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INFINITE PRODUCTS OF FINITE SIMPLE GROUPS

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ABSTRACT. We classify the sequences $\langle S_n \mid n \in \mathbb{N} \rangle$ of finite simple nonabelian groups such that $\prod_n S_n$ has uncountable cofinality.

1. INTRODUCTION

Suppose that G is a group that is not finitely generated. Then G can be expressed as the union of a chain of proper subgroups. The cofinality of G, written $c(G)$, is defined to be the least cardinal λ such that G can be expressed as the union of a chain of λ proper subgroups. Groups of uncountable cofinality were first considered by Serre in his study of groups acting on trees.

Definition 1.1. [Se, p. 58] A group H has property (FA) if and only if whenever H acts without inversion on a tree T, then there exists a vertex $t \in T$ such that $h(t) = t$ for all $h \in H$.

In [Se], Serre characterised the groups which have property (FA).

Theorem 1.2. [Se] *The group H has property* (FA) *if and only if the following three conditions are satisfied.*

1. *H is not a nontrivial free product with amalgamation.*
2. *$\mathbb{Z}$ is not a homomorphic image of H.*
3. *If H is not finitely generated, then $c(H) > \omega$.*

This result led to the question of whether there exist any natural examples of uncountable groups with property (FA). Let $\langle G_n \mid n \in \mathbb{N} \rangle$ be a sequence of nontrivial finite groups. Then $\prod_n G_n$ denotes the full direct product of the groups G_n, $n \in \mathbb{N}$. By Bass [Ba], if H is a profinite group and H acts without inversion on the tree T, then for every $h \in H$ there exists $t \in T$ such that $h(t) = t$. This implies that H satisfies conditions 1.2(1) and 1.2(2). In particular, we see that the profinite group $\prod_n G_n$ has property (FA) if and only if $c(\prod_n G_n) > \omega$. The following result, which was proved by Koppelberg and Tits, provided the first examples of uncountable groups with property (FA).

Theorem 1.3. [KT] *Let F be a nontrivial finite group and let $G_n = F$ for all $n \in \mathbb{N}$. Then $c(\prod_n G_n) > \omega$ if and only if F is perfect.*

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Suppose that F is perfect. Since $|\prod_n G_n| = 2^\omega$, Theorem 1.3 yields that

$$\omega_1 \leq c\left(\prod_n G_n\right) \leq 2^\omega.$$

This suggests the problem of trying to compute the exact value of $c(\prod_n G_n)$. (Of course, this problem is only interesting if $2^\omega > \omega_1$.) The following result is an immediate consequence of a theorem of Koppelberg [Ko].

Theorem 1.4. *If F is a nontrivial finite perfect group and $G_n = F$ for all $n \in \mathbb{N}$, then $c(\prod_n G_n) = \omega_1$.*

Proof. If $\langle g(n)\rangle_n \in \prod_n G_n$ and $\pi \in F$, let $X_\pi(g) = \{n \in \mathbb{N} \mid g(n) = \pi\}$. Then $\{X_\pi(g) \mid \pi \in F\}$ yields a partition of $\mathbb{N}$ into finitely many pieces. Consider the powerset $\mathcal{P}(\mathbb{N})$ as a Boolean algebra. By Koppelberg [Ko], we can express

$$\mathcal{P}(\mathbb{N}) = \bigcup_{\alpha<\omega_1} B_\alpha$$

as the union of a chain of ω_1 proper Boolean subalgebras. For each $\alpha < \omega_1$, define

$$H_\alpha = \{g \in \prod_n G_n \mid X_\pi(g) \in B_\alpha \text{ for all } \pi \in F\}.$$

Then it is easily checked that H_α is a proper subgroup of $\prod_n G_n$. Clearly $\prod_n G_n = \bigcup_{\alpha<\omega_1} H_\alpha$, and so $c(\prod_n G_n) \leq \omega_1$. □

The above results suggest the following questions.

Question 1.5. For which sequences $\langle S_n \mid n \in \mathbb{N}\rangle$ of finite simple nonabelian groups do we have $c(\prod_n S_n) > \omega$?

Question 1.6. Suppose that $\langle S_n \mid n \in \mathbb{N}\rangle$ is a sequence of finite simple nonabelian groups such that $c(\prod_n S_n) > \omega$. Is it possible to compute the exact value of $c(\prod_n S_n)$?

It may be helpful to give a word of explanation concerning Question 1.6. The point is that it may be impossible to compute the exact value of $c(\prod_n S_n)$ in *ZFC*. For example, consider the group $Sym(\mathbb{N})$ of all permutations of $\mathbb{N}$. In [MN], Macpherson and Neumann showed that $c(Sym(\mathbb{N})) > \omega$. Later Sharp and Thomas [ST1] proved that it is consistent that $c(Sym(\mathbb{N}))$ and 2^ω can be any two prescribed regular uncountable cardinals subject only to the requirement that $c(Sym(\mathbb{N})) \leq 2^\omega$. Hence it is impossible to compute the exact value of $c(Sym(\mathbb{N}))$ in *ZFC*. (The theorem of Macpherson and Neumann suggests that $Sym(\mathbb{N})$ is probably another natural example of an uncountable group with property (FA). In the final section of this paper, we shall confirm that this is true.)

The following result shows that there exist sequences $\langle S_n \mid n \in \mathbb{N}\rangle$ of finite simple nonabelian groups such that $c(\prod_n S_n) = \omega$.

Theorem 1.7. *Let $\langle S_n \mid n \in \mathbb{N}\rangle$ be a sequence of finite simple nonabelian groups. Suppose that there exists an infinite subset I of $\mathbb{N}$ such that the following conditions are satisfied.*

1. *There exists a fixed (possibly twisted) Lie type L such that for all $n \in I$, $S_n = L(q_n)$ for some prime power q_n.*
2. *If n, $m \in I$ and $n < m$, then $q_n < q_m$.*

Then $c(\prod_n S_n) = \omega$.

Here $L(q_n)$ denotes the group of Lie type L over the finite field $GF(q_n)$. The proof of Theorem 1.7 makes use of the following easy observation.

Lemma 1.8. *Suppose that $N \lhd G$ and that G/N is not finitely generated. Then $c(G) \leq c(G/N)$.* □

Proof of Theorem 1.7. By Lemma 1.8, we can suppose that $I = \mathbb{N}$. Let $\mathcal{D}$ be a nonprincipal ultrafilter on $\mathbb{N}$, and let N be the set of elements $g = \langle g(n)\rangle_n \in \prod_n S_n$ such that $\{n \in \mathbb{N} \mid g(n) = 1\} \in \mathcal{D}$. Then N is a normal subgroup of $\prod_n S_n$, and $\prod_n S_n/N$ is the ultraproduct $G = \prod_n S_n/\mathcal{D}$. (See Section 9.5 of Hodges [H].) By Lemma 1.8, it is enough to show that $c(G) = \omega$.

There exists a fixed integer d such that each of the groups $L(q_n)$ has a faithful d-dimensional linear representation over the field $GF(q_n)$. Since the class of groups with a faithful d-dimensional linear representation is first-order axiomatisable, it follows that G has a faithful d-dimensional linear representation over some field K. (For example, see Section 6.6 of Hodges [H]. It is perhaps worth mentioning that every known proof only yields the existence of a set of axioms for this class. The problem of finding an explicit intelligible set of axioms remains open.) To simplify notation, we shall suppose that $G \leqslant GL(d, K)$. We also suppose that K has been chosen so that $G \cap GL(d, K')$ is a proper subgroup of G for every proper subfield K' of K. By Exercise 9.5.5 of Hodges [H], $|G| = 2^\omega$. It follows that $|K| = 2^\omega$, and hence K has transcendence dimension 2^ω over its prime subfield k. Let B be a transcendence basis of K over k. Express $B = \bigcup_{n<\omega} B_n$ as the union of a chain of proper subsets. For each $n < \omega$, let K_n be the algebraic closure of B_n in K. Then each K_n is a proper subfield of K, and $K = \bigcup_{n<\omega} K_n$. For each $n < \omega$, let $G_n = G \cap GL(d, K_n)$. Then $G = \bigcup_{n<\omega} G_n$, and each G_n is a proper subgroup of G. Hence $c(G) = \omega$. □

The main result of this paper is that the converse of Theorem 1.7 is also true.

Theorem 1.9. *Suppose that $\langle S_n \mid n \in \mathbb{N}\rangle$ is a sequence of finite simple nonabelian groups such that $c(\prod_n S_n) = \omega$. Then there exists an infinite subset I of $\mathbb{N}$ such that conditions 1.7(1) and 1.7(2) are satisfied.*

Now suppose that $\langle S_n \mid n \in \mathbb{N}\rangle$ is a sequence of finite simple nonabelian groups such that $c(\prod_n S_n) > \omega$. If there exists an infinite subset J of $\mathbb{N}$ such that $S_n = S_m$ for all n, $m \in J$, then Lemma 1.8 and Theorem 1.4 imply that $c(\prod_n S_n) = \omega_1$. This is the only case in which we have been able to compute the exact value of $c(\prod_n S_n)$ in ZFC.

Question 1.10. Is it consistent that there exists a sequence $\langle S_n \mid n \in \mathbb{N}\rangle$ of finite simple nonabelian groups such that $c(\prod_n S_n) > \omega_1$?

We hope that Question 1.10 has a positive answer, as this would lead to some very attractive problems. For example, consider the following question. (We suspect that it cannot be answered in ZFC.)

Question 1.11. Is it true that $c(\prod_n Alt(n+5)) = c(\prod_n PSL(n+3, 2))$?

In Section 5, we shall prove the following consistency result. Among other things, it shows that it is impossible to prove in ZFC that $c(Sym(\mathbb{N})) = c(\prod_n S_n)$ for some sequence $\langle S_n \mid n \in \mathbb{N}\rangle$ of finite simple nonabelian groups.

Theorem 1.12. *It is consistent that both of the following statements are true.*

1. $c(Sym(\mathbb{N})) = \omega_2 = 2^\omega$.
2. $c(\prod_n G_n) \leq \omega_1$ *for* every *sequence* $\langle G_n \mid n \in \mathbb{N}\rangle$ *of nontrivial finite groups.*

The following problem is also open. (Of course, a negative answer to Question 1.10 would yield a negative answer to Question 1.13.)

Question 1.13. Is it consistent that there exists a sequence $\langle S_n \mid n \in \mathbb{N}\rangle$ of finite simple nonabelian groups such that $c(\prod_n S_n) > c(Sym(\mathbb{N}))$?

This paper is organised as follows. In Section 2, we shall prove that if $\langle S_n \mid n \in \mathbb{N}\rangle$ is a sequence of finite alternating groups, then $c(\prod_n S_n) > \omega$. In Section 3, we shall prove Theorem 1.9 in the special case when each S_n is a projective special linear group. In Section 4, we shall complete the proof of Theorem 1.9. Our proof makes use of the fact that there are only finitely many sporadic finite simple groups, and thus relies on the classification of the finite simple groups. Section 5 contains the proof of Theorem 1.12. In Section 6, we shall prove that $Sym(\mathbb{N})$ has property (FA).

Our notation is standard, but a couple of points should be mentioned. Suppose that G is a subgroup of $Sym(\Omega)$. If each nonidentity element $g \in G$ is fixed-point-free, then G is said to act *semiregularly* on Ω. If G acts transitively and semiregularly, then G is said to act *regularly* on Ω. In this paper, permutation groups and linear groups always act on the left. Thus, for example, we have that

$$(\,1\,2\,3\,)(\,1\,3\,5\,7\,)(\,1\,2\,3\,)^{-1} = (\,2\,1\,5\,7\,).$$

We follow the usual convention of regarding each ordinal as the set of its predecessors. Thus $\omega = \mathbb{N}$. Also if a, b are natural numbers such that $a > b$, then their set-theoretic difference is $a \smallsetminus b = \{b, b+1, \dots, a-1\}$. If A is a matrix, then A^T denotes the transpose of A.

2. Infinite products of alternating groups

In this section, we shall prove the following special case of Theorem 1.9.

Theorem 2.1. *Let* $\langle S_n \mid n \in \mathbb{N}\rangle$ *be a sequence of finite simple nonabelian groups. If each* S_n *is an alternating group, then* $c(\prod_n S_n) > \omega$.

We shall make use of the following two results, which will be used repeatedly throughout this paper.

Proposition 2.2. [Th] *Suppose that* G *is not finitely generated and that* H *is a subgroup of* G*. If* G *is finitely generated over* H*, then* $c(H) \leq c(G)$.

Proof. Let $c(G) = \lambda$. Express $G = \bigcup_{\alpha<\lambda} G_\alpha$ as the union of a chain of λ proper subgroups. Let $H_\alpha = H \cap G_\alpha$. Then $H = \bigcup_{\alpha<\lambda} H_\alpha$. Since G is finitely generated over H, each H_α is a proper subgroup of H. Thus $c(H) \leq \lambda$. □

Proposition 2.3. *Let* $\langle S_n \mid n \in \mathbb{N}\rangle$ *be a sequence of nontrivial finite perfect groups. Suppose that there exists a finite set* $\mathcal{F}$ *of groups such that* $S_n \in \mathcal{F}$ *for all* $n \in \mathbb{N}$*. Then* $c(\prod_n S_n) > \omega$.

Proof. By Proposition 2.2, we can suppose that for each group $S \in \mathcal{F}$, the set $\{n \in \mathbb{N} \mid S_n = S\}$ is either infinite or empty. Since the class of groups of uncountable cofinality is closed under taking finite direct products, Theorem 1.4 implies that $c(\prod_n S_n) > \omega$. $\square$

We shall begin the proof of Theorem 2.1 by making a couple of easy reductions. For each $m \geq 5$, let $P_m = \prod_n S_n^m$, where $S_n^m = \mathit{Alt}(m)$ for all $n \in \mathbb{N}$. Let $G_0 = \prod_{m \geq 5} P_m$. Then Lemma 1.8 implies that it is enough to prove that $c(G_0) > \omega$. Let $G_1 = \prod_{m \geq 8} P_m$. Then $G_0 = P_5 \times P_6 \times P_7 \times G_1$. By Theorem 1.4, $c(P_m) = \omega_1$ for all $m \geq 5$. Hence it is enough to prove that $c(G_1) > \omega$. Finally let $G_2 = \prod_{m \geq 3} P_{2^m}$. Then Theorem 2.1 is an immediate consequence of the following two results.

Lemma 2.4. $c(G_1) = c(G_2)$.

Theorem 2.5. $c(G_2) > \omega$.

First we shall prove Lemma 2.4. Note that Lemma 1.8 implies that $c(G_1) \leq c(G_2)$. Our proof that $c(G_2) \leq c(G_1)$ is based upon Proposition 2.2.

Let $I = \{\langle m, n\rangle \mid 8 \leq m \in \mathbb{N}, n \in \mathbb{N}\}$. Then $G_1 = \prod_{\langle m,n\rangle \in I} S_n^m$, where $S_n^m = \mathit{Alt}(m)$. For each $\langle m, n\rangle \in I$, let t be the integer such that $2^t \leq m < 2^{t+1}$ and let $T_n^m = \mathit{Alt}(2^t) \leqslant S_n^m$. Then we can identify G_2 with the subgroup $\prod_{\langle m,n\rangle \in I} T_n^m$ of G_1. By Proposition 2.2, it is enough to prove the following result.

Lemma 2.6. *G_1 is finitely generated over G_2.*

This is the first of the many places in this paper where we need to prove that an infinite product of groups is finitely generated over an infinite product of subgroups. A moment's thought shows that such results require "uniform generation" results for the corresponding sequences of groups. We shall make repeated use of the following easy observation.

Proposition 2.7. *Let $\langle H_n \mid n \in \mathbb{N}\rangle$ and $\langle G_n \mid n \in \mathbb{N}\rangle$ be sequences of groups such that $H_n \leqslant G_n$ for all $n \in \mathbb{N}$. Suppose that there exists a word $w(x_1, \dots, x_s, y_1, \dots, y_t)$ from the free group on $\{x_1, \dots, x_s, y_1, \dots, y_t\}$ such that the following condition is satisfied.*

(2.7) *For all $n \in \mathbb{N}$, there exist elements $\theta_1, \dots, \theta_t \in G_n$ such that each $\phi \in G_n$ can be expressed as $\phi = w(\psi_1, \dots, \psi_s, \theta_1, \dots, \theta_t)$ for some $\psi_1, \dots, \psi_s \in H_n$.*

Then there exist elements $g_1, \dots, g_t \in \prod_n G_n$ such that

$$\prod_n G_n = \langle \prod_n H_n, g_1, \dots, g_t\rangle. \quad \square$$

Lemma 2.6 is a consequence of the following "uniform generation" results for the finite alternating groups, which will also be needed in the proof of Theorem 2.5.

