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ESSENTIALLY NORMAL OPERATOR + COMPACT OPERATOR = STRONGLY IRREDUCIBLE OPERATOR

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Abstract. It is shown that given an essentially normal operator T with connected spectrum, there exists a compact operator K such that $T+K$ is strongly irreducible.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on a complex, separable, infinite dimensional Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *strongly irreducible* if it does not commute with any nontrivial idempotent [10], [15]. The strong irreducibility is invariant under similarity. Chun-lan Jiang and Zong-yao Wang proved that for each operator T with connected spectrum $\sigma(T)$, there exists a strongly irreducible operator A and a sequence $\{X_n\}$ of invertible operators such that $||X_n A X_n^{-1} - T|| \to 0$ $(n \to \infty)$ [13]. This result strengthened the following result of D. A. Herrero and Chun-lan Jiang: The norm closure of the set of strongly irreducible operators = the set of the operators with connected spectra.

An operator $T \in \mathcal{L}(\mathcal{H})$ is *irreducible* if it does not commute with any nontrivial orthogonal projection. P. R. Halmos proved that for each $T \in \mathcal{L}(\mathcal{H})$ and $\epsilon > 0$, there exists a K compact with $||K|| < \epsilon$ such that $T + K$ is irreducible [11]. It is obvious that if $\sigma(T)$ is not connected, then for each K compact, $T + K$ is still strongly reducible. Thus D. A. Herrero asked the following questions in personal communications:

1. Given $T \in \mathcal{L}(\mathcal{H})$ with connected $\sigma(T)$, does there exist a compact operator K such that $T + K$ is strongly irreducible?
2. Given an essentially normal operator T with connected $\sigma(T)$, does there exist a compact operator K such that $T + K$ is strongly irreducible? An operator T is *essentially normal* if $T^*T - TT^*$ is compact, where T^* denotes the dual of T.

This article proves that the second question has an affirmative answer.

Main Theorem. *Given an essentially normal operator T with connected spectrum, there exists a compact operator K such that $T + K$ is strongly irreducible.*

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If we can construct a strongly irreducible essentially normal operator S such that $\sigma_e(S) = \sigma_e(T)$ and $\text{ind}(\lambda - S) = \text{ind}(\lambda - T)$ for all $\lambda \notin \sigma_e(S)$, then by Brown-Douglas-Fillmore Theorem S is unitarily equivalent to some compact perturbation $T + K$ of T. Thus, the proof of the main theorem is reduced to constructing a strongly irreducible essentially normal operator with the right spectral picture. A more interesting question is if we can manage to obtain a small compact perturbation in the main theorem. Here the Brown-Douglas-Fillmore Theorem does not work. We have to develop more special techniques. In another paper, the authors and Y. Q. Ji have obtained the following result.

Theorem (J-J-W). *Given an essentially normal operator T with connected spectrum. If $\text{ind}(\lambda I - T) > 0$ for all $\lambda \in \rho_F(T)$, then for each $\epsilon > 0$, there exists K compact with $||K|| < \epsilon$ such that $T + K$ is strongly irreducible.*

2. Preparation

In what follows, $T \in (\text{SI})$ means that T is a strongly irreducible operator on its acting space and Ω always denotes a bounded, connected open set in C. Recall that for natural number n, $\mathcal{B}_n(\Omega)$, the set of *Cowen-Douglas operators of index n*, is the set of all operators B on $\mathcal{H}$ satisfying

(i) $\sigma(B) \supset \Omega$;
(ii) $\text{nul}\,(\lambda - B) = \text{ind}(\lambda - B) = n$ $(\lambda \in \Omega)$;
(iii) $\bigvee\{\ker(\lambda - B) \,:\, \lambda \in \Omega\} = \mathcal{H}$ [5], [6];

Note that (iii) can be replaced by (iii)' [6]:

(iii)' $\bigvee\{\ker(\lambda_0 - B)^k : k = 1, 2, \cdots\} = \mathcal{H}$ for each $\lambda_0 \in \Omega$.

