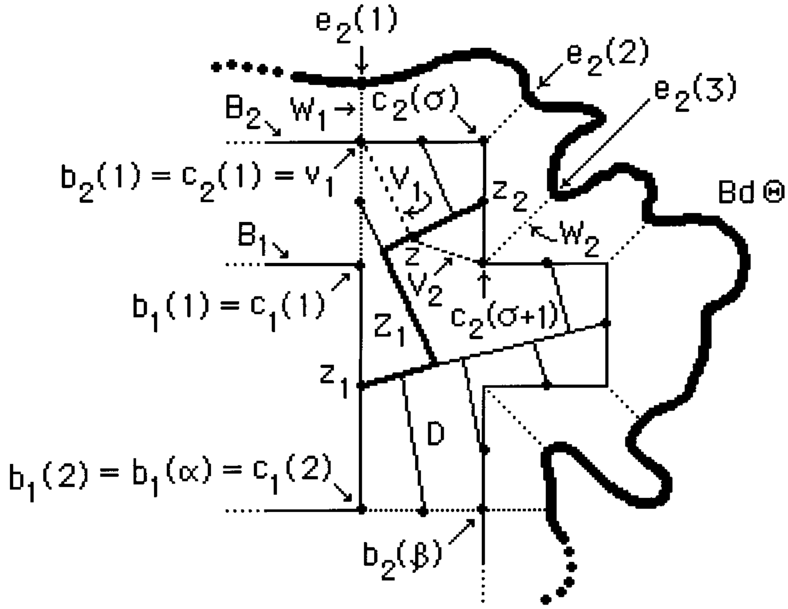

FIGURE 3

Let B be the polygonal simple closed curve in $\mathbb{R}^2$ with the set of vertices $\Gamma = \{b_2(1), b_2(2), \ldots, b_2(\beta), b_1(\alpha), b_1(\alpha-1), \ldots, b_1(1)\}$.

By (6.5), $\mathbb{T}(B)$ is 2^{-1}-lanky. By (6.7) and (6.8), each edge of B has diameter less than 2^{-1}.

Let $\Lambda = \{p \in B : p$ is the center point of an edge of $B\}$.

By [S, Lemma 5.4], there is a finite dendrite D in $\mathbb{T}(B)$ such that

(6.22) $$\Lambda = B \cap D = \{p \in D : p \text{ is an end point of } D\}$$

and

(6.23) if Z is an arc segment in D and z is a point of Z, then there exist two arcs V_1 and V_2 from z to Γ such that

(a) $(V_1 \cup V_2) \setminus \Gamma \subset \operatorname{Int} \mathbb{T}(B)$,

(b) $V_1 \cap V_2 = V_1 \cap D = V_2 \cap D = \{z\}$,

(c) V_1 and V_2 abut Z from opposite sides at z, and

(d) the diameter of $V_1 \cup V_2$ is less than 24.

Let Z_1 be an arc segment in D from a point z_1 of $\Lambda \cap B_1$ to a point z_2 of $\Lambda \cap (c_2(\sigma), c_2(\sigma+1))$ (see Figure 3).

For each point z of Z_1, we define an arc $\mathbb{U}(z)$ as follows. Let V_1 and V_2 be arcs that satisfy the conditions of (6.23) relative to Z_1. For $i = 1$ and 2, let v_i be the end point of V_i that belongs to Γ. For $i = 1$ and 2, let W_i be an arc of diameter less than 1 in $(B \cup N_2) \setminus \operatorname{Cl} Z_1$ from v_i to a point of $\{e_2(1), e_2(2), \ldots, e_2(\gamma)\}$ such that $V_i \cap W_i = \{v_i\}$. Let $\mathbb{U}(z) = V_1 \cup V_2 \cup W_1 \cup W_2$.

For each point z of Z_1,

(6.24) $$\mathbb{U}(z) \text{ is a 26-cut across the ray } \{z_1\} \cup Z_1 \text{ at } z.$$

For each point z of Z_1, let $E(z)$ be the component of $\Theta \setminus \mathbb{U}(z)$ that misses $\mathbb{T}(B_1)$.

For convenience define $E(z_1) = E_1(1), \mathbb{U}(z_2) = \mathbb{U}(E_2(\sigma))$, and $E(z_2) = E_2(\sigma)$.

Suppose $\mathrm{Cl}\, Z_1$ is ordered from z_1 to z_2 by $<$.

Sieklucki's construction has the following additional property. For each two points y and z of $\mathrm{Cl}\, Z_1$, no arc in $\mathbb{U}(y)$ crosses $\mathbb{U}(z)$ [Mo, p. 182]. Thus if $y < z$, then $\mathrm{Cl}(E(y)) \supset \mathbb{U}(z)$ [Mo, Theorem 36, p. 184]. Consequently $E(y) \supset E(z)$.

Hence for each two points y and z of $\mathrm{Cl}\, Z_1$,

$$\text{if } y < z, \text{ then } \mathbb{V}(E(y)) \supset \mathbb{V}(E(z)). \tag{6.25}$$

Note that

$$f[\mathbb{U}(z)] \subset \mathbb{V}(E(z)) \quad \text{for each point } z \text{ of } Z_1. \tag{6.26}$$

To see this, assume the contrary. By (6.20), $f[\mathbb{U}(z_2)] \subset \mathbb{V}(E(z_2))$. Thus by (6.1) and (6.2), there exists an arc Y in Z_1 with end points y and z ($y < z$) such that (a) $f[\mathbb{U}(y)] \not\subset \mathbb{V}(E(y))$, (b) $f[\mathbb{U}(z)] \subset \mathbb{V}(E(z))$, and (c) $\mathbb{U}(z) \cap f[Y \cup \mathbb{U}(y) \cup \mathbb{U}(z)] = \varnothing$. It follows from the argument for (6.14) that $f[Y \cup \mathbb{U}(y) \cup \mathbb{U}(z)] \subset \mathbb{V}(E(z))$. Therefore, by (6.25), $f[\mathbb{U}(y)] \subset \mathbb{V}(E(y))$, a contradiction. Hence (6.26) is true.

Continue this process. For $n = 2, 3, \ldots$ define an arc segment Z_n in $(\mathrm{Int}\, Q_n) \setminus Q_{n+1}$ from a point z_n of B_n to a point z_{n+1} of B_{n+1} with the following property. For each point z of $\{z_n\} \cup Z_n$, there exists a $26n^{-1}$-cut $\mathbb{U}(z)$ across the ray $(\bigcup\{\mathrm{Cl}\, Z_i : i = 1, 2, \ldots, n\}) \setminus \{z_{n+1}\}$ at z such that $f[\mathbb{U}(z)] \subset \mathbb{V}(E(z))$.

Define $\mathbb{P}_{z_1}$ to be the ray $\bigcup\{\mathrm{Cl}\, Z_n : n = 1, 2, \ldots\}$ in Θ, and note that

$$\text{the limit set of } \mathbb{L}_{z_1} \text{ of } \mathbb{P}_{z_1} \text{ is not degenerate.} \tag{6.27}$$

To see this, suppose $\mathbb{L}_{z_1}$ consists of one point p. Then $\mathbb{P}_{z_1}$ is a half-open arc in Θ from z_1 to p. Thus p belongs to $\mathrm{Ac}\, \Theta$. Since p is not a departure point of Θ, every arc segment in $\mathbb{D}$ from $f(p)$ to p intersects $\mathrm{Ac}\, \Theta$. It follows from the arcwise connectivity of $\mathbb{D}$ and Θ that there is an arc segment S in $\mathbb{D} \setminus \mathrm{Cl}\, \mathbb{P}_{z_1}$ from $f(p)$ to z_1.

Since the diameter of $\mathbb{U}(z)$ approaches zero as z approaches p, by (6.1), there is a point y of $\mathbb{P}_{z_1}$ such that for each point z of $\mathbb{P}_{z_1}(y)$,

$$\mathrm{Cl}\, S \cap \mathbb{U}(z) = \varnothing \quad \text{and} \quad \mathbb{U}(y) \cap f[\mathrm{Cl}\, \mathbb{P}_{z_1}(z)] = \varnothing. \tag{6.28}$$

Let z be a point of $\mathbb{P}_{z_1}(y)$ such that $\mathbb{U}(y) \cap \mathbb{U}(z) = \varnothing$. Since $f[\mathbb{U}(z)] \subset \mathbb{V}(E(z))$, there exists an arc T in $\mathbb{D} \setminus \mathbb{U}(y)$ from $f(z)$ to z. By (6.28), there is a simple closed curve C in $S \cup \mathbb{P}_{z_1} \cup T \cup f[\mathrm{Cl}\, \mathbb{P}_{z_1}(z)]$ such that $\mathrm{Int}\, \mathbb{T}(C)$ contains an end point of $\mathbb{U}(y)$, and this contradicts the assumption that $\mathbb{D}$ is simply connected. Hence (6.27) is true.

For each point x of $\mathbb{D}$, define the ray $\mathbb{P}_x$ as follows:

If $x \in \mathbb{P}_{z_1}$, define $\mathbb{P}_x = \mathbb{P}_{z_1}(x)$.

If $x \in \mathbb{D} \setminus \mathbb{P}_{z_1}$, let A be an arc in $\mathbb{D}$ from x that intersects $\mathbb{P}_{z_1}$.