Lemma 2.8. *Let $m \geq 3$ and let $\theta = (\ m-2\ \ m-1\ )(\ m\ \ m+1\ ) \in \mathit{Alt}(m+1)$. Then for every $\phi \in \mathit{Alt}(m+1)$, there exist ψ_1, ψ_2, $\psi_3 \in \mathit{Alt}(m)$ such that*

$$\phi = \psi_1 \theta \psi_2 \theta \psi_3.$$

Proof. If $\phi \in \mathit{Alt}(m)$, then we can take $\psi_1 = \phi$ and $\psi_2 = \psi_3 = id$. So suppose that $\phi \in \mathit{Alt}(m+1) \smallsetminus \mathit{Alt}(m)$. Let $\psi_2 = (\ m-2\ \ m-1\ \ m\ )$. Then $\tau = \theta\psi_2\theta =$

$(\ m-1\ m-2\ m+1\ )$. Since $Alt(m+1)$ acts 2-transitively on $\{1,\dots,m+1\}$, we have the double coset decomposition

$$Alt(m+1) = Alt(m) \cup Alt(m)\tau Alt(m).$$

Thus $\phi \in Alt(m)\tau Alt(m)$, and so there exist $\psi_1,\ \psi_3 \in Alt(m)$ such that $\phi = \psi_1\theta\psi_2\theta\psi_3$. □

Let $m = 4n$ for some $n \geq 2$. A permutation $\pi \in Alt(m)$ is said to have type 2^{2n} if π is the product of $2n$ disjoint transpositions. Thus $\pi^2 = id$ and π is fixed point free. The set of permutations $\pi \in Alt(m)$ of type 2^{2n} forms a single conjugacy class in $Alt(m)$.

Lemma 2.9 (Brenner [Br])**.** *Let $m = 4n$ for some $n \geq 2$. Let C be the conjugacy class of $Alt(m)$ consisting of all permutations of type 2^{2n}. Then for every $\phi \in Alt(m)$, there exist $\pi_1,\dots,\pi_4 \in C$ such that $\phi = \pi_1\dots\pi_4$.* □

Lemma 2.10. *Suppose that $m = 8n$ for some $n \geq 1$. Let $\Delta_0 = \{1,\dots,4n\}$, $\Delta_1 = \{4n+1,\dots,8n\}$ and let $\Gamma = Alt(\Delta_0)\times Alt(\Delta_1)$. Let $\theta = \prod_{i=1}^{4n}(\ i\ 2i\ )(\ 4n+i\ 2i-1\ )$. Then every $\phi \in Alt(m)$ can be expressed as a product*

$$\phi = \psi_1\theta\psi_2\theta\psi_3\theta\psi_4\theta\psi_5\theta\psi_6\theta\psi_7\theta\psi_8\theta\psi_9$$

for some $\psi_1,\dots,\psi_9 \in \Gamma$.

Proof. By Lemma 2.9, it is enough to show that each permutation $\phi \in Alt(m)$ of type 2^{4n} can be expressed as a product $\phi = \psi_1\theta\psi_2\theta\psi_3$ for some $\psi_1, \psi_2, \psi_3 \in \Gamma$. Let $A = \{\ell \in \Delta_0 \mid \phi(\ell) \in \Delta_0\}$ and $B = \{\ell \in \Delta_1 \mid \phi(\ell) \in \Delta_1\}$. Then $|A| = |B| = 2s$ for some $0 \leq s \leq 4n$. Let $C = \{1,\dots,8n\}\smallsetminus(A\cup B)$. Then $|C\cap\Delta_0| = |C\cap\Delta_1| = 4n-2s$. Let $t = 4n-2s$. Let $\Delta_2 = \{2i \mid 1 \leq i \leq 4n\}$ and $\Delta_3 = \{2i-1 \mid 1 \leq i \leq 4n\}$.

Case 1. Suppose that $t \geq 2n$. Choose a subset $D \subseteq C \cap \Delta_0$ of size $2n$, and let $E = \phi[D]$. Then there exists $\psi_1 \in \Gamma$ such that $\psi_1[D] = \Delta_2 \cap \Delta_0$ and $\psi_1[E] = \Delta_2 \cap \Delta_1$. This implies that

$$\psi_1\phi\psi_1^{-1} \in Alt(\Delta_2) \times Alt(\Delta_3) = \theta\,(Alt(\Delta_0) \times Alt(\Delta_1))\,\theta.$$

Thus we have that

$$\psi_2 = \theta\psi_1\phi\psi_1^{-1}\theta \in \Gamma,$$

and so

$$\phi = \psi_1^{-1}\theta\psi_2\theta\psi_1$$

is a suitable product.

Case 2. Suppose that $t < 2n$. Then $s > n$. Choose ϕ-invariant subsets $D \subseteq \Delta_0$ and $E \subseteq \Delta_1$ such that $|D| = |E| = 2n$. Then there exists $\psi_1 \in \Gamma$ such that $\psi_1[D] = \Delta_2 \cap \Delta_0$ and $\psi_1[E] = \Delta_2 \cap \Delta_1$. Arguing as in Case 1, we see that there exists $\psi_2 \in \Gamma$ such that $\phi = \psi_1^{-1}\theta\psi_2\theta\psi_1$. □

Proof of Lemma 2.6. For each $0 \leq i \leq 7$, let

$$H_i = \prod\{S_n^m \mid \langle m,n\rangle \in I, m \equiv i \pmod 8\ \}$$

and

$$K_i = \prod\{T_n^m \mid \langle m,n\rangle \in I, m \equiv i \pmod 8\ \}.$$

Then $G_1 = \prod_{i=0}^{7} H_i$ and $G_2 = \prod_{i=0}^{7} K_i$. Clearly it is enough to show that H_i is finitely generated over K_i for each $0 \leq i \leq 7$.

First consider the case when $i = 0$. Let $\langle m, n\rangle \in I$ satisfy $m = 8s$ for some $s \geq 1$, and let t be the integer such that $2^t \leq m < 2^{t+1}$. Define $\Delta_0^{m,n} = \{1, \dots, 4s\}$ and $\Delta_1^{m,n} = \{4s+1, \dots, 8s\}$. Then $Alt(\Delta_0^{m,n}) \leqslant T_n^m = Alt(2^t)$. There exists an element $\phi \in Alt(m) = S_n^m$ such that $\phi Alt(\Delta_0^{m,n})\phi^{-1} = Alt(\Delta_1^{m,n})$. Hence there exists $g_1 \in H_0$ such that

$$\prod \{ Alt(\Delta_0^{m,n}) \times Alt(\Delta_1^{m,n}) \mid \langle m, n\rangle \in I, m \equiv 0 \pmod 8 \} \leqslant \langle K_0, g_1\rangle.$$

Now Lemma 2.10 implies that there exists an element $g_2 \in H_0$ such that $H_0 = \langle K_0, g_1, g_2\rangle$.

Next consider the case when $i = 1$. For each $\langle m, n\rangle \in I$ with $m \equiv 1 \pmod 8$, let $U_n^m = Alt(m-1) \leqslant S_n^m$. By the previous paragraph, there exist elements g_1, $g_2 \in H_1$ such that

$$\prod \{ U_n^m \mid \langle m, n\rangle \in I, m \equiv 1 \pmod 8 \} \leqslant \langle K_1, g_1, g_2\rangle.$$

Now Lemma 2.8 implies that there exists an element $g_3 \in H_1$ such that $H_1 = \langle K_1, g_1, g_2, g_3\rangle$. Continuing in this fashion, we can successively deal with the remaining cases. □

The rest of this section is devoted to the proof of Theorem 2.5. Suppose that $c(G_2) = \omega$. Express $G_2 = \bigcup_{t<\omega} H_t$ as the union of a chain of ω proper subgroups. Our strategy will be to define by induction on $t < \omega$

1. a sequence of elements $f_t \in G_2$;
2. a strictly increasing sequence of integers i_t such that $f_t \in H_{i_t}$;
3. a sequence of elements $g_t \in G_2 \smallsetminus H_{i_t}$.

These sequences will be chosen so that there exists an element $h \in G_2$ such that $hf_th^{-1} = g_t$ for all $t < \omega$. But this implies that $h \notin \bigcup_{t<\omega} H_t$, which is the desired contradiction.

Let $J = \{\langle m, n\rangle \mid 3 \leq m \in \mathbb{N}, n \in \mathbb{N}\}$. Then $G_2 = \prod_{\langle m,n\rangle \in J} A_n^m$, where $A_n^m = Alt(2^m)$ for all $n \in \mathbb{N}$. The elements $g_t = \langle g_t(m, n)\rangle_{m,n} \in \prod_{m,n} A_n^m$, $t < \omega$, will be chosen so that for each $\langle m, n\rangle \in J$, the sequence

$$g_0(m, n),\ g_1(m, n), \dots,\ g_t(m, n)$$

is a generic sequence of elements of $Alt(2^m)$, in the following sense.

Definition 2.11. If $0 \leq t \leq m-1$, then the sequence $\pi_0, \dots, \pi_t$ of elements of $Alt(2^m)$ is a *generic sequence* if

1. the subgroup $\langle \pi_0, \dots, \pi_t\rangle$ is elementary abelian of order 2^{t+1};
2. if $id \neq \phi \in \langle \pi_0, \dots, \pi_t\rangle$, then ϕ is a permutation of type $2^{2^{m-1}}$. (In other words, $\langle \pi_0, \dots, \pi_t\rangle$ acts semiregularly on $\{1, \dots, 2^m\}$.)

If $m-1 \leq t < \omega$, then the sequence $\pi_0, \dots, \pi_t$ of elements of $Alt(2^m)$ is a generic sequence if

(a) $\pi_0, \dots, \pi_{m-1}$ is a generic sequence;
(b) $\pi_\ell = \pi_{m-1}$ for all $m-1 \leq \ell \leq t$.

It is an easy exercise to show that for each $t < \omega$, there exists a unique generic sequence $\pi_0, \dots, \pi_t$ in $Alt(2^m)$ up to conjugacy within $Sym(2^m)$; and two such generic sequences up to conjugacy within $Alt(2^m)$ if $m \geq 3$ and $2 \leq t < \omega$. (We shall not make use of this observation in the proof of Theorem 2.5.)

To begin the induction, choose any element $f_0 = \langle f_0(m,n)\rangle_{m,n} \in G_2 = \prod_{m,n} A^m_n$ such that $f_0(m,n)$ is a permutation of type $2^{2^{m-1}}$ for each $\langle m,n\rangle \in J$; and let i_0 be an integer such that $f_0 \in H_{i_0}$. Let $f_0^{G_2}$ be the conjugacy class of f_0 in G_2. Then Lemma 2.9 implies that $G_2 = \langle f_0^{G_2}\rangle$. Hence there exists an element $h_0 \in G_2$ such that $g_0 = h_0 f_0 h_0^{-1} \notin H_{i_0}$. Now suppose that $t \geq 0$ and that we have defined

1. a sequence of elements $f_j \in G_2$,
2. a strictly increasing sequence of integers i_j such that $f_j \in H_{i_j}$,
3. a sequence of elements $g_j \in G_2 \setminus H_{i_j}$, and
4. a sequence of elements $h_j \in G_2$

for $0 \leq j \leq t$ such that the following conditions hold.

(a) $f_0(m,n), \dots, f_t(m,n)$ is a generic sequence in $A^m_n = Alt(2^m)$ for all m, n.
(b) If $0 \leq j \leq k \leq t$, then $h_k f_j h_k^{-1} = g_j$.
(c) If $m-1 \leq j \leq t$ and $n \in \mathbb{N}$, then $h_j(m,n) = h_{m-1}(m,n)$.

First we shall define f_{t+1}.

Case 1. Suppose that $m-1 \leq t$ and $n \in \mathbb{N}$. Then we define $f_{t+1}(m,n) = f_{m-1}(m,n)$.

Case 2. Suppose that $t < m-1$ and $n \in \mathbb{N}$. We shall set up some notation which will be used during the rest of this section. Let

$$E(m,n) = \langle g_0(m,n), \dots, g_t(m,n)\rangle.$$

Then $E(m,n)$ is an elementary abelian group of order 2^{t+1} acting semiregularly on $\{1, \dots, 2^m\}$. Let

$$\{1, \dots, 2^m\} = \Phi^{m,n}_1 \cup \dots \cup \Phi^{m,n}_{2^{m-t-1}}$$

be the decomposition into $E(m,n)$-orbits. Then $E(m,n)$ acts regularly on $\Phi^{m,n}_i$ for each $1 \leq i \leq 2^{m-t-1}$. Choose $\alpha_i \in \Phi^{m,n}_i$ for each $1 \leq i \leq 2^{m-t-1}$. Let $E(m,n) = \{\pi_k \mid 1 \leq k \leq 2^{t+1}\}$, where $\pi_1 = id$, and define

$$\Delta^{m,n}_k = \{\pi_k(\alpha_i) \mid 1 \leq i \leq 2^{m-t-1}\}$$

for each $1 \leq k \leq 2^{t+1}$. Then the diagonal subgroup

$$\begin{aligned} D(m,n) &= Diag(Alt(\Delta^{m,n}_1) \times \dots \times Alt(\Delta^{m,n}_{2^{t+1}})) \\ &= \{\prod_{i=1}^{2^{t+1}} \pi_i \phi \pi_i \mid \phi \in Alt(\Delta^{m,n}_1)\} \end{aligned}$$

is contained in the centraliser of $E(m,n)$ in $Alt(2^m)$. Let $\tau(m,n) \in D(m,n)$ be any permutation of type $2^{2^{m-1}}$, and define

$$f_{t+1}(m,n) = h_t(m,n)^{-1}\tau(m,n)h_t(m,n).$$

This completes the definition of $f_{t+1} = \langle f_{t+1}(m,n)\rangle_{m,n}$.

Next we choose i_{t+1} to be an integer such that

(i) $i_t < i_{t+1}$ and $f_{t+1} \in H_{i_{t+1}}$; and
(ii) $P^* = \prod\{A^m_n \mid 3 \leq m \leq t+3, n \in \mathbb{N}\} \leqslant H_{i_{t+1}}$.

(Proposition 2.3 implies that $c(P^*) > \omega$. Hence i_{t+1} can be chosen so that (ii) also holds.)

Finally we shall define h_{t+1} and g_{t+1}.

Case 1. Suppose that $m - 1 \leq t$ and $n \in \mathbb{N}$. Then we define $h_{t+1}(m,n) = h_{m-1}(m,n)$ and $g_{t+1}(m,n) = g_{m-1}(m,n)$.

Case 2. Suppose that $t < m - 1$ and $n \in \mathbb{N}$. Then we choose a suitable element $\sigma(m,n) \in D(m,n)$ and define

$$h_{t+1}(m,n) = \sigma(m,n)h_t(m,n)$$

and

$$\begin{aligned} g_{t+1}(m,n) &= h_{t+1}(m,n)f_{t+1}(m,n)h_{t+1}(m,n)^{-1} \\ &= \sigma(m,n)\tau(m,n)\sigma(m,n)^{-1}. \end{aligned}$$

Of course, a suitable choice means one such that $g_{t+1} = \langle g_{t+1}(m,n)\rangle_{m,n} \notin H_{i_{t+1}}$. This completes the successor step of the induction, *provided* that a suitable choice exists.

Claim 2.12. *There exists a choice of* $\sigma(m,n)$ *for* $m > t+1$ *and* $n \in \mathbb{N}$ *such that* $g_{t+1} \notin H_{i_{t+1}}$.

Proof. Suppose that for every choice of the sequence

$$\langle \sigma(m,n) \mid t < m-1, n \in \mathbb{N}\rangle$$

we have that $g_{t+1} \in H_{i_{t+1}}$. Then we shall prove that G_2 is finitely generated over $H_{i_{t+1}}$. But this means that there exists $r \in \mathbb{N}$ with $i_{t+1} \leq r$ such that $H_r = G_2$, which is a contradiction.