Also note that each operator B in $\mathcal{B}_1(\Omega)$ is strongly irreducible [6], and recall that if $N(\Omega)$ is the "multiplication by λ" operator acting on $L^2(\Omega)$ and $N_+(\Omega)$, the Bergman operator, is the restriction of $N(\Omega)$ to $L^2_a(\Omega)$, the subspace of $L^2(\Omega)$ spanned by the rational functions with poles outside $\bar{\Omega}$, then

(i) $\mathcal{A}'(N_+(\Omega)) = \mathcal{H}^\infty(\Omega)$ [12] and $N_+(\Omega)$ is subnormal and essentially normal, where $\mathcal{A}'(N_+(\Omega))$ denotes the commutant of $N_+(\Omega)$;
(ii) If $N = N_+(\Omega^*)^*$, then $N \in \mathcal{B}_1(\Omega)$, $\sigma_l(N) = \sigma(N) = \overline{\Omega}$, $\sigma_e(N) = \sigma_r(N) = \partial\overline{\Omega}$, where $\Omega^* = \{\lambda \in C \,:\, \bar{\lambda} \in \Omega\}$ and $\partial\overline{\Omega}$ denotes the boundary of $\overline{\Omega}$.

Given $A, B \in \mathcal{L}(\mathcal{H})$, the *Rosenblum operator* $\tau_{AB} \in \mathcal{L}(\mathcal{L}(\mathcal{H}))$ is defined by $\tau_{AB}(X) = AX - XB$. It is well known that

(i) $\sigma(\tau_{AB}) = \sigma(A) - \sigma(B)$;
(ii) τ_{AB} is surjective if and only if $\sigma_r(A) \cap \sigma_l(B) = \varnothing$ [12].

Lemma 2.1 ([13, Lemma 2]). *Let $A, B \in \mathcal{L}(\mathcal{H})$. Assume that*

$$\mathcal{H} = \bigvee\{\ker(\lambda - B)^k \,:\, \lambda \in \Gamma,\ k \geq 1\}$$

for certain subset Γ of the point spectrum $\sigma_p(B)$ of B and $\sigma_p(A) \cap \Gamma = \varnothing$; then $\ker \tau_{AB} = \{0\}$.

Lemma 2.2. *Given $A \in \mathcal{B}_1(\Omega)$ and $\lambda_0 \in \Omega$, then there exist an ONB$\{e_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ and $r > 0$ such that*

$$A = \begin{pmatrix} \lambda_0 & a_{12} & a_{13} & \dots \\ & \lambda_0 & a_{23} & \dots \\ & & \lambda_0 & \dots \\ 0 & & & \dots \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_2 \\ \vdots \end{matrix}$$

and $|a_{k,k+1}| > r > 0$ $(k = 1, 2, \cdots)$.

Proof. Assume that $e_1 \in \ker(A - \lambda_0)$ and $||e_1|| = 1$. Let B be the right inverse of $A - \lambda_0$. Since $\ker B^k = \{0\}$ $(k = 1, 2, \cdots)$ and $e_1 \notin \operatorname{ran} B$, $\{e_1, Be_1, B^2e_1, \cdots\}$ is linearly independent. Since $B^{k-1}e_1 \in \ker(A - \lambda_0)^k$ and $\operatorname{nul}(A - \lambda_0)^k = k$, $\ker(A - \lambda_0)^k = \bigvee\{e_1, Be_1, B^2e_1, \cdots, B^{k-1}e_1\}$ $(k = 1, 2, \cdots)$. Since $\bigvee\{\ker(A - \lambda_0)^k,\ k = 1, 2, \cdots\} = \mathcal{H}$, $\bigvee\{B^k e_1 : k = 0, 1, 2, \cdots\} = \mathcal{H}$. Let $\{e_k\}_{k=1}^{\infty}$ be the Gram-Schmidt orthonormalization of $\{B^k e_1\}_{k=0}^{\infty}$. Then A has an upper triangular matrix representation

$$A = \begin{pmatrix} \lambda_0 & a_{12} & a_{13} & \dots \\ & \lambda_0 & a_{23} & \dots \\ & & \lambda_0 & \dots \\ 0 & & & \dots \end{pmatrix}$$

with respect to the ONB $\{e_k\}_{k=1}^{\infty}$. Note that $A - \lambda_0$ is bounded from below; thus there exists $r > 0$ such that $||(A - \lambda_0)y|| \geq r||y||$ for each $y \in [e_1]^{\perp}$. Set $x_k = a_{k,k+1}e_k$ and $x_k^1 = -\sum_{i=1}^{k-1} a_{i,k+1}e_i$ $(k = 1, 2, \cdots)$. Since $A - \lambda_0$ is onto, there is a vector $y_k \in \bigvee\{e_i : i = 2, 3, \cdots, k\}$ such that $x_k^1 = (A - \lambda_0)y_k$. Thus $x_k = (A - \lambda_0)(e_{k+1} + y_k)$ and

$$\begin{aligned} |a_{k,k+1}| = ||x_k|| &= ||(A - \lambda_0)(e_{k+1} + y_k)|| \\ &\geq r||e_{k+1} + y_k|| \\ &= r\sqrt{||e_{k+1}||^2 + ||y_k||^2} \geq r \quad (k = 1, 2, \cdots). \end{aligned}$$ □