Note that

$$\text{there is a first point } t \text{ of } A \cap \mathbb{P}_{z_1} \text{ with respect to the order of } A. \tag{6.29}$$

To see this, assume the contrary. Since $\mathbb{L}_{z_1}$ is not degenerate, there exist a small subarc B of A (close to the greatest lower bound of $A \cap \mathbb{P}_{z_1}$ with respect to the order of A) and an arc C in $\mathbb{P}_{z_1}$ such that $B \cup C$ is a simple closed curve and C contains a point z with the property that $B \cap \mathbb{U}(z) = \varnothing$. It follows that an end point of $\mathbb{U}(z)$ belongs to $\mathrm{Int}\, \mathbb{T}(B \cup C)$, and this contradicts the assumption that $\mathbb{D}$ is simply connected. Hence (6.29) is true.

In this case, define $\mathbb{P}_x$ to be the union of $\mathbb{P}_{z_1}(t)$ and the arc in A with end points x and t. This completes the definition of the ray $\mathbb{P}_x$ for each point x of $\mathbb{D}$.

Let x be a given point of $\mathbb{D}$.

There exists a point y of $\mathbb{P}_{z_1}$ such that

$$\mathbb{U}(z) \cap (\mathbb{P}_x \setminus \mathbb{P}_x(z)) = \varnothing \quad \text{for each point } z \text{ of } \mathbb{P}_x(y). \tag{6.30}$$

To see this, assume the contrary. Note that $\mathbb{U}(z) \cap \mathbb{P}_{z_1} = \{z\}$ for each point z of $\mathbb{P}_{z_1} \setminus \{z_1\}$. Thus $x \in \mathbb{D} \setminus \mathbb{P}_{z_1}$, and $\mathbb{P}_x \setminus \mathbb{P}_{z_1}$ is a half-open arc. Since $\mathbb{L}_x$ is not degenerate, there is a positive number ζ and a sequence $p_1, p_2, \ldots$ of points of $\mathbb{P}_x$ such that for each positive integer i, (a) $p_i \ll p_{i+1}$, (b) $\mathbb{U}(p_i) \cap (\mathbb{P}_x \setminus \mathbb{P}_{z_1}) \neq \varnothing$, and (c) the diameter of the arc from p_i to p_{i+1} in $\mathbb{P}_x$ is greater than ζ. For each i, let q_i be a point of $\mathbb{U}(p_i) \cap (\mathbb{P}_x \setminus \mathbb{P}_{z_1})$. Let K be a subarc of $\mathrm{Cl}(\mathbb{P}_x \setminus \mathbb{P}_{z_1})$ that contains infinitely many points of $q_1, q_2, \ldots$ and does not contain $\mathbb{L}_x$. Let G be an open subset of $\mathbb{R}^2 \setminus K$ that intersects $\mathbb{L}_x$. There exist points p_i, z, and p_j of $\mathbb{P}_x$ such that (a) $p_i \ll z \ll p_j$, (b) $\mathbb{U}(z) \subset G$, and (c) $G \cap (\mathbb{U}(p_i) \cup \mathbb{U}(p_j)) = \varnothing$. It follows that $K \cup \mathbb{U}(p_i) \cup \mathbb{U}(p_j) \cup (\mathbb{P}_x(p_i) \setminus \mathbb{P}_x(p_j))$ contains a simple closed curve C such that an end point of $\mathbb{U}(z)$ belongs to $\mathrm{Int}\, \mathbb{T}(C)$, and this contradicts the simple connectivity of $\mathbb{D}$. Hence (6.30) is true.

For each point z of $\mathbb{P}_x(y)$,

(6.31) every arc segment in $\mathbb{D}$ from $f(z)$ to an end point of $\mathbb{U}(z)$ intersects $\mathrm{Ac}\,\Theta$.

To see this, assume there is an arc segment S in $\mathbb{D} \setminus \mathrm{Ac}\,\Theta$ from $f(z)$ to an end point e of $\mathbb{U}(z)$. Recall that $\mathbb{U}(z) \cap f[\mathbb{U}(z)] = \varnothing$. Note that $f[\mathbb{U}(z)] \cap \mathrm{Ac}\,\Theta = \varnothing$; for otherwise, $(S \cup f[\mathbb{U}(z)]) \setminus \mathrm{Ac}\,\Theta$ contains an arc segment from e to another point of $\mathrm{Ac}\,\Theta$, and this contradicts Lemma 5.1. Thus $(S \cup f[\mathbb{U}(z)]) \setminus \mathrm{Ac}\,\Theta$ contains an arc segment from e to $f(e)$, and this contradicts the assumption that Θ does not have a departure point. Hence (6.31) is true.

For each point z of $\mathbb{P}_x(y)$, since $f[\mathbb{U}(z)] \subset \mathbb{V}(E(z))$, it follows from (6.30), (6.31), and the simple connectivity of $\mathbb{D}$ that every arc in $\mathbb{D}$ from $f(z)$ to x intersects $\mathbb{U}(z)$. Hence $\mathbb{P}_x$ has the cut property. This completes the proof of Lemma 6.3. $\square$

7. Unrestricted rays

Suppose Z is an arc segment in $\mathbb{D}$ and z is a point of Z. Let X and Y be arcs in $\mathbb{D}$ such that $X \cap Y = X \cap \mathrm{Cl}\, Z = Y \cap \mathrm{Cl}\, Z = \{z\}$. The arc $\mathbb{U}(z) = X \cup Y$ is called a *cut* across Z at z if X and Y abut Z from opposite sides and the end points of $\mathbb{U}(z)$ belong to $\mathrm{Bd}\, \mathbb{D}$.

Lemma 7.1. *Suppose Θ is an ϵ-lanky interior domain of $\mathbb{D}$. Suppose p_1 and p_2 are points of $\mathrm{Ac}\,\Theta \cap \mathrm{Bd}\,\Theta$. Then there exists an arc segment Z in Θ with end points p_1 and p_2 such that for each point z of Z there is a 50ϵ-cut $\mathbb{U}(z)$ in $\mathrm{Ac}\,\Theta$ across Z at z. Furthermore, for each two points y and z of Z, no arc in $\mathbb{U}(y)$ crosses $\mathbb{U}(z)$.*

Proof. Let $Q_1, Q_2, \ldots$ be a sequence of polygonal disks with the properties described in the proof of Lemma 6.3. For $i = 1$ and 2, let A_i be an arc segment of diameter less than ϵ in Θ such that p_i is an end point of A_i. Assume without loss of generality that B_1 intersects A_1 and A_2. Also assume without loss of generality that for each component E of $(\Theta \cap \mathrm{Int}\, Q_1) \setminus O_1$, the diameter of $\mathbb{U}(E)$ is less than ϵ and $\{p_1, p_2\} \cap \mathbb{U}(E) = \varnothing$.

For $i = 1$ and 2, let E_i be a component of $(\Theta \cap \operatorname{Int} Q_1) \setminus O_1$ such that $p_i \in \operatorname{Ac} E_i$. Assume without loss of generality that $E_1 \neq E_2$ and the closure of each component of $(\Theta \cap \operatorname{Int} Q_1) \setminus O_1$ misses either $\mathbb{U}(E_1)$ or $\mathbb{U}(E_2)$.

Let Γ be the set of vertices of B_1, and let $\Lambda = \{q \in B_1 : q$ is the center point of an edge of $B_1\}$.

For $i = 1$ and 2, since the diameter of $A_i \cup \mathbb{U}(E_i)$ is less than 2ϵ, there exists an arc segment Z_i of diameter less than 2ϵ in E_i from p_i to a point q_i of Λ.

For each point z of Z_i ($i = 1$ or 2), there exist two arc segments $I(z)$ and $J(z)$ of diameter less than 2ϵ in $E_i \setminus \operatorname{Cl} Z_i$ from z to B_1 whose closures abut $\operatorname{Cl} Z_i$ from opposite sides at z. Define $\mathbb{U}(z)$ to be an arc in $\mathbb{U}(E_i) \cup \operatorname{Cl}(I(z) \cup J(z))$ that is a 5ϵ-cut in $\operatorname{Cl} E_i$ across Z_i at z.

Since $\mathbb{T}(B_1)$ is ϵ-lanky, by [S, Lemma 5.4], there is a finite dendrite D in $\mathbb{T}(B_1)$ such that $\Lambda = B_1 \cap D = \{q \in D : q$ is an end point of $D\}$ and

(7.2) for each point z of an arc segment A in D there exist two arc segments $I(z)$ and $J(z)$ in $\operatorname{Int} \mathbb{T}(B_1) \setminus D$ from z to Γ such that $\operatorname{Cl} I(z)$ and $\operatorname{Cl} J(z)$ abut A from opposite sides at z and the diameter of $\operatorname{Cl}(I(z) \cup J(z))$ is less than 48ϵ.

Let A be an arc segment in D from q_1 to q_2, and define Z to be the arc segment $Z_1 \cup Z_2 \cup \operatorname{Cl} A$.