Let $J' = \{\langle m,n\rangle \mid t+4 \leq m \in \mathbb{N}, n \in \mathbb{N}\}$ and let $P' = \prod\{A_n^m \mid \langle m,n\rangle \in J'\}$. Thus $G_2 = P^* \times P'$. Note that if $\pi(m,n) \in D(m,n)$ is any element of type $2^{2^{m-1}}$, then there exists $\sigma(m,n) \in D(m,n)$ such that $\sigma(m,n)\tau(m,n)\sigma(m,n)^{-1} = \pi(m,n)$. Using the fact that $P^* \leqslant H_{i_{t+1}}$, we see that the following statement holds.

(†) Suppose that $\pi = \langle \pi(m,n)\rangle_{m,n} \in P'$. If $\pi(m,n) \in D(m,n)$ is an element of type $2^{2^{m-1}}$ for all $\langle m,n\rangle \in J'$, then $\pi \in H_{i_{t+1}}$.

Using Lemma 2.9, we see that $\prod\{D(m,n) \mid \langle m,n\rangle \in J'\} \leqslant H_{i_{t+1}}$. Now let $\theta_1 = \langle \theta_1(m,n)\rangle_{m,n} \in P'$ be an element such that $\theta_1(m,n) \in Alt(\Delta_1^{m,n})$ is a permutation of type $2^{2^{m-t-2}}$ for each $\langle m,n\rangle \in J'$. Then $\{\psi\theta_1(m,n)\psi^{-1} \mid \psi \in D(m,n)\}$ is the conjugacy class in $Alt(\Delta_1^{m,n})$ of all permutations of type $2^{2^{m-t-2}}$. Using Lemma 2.9 again, we see that

$$\prod\{Alt(\Delta_1^{m,n}) \mid \langle m,n\rangle \in J'\} \leqslant \langle H_{i_{t+1}}, \theta_1\rangle.$$

Continuing in this fashion, we find that there exist $\theta_1, \ldots, \theta_{2^{t+1}} \in P'$ such that

$$\prod\{Alt(\Delta_1^{m,n}) \times \cdots \times Alt(\Delta_{2^{t+1}}^{m,n}) \mid \langle m,n\rangle \in J'\} \leqslant \langle H_{i_{t+1}}, \theta_1, \ldots, \theta_{2^{t+1}}\rangle.$$

By repeatedly applying Lemma 2.10, we now see that there exists a finite subset F of P' such that

$$P' \leqslant \langle H_{i_{t+1}}, \theta_1, \ldots, \theta_{2^{t+1}}, F\rangle.$$

Hence $G_2 = P^* \times P'$ is finitely generated over $H_{i_{t+1}}$, which is a contradiction. □

Thus the induction can be carried out for all $t < \omega$. Define the element $h = \langle h(m,n)\rangle_{m,n} \in G_2$ by $h(m,n) = h_{m-1}(m,n)$. Then we have that $hf_th^{-1} = g_t$ for all $t < \omega$, which is a contradiction. This completes the proof of Theorem 2.5.

3. Infinite products of special linear groups

In this section, we shall prove the following result.

Theorem 3.1. *Suppose that* $\langle SL(d_n, q_n) \mid n \in \mathbb{N}\rangle$ *is a sequence of finite special linear groups which satisfies the following conditions.*

1. *If* $d_n = 2$*, then* $q_n > 3$.
2. *There do* not *exist an infinite subset* I *of* $\mathbb{N}$ *and an integer* d *such that*
   - (a) $d_n = d$ *for all* $n \in I$*; and*
   - (b) *if* n*,* $m \in I$ *and* $n < m$*, then* $q_n < q_m$.

Then $c(\prod_n SL(d_n, q_n)) > \omega$.

Using Theorem 3.1 and Lemma 1.8, we see that Theorem 1.9 is true in the special case when each S_n is a projective special linear group.

Our strategy in the proof of Theorem 3.1 will be the same as that in the proof of Theorem 2.1. We shall begin by defining the notion of a generic sequence of elements in $SL(2^m, q)$. Let $V = V(n,q)$ be an n-dimensional vector space over $GF(q)$, and let $\mathcal{B}$ be a basis of V. Then $Sym(\mathcal{B})$ denotes the group of permutation matrices with respect to the basis $\mathcal{B}$. Note that for any finite field $GF(q)$, we have that $Alt(\mathcal{B}) \leqslant SL(n,q)$.

Definition 3.2. If $0 \leq t \leq m-1$, then the sequence $\pi_0, \dots, \pi_t$ of elements of $SL(2^m, q)$ is a *generic sequence* if there exists a basis $\mathcal{B}$ of $V(2^m, q)$ such that

1. the group $\langle \pi_0, \dots, \pi_t\rangle$ is an elementary abelian subgroup of $Alt(\mathcal{B})$ of order 2^{t+1};
2. $\langle \pi_0, \dots, \pi_t\rangle$ acts semiregularly on $\mathcal{B}$.

If $m-1 \leq t < \omega$, then the sequence $\pi_0, \dots, \pi_t$ of elements of $SL(2^m, q)$ is a generic sequence if

- (a) $\pi_0, \dots, \pi_{m-1}$ is a generic sequence;
- (b) $\pi_\ell = \pi_{m-1}$ for all $m-1 \leq \ell \leq t$.

First we shall prove an analogue of Lemma 2.9. Let $m = 4n$ for some $n \geq 1$. Then $C(m,q)$ denotes the conjugacy class in $SL(m,q)$ consisting of all elements π such that π is represented by a permutation matrix of type 2^{2n} with respect to some basis $\mathcal{B}$ of $V(m,q)$. (It is easily checked that the set of such elements forms a single conjugacy class in $SL(m,q)$.) Note that $SL(m,q)$ has a maximal torus T_1 of order $(q^{4n}-1)/(q-1)$. By Zsigmondy's theorem [Zs], there exists a primitive prime divisor $p > 2$ of $q^{4n}-1$. Let $\psi \in T_1$ be an element of order p. Then ψ is clearly a regular element of T_1. (A semisimple element $g \in SL(m,q)$ is *regular* if and only if it lies in a unique maximal torus.)

Lemma 3.3. *With the above hypotheses, there exist elements* π_1*,* $\pi_2 \in C(m,q)$ *such that* $\psi = \pi_1\pi_2$.

Proof. Regard $K = GF(q^{4n})$ as a $4n$-dimensional vector space over $GF(q)$. Let $\tau \in K$ generate a normal basis of K over $GF(q)$; i.e. $\mathcal{B} = \{\tau, \tau^q, \tau^{q^2}, \dots, \tau^{q^{4n-1}}\}$ is a basis of K over $GF(q)$. Let $f \in Aut(K)$ be the Frobenius automorphism; so that

$f(\alpha) = \alpha^q$ for all $\alpha \in K$. Let $g = f^{2n}$. Then g is represented by a permutation matrix of type 2^{2n} with respect to the basis $\mathcal{B}$. Thus $g \in C(m, q)$.

Let $K^* = \langle \alpha \rangle$, and let $\beta = \alpha^{q^{2n}-1}$. Then β has order $q^{2n} + 1$, and $g(\beta) = \beta^{-1}$. Consider the primitive prime divisor p of $q^{4n} - 1 = (q^{2n} - 1)(q^{2n} + 1)$. Then clearly p divides $q^{2n} + 1$. Thus there exists an element $\gamma \in \langle \beta \rangle$ of order p; and we can suppose that $\psi \in T_1$ is the linear transformation defined by $\psi(x) = \gamma x$ for all $x \in K$. Since $g(\gamma) = \gamma^{-1}$, we see that $g\psi g^{-1} = \psi^{-1}$. Since ψ has odd order, the involutions g and $g\psi$ are conjugate in the dihedral group $\langle g, g\psi \rangle$. The result follows. □

Theorem 3.4. *Suppose that* $m = 4n$ *for some* $n \geq 1$. *Then for every* $\phi \in SL(m, q)$, *there exist* $\pi_1, \dots, \pi_{10} \in C(m, q)$ *such that* $\phi = \pi_1 \dots \pi_{10}$.

Proof. Let $G = SL(4n, q)$ and let $\psi \in T_1$ be as above. Let $\tau \in G$ be an element of order $q^{4n-1} - 1$ and let T_2 be the maximal torus which contains τ. (Of course, τ is a regular element of T_2.) Let C_1, C_2 be the conjugacy classes of ψ, τ respectively. We claim that the product $C_1 C_2$ of these two classes covers all of $G \smallsetminus Z(G)$. Using Lemma 3.3, this implies that each element of C_2 is a product of 3 elements of $C(m, q)$; and hence every element of $G \smallsetminus Z(G)$ is a product of 5 elements of $C(m, q)$. The result follows.

The proof of the claim uses character theory and follows [MSW, pp. 96–99] very closely. For any conjugacy class C_3 of G and $\sigma \in C_3$, define

$$m(C_1, C_2, C_3) = \frac{|G|^2}{|C_G(\psi)|\,|C_G(\tau)|\,|C_G(\sigma)|} \sum \frac{\chi(\psi)\chi(\tau)\chi(\sigma)}{\chi(1)}$$

where the summation runs over the irreducible characters χ of G. By a well-known class formula, $m(C_1, C_2, C_3)$ is equal to the number of triples (a_1, a_2, a_3) such that $a_i \in C_i$ and $a_1 a_2 a_3 = 1$. It therefore suffices to show that the character sum involved in the formula for $m(C_1, C_2, C_3)$ is positive for any class C_3 of non-central elements of G.

Now the values of the irreducible characters of G on semisimple elements can be calculated from the values of the Deligne-Lusztig characters. (See [Ca2, Chapter 7].) The Deligne-Lusztig characters $R_{T,\theta}$ are parametrized by pairs (T, θ). The equivalence relation of geometric conjugacy on these pairs yields a partition of the irreducible characters of G into disjoint series as follows. The geometric conjugacy classes of pairs (T, θ) can be parametrized by the conjugacy classes (s) of semisimple elements in the dual group $\hat{G} = PGL(4n, q)$. Let $\mathcal{E}(s)$ be the set of irreducible characters occurring as a constituent in one of the $R_{T,\theta}$ with (T, θ) corresponding to (s). Then the sets $\mathcal{E}(s)$, where (s) runs over the set of conjugacy classes of semisimple elements of $\hat{G}$, form a partition of the set of irreducible characters of G. (See [Ca2, 7.3.8 and 7.5.8].)

The $R_{T,\theta}$ span the space of class functions restricted to semisimple elements. (See [Ca2, 7.5.7].) In particular, suppose that ρ is a semisimple element of G and that $\chi \in \mathcal{E}(s)$. Then if $R_{T,\theta}(\rho) = 0$ for all pairs (T, θ) corresponding to (s), we have that $\chi(\rho) = 0$. Now $R_{T,\theta} = R(s)$ vanishes on the regular elements of the torus T' if the element s is not conjugate in $\hat{G}$ to an element of the dual $\hat{T}'$ of T'. Hence the $R_{T,\theta}$ not vanishing on either of the classes C_1, C_2 will correspond to semisimple classes (s) in $\hat{G}$ such that $s \in \hat{T}_1 \cap \hat{T}_2$. Let $\bar{T}_i$ be the preimage of $\hat{T}_i$ in $GL(4n, q)$. Then $\bar{T}_1$ is cyclic of order $q^{4n} - 1$, and $\bar{T}_2$ is the product of two cyclic groups of orders $q^{4n-1} - 1$ and $q - 1$. Furthermore, both $\bar{T}_1$ and $\bar{T}_2$ contain $Z(GL(4n, q))$.

Since $\left(\left|\bar{T}_1\right|, \left|\bar{T}_2\right|\right) = (q-1)(4n, q-1)$, it follows that $\hat{T}_1 \cap \hat{T}_2 = 1$. Thus we need only consider the set $\mathcal{E}(1)$ of unipotent characters of G. It is well-known that if the degree of an irreducible character is divisible by the full power of a prime r dividing the order of a group, then the character vanishes on all r-singular elements of the group. Using this result, an inspection of the degrees of the unipotent characters of G in [Ca2, p. 465] shows that only two irreducible characters contribute to the character sum in the above formula; namely, the principal character and the Steinberg character St. It follows that

$$m(C_1, C_2, C_3) = \frac{|G|^2}{|C_G(\psi)|\,|C_G(\tau)|\,|C_G(\sigma)|}\left(1 + \frac{St(\psi)St(\tau)St(\sigma)}{St(1)}\right)$$

and so [Ca2, 6.4]

$$m(C_1, C_2, C_3) = \frac{|G|^2}{|C_G(\psi)|\,|C_G(\tau)|\,|C_G(\sigma)|}\left(1 - \frac{St(\sigma)}{St(1)}\right).$$

This is 0 precisely when $\sigma \in Z(G)$, as claimed. □

Next we shall prove the analogue of Lemma 2.8. It is easier to state the result in terms of infinite products of groups, rather than in terms of "uniform generation". Let $\langle SL(d_n, q_n) \mid n \in \mathbb{N}\rangle$ be a sequence of special linear groups. Fix some $n \in \mathbb{N}$. Let $SL(d_n, q_n)$ act on the vector space $V(d_n, q_n)$ in the natural manner. Extend this action to $V(d_n+1, q_n) = V(d_n, q_n) \oplus \langle v_{d_n+1}\rangle$ by specifying that $\pi(v_{d_n+1}) = v_{d_n+1}$ for all $\pi \in SL(d_n, q_n)$. Using this extended action, we can regard $SL(d_n, q_n)$ as a subgroup of $SL(d_n+1, q_n)$.

Lemma 3.5. *$\prod_n SL(d_n+1, q_n)$ is finitely generated over $\prod_n SL(d_n, q_n)$.*

Proof. We shall make use of the Bruhat decomposition

$$SL(d, q) = \bigcup_{w \in W} BwB$$

of the special linear group, where B is a Borel subgroup and W is the Weyl group. Fix some integer $n \in \mathbb{N}$. Choose a basis $\{v_i \mid 1 \leq i \leq d_n\}$ of $V(d_n, q_n)$. We shall regard each element of $SL(d_n+1, q_n)$ as a matrix with respect to the basis $\mathcal{B} = \{v_i \mid 1 \leq i \leq d_n + 1\}$. Note that we have identified $SL(d_n, q_n)$ with the subgroup

$$S_n = \left\{ \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \mid A \in SL(d_n, q_n) \right\}$$

of $SL(d_n+1, q_n)$. Define

$$T_n = \left\{ \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix} \mid A \in SL(d_n, q_n) \right\}.$$

Then there exists $\pi \in SL(d_n+1, q_n)$ such that $\pi S_n \pi^{-1} = T_n$. Hence there exists an element $g_0 \in \prod_n SL(d_n+1, q_n)$ such that $\prod_n T_n \leqslant G_0 = \langle \prod_n S_n, g_0\rangle$. Let U_n be the subgroup of strictly upper triangular matrices in $SL(d_n+1, q_n)$, and let H_n be the subgroup of diagonal matrices. Then $B_n = U_n \rtimes H_n$ is a Borel subgroup of $SL(d_n+1, q_n)$. We shall show that $\prod_n B_n \leqslant G_0$.

First we shall show that $\prod_n H_n \leqslant G_0$. Fix some $n \in \mathbb{N}$. Let

$$D = \operatorname{diag}(\lambda_1, \ldots, \lambda_{d_n+1}) \in H_n.$$

Then $D_1 = \operatorname{diag}(\lambda_1, \lambda_1^{-1}, 1, \dots, 1) \in H_n \cap S_n$, $D_2 = \operatorname{diag}(1, \lambda_1\lambda_2, \lambda_3, \dots, \lambda_{d_n+1}) \in H_n \cap T_n$ and $D = D_1D_2$. The result follows.

Next we shall show that $\prod_n U_n \leqslant G_0$. Fix some $n \in \mathbb{N}$. Note that

$$\begin{pmatrix} 1 & \mathbf{0} & 0 \\ \mathbf{0} & A & \boldsymbol{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} 1 & \boldsymbol{c} & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \boldsymbol{c} & 0 \\ \mathbf{0} & A & \boldsymbol{b} \\ 0 & \mathbf{0} & 1 \end{pmatrix}$$

for each $(d_n - 1) \times (d_n - 1)$-matrix A. Also note that if $Z \in U_n$ has the form

$$\begin{pmatrix} 1 & \mathbf{0} & d \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & \mathbf{0} & 1 \end{pmatrix}$$

then there exist $X \in U_n \cap S_n$ and $Y \in U_n \cap T_n$ such that $[X, Y] = Z$. (For example, this follows from Chevalley's commutator formula [Ca1, 5.2.2].) Hence if $\phi \in U_n$ is arbitrary, then there exist θ, $\tau \in U_n \cap S_n$ and ψ, $\sigma \in U_n \cap T_n$ such that $\phi = \psi\theta[\tau, \sigma]$. The result follows.