Lemma 2.3. *Let $A_1 \in \mathcal{B}_1(\Omega)$ and let n be a natural number. Given $\epsilon > 0$, there exist $A_k \in \mathcal{B}_1(\Omega)$ $(k = 2, 3, \cdots, n)$ and K_k compact with $||K_k|| < \frac{\epsilon}{2^k}$ $(k = 1, 2, \cdots, n - 1)$ such that $A_k = A_{k-1} + K_{k-1}$ $(k = 2, 3, \cdots, n)$, $\ker \tau_{A_iA_j} = \{0\}$ $(i < j)$ and A_k $(k = 1, 2, \cdots, n)$ admits an upper triangular representation with respect to an ONB $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{H}$.*

Proof. Let λ_0 be in Ω. From Lemma 2.2, A_1 admits an upper triangular matrix of the form

$$A_1 = \begin{pmatrix} \lambda_0 & a_{12}^1 & a_{13}^1 & \dots \\ & \lambda_0 & a_{23}^1 & \dots \\ & & \lambda_0 & \dots \\ 0 & & & \dots \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_2 \\ \vdots \end{matrix}$$

with respect to an ONB $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ and $|a_{k,k+1}^1| > r > 0$ for some $r > 0$. Fix a natural number N so that $||a_k A_1|| < \frac{\epsilon}{2}$ $(k > N)$, where $(\frac{k+1}{k})^{1/2} = 1 + a_k$.

Set

$$A_2 = \begin{pmatrix} \lambda_0 & a_{12}^2 & a_{13}^2 & \dots \\ & \lambda_0 & a_{23}^2 & \dots \\ & & \lambda_0 & \dots \\ 0 & & & \dots \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_2 \\ \vdots \end{matrix}$$

with respect to the ONB $\{e_n\}_{n=1}^{\infty}$, where

$$a_{k,k+1}^2 = \begin{cases} a_{k,k+1}^1, & k \leq N; \\ \left(\frac{k+1}{k}\right)^{1/2} a_{k,k+1}^1, & k > N, \end{cases}$$
$$a_{ij}^2 = a_{ij}^1, \quad j \neq i+1.$$

Thus $A_2 = A_1 + K_1$ and K_1 is compact with $||K_1|| < \frac{\epsilon}{2}$. Since $a_{k,k+1}^2 \neq 0$ ($k = 1, 2, \cdots$), $\dim\ker(A_2 - \lambda_0)^* = 0$ and $\bigvee\{\ker(A_2 - \lambda_0)^k : k = 1, 2, \cdots\} = \mathcal{H}$. Thus $A_2 \in \mathcal{B}_1(\Omega)$. If $X \in \ker\tau_{A_1A_2}$, then $(A_1 - \lambda_0)X = X(A_2 - \lambda_0)$. From $(A_1 - \lambda_0)Xe_1 = X(A_2 - \lambda_0)e_1 = 0$ and $\ker(A_1 - \lambda_0) = \bigvee\{e_1\}$, $Xe_1 = x_{11}e_1$ for some number x_{11}. Since $(A_1 - \lambda_0)^2Xe_2 = 0$, $Xe_2 \in \bigvee\{e_1, e_2\}$, i.e. $Xe_2 = x_{12}e_1 + x_{22}e_2$. In general, we can prove that $Xe_k = \sum_{i=1}^{k} x_{ik}e_i$, i.e. X admits an upper triangular matrix representation with respect to the ONB $\{e_n\}_{n=1}^{\infty}$. Computations show that

$$x_{kk} = \frac{\prod_{i=1}^{k-1} a_{i,i+1}^2}{\prod_{i=1}^{k-1} a_{i,i+1}^1} x_{11} = \begin{cases} \left(\frac{k}{N+1}\right)^{1/2} x_{11} & (k > N), \\ x_{11} & (k \leq N). \end{cases}$$

Since $|x_{k,k}| \leq ||X||$, $x_{k,k} = 0$ ($k = 1, 2, \cdots$).