For each point z of A, let $I(z)$ and $J(z)$ be arc segments that satisfy the conditions of (7.2). Define $\mathbb{U}(z)$ to be an arc in $B_1 \cup \operatorname{Cl}(O_1 \cup I(z) \cup J(z))$ that is a 50ϵ-cut across Z at z. For $i = 1$ and 2, define $\mathbb{U}(q_i)$ to be the ϵ-cut $\mathbb{U}(E_i)$ across Z at q_i.

For each point z of Z, the arc $\mathbb{U}(z)$ is an 50ϵ-cut in $\operatorname{Ac} \Theta$ across Z at z. By Sieklucki's construction, these cuts can be defined so for each two points y and z of Z, no arc in $\mathbb{U}(y)$ crosses $\mathbb{U}(z)$. □

Lemma 7.3. *Suppose every interior domain of $\mathbb{D}$ has a departure point. Then each point x of $\mathbb{D}$ is the starting point of a ray $\mathbb{P}_x$ with the cut property that has a nondegenerate limit set.*

Proof. Let $\mathcal{C}$ be the collection of interior domains of $\mathbb{D}$. Define $\mathcal{C}_0 = \{\Theta \in \mathcal{C} : \Theta$ is not 1-lanky$\}$. For $i = 1, 2, 3, \ldots$, define $\mathcal{C}_i = \{\Theta \in \mathcal{C} : \Theta$ is i^{-1}-lanky and not $(i+1)^{-1}$-lanky$\}$. The sets $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots$ are disjoint and finite. Note that $\mathcal{C} = \bigcup\{\mathcal{C}_i : i = 0, 1, 2, \ldots\}$.

Suppose J is an arc in $\mathbb{D}$ and Θ is an interior domain of $\mathbb{D}$. Then J is *acceptable relative to* Θ provided that

(7.4) $$J \cap \Theta \text{ is an arc segment,}$$

(7.5) $$\operatorname{Cl}(J \cap \Theta) = J \cap \operatorname{Ac} \Theta,$$

(7.6) if Θ belongs to $\mathcal{C}_i$ ($i > 0$), then for each point z of $J \cap \Theta$ there is a $50i^{-1}$-cut $\mathbb{U}(z)$ in $\operatorname{Ac} \Theta$ across $J \cap \Theta$ at z, and

(7.7) for each two points y and z of $J \cap \Theta$, no arc in $\mathbb{U}(y)$ crosses $\mathbb{U}(z)$.

The arc J is *acceptable* if J is acceptable relative to each interior domain Θ of $\mathbb{D}$ with the property that $J \cap \operatorname{Ac} \Theta$ has more than one point.

For each two points p and q of $\operatorname{Bd} \mathbb{D}$,

(7.8) there exists an acceptable arc from p to q in $\mathbb{D}$.

To see this, let H be an arc in $\mathbb{D}$ from p to q. Let Θ be an interior domain of $\mathbb{D}$ such that $H \cap \operatorname{Ac}\Theta$ has more than one point. By Lemma 5.1, $H \cap \operatorname{Ac}\Theta$ is an arc. By Lemma 7.1, there exists an arc segment Z in Θ that has the same end points as $H \cap \operatorname{Ac}\Theta$ such that the arc $(H \setminus \operatorname{Ac}\Theta) \cup \operatorname{Cl} Z$ is acceptable relative to Θ. If Θ belongs to $\mathcal{C}_i$ ($i > 0$), then each point z of Z is within $50i^{-1}$ of $H \cap \operatorname{Ac}\Theta$; for otherwise, by (7.6), an end point of a cut $\mathbb{U}(z)$ belongs to the bounded complementary domain of a simple closed curve in $Z \cup (H \cap \operatorname{Ac}\Theta)$, and this contradicts the assumption that $\mathbb{D}$ is simply connected.

Repeat this process with each such interior domain of $\mathbb{D}$. If infinitely many arcs in H are replaced, the replaced arcs form a null sequence, and since each $\mathcal{C}_i$ is finite, the replacing arcs also form a null sequence. Thus we obtain an acceptable arc from p to q in $\mathbb{D}$. Hence (7.8) is true.

For each point p of $\operatorname{Int}\mathbb{D}$, let $\Theta(p)$ denote the interior domain of $\mathbb{D}$ that contains p. For each point p of $\operatorname{Bd}\mathbb{D}$, let $\mathbb{H}(p)$ be an arc $\mathbb{D}$ from p to $f(p)$. If $f(p) \in \operatorname{Bd}\mathbb{D}$, define q to be $f(p)$. If $f(p) \in \operatorname{Int}\mathbb{D}$, define q to be the first point of $\mathbb{H}(p)$ in $\operatorname{Ac}\Theta(f(p))$ (Lemma 5.1).

Suppose $p = q$. Let $\mathbb{K}(p)$ be an arc from p to $f(p)$ such that $\mathbb{K}(p) \setminus \{p\} \subset \Theta(f(p))$.

Suppose $p \neq q$. If $q \neq f(p)$, let $\mathbb{I}(p)$ be an arc segment in $\Theta(f(p))$ from q to $f(p)$. If $q = f(p)$, let $\mathbb{I}(p) = \varnothing$. It follows from the proof of (7.8) that there is an acceptable arc $\mathbb{J}(p)$ from p to q in $\mathbb{D}$ such that $\mathbb{I}(p) \cap \mathbb{J}(p) = \varnothing$. Let $\mathbb{K}(p)$ be the arc $\mathbb{J}(p) \cup \operatorname{Cl}\mathbb{I}(p)$ from p to $f(p)$.

For each point p of $\mathbb{D}$, let $C(p)$ denote the circle in $\mathbb{R}^2$ of diameter δ centered on p and let $D(p)$ denote the open disk $\operatorname{Int}\mathbb{T}(C(p))$.

Our first step in constructing the ray $\mathbb{P}(x)$ will be to define an arc A_0 from the given point x to a point x_1 of $\operatorname{Bd}\mathbb{D}$.

Case 7.9. Suppose $x \in \operatorname{Int}\mathbb{D}$. Let A be a half-open arc in $\Theta(x)$ from x to a departure point x_1 of $\Theta(x)$. Define $\Theta(x) = \Theta_1$ and $A_0 = \operatorname{Cl} A$.

Case 7.10. Suppose $x \in \operatorname{Bd}\mathbb{D}$. By (6.1), $\mathbb{K}(x) \cap C(x) \neq \varnothing$. Let A be the half-open arc in $\mathbb{K}(x) \cap D(x)$ from x to a point u_1 of $C(x)$.

Case 7.10A. Suppose u_1 is not accessible from an interior domain of $\mathbb{D}$ that intersects A. Define $x_1 = u_1$ and $A_0 = \operatorname{Cl} A$.

Case 7.10B. Suppose there is an interior domain Θ_1 of $\mathbb{D}$ such that $u_1 \in \operatorname{Ac}\Theta_1$ and $A \cap \Theta_1 \neq \varnothing$. Let v_1 be the first point of A in $\operatorname{Ac}\Theta_1$.

Note that

(7.11) $\quad$ v_1 is not a departure point of Θ_1.

To see this, assume the contrary. Let R be an arc segment in $\mathbb{D} \setminus \operatorname{Ac}\Theta_1$ from v_1 to $f(v_1)$. Let S be the arc in $\mathbb{K}(x)$ from v_1 to $f(x)$. By (6.1), $A \cap f[A] = \varnothing$. Since $\mathbb{J}(x)$ is acceptable and $\mathbb{I}(x) \subset \Theta(f(x))$, there is an arc segment in $\Theta_1 \cap S$ that has v_1 as an end point. Let T be the half-open arc that is the v_1-component of $(\operatorname{Cl} R) \setminus f[A]$. There exists a simple closed curve C in $S \cup T \cup f[A]$ containing T such that $\operatorname{Int}\mathbb{T}(C) \cap \Theta_1 \neq \varnothing$. Since $\mathbb{D}$ is simply connected and Θ_1 is an interior domain of $\mathbb{D}$, it follows that $\operatorname{Int}\mathbb{T}(C) \subset \Theta_1$, and this contradicts the fact that $T \not\subset \operatorname{Ac}\Theta_1$. Hence (7.11) is true.

Let x_1 be a departure point of Θ_1. By (7.11), $x_1 \neq v_1$. It follows that $x_1 \neq x$. By Lemma 7.1, there exists an arc segment Z in Θ_1 from v_1 to x_1 such that $\operatorname{Cl} Z$ is acceptable relative to Θ_1. Define A_0 to be the union of $\mathrm{Cl}\, Z$ and the x-component of $A \setminus \{v_1\}$.

We show that for each positive integer n there exists an arc A_n in $\mathbb{D}$ with end points x_n and x_{n+1} such that

(7.12$_\mathrm{n}$) either x_{n+1} is a departure point of an interior domain Θ_{n+1} of $\mathbb{D}$ and an end point of an arc segment in $A_n \cap \Theta_{n+1}$, or x_{n+1} is not accessible from an interior domain of $\mathbb{D}$ that intersects A_n,

$$A_n \cap \bigcup\{A_i : 0 \leq i < n\} = \{x_n\}, \tag{7.13$_\mathrm{n}$}$$

(7.14$_\mathrm{n}$) no arc segment in $A_{n-1} \cup A_n$ that contains x_n lies in a simple closed curve in $\mathbb{D}$,

(7.15$_\mathrm{n}$) $\bigcup\{A_i : 1 \leq i \leq n\}$ is an acceptable arc,

(7.16$_\mathrm{n}$) if the diameter of A_n is less than $\delta/4$, then x_{n+1} is a departure point of an interior domain Θ_{n+1} of $\mathbb{D}$ that intersects A_n, and Θ_{n+1} belongs to $\mathcal{C}_0$ or some $\mathcal{C}_i$ such that $50i^{-1} \geq \delta/4$, and

(7.17$_\mathrm{n}$) every arc in $\mathbb{D}$ from x to $f(x_n)$ contains x_n.