Let N_n be the subgroup of $SL(d_n + 1, q_n)$ consisting of the elements which stabilise the frame $\{\langle v_i \rangle \mid 1 \leq i \leq d_n + 1\}$. Then the Weyl group of $SL(d_n + 1, q_n)$ is $W_n = N_n / B_n \cap N_n$; and W_n is isomorphic to $Sym(d_n + 1)$ acting on the set $\{v_i \mid 1 \leq i \leq d_n + 1\}$. Note that $N_n \cap S_n$ corresponds to the subgroup $Sym(d_n)$ of W_n. Let $\theta = (\, d_n \; d_n + 1\,)$. Arguing as in the proof of Lemma 2.8, we see that for every $\phi \in Sym(d_n + 1)$, there exist ψ_1, ψ_2, $\psi_3 \in Sym(d_n)$ such that $\phi = \psi_1\theta\psi_2\theta\psi_3$. Hence there exists $g_1 \in \prod_n SL(d_n + 1, q_n)$ such that $\prod_n N_n \leqslant G_1 = \langle G_0, g_1 \rangle$. It follows that $G_1 = \prod_n SL(d_n + 1, q_n)$. ☐

Finally we shall prove the analogue of Lemma 2.10. Consider a product of the form $\prod_n SL(8d_n, q_n)$. Fix some $n \in \mathbb{N}$. Let $SL(8d_n, q_n)$ act on the vector space $V_n = V(8d_n, q_n)$ in the natural manner, and let $\mathcal{B}_n = \{v_i \mid 1 \leq i \leq 8d_n\}$ be a basis of V_n. Let $E_0 = \langle v_i \mid 1 \leq i \leq 4d_n \rangle$ and $E_1 = \langle v_i \mid 4d_n + 1 \leq i \leq 8d_n \rangle$. We regard $SL(E_0)$ as the subgroup of $SL(8d_n, q_n)$ consisting of the elements π such that $\pi[E_0] = E_0$ and such that $\pi(v_i) = v_i$ for all $4d_n + 1 \leq i \leq 8d_n$. We also regard $SL(E_1)$ as a subgroup of $SL(8d_n, q_n)$ in the obvious fashion. Let $\Gamma_n = SL(E_0) \times SL(E_1) \leqslant SL(8d_n, q_n)$.

Lemma 3.6. *$\prod_n SL(8d_n, q_n)$ is finitely generated over $\prod_n \Gamma_n$.*

Proof. Once again, we shall make use of the Bruhat decomposition of the special linear group. Fix some $n \in \mathbb{N}$. We shall regard $SL(8d_n, q_n)$ as a group of matrices with respect to the ordered basis $(v_1, \dots, v_{8d_n})$ of V_n. Let $B_n = U_n \rtimes H_n$ be the Borel subgroup consisting of the upper triangular matrices of $SL(8d_n, q_n)$ First we shall show that there exists a subgroup G_0 of $\prod_n SL(8d_n, q_n)$ such that

1. G_0 is finitely generated over $\prod_n \Gamma_n$, and
2. $\prod_n U_n \leqslant G_0$.

Fix some $n \in \mathbb{N}$. Let M_n be the ring of all $4d_n \times 4d_n$-matrices over $GF(q_n)$, and let

$$T_n = \left\{ \begin{pmatrix} \mathbf{1} & S \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \mid S \in M_n \right\}.$$

Then it is enough to find G_0 such that $\prod_n T_n \leqslant G_0$. Note that for each $A \in SL(4d_n, q_n)$, we have that

$$\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & S \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} A^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & AS \\ \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

Regard M_n as an $SL(4d_n, q_n)$-module with the natural action, $S \overset{A}{\longmapsto} AS$. Then the existence of a suitable subgroup G_0 is an immediate consequence of the following claim.

Claim 3.7. *$\prod_n M_n$ is finitely generated as a $\prod_n SL(4d_n, q_n)$-module.*

Proof of Claim. We shall prove that M_n is "uniformly generated" as an $SL(4d_n, q_n)$-module. The result will then follow. Fix some $n \in \mathbb{N}$. Throughout this proof, each of the matrices will be expressed in terms of $2d_n \times 2d_n$ blocks. Let

$$J_1 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad J_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \qquad J_3 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad J_4 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}.$$

If $B \in GL(2d_n, q_n)$, then $\left(\begin{smallmatrix} B & \mathbf{0} \\ \mathbf{0} & B^{-1} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} B^{-1} & \mathbf{0} \\ \mathbf{0} & B \end{smallmatrix}\right) \in SL(4d_n, q_n)$; and

$$\begin{pmatrix} B & \mathbf{0} \\ \mathbf{0} & B^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} B & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \text{ and } \begin{pmatrix} B^{-1} & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ B & \mathbf{0} \end{pmatrix}, \text{ etc.}$$

Thus if $B_1, \dots, B_4 \in GL(2d_n, q_n)$, then there exist $C_1, \dots, C_4 \in SL(4d_n, q_n)$ such that

$$\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \sum_{i=1}^{4} C_i J_i.$$

Now suppose that $\left(\begin{smallmatrix} S_1 & S_2 \\ S_3 & S_4 \end{smallmatrix}\right) \in M_n$ is arbitrary. By [Ze], each of the matrices S_i is the sum of two non-singular ones. Hence there exist $C_1, \dots, C_4, D_1, \dots, D_4 \in SL(4d_n, q_n)$ such that

$$\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} = \sum_{i=1}^{4} C_i J_i + \sum_{i=1}^{4} D_i J_i.$$

Thus each M_n is "uniformly generated" from the generators $J_1, \dots, J_4$. □

Next we shall show that there exists an element $g_0 \in \prod_n SL(8d_n, q_n)$ such that $\prod_n H_n \leqslant G_1 = \langle G_0, g_0 \rangle$; and hence $\prod_n B_n \leqslant G_1$. For each $\lambda \in GF(q_n)^*$, let $D_\lambda = \operatorname{diag}(\lambda, 1, \dots, 1) \in GL(4d_n, q_n)$. Define

$$F_n = \left\{ \begin{pmatrix} D_\lambda & \mathbf{0} \\ \mathbf{0} & D_\lambda^{-1} \end{pmatrix} \mid \lambda \in GF(q_n)^* \right\}.$$

Since $\prod_n \Gamma_n \leqslant G_0$, it is enough to find an element g_0 such that $\prod_n F_n \leqslant \langle G_0, g \rangle$. For each $\lambda \in GF(q_n)^*$, let $E_\lambda = \operatorname{diag}(\lambda, \lambda^{-1}, 1, \dots, 1) \in GL(4d_n, q_n)$. Define

$$K_n = \left\{ \begin{pmatrix} E_\lambda & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \mid \lambda \in GF(q_n)^* \right\}.$$

Then $\prod_n K_n \leqslant \prod_n \Gamma_n \leqslant G_0$. Also there exists an element $\pi \in N_n$ such that $\pi K_n \pi^{-1} = F_n$. The existence of a suitable element g_0 follows easily.

Finally we shall show that there exists an element $g_1 \in \prod_n SL(8d_n, q_n)$ such that $\prod_n N_n \leqslant G_2 = \langle G_1, g_1 \rangle$. Fix some $n \in \mathbb{N}$. Let $\mathcal{E}_0^n = \{v_i \mid 1 \le i \le 4d_n\}$ and $\mathcal{E}_1^n = \{v_i \mid 4d_n + 1 \le i \le 8d_n\}$, so that $\mathcal{B}_n = \mathcal{E}_0^n \cup \mathcal{E}_1^n$. Then the groups of permutation matrices $Alt(\mathcal{E}_0^n)$, $Alt(\mathcal{E}_1^n)$ are subgroups of Γ_n. Lemma 2.10 implies that there exists an element $g_1 \in \prod_n Alt(\mathcal{B}_n)$ such that $\prod_n Alt(\mathcal{B}_n) = \langle \prod \left(Alt(\mathcal{E}_0^n) \times Alt(\mathcal{E}_1^n) \right), g_1 \rangle$. Now let $\pi = \langle \pi(n) \rangle_n \in \prod_n N_n$ be an arbitrary element. Let X be the subset of $\mathbb{N}$ consisting of those n such that $\pi(n)$ corresponds to an odd permutation of $\mathcal{B}_n$. Then there exists an element $\psi_X = \langle \psi_X(n) \rangle_n \in \prod_n (N_n \cap \Gamma_n)$ such that

1. $\psi_X(n)$ corresponds to the odd permutation $(\, v_1 \, v_2 \,)$ if $n \in X$, and
2. $\psi_X(n) = 1$ if $n \notin X$.

Since $\prod_n H_n$, $\prod_n Alt(\mathcal{B}_n) \leqslant \langle G_1, g_1 \rangle$, it follows that $\pi \psi_X \in \langle G_1, g_1 \rangle$, and hence $\pi \in \langle G_1, g_1 \rangle$. Thus $\prod_n N_n \leqslant \langle G_1, g_1 \rangle$. □

Now we are ready to begin the proof of Theorem 3.1. Since the proof is very similar to that of Theorem 2.1, we shall just sketch the main points. Suppose that $\langle SL(d_n, q_n) \mid n \in \mathbb{N} \rangle$ is a sequence of finite special linear groups which satisfies the hypotheses of Theorem 3.1. By Proposition 2.3, we can suppose that $\{d_n \mid n \in \mathbb{N}\}$ is an infinite subset of $\mathbb{N}$. Arguing as in the proof of Lemma 2.4, we can reduce to the case when each d_n has the form 2^{m_n} for some $m_n \geq 2$. (Since the sequence satisfies condition 3.1(2), there exists a finite set $\mathcal{F}$ of groups such that if $d_n \leq 3$, then $SL(d_n, q_n) \in \mathcal{F}$. By Propositions 2.2 and 2.3, we can safely ignore these factors.) Let $G = \prod_n SL(2^{m_n}, q_n)$ and suppose $c(G) = \omega$. Express $G = \bigcup_{t<\omega} H_t$ as the union of a chain of ω proper subgroups. Now suppose that $t \geq 0$ and that we have defined

1. a sequence of elements $f_j = \langle f_j(n) \rangle_n \in G$,
2. a strictly increasing sequence of integers i_j such that $f_j \in H_{i_j}$,
3. a sequence of elements $g_j = \langle g_j(n) \rangle_n \in G \smallsetminus H_{i_j}$, and
4. a sequence of elements $h_j = \langle h_j(n) \rangle_n \in G$

for $0 \leq j \leq t$ such that the following conditions hold.

(a) $f_0(n), \dots, f_t(n)$ is a generic sequence in $SL(2^{m_n}, q_n)$ for each $n \in \mathbb{N}$.
(b) If $0 \leq j \leq k \leq t$, then $h_k f_j h_k^{-1} = g_j$.
(c) If $m_n - 1 \leq j \leq t$, then $h_j(n) = h_{m_n - 1}(n)$.

We must show that it is possible to continue the induction. There is no difficulty in defining f_{t+1} and i_{t+1}. The problem is to show that there exist suitable elements h_{t+1} and $g_{t+1} = h_{t+1} f_{t+1} h_{t+1}^{-1}$ such that $g_{t+1} \notin H_{i_{t+1}}$. As in the proof of Theorem 2.1, we shall show that if no such elements exist, then G is finitely generated over $H_{i_{t+1}}$; which is a contradiction. So suppose that no such elements exist. Let $P^* = \prod \{SL(2^{m_n}, q_n) \mid m_n \leq t+3\}$ and $P' = \prod \{SL(2^{m_n}, q_n) \mid m_n \geq t+4\}$, so that $G = P^* \times P'$. Since $\langle SL(2^{m_n}, q_n \mid n \in \mathbb{N} \rangle$ satisfies condition 3.1(2), either $c(P^*) > \omega$ or P^* is finite. Thus we can suppose that i_{t+1} was chosen so that $P^* \leqslant H_{i_{t+1}}$. Fix some $n \in \mathbb{N}$ such that $m_n \geq t+4$. Let $\mathcal{B}_n$ be a basis of $V(2^{m_n}, q_n)$ chosen so that the group $E(n) = \langle g_0(n), \dots, g_t(n) \rangle$ is an elementary abelian subgroup of $Alt(\mathcal{B}_n)$ of order 2^{t+1}, which acts semiregularly on $\mathcal{B}_n$. Let $\{\Phi_i^n \mid 1 \leq i \leq 2^{m_n - t - 1}\}$ be the set of orbits of $E(n)$ on $\mathcal{B}_n$. For each $1 \leq i \leq 2^{m_n - t - 1}$, choose $v_i^1 \in \Phi_i^n$. Let $E(n) = \{\psi_k \mid 1 \leq k \leq 2^{t+1}\}$. For each $1 \leq k \leq 2^{t+1}$ and $1 \leq i \leq 2^{m_n - t - 1}$, define $v_i^k = \psi_k(v_i^1)$. For each $1 \leq k \leq 2^{t+1}$, let $V_k^n = \langle v_i^k \mid 1 \leq i \leq 2^{m_n - t - 1} \rangle$. Then $V(2^{m_n}, q_n) = V_1^n \oplus \cdots \oplus V_{2^{t+1}}^n$, and the diagonal subgroup

$$D_n = Diag\left(SL(V_1^n) \times \cdots \times SL(V_{2^{t+1}}^n) \right)$$

is contained in the centraliser of $\langle g_0(n), \dots, g_t(n) \rangle$ in $SL(2^{m_n}, q_n)$. Since each candidate π for g_{t+1} satisfies $\pi \in H_{i_{t+1}}$, we find that the following statement holds.

(†) Suppose that $\pi \in P'$. If $\pi(n) \in D_n \cap C(2^{m_n}, q_n)$ for all n such that $m_n \geq t+4$, then $\pi \in H_{i_{t+1}}$.

Using Theorem 3.4, this implies that $\prod\{D_n \mid m_n \geq t+4\} \leqslant H_{i_{t+1}}$. Arguing as in the proof of Theorem 2.1, we see that there exists a subgroup Γ_0 of G such that

1. Γ_0 is finitely generated over $H_{i_{t+1}}$, and
2. $\prod\{SL(V_1^n) \times \cdots \times SL(V_{2^{t+1}}^n) \mid m - n \geq t+4\} \leqslant \Gamma_0$.

By repeatedly applying Lemma 3.6, we next see that there exists a subgroup Γ_1 of G such that

1. Γ_1 is finitely generated over Γ_0, and
2. $P' = \prod\{SL(2^{m_n}, q_n) \mid m_n \geq t+4\} \leqslant \Gamma_1$.

But this means that $\Gamma_1 = G$, and so G is finitely generated over $H_{i_{t+1}}$. This contradiction shows that the induction can be carried out for all $t < \omega$. But this yields an element $h \in G$ such that $hf_th^{-1} = g_t$ for all $t < \omega$, which is impossible. Thus $c(G) > \omega$. This completes the proof of Theorem 3.1.

4. The proof of Theorem 1.9

In this section, we shall complete the proof of Theorem 1.9. Most of our work will go into proving the special cases of Theorem 1.9 in which each S_n is a classical group of a fixed kind. We shall deal successively with the symplectic groups, the unitary groups and the orthogonal groups over finite fields. The general result will then follow easily. (Clear accounts of the classical groups can be found in [Ca1] and [Ta].)