Similarly,

$$x_{k,k+1} = \frac{a_{k,k+1}^2 \prod_{i=2}^{k-1} a_{i,i+1}^2}{a_{12}^1 \prod_{i=2}^{k-1} a_{i,i+1}^1} x_{12} = \begin{cases} \frac{a_{k,k+1}^2}{a_{12}^1} \left(\frac{k}{N+1}\right)^{1/2} x_{12} & (k > N), \\ \frac{a_{k,k+1}^2}{a_{12}^1} x_{12} & (k \leq N). \end{cases}$$

Since $|a_{k,k+1}^2| > r > 0$ and $|x_{k,k+1}| \leq ||X||$, $x_{k,k+1} = 0$ ($k = 1, 2, \cdots$). The same argument shows that $x_{ij} = 0$ ($j - i \geq 2$), i.e. $X = 0$ and $\ker\tau_{A_1A_2} = \{0\}$.

Define $A_3, \cdots, A_n$ inductively such that $A_k = A_{k-1} + K_{k-1}$ where K_k is compact with $||K_k|| < \frac{\epsilon}{2^k}$. The similar arguments indicate that $\{A_k\}_{k=1}^{n}$, $\{K_k\}_{k=1}^{n-1}$ satisfy the requirements of the lemma. □

Lemma 2.4. *Let $\{A_k\}_{k=1}^{n}$ be given in Lemma 2.3 and let $\epsilon > 0$; then there exists $Q_{k,k-1}$ compact with $||Q_{k,k-1}|| < \epsilon$ such that $Q_{k,k-1} \notin \operatorname{ran}\tau_{A_kA_{k-1}}$ ($k = 2, 3, \cdots, n$).*

Proof. Set

$$Q_{21} = \begin{pmatrix} 0 & a_1 & 0 & 0 & \dots \\ 0 & 0 & a_2 & 0 & \dots \\ 0 & 0 & 0 & a_3 & \dots \\ & & & & \ddots \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ \vdots \end{matrix}$$

where $a_i = \frac{\epsilon}{2\sqrt{i}||A_2||^2} a_{i,i+1}^2$ ($i = 1, 2, \cdots$); then Q_{21} is compact with $||Q_{21}|| < \epsilon$. If $X \in \mathcal{L}(\mathcal{H})$ satisfies $A_2X - XA_1 = Q_{21}$, then

$$(A_2 - \lambda_0)Xe_1 = Q_{21}e_1 + X(A_1 - \lambda_0)e_1 = 0.$$

Thus $Xe_1 \in \ker(A_2 - \lambda_0)$, i.e. $Xe_1 = x_{11}e_1$ for some number x_{11}. Again, $(A_2 - \lambda_0)Xe_2 = a_1e_1 + X(A_1 - \lambda_0)e_2$ and $Xe_2 \in \ker(A_2 - \lambda_0)^2$. i.e. $Xe_2 = x_{12}e_1 + x_{22}e_2$. In general, $Xe_k = x_{1k}e_1 + \cdots + x_{kk}e_k$ $(k = 1, 2, \cdots)$ and X has an upper triangular form with respect to $\{e_n\}_{n=1}^{\infty}$.

Computations indicate that

$$\begin{aligned} x_{22} &= \frac{a_{12}^1}{a_{12}^2}x_{11} + \frac{a_1}{a_{12}^2}, \\ x_{33} &= \frac{a_{12}^1 a_{23}^1}{a_{12}^2 a_{23}^2}x_{11} + \frac{a_1}{a_{12}^2} \cdot \frac{a_{23}^1}{a_{23}^2} + \frac{a_2}{a_{23}^2}, \\ &\cdots\cdots\cdots\cdots \\ x_{k+1,k+1} &= \prod_{i=1}^{k} \frac{a_{i,i+1}^1}{a_{i,i+1}^2}x_{11} + \frac{a_1}{a_{12}^2}\prod_{i=2}^{k} \frac{a_{i,i+1}^1}{a_{i,i+1}^2} + \frac{a_2}{a_{23}^2}\prod_{i=3}^{k} \frac{a_{i,i+1}^1}{a_{i,i+1}^2} + \cdots \\ &\quad + \frac{a_{k-1}}{a_{k-1,k}^2}\frac{a_{k,k+1}^1}{a_{k,k+1}^2} + \frac{a_k}{a_{k,k+1}^2}. \end{aligned}$$