We begin by defining an arc A_1 in $\mathbb{D}$ from x_1 to a point x_2 of $\mathrm{Bd}\, \mathbb{D}$ that satisfies the conditions (7.12$_1$)–(7.17$_1$).

Either x_1 is a departure point of an interior domain Θ_1 of $\mathbb{D}$ and an end point of an arc segment in $A_0 \cap \Theta_1$ (Cases 7.9 and 7.10B), or x_1 is not accessible from an interior domain of $\mathbb{D}$ that intersects A_0 (Case 7.10A).

Case 7.18. Suppose $f(x_1) \in \mathrm{Int}\, \mathbb{D}$ and $x_1 \in \mathrm{Ac}\, \Theta(f(x_1))$. Let x_2 be a departure point of $\Theta(f(x_1))$. Since x_1 is not a departure point of $\Theta(f(x_1))$, it follows that $x_1 \neq x_2$. Let Z be an arc segment in $\Theta(f(x_1))$ from x_1 to x_2 such that $\mathrm{Cl}\, Z$ is acceptable relative to $\Theta(f(x_1))$. Define $\Theta_2 = \Theta(f(x_1))$ and $A_1 = \mathrm{Cl}\, Z$.

Case 7.19. Suppose $f(x_1)$ does not belong to an interior domain of $\mathbb{D}$ from which x_1 is accessible.

Case 7.19A. Suppose x_1 is a departure point of an interior domain Θ_1 of $\mathbb{D}$ that intersects A_0 (Cases 7.9 and 7.10B). By Lemma 5.1, $\mathbb{K}(x_1) \cap (A_0 \cup \mathrm{Ac}\, \Theta_1) = \{x_1\}$. Let A be the half-open arc in $\mathbb{K}(x_1) \cap D(x_1)$ from x_1 to a point u_2 of $C(x_1)$.

Case 7.19A1. Suppose u_2 is not accessible from an interior domain of $\mathbb{D}$ that intersects A. Define $x_2 = u_2$ and $A_1 = \mathrm{Cl}\, A$.

Case 7.19A2. Suppose there is an interior domain Θ_2 of $\mathbb{D}$ such that $u_2 \in \mathrm{Ac}\, \Theta_2$ and $A \cap \Theta_2 \neq \varnothing$. Let v_2 be the first point of A in $\mathrm{Ac}\, \Theta_2$. Let x_2 be a departure point of Θ_2. By the argument for (7.11), v_2 is not a departure point of Θ_2. Thus $x_2 \neq v_2$ and $x_2 \neq x_1$. Let Z be an arc segment in Θ_2 from v_2 to x_2 such that $\mathrm{Cl}\, Z$ is acceptable relative to Θ_2. Define A_1 to be the union of $\mathrm{Cl}\, Z$ and the x_1-component of $A \setminus \{v_2\}$.

Case 7.19B. Suppose x_1 is not accessible from an interior domain of $\mathbb{D}$ that intersects A_0. This can only occur if Case 7.10A holds. Hence the diameter of A_0 is less than δ. As in Case 7.19A, let A be the half-open arc in $\mathbb{K}(x_1) \cap D(x_1)$ from x_1 to a point u_2 of $C(x_1)$.

Case 7.19B1. Suppose u_2 is not accessible from an interior domain of $\mathbb{D}$ that intersects A. Define $x_2 = u_2$ and $A_1 = \operatorname{Cl} A$.

Case 7.19B2. Suppose there is an interior domain Θ_2 of $\mathbb{D}$ such that $u_2 \in \operatorname{Ac}\Theta_2$ and $A \cap \Theta_2 \neq \varnothing$. Use the procedure given in Case 7.19A2 to define x_2 and A_1.

Either x_2 is a departure point of an interior domain Θ_2 of $\mathbb{D}$ and an end point of an arc segment in $A_1 \cap \Theta_2$ (Cases 7.18, 7.19A2, and 7.19B2), or x_2 is not accessible from an interior domain of $\mathbb{D}$ that intersects A_1 (Cases 7.19A1 and 7.19B1). Thus (7.12_1) is satisfied.

Note that (7.13_1) is true ($A_0 \cap A_1 = \{x_1\}$). To see this we must consider two cases.

Case 7.20. Suppose x_1 is a departure point of an interior domain Θ_1 of $\mathbb{D}$ and there is an arc segment S in $A_0 \cap \Theta_1$ that has x_1 as an end point (Cases 7.9 and 7.10B).

If Case 7.18 holds, then A_1 is the closure of an arc segment Z in $\Theta(f(x_1))$ from x_1 to x_2. Since $\Theta_1 \neq \Theta(f(x_1))$, it follows that $S \cap A_1 = \varnothing$. Thus $A_0 \cap A_1 = \{x_1\}$; for otherwise, by Lemma 5.1, $S \subset \operatorname{Ac}\Theta(f(x_1))$, and this contradicts the fact that $S \subset \Theta_1$.

If Case 7.19 holds, then $\mathbb{K}(x_1) \cap \operatorname{Ac}\Theta_1 = \{x_1\}$. Thus $\mathbb{K}(x_1) \cap S = \varnothing$. By the argument in the preceding paragraph, $A_0 \cap A_1 = \{x_1\}$.

Case 7.21. Suppose x_1 is not accessible from an interior domain of $\mathbb{D}$ that intersects A_0 (Case 7.10A). Then A_0 is the arc in $\mathbb{J}(x) \cap \operatorname{Cl} D(x)$ that goes from x to the point x_1 of $C(x)$.

If Case 7.18 holds, then A_1 is the closure of an arc segment in $\Theta(f(x_1))$ from x_1 to x_2. Since $\mathbb{J}(x)$ is acceptable, it follows that x_1 is the first point of $\mathbb{K}(x)$ in $\operatorname{Ac}\Theta(f(x_1))$. Hence $A_0 \cap A_1 = \{x_1\}$.

Suppose Case 7.19B holds and $A_0 \cap A_1 \neq \{x_1\}$. Let z be the last point of $\mathbb{K}(x_1)$ that belongs to $A_0 \setminus \{x_1\}$. Let J be the arc in $\mathbb{K}(x)$ from z to $f(x)$. Let K be the arc in $\mathbb{K}(x_1)$ from z to $f(x_1)$. By (6.1), $f[A_0] \cap \operatorname{Cl} D(x) = \varnothing$. Thus there exists a simple closed curve C in $J \cup K \cup f[A_0]$ such that x_1 belongs to an arc segment in $C \cap J$, and this contradicts the fact that $\mathbb{J}(x)$ is acceptable. Hence (7.13_1) is true.

Observe that (7.14_1) is satisfied. To see this, assume there exist an arc segment R in $A_0 \cup A_1$ and a simple closed curve C in $\mathbb{D}$ such that $x_1 \in R \subset C$. Note that $\operatorname{Int}\mathbb{T}(C)$ is contained in an interior domain of $\mathbb{D}$. Case 7.10A does not hold, since $\mathbb{J}(x)$ is acceptable, $\mathbb{I}(x) \subset \Theta(f(x))$, and (in Case 7.10A) $A_0 \subset \mathbb{J}(x) \cup \operatorname{Cl}\mathbb{I}(x)$ and x_1 is not accessible from an interior domain of $\mathbb{D}$ that intersects A_0. Consequently Case 7.9 or Case 7.10B holds. Therefore x_1 is a departure point of an interior domain Θ_1 of $\mathbb{D}$, and there is an arc segment S in $A_0 \cap \Theta_1$ that has x_1 as an end point. Since $R \cap S \neq \varnothing$, it follows that $\Theta_1 \cap R \neq \varnothing$. Hence $\operatorname{Int}\mathbb{T}(C) \subset \Theta_1$ and $R \subset \operatorname{Ac}\Theta_1$.

Case 7.18 does not hold; for otherwise, $\operatorname{Int}\mathbb{T}(C) \subset \Theta(f(x_1))$ and $\Theta_1 = \Theta(f(x_1))$, contradicting the fact that x_1 is a departure point of Θ_1.

Since Case 7.10A does not hold, Case 7.19B does not hold.

Evidently Case 7.19A holds. However, in Case 7.19A, $\mathbb{K}(x_1) \cap \operatorname{Ac}\Theta_1 = \{x_1\}$. It follows from Lemma 5.1 that in Cases 7.19A1 and 7.19A2, $A_1 \cap \operatorname{Ac}\Theta_1 = \{x_1\}$, and this contradicts the fact that $R \subset \operatorname{Ac}\Theta_1$. Hence (7.14_1) is true.

Since A_1 is acceptable, (7.15_1) is satisfied.