4.1. **Symplectic groups.** Suppose that $\langle Sp(2d_n, q_n) \mid n \in \mathbb{N}\rangle$ is a sequence of finite symplectic groups such that $d_n \geq 2$ for each $n \in \mathbb{N}$. Fix some $n \in \mathbb{N}$. Then there exists a basis $\boldsymbol{e}\frown\boldsymbol{f} = (e_i \mid 1 \leq i \leq d_n)\frown(f_i \mid 1 \leq i \leq d_n)$ of the corresponding symplectic space such that $(e_i, f_j) = \delta_{ij}$ and $(e_i, e_j) = (f_i, f_j) = 0$ for all $1 \leq i, j \leq d_n$. (Such a basis is called a *normal basis*.) We shall consider $Sp(2d_n, q_n)$ as a group of matrices with respect to the ordered basis $\boldsymbol{e}\frown\boldsymbol{f}$. Let $E_{d_n} = \langle e_1, \dots, e_{d_n}\rangle$ and $F_{d_n} = \langle f_1, \dots, f_{d_n}\rangle$. Then the setwise stabiliser of the subspaces E_{d_n} and F_{d_n} in $Sp(2d_n, q_n)$ contains the subgroup

$$G_n = \left\{ \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & (A^{-1})^T \end{pmatrix} \mid A \in SL(d_n, q_n) \right\}.$$

Theorem 4.1. *Suppose that $d_n \geq 3$ for all $n \in \mathbb{N}$. Then $\prod_n Sp(2d_n, q_n)$ is finitely generated over the subgroup $\prod_n G_n$.*

Corollary 4.2. *Suppose that $\langle S_n \mid n \in \mathbb{N}\rangle$ is a sequence of finite simple symplectic groups such that there does* not *exist an infinite subset I of $\mathbb{N}$ for which conditions 1.7(1) and 1.7(2) are satisfied. Then $c(\prod_n S_n) > \omega$.*

Proof of Corollary 4.2. For each $n \in \mathbb{N}$, let $S_n = PSp(2d_n, q_n)$. Put $J = \{n \in \mathbb{N} \mid d_n < 3\}$, so that $\prod_{n\in\mathbb{N}} S_n = \left(\prod_{n\in J} S_n\right) \times \left(\prod_{n\notin J} S_n\right)$. By assumption, there exists a finite set $\mathcal{F}$ of groups such that $S_n \in \mathcal{F}$ for all $n \in J$. By Proposition 2.3, either $c(\prod_{n\in J} S_n) > \omega$ or $\prod_{n\in J} S_n$ is finite. Hence if $\prod_{n\notin J} S_n$ is finite, then the result follows from Proposition 2.2. So we can suppose that $\prod_{n\notin J} S_n$ is infinite; and it is enough to prove that $c(\prod_{n\notin J} S_n) > \omega$. To simplify notation, we shall suppose that $J = \emptyset$. Let $\prod_n G_n$ be the subgroup of $\prod_n Sp(2d_n, q_n)$ defined above.

By Theorem 3.1, $c(\prod_n G_n) > \omega$. So using Theorem 4.1 and Proposition 2.2, we see that $c(\prod_n Sp(2d_n, q_n)) > \omega$. Hence $c(\prod_n PSp(2d_n, q_n)) > \omega$. $\square$

We shall approach Theorem 4.1 via the Bruhat decomposition

$$Sp(2d, q) = \bigcup_{w \in W} BwB$$

of the symplectic group, where B is a Borel subgroup and W is the Weyl group. Fix some $n \in \mathbb{N}$. Let $\boldsymbol{e}\frown\boldsymbol{f} = (e_i \mid 1 \leq i \leq d_n)\frown(f_i \mid 1 \leq i \leq d_n)$ be our distinguished normal basis. For each $1 \leq i \leq d_n$, let $E_i = \langle e_1, \dots e_i \rangle$. Then the stabiliser B_n of the flag of totally isotropic subspaces

$$E_1 \leqslant E_2 \leqslant \cdots \leqslant E_{d_n}$$

is a Borel subgroup of $Sp(2d_n, q_n)$. Let N_n be the subgroup of $Sp(2d_n, q_n)$ which stabilises the symplectic frame $\{\langle e_i \rangle, \langle f_i \rangle \mid 1 \leq i \leq d_n\}$. Then the Weyl group of $Sp(2d_n, q_n)$ is $N_n / B_n \cap N_n$. Let $H_n = B_n \cap N_n$. Then H_n consists of the matrices of the form

$$\begin{pmatrix} D & \mathbf{0} \\ \mathbf{0} & D^{-1} \end{pmatrix}$$

where $D \in GL(d_n, q_n)$ is a diagonal matrix. Let UT_n be the subgroup of strictly upper triangular matrices in $SL(d_n, q_n)$, and define

$$U_n = \left\{ \begin{pmatrix} P & PS \\ \mathbf{0} & \left(P^{-1}\right)^T \end{pmatrix} \mid P \in UT_n, S^T = S \right\}.$$

Then $B_n = U_n \rtimes H_n$.

First we shall show that there exists a subgroup Γ_0 of $\prod_n Sp(2d_n, q_n)$ such that

1. Γ_0 is finitely generated over $\prod_n G_n$, and
2. $\prod_n U_n \leqslant \Gamma_0$.

Note that for each $A \in SL(d_n, q_n)$, we have

$$\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & \left(A^{-1}\right)^T \end{pmatrix} \begin{pmatrix} \mathbf{1} & S \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} A^{-1} & \mathbf{0} \\ \mathbf{0} & A^T \end{pmatrix} = \begin{pmatrix} \mathbf{1} & ASA^T \\ \mathbf{0} & \mathbf{1} \end{pmatrix}.$$

Let M_n be the left $SL(d_n, q_n)$-module of symmetric $d_n \times d_n$-matrices, with the action

$$S \xmapsto{A} ASA^T.$$

Then it is enough to prove that $\prod_n M_n$ is finitely generated as a $\prod_n SL(d_n, q_n)$-module. We shall consider M_n in three different cases, and show that in each case M_n is "uniformly generated" as a $SL(d_n, q_n)$-module. The result will then follow. Let $p = \text{char}(GF(q_n))$.

Case 1. Suppose that $p > 3$. Since p is odd, every $S \in M_n$ is congruent to a diagonal matrix. This easily implies that there exists $A \in SL(d_n, q_n)$ such that ASA^T is a diagonal matrix. Thus we need only consider diagonal matrices $D = \text{diag}(\lambda_1, \dots, \lambda_{d_n}) \in M_n$. Let $D_1 = \text{diag}(1, 0, \dots, 0)$ and $D_2 = \text{diag}(0, 1, \dots, 1)$, so that $\mathbf{1} = D_1 + D_2$. If $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{d_n}) \in (GF(q_n)^*)^{d_n}$, let

$$R_{\boldsymbol{\alpha}} = \text{diag}(\alpha_1, 1, \dots, 1, \alpha_1^{-1})$$

and

$$S_{\boldsymbol{\alpha}} = \text{diag}((\alpha_2 \dots \alpha_{d_n})^{-1}, \alpha_2, \dots, \alpha_{d_n}).$$

Then $R_{\boldsymbol{\alpha}} D_1 R_{\boldsymbol{\alpha}}^T = \mathrm{diag}(\alpha_1^2, 0, \dots, 0)$ and $S_{\boldsymbol{\alpha}} D_2 S_{\boldsymbol{\alpha}}^T = \mathrm{diag}(0, \alpha_2^2, \dots, \alpha_{d_n}^2)$. Since $p > 3$, for each $\lambda \in GF(q_n)$, there exist $\beta_1, \dots, \beta_4 \in GF(q_n)^*$ such that $\lambda = \sum_{i=1}^4 \beta_i^2$. (For example, see Chapter 4 [Sm].) Thus we can "uniformly generate" each diagonal matrix $D \in M_n$ from the generators D_1 and D_2.

Case 2. Suppose that $p = 3$. Once again, we need only consider diagonal matrices $D \in M_n$. Let $D_1 = \mathrm{diag}(1, 1, 0, \dots, 0)$ and $D_2 = \mathrm{diag}(\delta, 1, 1, \dots, 1)$, where $\delta = 1$ if d_n is even and $\delta = 0$ if d_n is odd. For each subset X of $\{1, \dots, d_n\}$, let $D_1^X = \mathrm{diag}(\chi_X(1), 0, \dots, 0)$ and $D_2^X = \mathrm{diag}(0, \chi_X(2), \dots, \chi_X(d_n))$, where χ_X is the characteristic function of X. It is easy to check that if

$$S \in \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

then there exist $A_1, \dots, A_6 \in SL(2, 3)$ such that $S = \sum_{i=1}^6 A_i A_i^T$. Hence there exist B_i, $C_i \in SL(d_n, 3)$ for $1 \leq i \leq 6$ such that $\sum_{i=1}^6 B_i D_1 B_i^T = D_1^X$ and $\sum_{i=1}^6 C_i D_2 C_i^T = D_2^X$. Now it is easy to complete the proof of this case. Let $D = \mathrm{diag}(\lambda_1, \dots, \lambda_{d_n}) \in M_n$. Then for each $1 \leq i \leq d_n$, there exist α_i, $\beta_i \in GF(q_n)$ such that $\lambda_i = \alpha_i^2 + \beta_i^2$. (It is well-known that if $\mathbb{F}$ is *any* finite field, then every element of $\mathbb{F}$ is a sum of two squares. Unfortunately it is often not possible to express an element as the sum of two nonzero squares.) Let $X = \{i \mid \alpha_i \neq 0\}$ and $Y = \{i \mid \beta_i \neq 0\}$. Then there exist diagonal matrices $R_j \in SL(d_n, q_n)$ for $1 \leq j \leq 4$ such that $R_1 D_1^X R_1^T = \mathrm{diag}(\alpha_1^2, 0, \dots, 0)$, $R_2 D_2^X R_2^T = \mathrm{diag}(0, \alpha_2^2, \dots, \alpha_{d_n}^2)$, $R_3 D_1^Y R_3^T = \mathrm{diag}(\beta_1^2, 0, \dots, 0)$ and $R_4 D_2^Y R_4^T = \mathrm{diag}(0, \beta_2^2, \dots, \beta_{d_n}^2)$. Thus we can "uniformly generate" each diagonal matrix from the generators D_1 and D_2.

Case 3. Suppose that $p = 2$. If $S = (s_{ij}) \in M_n$, then S is said to be *alternating* if $s_{ii} = 0$ for all $1 \leq i \leq d_n$. If S is *not* alternating, then S is congruent to a diagonal matrix. Clearly for each $S \in M_n$, there exist $B_k \in M_n$ for $1 \leq k \leq 3$ such that $S = \sum_{k=1}^3 B_k$ and none of the B_k are alternating. Thus we need only consider diagonal matrices $D \in M_n$. It is easily checked that if C is a diagonal 3×3-matrix over $GF(2)$, then there exist $A_i \in SL(3, 2)$ for $1 \leq i \leq 4$ such that $C = \sum_{i=1}^4 A_i A_i^T$. It is now easy to adapt the argument of Case 2. (In fact, the argument is even simpler in this case, as every element in $GF(q_n)$ is a square.) This completes the proof of the existence of the subgroup Γ_0 of $\prod_n Sp(2d_n, q_n)$.

Next we shall show that there exists a subgroup Γ_1 of $\prod_n Sp(2d_n, q_n)$ such that

1. Γ_1 is finitely generated over Γ_0, and
2. $\prod_n H_n \leqslant \Gamma_1$.

For each $n \in \mathbb{N}$, let LT_n be the subgroup of strictly lower triangular matrices in $SL(d_n, q_n)$, and define

$$V_n = \left\{ \begin{pmatrix} Q & \mathbf{0} \\ SQ & (Q^{-1})^T \end{pmatrix} \mid Q \in LT_n, S^T = S \right\}.$$

Then there exists an element $\pi \in Sp(2d_n, q_n)$ such that $\pi U_n \pi^{-1} = V_n$. Hence there exists $g_1 \in \prod_n Sp(2d_n, q_n)$ such that $\prod_n V_n \leqslant \Gamma_1 = \langle \Gamma_0, g_1 \rangle$. We shall prove that $\prod_n H_n \leqslant \Gamma_1$. For each $\lambda \in GF(q_n)^*$, let $D_\lambda = \mathrm{diag}(\lambda, 1, \dots, 1) \in GL(d_n, q_n)$. Define

$$F_n = \left\{ \begin{pmatrix} D_\lambda & \mathbf{0} \\ \mathbf{0} & D_\lambda^{-1} \end{pmatrix} \mid \lambda \in GF(q_n)^* \right\}.$$

Since $\prod_n G_n \leqslant \Gamma_1$, it suffices to show that $\prod_n F_n \leqslant \Gamma_1$. For each $t \in GF(q_n)$, let $S(t)$ be the symmetric $d_n \times d_n$-matrix with t in the upper left position and 0 elsewhere. Define

$$X(t) = \begin{pmatrix} \mathbf{1} & S(t) \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \in U_n \text{ and } T(t) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ S(t) & \mathbf{1} \end{pmatrix} \in V_n.$$

Then it is easily checked that for each $\lambda \in GF(q_n)^*$,

$$\begin{pmatrix} D_\lambda & \mathbf{0} \\ \mathbf{0} & D_\lambda^{-1} \end{pmatrix} = X(\lambda)Y(-\lambda^{-1})X(\lambda)X(-1)Y(1)X(-1).$$

(This is essentially [Ca1, 6.4.4].) Hence we have that $\prod_n F_n \leqslant \Gamma_1$, as required.

Finally we shall show that there exists a subgroup Γ_2 of $\prod_n Sp(2d_n, q_n)$ such that

1. Γ_2 is finitely generated over Γ_1; and
2. $\prod_n N_n \leqslant \Gamma_2$.

This implies that $\Gamma_2 = \prod_n Sp(2d_n, q_n)$, and hence completes the proof of Theorem 4.1. Again fix some $n \in \mathbb{N}$. Then W_n is generated by the images of the elements $\{w_i \mid 1 \leq i \leq d_n\}$ of N_n defined as follows.

(a) If $1 \leq i < d_n$, then w_i is the permutation matrix corresponding to the permutation $(\, e_i \, e_{i+1}\,)(\, f_i \, f_{i+1}\,)$.

(b) $w_{d_n}(e_{d_n}) = -f_{d_n}$, $w_{d_n}(f_{d_n}) = e_{d_n}$ and w_{d_n} fixes the remaining elements of $\boldsymbol{e}\,\hat{}\,\boldsymbol{f}$. (Thus w_{d_n} corresponds to the odd permutation $(\, e_{d_n} \, f_{d_n}\,)$.)

It follows that W_n is isomorphic to $\mathbb{Z}_2^{d_n} \rtimes Sym(d_n)$, where $\mathbb{Z}_2^{d_n}$ is the natural permutation module for $Sym(d_n)$. Let $t = \lfloor d_n/2 \rfloor$ and let $v \in \mathbb{Z}_2^{d_n}$ be a vector of weight t. Let E_n be the submodule of $\mathbb{Z}_2^{d_n}$ consisting of the vectors of even weight. Then for every $u \in E_n$, there exist π, $\phi \in Sym(d_n)$ such that $u = \pi(v) + \phi(v)$.

Let W_n^+ be the subgroup of W_n consisting of the even permutations of the set $\{e_i, f_i \mid 1 \leq i \leq d_n\}$. Then W_n^+ can be regarded as a subgroup of $Sp(2d_n, q_n)$. Also notice that W_n^+ corresponds to the subgroup $E_n \rtimes Sym(d_n)$ of $\mathbb{Z}_2^{d_n} \rtimes Sym(d_n)$. So the argument of the previous paragraph shows that there exists $g_2 \in \prod_n Sp(2d_n, q_n)$ such that $\prod_n W_n^+ \leqslant \langle \Gamma_1, g_2 \rangle$. We shall show that $\prod_n N_n \leqslant \langle \Gamma_1, g_2 \rangle$. Once again fix some $n \in \mathbb{N}$. Consider the element $w = w_{d_n} w_{d_n - 1} w_{d_n}$. Since w corresponds to an even permutation of $\{e_i, f_i \mid 1 \leq i \leq d_n\}$, it follows that $w \in \langle W_n^+, B_n \rangle$. Using the standard properties of groups with BN-pairs [Ca1, Section 8.2], we have that

$$(B_n w_{d_n} B_n)\,(B_n w B_n) = B_n w_{d_n} w B_n \cup B_n w B_n.$$

In particular, $w \in (B_n w_{d_n} B_n)\,(B_n w B_n)$. Hence there exist elements b_1, b_2, $b_3 \in B_n$ such that $w = b_1 w_{d_n} b_2 w b_3$, and so $w_{d_n} = b_1^{-1} w b_3^{-1} w^{-1} b_2^{-1}$. Obviously if we let $1 = c_1 = c_2 = c_3 \in B_n$, then $1 = c_1^{-1} w c_3^{-1} w^{-1} c_2^{-1}$. Hence for each subset X of $\mathbb{N}$, $\psi_X = \langle \psi_X(n) \rangle_n \in \langle \Gamma_1, g_2 \rangle$, where $\psi_X(n) = w_{d_n}$ if $n \in X$ and $\psi_X(n) = 1$ if $n \notin X$. It follows easily that $\prod_n N_n \leqslant \langle \Gamma_1, g_2 \rangle$. (Cf. the final paragraph of the proof of Lemma 3.6.)