From the proof of Lemma 2.3,

$$a_{i,i+1}^2 = \begin{cases} a_{i,i+1}^1, & i \le N; \\ \left(\frac{i+1}{i}\right)^{1/2} a_{i,i+1}^1, & i > N. \end{cases}$$

Thus, if $k > N$ we have

$$\prod_{j=i}^{k} \frac{a_{j,j+1}^1}{a_{j,j+1}^2} = \begin{cases} \prod_{j=N+1}^{k} \left(\frac{j}{j+1}\right)^{1/2} = \left(\frac{N+1}{k+1}\right)^{1/2}, & i \le N; \\ \prod_{j=i}^{k} \left(\frac{j}{j+1}\right)^{1/2} = \left(\frac{i}{k+1}\right)^{1/2}, & i > N, \end{cases}$$

and

$$\frac{a_i}{a_{i,i+1}^2}\prod_{j=i}^{k} \frac{a_{j,j+1}^1}{a_{j,j+1}^2} \ge \frac{\epsilon}{2||A_2||(k+1)^{1/2}} \qquad (i = 2, \cdots, k).$$

Thus $|x_{k+1,k+1}| \ge \frac{k\epsilon}{2||A_2||(k+1)^{1/2}} \to \infty \quad (k \to \infty)$. The contradiction indicates that $Q_{21} \notin \operatorname{ran}\tau_{A_2A_1}$. By the same arguments we can prove the conclusion for $k = 3, \cdots, n$. ☐

Lemma 2.5. *Let* $A_1 \in \mathcal{B}_1(\Omega)$ *and let* n *be a natural number,* $\epsilon > 0$*; then there exist* $\bar{A}(\Omega) \in$ *(SI),* K*,* $Q(\Omega)$ *compact with* $||K|| < \epsilon$ *and* $||Q(\Omega)|| < \epsilon$ *such that*

$$\bar{A}(\Omega) = A_1^{(n-1)} + K \in \mathcal{B}_{n-1}(\Omega),\ Q(\Omega) \notin \operatorname{ran}\tau_{\bar{A}(\Omega),A_1}$$

and $\ker\tau_{A_1,\bar{A}(\Omega)} = \{0\}$.

Proof. Assume that $A_2, \cdots, A_n$ are given in Lemma 2.3 such that

$$A_k = A_{k-1} + K_{k-1} = A_1 + \sum_{i=1}^{k-1} K_i = A_1 + K_k'.$$

Thus K_k' is compact with $||K_k'|| < \epsilon$. Assume that $Q_{2,1}, \cdots, Q_{n,n-1}$ are given in Lemma 2.4 such that $Q_{k,k-1}$ is compact with $||Q_{k,k-1}|| < \epsilon$ and $Q_{k,k-1} \notin \operatorname{ran}\tau_{A_kA_{k-1}}$ $(k = 2, 3, \cdots, n)$.

Set

$$\bar{A}(\Omega) = \begin{pmatrix} A_n & Q_{n,n-1} & & & \\ & A_{n-1} & Q_{n-1,n-2} & & \\ & & A_{n-2} & \ddots & \\ & & & \ddots & Q_{3,2} \\ 0 & & & & A_2 \end{pmatrix}$$

on $\mathcal{H}^{(n-1)}$. Then $\bar{A}(\Omega) = A_1^{(n-1)} + K$, where

$$K = \begin{pmatrix} K'_n & Q_{n,n-1} & & & \\ & K'_{n-1} & Q_{n-1,n-2} & & \\ & & K'_{n-2} & \ddots & \\ & & & \ddots & Q_{3,2} \\ 0 & & & & K'_2 \end{pmatrix}$$

Thus it is obvious that K is compact with $||K|| < \epsilon$.
Suppose that $P \in \mathcal{A}'(\bar{A}(\Omega))$ is an idempotent and

$$P = \begin{pmatrix} P_{nn} & \dots & P_{n2} \\ & \dots & \\ P_{2n} & \dots & P_{22} \end{pmatrix} \begin{matrix} \mathcal{H} \\ \vdots \\ \mathcal{H} \end{matrix}.$$