Observe that

(7.22) the diameter of A_1 in Cases 7.18, 7.19A1, and 7.19B1 is at least $\delta/2$.

To see this, assume the diameter of A_1 is less than $\delta/2$. Since $A_1 \subset D(x_1)$, neither Case 7.19A1 nor Case 7.19B1 holds. Thus A_1 is defined in Case 7.18. By (6.1), $A_1 \cap f[A_1] = \varnothing$. Let R be an arc segment in $\mathbb{D} \setminus \mathrm{Ac}\,\Theta(f(x_1))$ from x_2 to $f(x_2)$. Let S be an arc segment in $\Theta(f(x_1))$ from $f(x_1)$ to x_2. Let T be the half-open arc that is the x_2-component of $(\mathrm{Cl}\,R) \setminus f[A_1]$. There exists a simple closed curve C in $S \cup T \cup f[A_1]$ containing T such that $\Theta(f(x_1)) \cap \mathrm{Int}\,\mathbb{T}(C) \neq \varnothing$. Since $\Theta(f(x_1))$ is an interior domain of $\mathbb{D}$, it follows that $\mathrm{Int}\,\mathbb{T}(C) \subset \Theta(f(x_1))$, and this contradicts the fact that $T \not\subset \mathrm{Ac}\,\Theta(f(x_1))$. Hence (7.22) is true.

Suppose A_1 is defined in Case 7.19A2 or Case 7.19B2. Suppose i is a positive integer such that $\Theta_2 \in \mathcal{C}_i$.

Note that

(7.23) if $50i^{-1} < \delta/4$, then the diameter of A_1 is at least $\delta/4$.

To see this, assume the contrary. The point x_2 is not in $\mathbb{K}(x_1)$; for otherwise, since $\rho(u_2, \mathrm{Cl}\,Z) > \delta/4$, there is a $50i^{-1}$-cut across $\mathbb{K}(x_1) \cap \Theta_2$ at u_2 with an end point in the bounded complementary domain of a simple closed curve in $Z \cup \mathbb{K}(x_1)$, contradicting the simple connectivity of $\mathbb{D}$.

Let v be the last point of Z that belongs to $\mathbb{K}(x_1)$. Let V be the arc segment in Z from v to x_2. Since x_2 is a departure point of Θ_2, there is an arc segment R in $\mathbb{D} \setminus \mathrm{Ac}\,\Theta_2$ from x_2 to $f(x_2)$. Let S be the arc in $\mathbb{K}(x_1)$ from v to $f(x_1)$. Let T be the half-open arc that is the x_2-component of $\mathrm{Cl}\,R \setminus f[A_1]$. There is a simple closed curve C in $V \cup S \cup T \cup f[A_1]$ containing $V \cup T$ such that $\Theta_2 \cap \mathrm{Int}\,\mathbb{T}(C) \neq \varnothing$. Since Θ_2 is an interior domain, it follows that $\mathrm{Int}\,\mathbb{T}(C) \subset \Theta_2$, and this contradicts the fact that $T \not\subset \mathrm{Ac}\,\Theta_2$. Hence (7.23) is true.

By (7.22) and (7.23), if the diameter of A_1 is less than $\delta/4$, then x_2 is a departure point of an interior domain Θ_2 of $\mathbb{D}$ that intersects A_1, and Θ_2 belongs to $\mathcal{C}_0$ or some $\mathcal{C}_i$ such that $50i^{-1} \geq \delta/4$. Thus (7.16_1) is satisfied.

Note that in Case 7.19B2, by (7.13_1) and Lemma 5.1, $(A_0 \setminus \{x_1\}) \cap \mathrm{Ac}\,\Theta_2 = \varnothing$. Thus in Case 7.19B2, by Lemma 5.1, there is an arc in $\mathbb{K}(x_1) \cup \mathrm{Ac}\,\Theta_2$ from x_2 to $f(x_1)$ that misses A_0.

In fact, in every case,

(7.24) there exists an arc in $\mathbb{D} \setminus A_0$ from x_2 to $f(x_1)$.

By (7.13_1) (7.14_1), and (7.24), every arc in $\mathbb{D}$ from x to $f(x_1)$ contains x_1. Thus (7.17_1) is satisfied.

Proceeding inductively, we let n be an integer greater than 1.

Assume that for each positive integer m less than n, an arc A_m in $\mathbb{D}$ with end points x_m and x_{m+1} has been defined that satisfies conditions (7.12_{m})–(7.17_{m}). To complete the inductive step, use the preceding arguments to define an arc A_n in $\mathbb{D}$ with end points x_n and x_{n+1} that satisfies conditions (7.12_{n})–(7.17_{n}). Note that (7.15_{n}) follows from (7.14_{n}), ($7.15_{\mathrm{n-1}}$), and the acceptability of A_n.

Define $\mathbb{P}_x = \bigcup\{A_n : n = 0, 1, 2, \dots\}$. By ($7.13_{\mathrm{n}}$), $\mathbb{P}_x$ is a ray. By (7.14_{n}),

(7.25) no interior domain of $\mathbb{D}$ intersects more than one element of $\{A_n : n = 1, 2, \dots\}$.

Since each $\mathcal{C}_i$ is finite and only finitely many positive integers i have the property that $50i^{-1} \geq \delta/4$, by (7.16_{n}) and (7.25), only finitely many elements of $\{A_n : n = 1, 2, \dots\}$ have diameters less than $\delta/4$. Hence $\mathbb{L}_x$ is not degenerate.

To see that $\mathbb{P}_x$ has the cut property, let ϵ be a given positive number. Let j be a positive integer such that $50j^{-1}$ is less than ϵ and δ. Since each $\mathcal{C}_i$ is finite, by (7.25), there is an integer $m > 1$ such that $\mathbb{P}_x(x_m) \cap \cup\{\mathcal{C}_i : 0 \leq i < j\} = \varnothing$.

For each point z of $\mathbb{P}_x(x_m) \cap \operatorname{Int}\mathbb{D}$, the arc $\operatorname{Cl}(\mathbb{P}_x(x_m) \cap \Theta(z))$ is acceptable relative to $\Theta(z)$. By (7.6), there is a $50j^{-1}$-cut $\mathbb{U}(z)$ in $\operatorname{Ac}\Theta(z)$ across $\mathbb{P}_x(x_m) \cap \Theta(z)$ at z. Furthermore, by (7.7), for each two points y and z of $\mathbb{P}_x(x_m) \cap \Theta(z)$, no arc in $\mathbb{U}(y)$ crosses $\mathbb{U}(z)$. Let $E(z)$ denote the component of $\Theta(z) \setminus \mathbb{U}(z)$ that intersects $\mathbb{P}_x(z)$.

For each point z of $\mathbb{P}_x(x_m) \cap \operatorname{Int}\mathbb{D}$, let $\mathbb{W}(z)$ denote $\operatorname{Ac}E(z) \cup \{p \in \mathbb{D}$: an arc segment in $\mathbb{D} \setminus \operatorname{Ac}\Theta(z)$ goes from p to a point of $\operatorname{Ac}E(z) \setminus \mathbb{U}(z)\}$.

For each point z of $\mathbb{P}_x(x_m) \setminus \operatorname{Int}\mathbb{D}$, let $\mathbb{U}(z) = \{z\}$ and let $\mathbb{W}(z) = \{p \in \mathbb{D}$: an arc in $\mathbb{D} \setminus (\mathbb{P}_x \setminus \mathbb{P}_x(z))$ goes from p to $z\}$.

For each two points y and z of $\mathbb{P}_x(x_m)$,

$$\text{if } y \ll z, \text{ then } \mathbb{W}(y) \supset \mathbb{W}(z). \tag{7.26}$$

Let n be an integer greater than m. For each point z of $\mathbb{P}_x(x_m) \setminus \mathbb{P}_x(x_n)$,

$$f(z) \in \mathbb{W}(z). \tag{7.27}$$

To see this, assume the contrary. It follows from (7.17_{n}) and the arcwise connectivity of $\mathbb{D}$ that $f(x_n) \in \mathbb{W}(x_n)$. Thus by (6.1), there exists an arc A in $\operatorname{Cl}(\mathbb{P}_x(x_m) \setminus \mathbb{P}_x(x_n))$ with end points p and q ($p \ll q$) such that (a) $f(p) \in \mathbb{D} \setminus \mathbb{W}(p)$, (b) $f(q) \in \mathbb{W}(q)$, and (c) $\mathbb{U}(q) \cap f[A] = \varnothing$. It follows from the argument for (6.14) that $f[A] \subset \mathbb{W}(q)$. Therefore by (7.26), $f(p) \in \mathbb{W}(p)$, a contradiction. Hence (7.27) is true.

By (7.5), (7.15_{n}), and the simple connectivity of $\mathbb{D}$, no simple closed curve in $\mathbb{D}$ contains an arc segment in $\mathbb{P}_x$ that intersects $\mathbb{P}_x(x_m) \setminus \operatorname{Int}\mathbb{D}$. By (7.27) and the simple connectivity of $\mathbb{D}$, for each point z of $\mathbb{P}_x(x_m)$, every arc in $\mathbb{D}$ from x to $f(z)$ intersects $\mathbb{U}(z)$. Hence $\mathbb{P}_x$ has the cut property. This completes the proof of Lemma 7.3. □

8. The general theorem

We are ready to generalize Theorem 3.1.