4.2. **Unitary groups.** In this subsection, we consider products $\prod_n SU(d_n, q_n)$ of finite special unitary groups. First consider the case when d_n is even. Then the corresponding unitary space has a normal basis $\boldsymbol{e}\,\hat{}\,\boldsymbol{f}$. Arguing as in Subsection 4.1, we obtain the following result.

Theorem 4.3. *Suppose that $\langle SU(2d_n, q_n) \mid n \in \mathbb{N} \rangle$ is a sequence of special unitary groups which satisfies the following conditions.*

(1) *If* $d_n = 1$, *then* $q_n > 3$.
(2) *There do* not *exist an infinite subset* I *of* $\mathbb{N}$ *and an integer* d *such that*
  (a) $d_n = d$ *for all* $n \in I$; *and*
  (b) *if* $n, m \in I$ *and* $n < m$, *then* $q_n < q_m$.

Then $c(\prod_n SU(2d_n, q_n)) > \omega$. □

Next consider a product of the form $\prod_n SU(2d_n + 1, q_n)$, where $d_n \geq 2$ for all $n \in \mathbb{N}$. Fix some $n \in \mathbb{N}$. Then there exists a basis

$$\boldsymbol{e}^\frown \boldsymbol{f}^\frown(w) = (e_i \mid 1 \leq i \leq d_n)^\frown(f_i \mid 1 \leq i \leq d_n)^\frown(w)$$

of the corresponding unitary space such that $(e_i, f_j) = \delta_{ij}$ and $(e_i, e_j) = (f_i, f_j) = 0$ for all $1 \leq i, j \leq d_n$; and $(w, w) = 1$ and $(w, e_i) = (w, f_i) = 0$ for all $1 \leq i \leq d_n$. Then we can regard $SU(2d_n, q_n)$ as the subgroup of $SU(2d_n + 1, q_n)$ consisting of the elements π such that $\pi(w) = w$.

Theorem 4.4. *Suppose that* $d_n \geq 2$ *for all* $n \in \mathbb{N}$. *Then* $\prod_n SU(2d_n + 1, q_n)$ *is finitely generated over* $\prod_n SU(2d_n, q_n)$.

Proof. We shall make use of the Bruhat decomposition

$$SU(2d + 1, q) = \bigcup_{w \in W} BwB$$

of the special unitary group, where B is a Borel subgroup and W is the Weyl group. Fix some $n \in \mathbb{N}$. For each $1 \leq i \leq d_n$, let $E_i = \langle e_1, \dots, e_i \rangle$. Then the stabiliser B_n of the flag of totally isotropic subspaces $E_1 \leqslant E_2 \leqslant \cdots \leqslant E_{d_n}$ is a Borel subgroup of $SU(2d_n + 1, q_n)$. Let N_n be the subgroup of $SU(2d_n + 1, q_n)$ which stabilises the polar frame $\{\langle e_i \rangle, \langle f_i \rangle \mid 1 \leq i \leq d_n\}$. Then the Weyl group of $SU(2d_n + 1, q_n)$ is $W_n = N_n / B_n \cap N_n$. Note that $N_n \cap SU(2d_n, q_n)$ already contains representatives of each element of W_n. Thus it suffices to prove that there exists an element g such that $\prod_n B_n \leqslant \langle \prod_n SU(2d_n, q_n), g \rangle$. We shall regard $SU(2d_n + 1, q_n)$ as a group of matrices with respect to the ordered basis $(e_1, \dots, e_{d_n}, w, f_{d_n}, \dots, f_1)$. Thus B_n is the subgroup of the upper triangular matrices which are contained in $SU(2d_n + 1, q_n)$. Let U_n be the subgroup of B_n consisting of the strictly upper triangular matrices. Let $H_n = B_n \cap N_n$. Then H_n consists of the diagonal matrices of the form

$$\begin{pmatrix} D & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \lambda^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D^* \end{pmatrix}$$

where

1. $D = \operatorname{diag}(\lambda_1, \dots, \lambda_{d_n}) \in GL(d_n, q_n^2)$;
2. $D^* = \operatorname{diag}(\bar{\lambda}_{d_n}^{-1}, \dots, \bar{\lambda}_1^{-1})$; and
3. $\lambda = \det(DD^*)$.

Here $\sigma \longmapsto \bar{\sigma}$ is the automorphism of $GF(q_n^2)$ of order 2. (Notice that for all diagonal matrices $D \in GL(d_n, q_n^2)$, we have that $\lambda \bar{\lambda} = 1$ and hence $(\lambda^{-1} w, \lambda^{-1} w) = (w, w)$.) We have that $B_n = U_n \rtimes H_n$.

First we shall show that there exists an element $g \in \prod_n SU(2d_n + 1, q_n)$ such that $\prod_n U_n \leqslant \langle \prod_n SU(2d_n, q_n), g \rangle$. Later we shall see that also

$$\prod_n H_n \leqslant \langle \prod_n SU(2d_n, q_n), g \rangle;$$

and so g satisfies our requirements. Once more, fix some $n \in \mathbb{N}$. Let $E^+_{d_n} = \langle E_{d_n}, w\rangle$, and let Γ_n be the setwise stabiliser of $E^+_{d_n}$ in $SU(2d_n+1, q_n)$. Let $\rho : \Gamma_n \to GL(E^+_{d_n})$ be the restriction map. We shall regard $GL(E^+_{d_n})$ as a group of matrices with respect to the ordered basis $(e_1, \dots, e_{d_n}, w)$. Note that for each $A \in SL(d_n, q_n^2)$, we have that

$$\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \in \rho\,[SU(2d_n, q_n) \cap \Gamma_n]\,;$$

and that for each $\boldsymbol{x} \in GF(q_n^2)^{d_n}$, we have that

$$\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} \mathbf{1} & \boldsymbol{x} \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} A^{-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & A\boldsymbol{x} \\ \mathbf{0} & 1 \end{pmatrix}.$$

Now choose $\theta \in U_n$ such that $\rho(\theta) = \begin{pmatrix} \mathbf{1} & \boldsymbol{x}_0 \\ \mathbf{0} & 1 \end{pmatrix}$, where $\boldsymbol{x}_0 \in GF(q_n^2)^{d_n}$ is any nonzero vector. Let $\phi \in U_n$ be an arbitrary element. Then $\rho(\phi) = \begin{pmatrix} B & \boldsymbol{y} \\ \mathbf{0} & 1 \end{pmatrix}$ for some $\boldsymbol{y} \in GF(q_n^2)^{d_n}$ and some strictly upper triangular matrix $B \in GL(d_n, q_n^2)$. Clearly there exist A_1, $A_2 \in SL(d_n, q_n^2)$ such that

$$\begin{pmatrix} \mathbf{1} & -\boldsymbol{y} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & A_1\boldsymbol{x}_0 + A_2\boldsymbol{x}_0 \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{1} & A_1\boldsymbol{x}_0 \\ \mathbf{0} & 1 \end{pmatrix}\begin{pmatrix} \mathbf{1} & A_2\boldsymbol{x}_0 \\ \mathbf{0} & 1 \end{pmatrix}.$$

Hence there exist ψ_1, $\psi_2 \in SU(2d_n, q_n) \cap \Gamma_n$ such that $\rho(\psi_1\theta\psi_1^{-1}\psi_2\theta\psi_2^{-1}\phi) = \begin{pmatrix} B & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}$ and hence $\psi_1\theta\psi_1^{-1}\psi_2\theta\psi_2^{-1}\phi \in SU(2d_n, q_n)$. Thus we can "uniformly generate" U_n using the element θ. It follows that there exists an element $g \in \prod_n SU(2d_n+1, q_n)$ such that $\prod_n U_n \leqslant \langle \prod_n SU(2d_n, q_n), g\rangle$.

Let $\psi \in N_n \cap SU(2d_n, q_n)$ correspond to the permutation $(\,e_1\, f_1\,)\dots(\,e_{d_n}\, f_{d_n}\,)$. Then $V_n = \psi U_n \psi^{-1}$ is the unipotent subgroup of strictly lower triangular matrices of $SU(2d_n+1, q_n)$; and also $\prod_n V_n \leqslant \langle \prod_n SU(2d_n, q_n), g\rangle$.

We can regard $SU(3, q_n)$ as the subgroup of $SU(2d_n+1, q_n)$ consisting of those elements π such that $\pi(e_i) = e_i$ and $\pi(f_i) = f_i$ for all $1 \le i \le d_n - 1$. Now let $h \in \prod_n H_n$ be an arbitrary element. Then there exists $g \in \prod_n (H_n \cap SU(2d_n, q_n))$ such that $hg \in \prod_n (H_n \cap SU(3, q_n))$. Consequently, in order to show that $\prod_n H_n$ is contained in $\langle \prod_n SU(2d_n, q_n), g\rangle$, it is enough to show that $\prod (H_n \cap SU(3, q_n)) \leqslant \langle \prod_n SU(2d_n, q_n), g\rangle$. To accomplish this, we shall use a slightly modified form of [Ca1, pp. 239–242]. For the rest of this proof, we shall write the elements of $SU(3, q_n)$ as 3×3-matrices with respect to the ordered basis (e_{d_n}, w, f_{d_n}). Fix an element $\epsilon \in GF(q_n^2)$ such that $\epsilon\bar{\epsilon} = -1$. Suppose that λ, $t \in GF(q_n^2)$ satisfy $\lambda^{-1} + \bar{\lambda}^{-1} = t\bar{t}$. Then the matrices

$$A_1 = \begin{pmatrix} 1 & \epsilon^{-1}\lambda t & \lambda \\ 0 & 1 & \epsilon\bar{\lambda}\bar{t} \\ 0 & 0 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1 & \epsilon^{-1}\bar{\lambda} t & \lambda \\ 0 & 1 & \epsilon\lambda\bar{t} \\ 0 & 0 & 1 \end{pmatrix}$$

are elements of $U_n \cap SU(3, q_n)$, and the matrix

$$B = \begin{pmatrix} 1 & 0 & 0 \\ -\epsilon\bar{t} & 1 & 0 \\ \bar{\lambda}^{-1} & -\epsilon^{-1}t & 1 \end{pmatrix}$$

is an element of $V_n \cap SU(3, q_n)$. The product of these matrices is

$$A_1 B A_2 = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & -\lambda^{-1}\bar{\lambda} & 0 \\ \bar{\lambda}^{-1} & 0 & 0 \end{pmatrix}.$$

Let L_n be the subset of $GF(q_n^2)^*$ consisting of those elements λ such that there exists $t \in GF(q_n^2)$ such that $\lambda^{-1} + \bar{\lambda}^{-1} = t\bar{t}$. By [Ca1, 13.7.3], each $\lambda \in GF(q_n^2)^*$ can be expressed as $\lambda = \lambda_1 \bar{\lambda}_2^{-1}$ for some λ_1, $\lambda_2 \in L_n$. Hence we can "uniformily generate" each element of $H_n \cap SU(3, q_n)$ via the equation

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1}\bar{\lambda} & 0 \\ 0 & 0 & \bar{\lambda}^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & -\lambda_1^{-1}\bar{\lambda}_1 & 0 \\ \bar{\lambda}_1^{-1} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \lambda_2 \\ 0 & -\lambda_2^{-1}\bar{\lambda}_2 & 0 \\ \bar{\lambda}_2^{-1} & 0 & 0 \end{pmatrix}.$$

□

We can now easily obtain the following result.

Corollary 4.5. *Suppose that $\langle S_n \mid n \in \mathbb{N} \rangle$ is a sequence of finite simple unitary groups such that there does* not *exist an infinite subset I of $\mathbb{N}$ for which conditions 1.7(1) and 1.7(2) are satisfied. Then $c(\prod_n S_n) > \omega$.* □

4.3. **Orthogonal groups.** In this subsection, we shall consider products of finite orthogonal groups. First consider the case when each group has the form $\Omega^+(2d, q)$. Then the corresponding orthogonal space has a normal basis $\boldsymbol{e}^\frown\boldsymbol{f}$. Arguing as in Subsection 4.1, we obtain the following result.

Theorem 4.6. *Suppose that $\langle \Omega^+(2d_n, q_n) \mid n \in \mathbb{N} \rangle$ is a sequence of orthogonal groups which satisfies the following conditions.*

(1) *$d_n \geq 3$ for each $n \in \mathbb{N}$.*
(2) *There does* not *exist an infinite subset I of $\mathbb{N}$ and an integer d such that*
  (a) *$d_n = d$ for all $n \in I$; and*
  (b) *if n, $m \in I$ and $n < m$, then $q_n < q_m$.*

Then $c(\prod_n \Omega^+(2d_n, q_n)) > \omega$. □

Now we shall consider products of the form $\prod_n \Omega(2d_n + 1, q_n)$, where $d_n \geq 2$ for each $n \in \mathbb{N}$. Fix some $n \in \mathbb{N}$. Let Q be the quadratic form on the corresponding orthogonal space, and let $(u, v) = Q(u+v) - Q(u) - Q(v)$ be the associated bilinear map. We can suppose that there exists a basis

$$\boldsymbol{e}^\frown\boldsymbol{f}^\frown(w) = (e_i \mid 1 \leq i \leq d_n)^\frown(f_i \mid 1 \leq i \leq d_n)^\frown(w)$$

of the orthogonal space such that

$$(e_i, f_j) = \delta_{ij} \text{ and } (e_i, e_j) = (f_i, f_j) = Q(e_i) = Q(f_i) = 0$$

for all $1 \leq i, j \leq d_n$; and

$$Q(w) = 1 \text{ and } (w, e_i) = (w, f_i) = 0$$

for all $1 \leq i \leq d_n$. Then we can regard $\Omega^+(2d_n, q_n)$ as the subgroup of $\Omega(2d_n+1, q_n)$ consisting of the elements π such that π stabilises the subspace $\langle e_i, f_i \mid 1 \leq i \leq d_n \rangle$ setwise and $\pi(w) = w$. Clearly this situation is very similar to that which we considered in Subsection 4.2. The main difference is that the Weyl group gets larger in the passage from $\Omega^+(2d_n, q_n)$ to $\Omega(2d_n + 1, q_n)$. The Weyl group W_n of $\Omega(2d_n + 1, q_n)$ is $\mathbb{Z}_2^{d_n} \rtimes Sym(d_n)$, acting on the set $\{e_i, f_i \mid 1 \leq i \leq d_n\}$ with blocks of imprimitivity $\{e_i, f_i \mid 1 \leq i \leq d_n\}$. The Weyl group of $\Omega^+(2d_n, q_n)$ is the subgroup W_n^+ of W_n consisting of the even permutations of $\{e_i, f_i \mid 1 \leq i \leq d_n\}$. But this point has already been dealt with during our treatment of the symplectic groups in Subsection 4.1. Hence we can easily obtain the following result.

Theorem 4.7. *Suppose that* $d_n \geq 2$ *for all* $n \in \mathbb{N}$. *Then* $\prod_n \Omega(2d_n + 1, q_n)$ *is finitely generated over* $\prod_n \Omega^+(2d_n, q_n)$. □

Finally we shall consider products of the form $\prod_n \Omega^-(2d_n+2, q_n)$, where $d_n \geq 3$ for each $n \in \mathbb{N}$. In this case, there exists a basis $\boldsymbol{e}^\frown\boldsymbol{f}^\frown(w, z)$ of the corresponding orthogonal space such that

$$(e_i, f_j) = \delta_{ij} \text{ and } (e_i, e_j) = (f_i, f_j) = Q(e_i) = Q(f_i) = 0$$

for all $1 \leq i, j \leq d_n$; and

$$(w, e_i) = (w, f_i) = (z, e_i) = (z, f_i) = 0$$

for all $1 \leq i \leq d_n$; and the subspace $\langle w, z\rangle$ does not contain any singular vectors. So we can regard $\Omega^+(2d_n, q_n)$ as the subgroup of $\Omega^-(2d_n + 2, q_n)$ consisting of the elements π such that π stabilises the subspace $\langle e_i, f_i \mid 1 \leq i \leq d_n\rangle$ setwise, $\pi(w) = w$ and $\pi(z) = z$.

Theorem 4.8. *Suppose that* $d_n \geq 3$ *for each* $n \in \mathbb{N}$. *Then* $\prod_n \Omega^-(2d_n + 2, q_n)$ *is finitely generated over* $\prod_n \Omega^+(2d_n, q_n)$.