Theorem 8.1. *Suppose M is a plane continuum, $\mathcal{D}$ is a decomposition of M, and each element of $\mathcal{D}$ is simply connected. Then every map of M that sends each element of $\mathcal{D}$ into itself has a fixed point.*

Proof. We summarize the proof of Theorem 3.1 that is given in [H7] and present the changes required to prove Theorem 8.1. Our summary of [H7] is only a sketch with the key ideas reordered to give an overview. To be more comprehensive, the reader may choose to make Changes 1–7 (given below) and then read [H7] from beginning to end.

We assumed in [H7] that there exists a fixed-point-free map f that sends each element of $\mathcal{D}$ into itself. In Section 10 of [H7], we used the unique arcwise connectivity of the elements of $\mathcal{D}$ to define, for each point x of M, a Borsuk ray $\mathbb{P}_x$ with a nondegenerate limit set $\mathbb{L}_x$. This is impossible when the elements of $\mathcal{D}$ are only assumed to be simply connected. □ □

Change 1. Replace (10.2)–(10.8) of [H7] with the following:

For each point x of M in Theorem 8.1, let $\mathbb{P}_x$ be the ray defined in Lemma 6.3 or Lemma 7.3 with the cut property that has a nondegenerate limit set $\mathbb{L}_x$. $\square\ \square\ \square$

The statements made at the end of Section 10 about images under f of points on rays must be modified in the more general setting. $\square\ \square$

Change 2. Replace line 2 from the bottom of page 153 through line 6 of page 154 in [H7] with the following:

Let ϵ be a positive number less than δ and the diameter of $\mathbb{D}_x$. Since $\mathbb{P}_x$ has the cut property, there is a point y of $\mathbb{P}_x$ such that

(a) for each point z of $\mathbb{P}_x(y) \cap \operatorname{Int} \mathbb{D}_x$, there exists an adherent $\epsilon/16$-cut $\mathbb{U}(z)$ across $\mathbb{P}_x$ at z, and

(b) for each point z of $\mathbb{P}_x(y) \setminus \operatorname{Int} \mathbb{D}_x$, every arc from x to $f(z)$ contains z and no arc segment in $\mathbb{P}_x$ that contains z lies in a simple closed curve in $\mathbb{D}_x$.

We call y a *control point* of $\mathbb{P}_x$.

For convenience, define the set $\mathbb{C}(z)$ for each point z of $\mathbb{P}_x(y)$ as follows: Let $\mathbb{C}(z) = \mathbb{U}(z)$ if $z \in \operatorname{Int} \mathbb{D}_x$, and let $\mathbb{C}(z) = \{z\}$ if $z \in \operatorname{Bd} \mathbb{D}_x$.

For each control point y of $\mathbb{P}_x$,

(10.9) there is a point z of $\mathbb{P}_x(y)$ such that an arc goes from x to $f(y)$ in $\mathbb{D}_x \setminus \mathbb{C}(z)$.

To see this, assume the contrary. Let A be an arc in $\mathbb{D}_x$ ordered from x to $f(y)$. Then $A \cap \mathbb{C}(z) \neq \varnothing$ for each point z of $\mathbb{P}_x(y)$.

Let $z_1, z_2, \dots$ be a sequence of points of $\mathbb{P}_x(y)$ with the following properties. For each positive integer i, (a) $z_i \ll z_{i+1}$, (b) $\rho(z_i, z_{i+1}) = \epsilon/4$, and (c) the diameter of the arc B_i in $\mathbb{P}_x$ from z_i to z_{i+1} is less than $\epsilon/2$.

It follows that $\rho(\mathbb{C}(z_i), \mathbb{C}(z_{i+1})) > \epsilon/8$ for each i. By the cut property, for each $i > 1$, there is an arc C_i in $\mathbb{D}_x$ from $\mathbb{C}(z_i)$ to $f(z_i)$ that misses $\mathbb{C}(z_{i-1})$.

For each i, no point of $A \cap \mathbb{C}(z_{i+1})$ precedes $A \cap \mathbb{C}(z_i)$ with respect to the order of A; for otherwise, $A \cup \mathbb{C}(z_{i+1}) \cup C_{i+1} \cup f[B_i]$ contains an arc from x to $f(z_i)$ that misses $\mathbb{C}(z_i)$, a violation of the cut property. Since $\rho(A \cap \mathbb{C}(z_i), A \cap \mathbb{C}(z_{i+1})) > \epsilon/8$ for each i, this contradicts the fact that A is an arc. Hence (10.9) is true.

We call z a *target point* of $\mathbb{P}_x$ relative to y.

Let y be a control point of $\mathbb{P}_x$ and let z be a target point of $\mathbb{P}_x$ relative to y. For each point w of $\mathbb{P}_x(z)$,

$$\mathbb{C}(w) \cap f[\mathbb{P}_x(y) \setminus \mathbb{P}_x(w)] \neq \varnothing. \tag{10.10}$$

To see this, assume the contrary. Since z is a target point of $\mathbb{P}_x$ relative to y, there is an arc A from x to $f(y)$ in $\mathbb{D}_x \setminus \mathbb{C}(z)$. By the cut property, $A \cap \mathbb{C}(w) = \varnothing$. Thus $A \cup f[\mathbb{P}_x(y) \setminus \mathbb{P}_x(w)]$ contains an arc from x to $f(w)$ that misses $\mathbb{C}(w)$, and this violates the cut property. Hence (10.10) is true. $\square\ \square\ \square$

In Section 11 of [H7], we used Zorn's lemma to define a subcontinuum L of M such that for each point x of L either $\mathbb{P}_x \not\subset L$ or $\mathbb{L}_x = L$. In Section 12 of [H7], we defined $\mathcal{E}$ to be the collection of elements of $\mathcal{D}$ that are contained in L. The proof breaks down to two cases.

Case 1. Suppose $\mathcal{E}$ is countable. By definition, L is either the intersection of a nested sequence $\mathbb{L}_{x_1}, \mathbb{L}_{x_2}, \dots$ of limits of Borsuk rays (Case 12.2a) or the limit of one Borsuk ray (Case 12.2b).

We focus on the argument given in [H7] that rules out Case 12.2a.

If $L = \bigcap\{\mathbb{L}_{x_n} : n = 1, 2, \dots\}$, define $\mathbb{Q}_{x_1}$ to be $\bigcup\{\mathbb{P}_{x_n} : n = 1, 2, \dots\}$ with the linear order induced by the order on each of the $\mathbb{P}_{x_n}$'s. For each positive integer n, we have $\operatorname{Cl}\mathbb{P}_{x_n} \supset \mathbb{L}_{x_n} \supset \operatorname{Cl}\mathbb{P}_{x_{n+1}}$.

We assume that M is embedded in a 2-sphere $\mathbb{S}^2$. According to Lemmas 6.1 and 6.47 of [H7], there exist a component Δ of $\mathbb{S}^2 \setminus L$ and a point y of $\mathbb{P}_{x_1}$ such that $y \in \Delta$ and $\mathbb{P}_y \subset \operatorname{Cl}\Delta$. The set $\mathbb{P}_y$ consists of y and all points of $\mathbb{P}_{x_1}$ that follow y. Recall that in Section 6 (above) we changed the notation for this subray to $\mathbb{P}_{x_1}(y)$. The set $\mathbb{Q}_y$ is defined in [H7] to be y and all points of $\mathbb{Q}_{x_1}$ that follow y. Note that $\mathbb{Q}_y \subset \operatorname{Cl}\Delta$.

To establish the existence of Δ, we first showed that $\mathbb{P}_{x_1}$ cannot enter a complementary domain Ω of L, leave $\operatorname{Cl}\Omega$, and then return to Ω. This follows from (12.5) of [H7]. However (12.5) follows from (12.4), a statement that must be modified in the more general setting. $\square\ \square$

Change 3. Replace (12.4) with two statements (12.4A) and (12.4B) by changing lines 11 through 15 of page 156 in [H7] to the following:

For each positive integer n, let y_n be a control point of $\mathbb{P}_{x_n}$. For each point z of $\mathbb{P}_{x_n}(y_n)$, let $\mathbb{C}[\mathbb{P}_{x_n}(z)]$ denote the set $\bigcup\{\mathbb{C}(p) : p \in \mathbb{P}_{x_n}(z)\}$.

For each arc J in an element of $\mathcal{D}$ and each positive integer n,

$$\text{(12.4A)} \qquad \text{there is a point } z \text{ of } \mathbb{P}_{x_n}(y_n) \text{ such that } J \cap \mathbb{C}[\mathbb{P}_{x_n}(z)] = \varnothing.$$

To see this, assume the contrary. Since the elements of $\mathcal{D}$ are disjoint, $J \subset \mathbb{D}_{x_n}$. Since $\mathbb{L}_{x_n}$ is not degenerate, there is a positive number ζ and a sequence $z_1, z_2, \dots$ of points of $\mathbb{P}_{x_n}(y_n)$ such that for each positive integer i, (a) $z_i \ll z_{i+1}$, (b) $\mathbb{C}(z_i) \cap J \neq \varnothing$, and (c) the diameter of the arc from z_i to z_{i+1} in $\mathbb{P}_{x_n}$ is greater than ζ. For each i, let p_i be a point of $\mathbb{C}(z_i) \cap J$. Let K be a subarc of J that contains infinitely many points of $p_1, p_2, \dots$ and does not contain $\mathbb{L}_{x_n}$. Let G be an open subset of $\mathbb{R}^2 \setminus K$ that intersects $\mathbb{L}_{x_n}$. By the cut property, there exist points z_i, z, and z_j of $\mathbb{P}_{x_n}$ such that (a) $z_i \ll z \ll z_j$, (b) $\mathbb{C}(z) \subset G$, (c) $G \cap (\mathbb{C}(z_i) \cup \mathbb{C}(z_j)) = \varnothing$, and (d) $\{p_i, p_j\} \subset K$. If $\mathbb{C}(z)$ is a cut, then $K \cup \mathbb{C}(z_i) \cup \mathbb{C}(z_j) \cup (\mathbb{P}_{x_n}(z_i) \setminus \mathbb{P}_{x_n}(z_j))$ contains a simple closed curve that does not bound a disk in $\mathbb{D}_{x_n}$. If $\mathbb{C}(z) = \{z\}$, then $\mathbb{C}(z)$ is in $\operatorname{Bd}\mathbb{D}_{x_n}$ and a simple closed curve in $K \cup \mathbb{C}(z_i) \cup \mathbb{C}(z_j) \cup (\mathbb{P}_{x_n}(z_i) \setminus \mathbb{P}_{x_n}(z_j))$ contains an arc segment in $\mathbb{P}_{x_n}$ that contains $\mathbb{C}(z)$, violating the cut property for $\mathbb{P}_{x_n}$. Hence (12.4A) is true.

For each arc J in an element of $\mathcal{D}$ and each positive integer n,

$$\text{(12.4B)} \qquad \text{there is a point } z \text{ of } \mathbb{P}_{x_n}(y_n) \text{ such that } J \cap f[\mathbb{P}_{x_n}(z)] = \varnothing.$$

To see this, assume the contrary. Let z_1 be a point of $\mathbb{P}_{x_n}(y_n)$ such that $f(z_1) \in J$. Let K be an arc in $\mathbb{D}_{x_n}$ from x_n to $f(z_1)$. By (12.4A), there is a point z_2 of $\mathbb{P}_{x_n}(y_n)$ such that $(J \cup K) \cap \mathbb{C}[\mathbb{P}_{x_n}(z_2)] = \varnothing$. Let z_3 be a point of $\mathbb{P}_{x_n}(z_2)$ such that $f(z_3) \in J$. Then $J \cup K$ contains an arc from x_n to $f(z_3)$ that misses $\mathbb{C}(z_3)$, and this violates the cut property. Hence (12.4B) is true.

The argument for (12.1) of [H7] holds when subrays are used in place of the $\mathbb{P}_{x_n}$'s. Thus every application of (12.4) in the proof of (12.1) of [H7] can be replaced by an application of either (12.4A) or (12.4B) in the more general setting. $\square\ \square\ \square$

If Δ does not exist, $\mathbb{P}_{x_1}$ runs through infinitely many complementary domains of L without returning to any one. Since $L \subset \mathbb{L}_{x_1}$, it follows from an argument in Section 6 of [H7] that $\mathbb{P}_{x_1}$ must keep doubling back. Since the elements of $\mathcal{D}$ are not assumed to be uniquely arcwise connected, we must modify this argument. $\square\ \square$

Change 4. Replace line 4 of page 129 through line 23 of page 130 in [H7] with the following:

For each positive integer α, let

$$H_\alpha = \{p \in L \cap \mathbb{D}_x : \text{ an arc in } \mathbb{D}_x \setminus \operatorname{Cl} F_\alpha \text{ goes from } p \text{ to } x\}.$$

For each positive integer α and each point σ of Σ, let

$$H_{\alpha,\sigma} = \{p \in L \cap \mathbb{D}_\sigma : \text{ an arc in } \mathbb{D}_\sigma \setminus \operatorname{Cl} F_\alpha \text{ goes from } p \text{ to } \sigma\}.$$

For each positive integer α and each element Γ of $\mathcal{G}_\alpha$, let

$$H_{\alpha,\Gamma} = \{p \in L \setminus \cup(\mathcal{E} \cup \{\mathbb{D}_x\}) : \text{ an arc in } \mathbb{D}_p \setminus \operatorname{Cl} F_\alpha \text{ goes from } p \text{ into } \Gamma\}.$$

Note that

$$L = \bigcup\{H_\alpha \cup H_{\alpha,\sigma} \cup H_{\alpha,\Gamma} : \alpha = 1, 2, \ldots,\ \sigma \in \Sigma,\ \text{and } \Gamma \in \mathcal{G}_\alpha\}. \tag{6.4}$$

To see this, let p be a point of L. We consider three cases.

Suppose p is a point of $L \cap \mathbb{D}_x$. Let A be an arc from p to x in $\mathbb{D}_x$. By (6.2), L is not an arc. Therefore A does not contain L. Thus there is an integer α such that $A \cap \operatorname{Cl} F_\alpha = \varnothing$. Hence $p \in H_\alpha$.

Suppose p is a point of $L \setminus \mathbb{D}_x$ and L contains $\mathbb{D}_p$. Let σ be a point of Σ such that $\mathbb{D}_p = \mathbb{D}_\sigma$. Let A be an arc from p to σ in $\mathbb{D}_\sigma$. Since A does not contain L, there is an integer α such that $A \cap \operatorname{Cl} F_\alpha = \varnothing$. Hence $p \in H_{\alpha,\sigma}$.

Suppose p is a point of $L \setminus \mathbb{D}_x$ and L does not contain $\mathbb{D}_p$. Let q be a point of $\mathbb{D}_p \setminus L$. Let A be an arc from p to q in $\mathbb{D}_p$. Since A does not contain L, there is an integer α such that $A \cap \operatorname{Cl} F_\alpha = \varnothing$. Note that q does not belong to $\operatorname{Cl}(\mathbb{P}_x \cup F_\alpha)$. Let Γ be the q-component of $\mathbb{S}^2 \setminus \operatorname{Cl}(\mathbb{P}_x \cup F_\alpha)$. Then $p \in H_{\alpha,\Gamma}$. Hence (6.4) is true.

By (6.4) and the Baire category theorem, there exist an integer α, a point σ of Σ, and a complementary domain Γ of $\operatorname{Cl}(\mathbb{P}_x \cup F_\alpha)$ such that either $H_\alpha, H_{\alpha,\sigma}$, or $H_{\alpha,\Gamma}$ is somewhere dense in L.

If H_α is somewhere dense in L, define H to be a subset of H_α such that $\operatorname{Cl} H$ contains an open subset of L.

If H_α is nowhere dense and $H_{\alpha,\sigma}$ is somewhere dense in L, define H to be a subset of $H_{\alpha,\sigma}$ such that $\operatorname{Cl} H$ contains an open subset of L.

If $H_\alpha \cup H_{\alpha,\sigma}$ is nowhere dense and $H_{\alpha,\Gamma}$ is somewhere dense in L, define H to be a subset of $H_{\alpha,\Gamma}$ such that $\operatorname{Cl} H$ contains an open subset of L.

Let Y be a disk in F_α such that

$$L \cap \operatorname{Int} Y \neq \varnothing. \tag{6.5}$$

Since $L = \mathbb{L}_x$, there exists a point x' of $\mathbb{P}_x$ in Y.

Since $\mathbb{P}_x$ has the cut property, we can assume

(6.5′) for each point z of $\mathbb{P}_x(x') \cap Y$, either there is a cut in F_α across $\mathbb{P}_x$ at z, or no simple closed curve in $\mathbb{D}_x$ contains an arc segment in $\mathbb{P}_x$ that contains z.

Note that

(6.6) each two points of H are the end points of an arc in $\mathbb{S}^2 \setminus (\mathbb{P}_x(x') \cup Y)$.

To see this, let a and b be distinct points of H. We consider three cases.

Suppose $H \subset H_\alpha$. Then there exist arcs A and B in $\mathbb{D}_x \setminus F_\alpha$ from x to a and b, respectively. Since $x' \in Y$, it follows from (6.5′) and the simple connectivity of $\mathbb{D}_x$ that $\mathbb{P}_x(x')$ and $A \cup B$ are disjoint. Hence there is an arc in $A \cup B$ with end points a and b that misses $\mathbb{P}_x(x') \cup Y$.

Suppose $H \subset H_{\alpha,\sigma}$. Let A and B be arcs in $\mathbb{D}_\sigma \setminus F_\alpha$ from σ to a and b, respectively. Since $\mathbb{D}_\sigma \cap \mathbb{P}_x = \varnothing$, it follows that $A \cup B \subset \mathbb{S}^2 \setminus (\mathbb{P}_x(x') \cup Y)$. Hence there is an arc in $A \cup B$ with end points a and b that misses $\mathbb{P}_x(x') \cup Y$.