Proof. As before, we shall make use of the Bruhat decomposition

$$\Omega^-(2d + 2, q) = \bigcup_{w\in W} BwB,$$

where B is a Borel subgroup and W is the Weyl group. Fix some $n \in \mathbb{N}$. For each $1 \leq i \leq d_n$, let $E_i = \langle e_1, \dots, e_i\rangle$. Then the stabiliser B_n of the flag of totally singular subspaces

$$E_1 \leqslant E_2 \leqslant \cdots \leqslant E_{d_n}$$

is a Borel subgroup of $\Omega^-(2d_n + 2, q_n)$. Let N_n be the subgroup of $\Omega^-(2d_n + 2, q_n)$ which stabilises the polar frame $\{\langle e_i\rangle, \langle f_i\rangle \mid 1 \leq i \leq d_n\}$. Then the Weyl group of $\Omega^-(2d_n + 2, q_n)$ is $W_n = N_n/B_n \cap N_n$. Once again, W_n is $\mathbb{Z}_2^{d_n} \rtimes Sym(d_n)$ acting on the set $\{e_i, f_i \mid 1 \leq i \leq d_n\}$ with blocks of imprimitivity $\{e_i, f_i\}$ for $1 \leq i \leq d_n$. As before, the main point is to show that there exists a subgroup G of $\prod_n \Omega^-(2d_n + 2, q_n)$ such that

1. G is finitely generated over $\prod_n \Omega^+(2d_n, q_n)$, and
2. $\prod_n B_n \leqslant G$.

Let U_n be the subgroup of unipotent elements of B_n and let $H_n = B_n \cap N_n$, so that $B_n = U_n \rtimes H_n$. First we shall show that there exists an element $g_0 \in \prod_n \Omega^-(2d_n + 2, q_n)$ such that $\prod_n U_n \leqslant G_0 = \langle \prod_n \Omega^+(2d_n, q_n), g_0\rangle$. Note that if $\pi \in U_n$, then there exist vectors $\boldsymbol{x}$, $\boldsymbol{y} \in E_{d_n}$ such that $\pi(w) = w + \boldsymbol{x}$ and $\pi(z) = z + \boldsymbol{y}$. Let $E_{d_n}^+ = \langle E_{d_n}, w, z\rangle$ and let Γ_n be the setwise stabiliser of $E_{d_n}^+$ in $\Omega^-(2d_n+2, q_n)$. Let $\rho : \Gamma_n \to GL(E_{d_n}^+)$ be the restriction map. We regard $GL(E_{d_n}^+)$ as a group of matrices with respect to the ordered basis $(e_1, \dots, e_{d_n}, w, z)$. So for each $A \in SL(d_n, q_n)$, we have that

$$\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \in \rho\left[\Omega^+(2d_n, q_n) \cap \Gamma_n\right].$$

Arguing as in the proof of Theorem 4.4, we see that the existence of a suitable element $g_0 \in \prod_n \Omega^-(2d_n+2, q_n)$ is a consequence of the following easy observation.

Claim 4.9. *Suppose that $d \geq 3$. Let $S = SL(d,q)$ and $V = V(d,q)$. Let S act on $V \times V$ via the action $A(\boldsymbol{x}, \boldsymbol{y}) = (A\boldsymbol{x}, A\boldsymbol{y})$. Suppose that $\boldsymbol{a}$, $\boldsymbol{b} \in V$ are linearly independent. Then for all $(\boldsymbol{x}, \boldsymbol{y}) \in V \times V$, there exist A, $B \in S$ such that $(\boldsymbol{x}, \boldsymbol{y}) = A(\boldsymbol{a}, \boldsymbol{b}) + B(\boldsymbol{a}, \boldsymbol{b})$.* $\square$

Finally we shall show that there exists an element $g_1 \in \prod_n \Omega^-(2d_n + 2, q_n)$ such that $\prod_n H_n \leqslant G_1 = \langle G_0, g_1 \rangle$. We shall regard $\Omega^-(4, q_n)$ as the subgroup of $\Omega^-(2d_n + 2, q_n)$ consisting of the elements π such that π stabilises the subspace $\langle e_{d_n}, f_{d_n}, w, z \rangle$ setwise and such that $\pi(e_i) = e_i$ and $\pi(f_i) = f_i$ for all $1 \leq i \leq d_n - 1$. Since $\prod_n \Omega^+(2d_n, q_n) \leqslant G_0$, it is enough to find an element g_1 such that $\prod_n (H_n \cap \Omega^-(4, q_n)) \leqslant \langle G_0, g_1 \rangle$. We shall make use of the fact that

$$\Omega^-(4, q_n) \simeq SL(2, q_n^2)/\{\pm \mathbf{1}\}.$$

(For example, see [Ta, 12.42].) Let $p_n = \text{char}(GF(q_n))$. Then $U_n \cap \Omega^-(4, q_n)$ is a group of order q_n^2, and hence is a Sylow p_n-subgroup of $\Omega^-(4, q_n)$. It is easily checked that $SL(2, q_n^2)$ is "uniformly generated" by the two subgroups $UT(2, q_n^2)$ and $LT(2, q_n^2)$, consisting of the strictly upper triangular matrices and strictly lower triangular matrices of $SL(2, q_n^2)$. (See [Ca1, 6.4.4].) Since $UT(2, q_n^2)$ and $LT(2, q_n^2)$ are Sylow p_n-subgroups of $SL(2, q_n^2)$, it follows that there exists an element $g_1 \in \prod_n \Omega^-(2d_n + 2, q_n)$ such that $\prod_n \Omega^-(4, q_n) \leqslant \langle G_0, g_1 \rangle$. $\square$

Corollary 4.10. *Suppose that $\langle S_n \mid n \in \mathbb{N} \rangle$ is a sequence of finite simple orthogonal groups such that there does* not *exist an infinite subset I of $\mathbb{N}$ for which conditions 1.7(1) and 1.7(2) are satisfied. Then $c(\prod_n S_n) > \omega$.*

4.4. **Conclusion.**

We can now complete the proof of Theorem 1.9. Suppose that $\langle S_n \mid n \in \mathbb{N} \rangle$ is sequence of finite simple nonabelian groups such that there does *not* exist an infinite subset I of $\mathbb{N}$ for which conditions 1.7(1) and 1.7(2) are satisfied. Let $G = \prod_n S_n$. Let

- $\mathcal{C}_0$ be the set of 26 sporadic finite simple groups,
- $\mathcal{C}_1$ be the set of finite simple alternating groups,
- $\mathcal{C}_2$ be the set of finite simple projective special linear groups,
- $\mathcal{C}_3$ be the set of finite simple symplectic groups,
- $\mathcal{C}_4$ be the set of finite simple unitary groups,
- $\mathcal{C}_5$ be the set of finite simple orthogonal groups, and
- $\mathcal{C}_6$ be the set of finite simple groups of Lie types E_6, E_7, E_8, F_4, G_2, ${}^2E_6$, ${}^2B_2$, ${}^2G_2$, ${}^2F_4$ and ${}^3D_4$.

By the classification of the finite simple groups, each finite simple nonabelian group lies in one of the above sets. Some groups lie in more than one of these sets. For example, $Alt(8) \simeq PSL(4, 2)$. For the rest of this argument, we shall suppose that we have slightly modified the above sets so that they yield a partition of the finite simple nonabelian groups.

For each $0 \leq i \leq 6$, let $J_i = \{n \in \mathbb{N} \mid S_n \in \mathcal{C}_i\}$ and let $P_i = \prod_{n \in J_i} S_n$. Then $G = \prod_{i=0}^6 P_i$. Using Proposition 2.2, it is enough to show that for each $0 \leq i \leq 6$, either $c(P_i) > \omega$ or P_i is finite. If $1 \leq i \leq 5$, this has been proved in Sections 2, 3 and 4. And if $i = 0$, this is an immediate consequence of Proposition 2.3. Finally consider P_6. Our hypothesis on $\langle S_n \mid n \in \mathbb{N} \rangle$ implies that there exists a finite set of simple groups $\mathcal{F} \subseteq \mathcal{C}_6$ such that $S_n \in \mathcal{F}$ for all $n \in J_6$. So the result once again follows from Proposition 2.3. This completes the proof of Theorem 1.9.

5. A CONSISTENCY RESULT

In this section, we shall prove Theorem 1.12. Our notation follows that of Kunen [Ku]. Thus if $\mathbb{P}$ is a notion of forcing and $p, q \in \mathbb{P}$, then $q \leq p$ means that q is a strengthening of p. If V is the ground model, then we denote the generic extension by $V^{\mathbb{P}}$ when we do not want to specify a particular generic filter $G \subseteq \mathbb{P}$.

Definition 5.1. A notion of forcing $\mathbb{P}$ is said to have the *Laver property* if the following holds. Suppose that

1. $\langle A_n \mid n \in \mathbb{N} \rangle$ is a sequence of finite sets;
2. $f : \mathbb{N} \to \mathbb{N}$ is a function such that $f(n) \geq 1$ for all $n \in \mathbb{N}$ and $f(n) \to \infty$ as $n \to \infty$;
3. $p \in \mathbb{P}$, $\tilde{g}$ is a $\mathbb{P}$-name and $p \Vdash \tilde{g} \in \prod_n A_n$.

Then there exist $q \leq p$ and a sequence $\langle B_n \mid n \in \mathbb{N} \rangle$ such that

(a) $B_n \subseteq A_n$ and $|B_n| \leq f(n)$;
(b) $q \Vdash \tilde{g} \in \prod_n B_n$.

Theorem 1.12 is an immediate consequence of the following two results.

Theorem 5.2. *Suppose that $V \models CH$, and that $\langle \mathbb{P}_\alpha, \tilde{\mathbb{Q}}_\alpha \mid \alpha < \omega_2 \rangle$ is a countable support iteration of proper notions of forcing such that for all $\alpha < \omega_2$*

(1) *$\Vdash_\alpha \tilde{\mathbb{Q}}_\alpha$ has the cardinality of the continuum; and*
(2) *$\Vdash_\alpha \tilde{\mathbb{Q}}_\alpha$ has the Laver property.*

Then in $V^{\mathbb{P}_{\omega_2}}$, $c\left(\prod_n G_n\right) \leq \omega_1$ for every *sequence $\langle G_n \mid n \in \mathbb{N} \rangle$ of nontrivial finite groups.*

Theorem 5.3. *Suppose that $V \models CH$. Then there exists a countable support iteration $\langle \mathbb{P}_\alpha, \tilde{\mathbb{Q}}_\alpha \mid \alpha < \omega_2 \rangle$ of proper notions of forcing such that*

(a) *$\langle \mathbb{P}_\alpha, \tilde{\mathbb{Q}}_\alpha \mid \alpha < \omega_2 \rangle$ satisfies conditions 5.2(1) and 5.2(2); and*
(b) *$V^{\mathbb{P}_{\omega_2}} \models c(Sym(\mathbb{N})) = \omega_2 = 2^\omega$.*

First we shall prove Theorem 5.2.

Definition 5.4. Let $\langle G_n \mid n \in \mathbb{N} \rangle$ be a sequence of nontrivial finite groups.

1. A *cover* is a function $c : \mathbb{N} \to \left[\bigcup_{n \in \mathbb{N}} G_n \right]^{<\omega}$ such that for all $n \in \mathbb{N}$
   (a) $\emptyset \neq c(n) \subseteq G_n$;
   (b) the identity element $1_{G_n} \in c(n)$;
   (c) if $a \in c(n)$, then $a^{-1} \in c(n)$.
2. If $g = \langle g(n) \rangle_n \in \prod_n G_n$, then c *covers* g if $g(n) \in c(n)$ for all $n \in \mathbb{N}$.
3. If c is a cover and $f : \mathbb{N} \to \mathbb{N}$, then c is an *f-cover* if $|c(n)| \leq f(n)$ for all $n \in \mathbb{N}$.
4. If c_1 and c_2 are covers, then the cover $c_1 * c_2$ is defined by

$$(c_1 * c_2)(n) = \{ab, (ab)^{-1} \mid a \in c_1(n), b \in c_2(n)\}.$$

Lemma 5.5. *If c_1 is an f_1-cover and c_2 is an f_2-cover, then $c_1 * c_2$ is a $2f_1f_2$-cover.*

Proof. Obvious. □

It is perhaps worth mentioning that $*$ is generally *not* an associative operation on the set of covers of $\prod_n G_n$.

Definition 5.6. If C is a set of covers of $\prod_n G_n$, then its *closure* $c\ell(C)$ is the least set of covers satisfying

1. $C \subseteq c\ell(C)$; and
2. if d_1, $d_2 \in c\ell(C)$ then $d_1 * d_2 \in c\ell(C)$.

Lemma 5.7. *Suppose that C is a set of covers of $\prod_n G_n$. Then*

$$\{g \in \prod_n G_n \mid \textit{There exists } d \in c\ell(C) \textit{ such that } g \textit{ is covered by } d\}$$

is a subgroup of $\prod_n G_n$.

Proof. Easy. □

From now on, let $f : \mathbb{N} \to \mathbb{N}$ be the function defined by $f(n) = 2^{n+2}$ for all $n \in \mathbb{N}$.

Lemma 5.8. *Suppose that $\langle G_n \mid n \in \mathbb{N}\rangle$ is a sequence of finite groups such that $|G_n| \geq 2^{(n+2)^2}$ for all $n \in \mathbb{N}$. If C is a countable set of f-covers of $\prod_n G_n$, then*

$$\{g \in \prod_n G_n \mid \textit{There exists } d \in c\ell(C) \textit{ such that } g \textit{ is covered by } d\}$$

is a proper *subgroup of $\prod_n G_n$.*

Proof. Suppose that $d \in c\ell(C)$ is an m-fold $*$-product of $c_1, \dots, c_m \in C$ in some order. (Remember that $*$ is not an associative operation.) Then Lemma 5.5 implies that d is a $2^{m-1}f^m$-cover. So we can enumerate $c\ell(C) = \{d_n \mid n \in \mathbb{N}\}$ in such a way that d_n is a ϕ_n-cover for all $n \in \mathbb{N}$, where $\phi_n = 2^n f^{n+1}$. In particular,

$$|d_n(n)| \leq 2^n f(n)^{n+1} = 2^{n^2+4n+2} < |G_n|.$$

Hence there exists $g = \langle g(n)\rangle_n \in \prod_n G_n$ such that $g(n) \in G_n \smallsetminus d_n(n)$ for all $n \in \mathbb{N}$. Clearly g is not covered by any element $d \in c\ell(C)$. □

Proof of Theorem 5.2. Suppose that $V \models CH$ and that $\langle \mathbb{P}_\alpha, \tilde{\mathbb{Q}}_\alpha \mid \alpha < \omega_2\rangle$ is a countable support iteration of proper notions of forcing such that for all $\alpha < \omega_2$

1. $\Vdash_\alpha \tilde{\mathbb{Q}}_\alpha$ has the cardinality of the continuum; and
2. $\Vdash_\alpha \tilde{\mathbb{Q}}_\alpha$ has the Laver property.

From now on, we shall work inside $V^{\mathbb{P}_{\omega_2}}$. Let $\langle G_n \mid n \in \mathbb{N}\rangle$ be a sequence of nontrivial finite groups. First suppose that there exists an infinite subset I of $\mathbb{N}$ and a finite group G such that $G_n = G$ for all $n \in I$. By Lemma 1.8 and Theorems 1.3 and 1.4, $c\left(\prod_n G_n\right) \leq c\left(\prod_{n\in I} G_n\right) \leq \omega_1$. Hence we can assume that no such subset I of $\mathbb{N}$ exists. Then there exists an infinite subset $J = \{j_n \mid n \in \mathbb{N}\}$ of $\mathbb{N}$ such that $|G_{j_n}| \geq 2^{(n+2)^2}$ for all $n \in \mathbb{N}$. By Lemma 1.8, $c\left(\prod_n G_n\right) \leq c\left(\prod_{n\in J} G_n\right)$. To simplify notation, we shall suppose that $|G_n| \geq 2^{(n+2)^2}$ for all $n \in \mathbb{N}$.

Since the sequence $\langle G_n \mid n \in \mathbb{N}\rangle$ of finite groups can be coded by a real number, there exists $\alpha < \omega_2$ such that $\langle G_n \mid n \in \mathbb{N}\rangle \in V^{\mathbb{P}_\alpha}$. By Shelah (III 4.1 of [Sh-b]), $V^{\mathbb{P}_\alpha} \models CH$. Let $\{c_\beta \mid \beta < \omega_1\}$ be an enumeration of the f-covers $c \in V^{\mathbb{P}_\alpha}$ of $\prod_n G_n$. For each $\gamma < \omega_1$, let $C_\gamma = \{c_\beta \mid \beta < \gamma\}$ and define

$$H_\gamma = \{g \in \prod_n G_n \mid \text{There exists } d \in c\ell(C_\gamma) \text{ such that } g \text{ is covered by } d\}.$$

By Lemma 5.8, H_γ is a proper subgroup of $\prod_n G_n$ for all $\gamma < \omega_1$. Thus it suffices to show that $\prod_n G_n = \bigcup_{\gamma<\omega_1} H_\gamma$.

Let $g \in \prod_n G_n$ be any element. By Shelah ([Sh-b, VI, Section 2] and [Sh-326, Appendix]), the Laver property is preserved by countable support iterations of proper notions of forcing. This implies that there exists a sequence $\langle B_n \mid n \in \mathbb{N}\rangle \in V^{\mathbb{P}_\alpha}$ such that

(a) $B_n \subseteq G_n$ and $|B_n| \leq 2^n$; and
(b) $g(n) \in B_n$ for all $n \in \mathbb{N}$.

Define the function c by

$$c(n) = B_n \cup \{a^{-1} \mid a \in B_n\} \cup \{1_{G_n}\}$$

for all $n \in \mathbb{N}$. Then $c \in V^{\mathbb{P}_\alpha}$ is an f-cover of $\prod_n G_n$, and so $c = c_\beta$ for some $\beta < \omega_1$. Hence $g \in \bigcup_{\gamma<\omega_1} H_\gamma$. □

The rest of this section will be devoted to the proof of Theorem 5.3. Each of the notions of forcing which we shall use in our iteration will satisfy Axiom A. It is well-known that if $\mathbb{P}$ satisfies Axiom A, then $\mathbb{P}$ is proper. (For example, see [J, p.101].)

Definition 5.9. A notion of forcing $\mathbb{P}$ satisfies *Axiom A* if there is a collection $\{\leq_n \mid n \in \omega\}$ of partial orderings of $\mathbb{P}$ which satisfies the following conditions.

(1) $p \leq_0 q$ iff $p \leq q$.
(2) If $p \leq_{n+1} q$, then $p \leq_n q$.
(3) If $\langle p_n \mid n \in \omega\rangle$ is a sequence such that $p_{n+1} \leq_n p_n$ for all $n \in \omega$, then there exists $q \in \mathbb{P}$ such that $q \leq_n p_n$ for all $n \in \omega$.
(4) For each $p \in \mathbb{P}$, $n \in \omega$ and an ordinal name $\tilde{\alpha}$, there exist a countable set B and a condition $q \in \mathbb{P}$ such that $q \leq_n p$ and $q \Vdash \tilde{\alpha} \in B$.

Definition 5.10. Fix a partition $\{I_n \mid n \in \mathbb{N}\}$ of $\mathbb{N}$ into infinitely many finite subsets such that the following conditions hold.

1. $|I_n| \geq 2$ for all $n \in \mathbb{N}$.
2. For each $t \geq 2$, there exist infinitely many $n \in \mathbb{N}$ such that $|I_n| = t$.
3. If $n < m$, then $\max(I_n) < \min(I_m)$. (Thus each I_n consists of a finite set of consecutive integers.)

The notion of forcing $\mathbb{B}$ consists of all functions p such that

(a) there exists a subset J of $\mathbb{N}$ such that $\operatorname{dom} p = \bigcup_{n\in J} I_n$;
(b) if $n \in J$, then $p \restriction I_n \in \mathit{Sym}(I_n)$;
(c) if $t \geq 2$, then there exist infinitely many $n \in \mathbb{N} \setminus J$ such that $|I_n| = t$.

If p, $q \in \mathbb{B}$, then we define $q \leq p$ if and only if $q \supseteq p$.

Lemma 5.11. *$\mathbb{B}$ satisfies Axiom A and has the Laver property.*

Proof. For each $p \in \mathbb{B}$ and $t \geq 2$, let

$$S^t(p) = \{m \in \mathbb{N} \mid m \notin \operatorname{dom} p \text{ and } |I_m| = t\},$$

and for each $n \geq 1$, let $S^t_n(p)$ be the set of the first n elements of $S^t(p)$. If $t \leq 1$ or $n = 0$, let $S^t_n(p) = \emptyset$. For each $n \in \omega$, define a partial ordering $\leq_n$ on $\mathbb{B}$ by setting $q \leq_n p$ if and only if

1. $q \supseteq p$, and

2. for each $t \leq n$, $S^t_n(p) \subseteq \mathbb{B} \smallsetminus \operatorname{dom} p$.

Then it is easily checked that the partial orderings $\{\leq_n \mid n \in \omega\}$ satisfy clauses (1)–(4) of Definition 5.9. It is also easy to verify that $\mathbb{B}$ has the Laver property. ☐

It follows that $\mathbb{B}$ satisfies conditions 5.2(1) and 5.2(2). (It is also easily seen that $\mathbb{B}$ is ${}^{\omega}\omega$-bounding. However, we shall not need this fact in the proof of Theorem 5.3.) After first introducing some group theoretic notation, we shall explain the relevance of $\mathbb{B}$ to the problem of computing $c(Sym(\mathbb{N}))$.

Definition 5.12. Suppose that $\{a_n \mid n \in \mathbb{N}\}$ is the increasing enumeration of the infinite subset A of $\mathbb{N}$. If $\pi \in Sym(\mathbb{N})$, then $\pi^A \in Sym(A)$ is defined by $\pi^A(a_n) = a_{\pi(n)}$ for all $n \in \mathbb{N}$. If Γ is a subgroup of $Sym(\mathbb{N})$, then $\Gamma^A = \{\pi^A \mid \pi \in \Gamma\}$.

Definition 5.13. If $g : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, then

$$P_g = \prod_n Sym(g(n) \smallsetminus g(n-1)).$$

(Here we use the convention that $g(-1) = 0$.)

Definition 5.14. 1. If f, $g : \mathbb{N} \to \mathbb{N}$, then $f \leq^* g$ iff there exists $n_0 \in \mathbb{N}$ such that $f(n) \leq g(n)$ for all $n \geq n_0$.

2. If $g : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, then

$$S_g = \langle \pi \in Sym(\mathbb{N} \mid \pi, \pi^{-1} \leq^* g \rangle.$$

$\mathbb{B}$ was designed so that the following density condition would hold.

Lemma 5.15. *Suppose that $g : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function and that $p \in \mathbb{B}$. Then there exists an infinite subset A of $\mathbb{N} \smallsetminus \operatorname{dom} p$ such that $p \cup \pi \in \mathbb{B}$ for all $\pi \in P^A_g$.*

Proof. Let $\operatorname{dom} p = \bigcup_{n \in J} I_n$. Then it is easy to find a suitable set A of the form $\bigcup_{n \in K} I_n$, where K is an appropriately chosen subset of $\mathbb{N} \smallsetminus J$. ☐

Arguing as in Section 2 of [ST2], we can now easily obtain the following result.

Lemma 5.16. *Let $V \vDash CH$ and let $\langle \mathbb{P}_\alpha, \tilde{\mathbb{Q}}_\alpha \mid \alpha < \omega_2 \rangle$ be a countable support iteration of proper notions of forcing such that for each $\alpha < \omega_2$, $\mathbb{P}_\alpha \Vdash |\tilde{\mathbb{Q}}_\alpha| = 2^\omega$. Suppose that $S \subseteq \{\alpha < \omega_2 \mid cf(\alpha) = \omega_1\}$ is a stationary subset of ω_2, and that $\tilde{\mathbb{Q}}_\alpha = \tilde{\mathbb{B}}$ for all $\alpha \in S$. (Here $\tilde{\mathbb{B}}$ is the notion of forcing $\mathbb{B}$ in the generic extension $V^{\mathbb{P}_\alpha}$.) Then the following statements are equivalent in $V^{\mathbb{P}_{\omega_2}}$.*

(1) $c(Sym(\mathbb{N})) = \omega_1$.

(2) *It is possible to express $Sym(\mathbb{N}) = \bigcup_{i<\omega_1} G_i$ as the union of a chain of proper subgroups such that for each strictly increasing function $g : \mathbb{N} \to \mathbb{N}$, there exists $i < \omega_1$ with $S_g \leqslant G_i$.* ☐

Definition 5.17. Laver forcing $\mathbb{L}$ consists of the set of all trees $T \subseteq {}^{<\omega}\omega$ with the following property. There exists an integer k such that

1. if $n < k$, then $|T \cap {}^n\omega| = 1$;
2. if $n \geq k$ and $\eta \in T \cap {}^n\omega$, then there exist infinitely many $i \in \omega$ such that $\eta^\frown\langle i \rangle \in T$.

If S, $T \in \mathbb{L}$, then $S \leq T$ iff $S \subseteq T$.

The following result is well-known.

Lemma 5.18. (1) *Suppose that* $V \vDash ZFC$. *Then there exists a function* $g \in {}^{\mathbb{N}}\mathbb{N} \cap V^{\mathbb{L}}$ *such that* $f \leq^* g$ *for all* $f \in {}^{\mathbb{N}}\mathbb{N} \cap V$.

(2) $\mathbb{L}$ *satisfies Axiom A and has the Laver property.* □

It is now easy to complete the proof of Theorem 5.3. Let $V \vDash CH$. Define a countable support iteration $\langle \mathbb{P}_\alpha, \tilde{\mathbb{Q}}_\alpha \mid \alpha < \omega_2 \rangle$ of proper notions of forcing with the Laver property inductively as follows. If $cf(\alpha) = \omega_1$, let $\tilde{\mathbb{Q}}_\alpha = \tilde{\mathbb{B}}$. Otherwise, let $\tilde{\mathbb{Q}}_\alpha = \tilde{\mathbb{L}}$. From now on, we work inside $V^{\mathbb{P}_{\omega_2}}$. Clearly $2^\omega = \omega_2$. Suppose that $c(Sym(\mathbb{N})) = \omega_1$. By Lemma 5.16, we can express $Sym(\mathbb{N}) = \bigcup\limits_{i<\omega_1} G_i$ as the union of a chain of proper subgroups such that for each strictly increasing function $g : \mathbb{N} \to \mathbb{N}$, there exists $i < \omega_1$ with $S_g \leqslant G_i$. Lemma 5.18 implies that there exists a sequence $\langle g_\alpha : \mathbb{N} \to \mathbb{N} \mid \alpha < \omega_2 \rangle$ of strictly increasing functions such that

1. if $\alpha < \beta < \omega_2$, then $g_\alpha \leq^* g_\beta$; and
2. for all $f : \mathbb{N} \to \mathbb{N}$, there exists $\alpha < \omega_2$ such that $f \leq^* g_\alpha$.

There exist $i < \omega_1$ and a cofinal subset C of ω_2 such that $S_{g_\alpha} \leqslant G_i$ for all $\alpha \in C$. But this means that $G_i = Sym(\mathbb{N})$, which is a contradiction. Hence $c(Sym(\mathbb{N})) = \omega_2$.

6. *Sym*($\mathbb{N}$) HAS PROPERTY (FA)

In this section, we shall prove that $Sym(\mathbb{N})$ has property (FA). By Macpherson and Neumann [MN], $c(Sym(\mathbb{N})) > \omega$. Also, since every proper normal subgroup of $Sym(\mathbb{N})$ is countable, $\mathbb{Z}$ is not a homomorphic image of $Sym(\mathbb{N})$. Thus it is enough to prove the following result.

Theorem 6.1. $Sym(\mathbb{N})$ *is not a nontrivial free product with amalgamation.*

Suppose that $Sym(\mathbb{N})$ is a nontrivial free product with amalgamation. Then there exists a tree T such that

1. $Sym(\mathbb{N})$ acts without inversion on T; and
2. there exists $\pi \in Sym(\mathbb{N})$ such that $\pi(t) \neq t$ for all $t \in T$.

(See Theorem 7 in [Se].) Thus it suffices to prove that whenever $Sym(\mathbb{N})$ acts without inversion on a tree T, then for every $\pi \in Sym(\mathbb{N})$ there exists a vertex $t \in T$ such that $\pi(t) = t$. (This also yields a second proof that $\mathbb{Z}$ is not a homomorphic image of $Sym(\mathbb{N})$.) We shall make use of the following theorems of Serre.

Theorem 6.2. [Se, Theorem 16] $SL(3, \mathbb{Z})$ *has property (FA).*

Theorem 6.3. [Se, Proposition 27] *Let* $G = \langle g_1, \dots, g_n \rangle$ *be a finitely generated nilpotent group acting without inversion on the tree* T. *Suppose that for each* $1 \leq i \leq n$, *there exists* $t_i \in T$ *such that* $g_i(t_i) = t_i$. *Then there exists* $t \in T$ *such that* $g(t) = t$ *for all* $g \in G$.

For the rest of this section, let $Sym(\mathbb{N})$ act without inversion on the tree T.

Lemma 6.4. *If* $\pi \in Sym(\mathbb{N})$ *contains no infinite cycles, then there exists* $t \in T$ *such that* $\pi(t) = t$.

Proof. There exists a sequence $\langle G_n \mid n \in \mathbb{N}\rangle$ of nontrivial finite cyclic groups such that $\pi \in \prod_n G_n \leqslant Sym(\mathbb{N})$. By Bass [Ba], whenever the profinite group $\prod_n G_n$ acts without inversion on a tree T, then for every $g \in \prod_n G_n$ there exists $t \in T$ such that $g(t) = t$. □

Lemma 6.5. *Suppose that $\pi \in Sym(\mathbb{N})$ contains infinitely many infinite cycles and no nontrivial finite cycles. Then there exists $t \in T$ such that $\pi(t) = t$.*

Proof. Consider the left regular action of $SL(3,\mathbb{Z})$ on itself. Let $h \in SL(3,\mathbb{Z})$ be an element of infinite order, and let $H = \langle h\rangle$. Then H has infinite index in $SL(3,\mathbb{Z})$. Hence in the left regular action on $SL(3,\mathbb{Z})$, h contains infinitely many infinite cycles and no finite cycles. Consequently if $\Omega = \{n \in \mathbb{N} \mid \pi(n) \neq n\}$, then there exists a subgroup G of $Sym(\mathbb{N})$ such that

1. $\pi \in G \leqslant Sym(\Omega) \leqslant Sym(\mathbb{N})$; and
2. the permutation group (G, Ω) is isomorphic to the left regular action of $SL(3,\mathbb{Z})$ on itself.

By Theorem 6.2, there exists $t \in T$ such that $g(t) = t$ for all $g \in G$. □

Lemma 6.6. *Suppose that $\pi \in Sym(\mathbb{N})$ contains finitely many infinite cycles and no nontrivial finite cycles, and that π fixes infinitely many $n \in \mathbb{N}$. Then there exists $t \in T$ such that $\pi(t) = t$.*

Proof. There exist ϕ_1, $\phi_2 \in Sym(\mathbb{N})$ such that the following conditions are satisfied.

1. ϕ_1 and ϕ_2 both contain infinitely many infinite cycles and no nontrivial finite cycles.
2. $[\phi_1, \phi_2] = 1$.
3. $\pi = \phi_1\phi_2$.

By Lemma 6.5 and Theorem 6.3, there exists $t \in T$ such that $g(t) = t$ for all $g \in G = \langle \phi_1, \phi_2\rangle$. □

Proof of Theorem 6.1. Let $\pi \in Sym(\mathbb{N})$ be any element. We shall show that π fixes a vertex of T. Express $\pi = \phi\psi$ as a product of disjoint permutations such that ψ has no infinite cycles and ϕ has no nontrivial finite cycles. By Lemma 6.4 and Theorem 6.3, it is enough to show that ϕ fixes a vertex of T. Suppose that $\phi \neq 1$. By Lemma 6.5, we can assume that ϕ contains only finitely many infinite cycles. Let $\theta = \phi^2$. Then θ contains ℓ infinite cycles for some $2 \leq \ell \in \mathbb{N}$. Hence there exist τ_1, $\tau_2 \in Sym(\mathbb{N})$ such that the following conditions are satisfied.

1. τ_1 and τ_2 both contain finitely many infinite cycles and no nontrivial finite cycles.
2. τ_1 and τ_2 are disjoint permutations.
3. $\theta = \tau_1\tau_2$.

By Lemma 6.6 and Theorem 6.3, $\theta = \phi^2$ fixes a vertex of T. By 6.3.4 of [Se], ϕ also fixes a vertex of T. □

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