Suppose $H \subset H_{\alpha,\Gamma}$. Let I and J be two arcs in $M \setminus (\mathbb{D}_x \cup \operatorname{Cl} F_\alpha)$ from a into Γ and b into Γ, respectively. Since Γ is a connected open subset of $\mathbb{S}^2 \setminus (\mathbb{P}_x(x') \cup \operatorname{Cl} F_\alpha)$, there exists an arc K in $I \cup J \cup \Gamma$ from a to b that misses $\mathbb{P}_x(x') \cup \operatorname{Cl} F_\alpha$. Since $Y \subset F_\alpha$, the arc K is in $\mathbb{S}^2 \setminus (\mathbb{P}_x(x') \cup Y)$. Hence (6.6) is true. □ □ □

Since $\mathbb{P}_{x_1}$ keeps doubling back, L contains an indecomposable continuum, and this contradicts Lemma 5.1 of [H7], establishing the existence of Δ. To prove Lemma 5.1 in the more general setting we must use the cut property. □ □

Change 5. Replace the last four sentences in the proof of Lemma 5.1 of [H7] with the following:

Since $\Phi \subset \mathbb{L}_x$ and $\mathbb{P}_x(y) \cap \Psi = \varnothing$, for each point v of $\mathbb{P}_x(y)$, the arc A intersects $\mathbb{P}_x(v)$. Since $A \cup \mathbb{P}_x \subset \mathbb{D}_x$ and $\mathbb{P}_x$ has the cut property, there is a point v of $\mathbb{P}_x(y)$ such that $(\mathbb{P}_x(v) \setminus A) \cap \operatorname{Bd} \mathbb{D}_x = \varnothing$. Let W be an open set in $\mathbb{S}^2 \setminus (A \cup B)$ that intersects $\Phi \setminus (A \cup B)$. By the cut property, there exists a cut $\mathbb{U}(z)$ in W across an arc segment Z in $\mathbb{P}_x(v)$ such that the end points of Z are in A. Hence $A \cup Z$ is a subset of $\mathbb{D}_x$ that contains a simple closed curve that does not bound a disk in $\mathbb{D}_x$, contradicting the assumption that $\mathbb{D}_x$ is simply connected. This completes the proof of Lemma 5.1. □ □ □

There exists an arc segment Σ in Δ contained in an element of $\mathcal{D}$ that has one endpoint in L. The nonseparating plane continuum $\mathbb{S}^2 \setminus \Delta$ is the intersection of a nested sequence of polygonal disks. Two of the polygonal disks are drawn in Figure 10 in Section 7 of [H7].

Let R be the annular region that is the complement of $\mathbb{S}^2 \setminus \Delta$ relative to a polygonal disk. The region R is divided into sections by a collection of disjoint half-open arcs that run from L to the boundary of the polygonal disk. This collection of sections is called a frame. One of the dividing half-open arcs is in $\operatorname{Cl} \Sigma$. According to Lemma 7.7 of [H7], we can assume that the intersection of L and the closure of a section is always a proper subcontinuum of L.

At most one ray in $\mathbb{Q}_y$ intersects Σ. Therefore, instead of circling around $\mathbb{S}^2 \setminus \Delta$, the rays of $\mathbb{Q}_y$ must run back and forth through the consecutive sections of a frame as they get closer and closer to L. A slight modification must be made to establish this condition in the more general setting. □ □

Change 6. Insert at the end of the paragraph that follows (12.8) of [H7] the following comment:

When the elements of $\mathcal{D}$ are only assumed to be simply connected, the establishment of an arc to replace $[f(v), f(w)]$ in this paragraph requires a straightforward application of the new statement (10.10) with the $\mathbb{C}(w)$ sets chosen in the end sections of $\mathcal{G}_n$. □ □ □

Property 12.9 involves a sequence of frames with this condition. In Bing's terminology, this condition asserts that the dog and the rabbit are repeatedly in one of the sections of the frame at the same time. Consequently there exists a proper subcontinuum of $\mathbb{S}^2 \setminus \Delta$ that has Property 12.9.

By the Brouwer reduction theorem, there exists a compact connected set X in $\mathbb{S}^2 \setminus \Delta$ that is irreducible with respect to Property 12.9. Either $f(\operatorname{Bd} X) \subset \operatorname{Bd} X$ or $\operatorname{Bd} X \subset f(\operatorname{Bd} X)$. Since f is fixed-point free, X is not a point. Therefore X is a nonseparating plane continuum, and the above argument for $\mathbb{S}^2 \setminus \Delta$ can be modified to show that X has a proper subcontinuum with Property 12.9. This contradiction of the irreducibility of X rules out Case 12.2a of [H7]. Only slight modification of this argument is required to eliminate Case 12.2b of [H7]. Hence Case 1 is impossible.

Case 2. Suppose $\mathcal{E}$ is uncountable. In [H7], we define disjoint open subsets J and G_n of $\mathbb{S}^2$ that intersect L and an uncountable subcollection $\mathcal{E}'$ of $\mathcal{E}$ with the following property. For each point x of $J \cap (\bigcup \mathcal{E}')$, the arc $[x, f(x)]$ is in $\mathbb{P}_x$ and misses G_n. Note that this can be done in the more general setting. □ □

Change 7. Change lines 17 and 18 of page 169 of [H7] to the following:

Let $\mathcal{E}' = \{E \in \mathcal{E} : E$ is not bridged and does not contain either a triod or an open subset of the $\mathbb{S}^2\}$. It follows from (12.1), (13.1), (13.4), and the second countability of $\mathbb{S}^2$ that $\mathcal{E}'$ is uncountable. Furthermore, each element of $\mathcal{E}'$ is uniquely-arcwise connected. □ □ □

For each point x of $L \cap (\bigcup \mathcal{E}')$, we have that $\mathbb{L}_x = L$. Hence each element E of $\mathcal{E}'$ contains an arc $A(E)$ that starts in J, ends in G_n, and runs back and forth five times between J and G_n. The order of each $A(E)$ agrees with the order of a Borsuk ray. Since there are uncountably many of these disjoint arcs, we can find three that run back and forth in the parallel manner pictured in Figure 15 in Section 14 of [H7]. Two of these arcs are the sides of a zig-zagging strip S that contains the third arc $A(E_0)$.

Let x be the first point of $A(E_0)$. Since S is very narrow and the rabbit cannot get too far ahead of the dog in S, each time $\mathbb{P}_x$ passes through S, it enters through the disk Q_1 at one end of S and leaves through the disk Q_{12} at the other end of S (see Figure 15 of [H7]). Thus $A(E_0)$ is not in $\mathbb{L}_x$, and this contradicts the fact that $\mathbb{L}_x = L$. Hence Case 2 is impossible. Therefore f has a fixed point. □

9. Applications

Theorem 9.1. *If M is a simply-connected plane continuum, then M has the fixed-point property.*

Proof. Let $\mathcal{D}$ in Theorem 8.1 be $\{M\}$. □

Corollary 9.2. *Every arcwise-connected nonseparating plane continuum has the fixed-point property* [H1].

Proof. Every simple closed curve bounds a disk in a nonseparating plane continuum. □

Corollary 9.3. *Every uniquely-arcwise-connected plane continuum has the fixed-point property* [H4].

Proof. There are no simple closed curves in a uniquely-arcwise-connected continuum. □

Theorem 9.4. *Suppose M is an arcwise-connected plane continuum. For M to have the fixed-point property it is necessary and sufficient that the fundamental group of M be trivial.*

Proof. Theorem 9.1 asserts that the group condition is sufficient. To see that it is also necessary, assume the fundamental group of M is not trivial. Then M contains a simple closed curve C that does not bound a disk in M. There is a retraction r of M onto C. Let f be a fixed-point-free map of C onto C. The composition $f \circ r$ is a fixed-point-free map of M into M. □

The following theorem generalizes [H5, Theorem 15.1].

Theorem 9.5. *Suppose M is a plane continuum with the property that every simple closed curve in M bounds a disk in M. Then every map of M that sends each arc component into itself has a fixed point.*

Proof. Let $\mathcal{D}$ in Theorem 8.1 be the collection of arc components of M. □

Corollary 9.6. *If M is a plane continuum and every simple closed curve in M bounds a disk in M, then every deformation of M has a fixed point.*

Proof. Let f be a deformation of M. Since the continuous image of an interval is arcwise connected, for each point p of M there is an arc in M from p to $f(p)$. Hence f sends each arc component of M into itself. □

10. Concluding remarks

The double $\sin 1/x$ circle is a separating plane continuum that satisfies the hypothesis of Theorem 9.5 and admits a fixed-point-free map. The conclusion to Theorem 9.5 was established for nonseparating plane continua in [H5, Theorem 4.1].

Still open is the following:

Question 10.1. Does every nonseparating plane continuum have the fixed-point property?

Although many related positive results have been established in the last 65 years (see [Bi2], [Ay], [Bo1], [Bo3], [Ha1], [Ha2], [C-L], [Y], [St], [A], [B1], [B2], [S], [I], [H2], [H3], [H6], [Kr], [Ma], and [Mi1]), recent examples indicate that Question 10.1 might soon be answered in the negative (see [Be], [O-R1], [O-R2], [Mi2], and [Mi3]). A fundamental exposition on Question 10.1 is given in [K-W, pp. 66 and 145].

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Department of Mathematics, California State University, Sacramento, California 95819-6051

E-mail address: hagopian@csus.edu