Then $P\bar{A}(\Omega) = \bar{A}(\Omega)P$ implies that

$$P_{2n}A_n = A_2 P_{2n}.$$

Since $\ker \tau_{A_2 A_n} = \{0\}$, $P_{2n} = 0$. By the same argument we can prove

$$P_{ij} = 0 \quad (i < j).$$

Thus $P_{ii} \in \mathcal{A}'(A_i)$ $(i = 2, \cdots, n)$. Since each $\mathcal{B}_1(\Omega)$ operator is in (SI), $P_{ii} = \delta_i I$, where $\delta_i = 0$ or 1 and I is the identity of $\mathcal{H}$. Assume that $\delta_n = 0$ (if $\delta_n = 1$, consider $I^{(n-1)} - P$); then

$$P_{n,n-1}A_{n-1} = A_n P_{n,n-1} + \delta_{n-1} Q_{n,n-1}.$$

Since $Q_{n,n-1} \notin \operatorname{ran} \tau_{A_n A_{n-1}}$, $\delta_{n-1} = 0$. Similarly $\delta_i = 0$ $(i = 2, 3, \cdots, n-2)$. Thus $P = 0$ and $\bar{A}(\Omega) \in$ (SI). It is a routine exercise to check that $\bar{A}(\Omega) \in \mathcal{B}_{n-1}(\Omega)$.
Set

$$Q(\Omega) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ Q_{21} \end{pmatrix} \in \mathcal{L}(\mathcal{H}, \mathcal{H}^{(n-1)}).$$

If $X \in \mathcal{L}(\mathcal{H}, \mathcal{H}^{(n-1)})$ satisfies $\bar{A}(\Omega)X - XA_1 = Q(\Omega)$ and assuming that

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_{n-1} \end{pmatrix},$$

then $A_2 X_{n-1} - X_{n-1}A_1 = Q_{21}$, i.e. $Q_{21} \in \operatorname{ran} \tau_{A_2 A_1}$. Thus $Q(\Omega) \notin \operatorname{ran} \tau_{\bar{A}(\Omega) A_1}$.

If $Y \in \mathcal{L}(\mathcal{H}^{(n-1)}, \mathcal{H})$ satisfies $A_1 Y = Y\bar{A}(\Omega)$ and assume that

$$Y = (Y_1, Y_2, \cdots, Y_{n-1}).$$

Since $\ker \tau_{A_i A_j} = \{0\}$ $(1 \le i < j \le n-1)$, $Y_i = 0$ $(i = 1, 2, \cdots, n-1)$, i.e. $\ker \tau_{A_1 \bar{A}(\Omega)} = \{0\}$. □

Lemma 2.6. *Given* $A, B \in \mathcal{L}(\mathcal{H})$. *Let* $\Gamma = \{a_n\}_{n=1}^{\infty} \subset \sigma_p(A)$ *and* $\beta \in \sigma_p(B)$ *satisfy*

(i) $\beta \notin \sigma_p(A)$ *but* $\beta \in \bar{\Gamma}$;
(ii) $\dim \ker(A - a_n) < \infty$ $(n = 1, 2, \cdots)$, $\dim \ker(B - \beta) < \infty$;
(iii) $\bigvee\{\ker(A - a_n) : n = 1, 2, \cdots\} = \bigvee\{\ker(B - \beta)^n : n = 1, 2, \cdots\} = \mathcal{H}$.

Then for each $\epsilon > 0$, *there exists* K *compact with* $||K|| < \epsilon$ *such that* $K \notin \operatorname{ran} \tau_{AB}$.

Proof. From (ii), (iii), there is an ONB $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ such that

$$A = \begin{pmatrix} a_1 & & & * \\ & a_2 & & \\ & & \ddots & \\ 0 & & & \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \\ \vdots \end{matrix},$$

$$UBU^* = B' = \begin{pmatrix} \beta & & & * \\ & \beta & & \\ & & \ddots & \\ 0 & & & \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \\ \vdots \end{matrix}$$

where U is an unitary. Assume that subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ satisfies

$$|a_{n_k} - \beta| < \frac{1}{2^k} \quad (k = 1, 2, \cdots).$$

Let $K' \in \mathcal{L}(\mathcal{H})$ be defined by $K' e_n = \lambda_n e_n$, where

$$\lambda_n = \begin{cases} \frac{\epsilon}{2k}, & \text{if } n = n_k \text{ for some } k; \\ 0, & \text{otherwise.} \end{cases}$$

Then K' is compact and $||K'|| < \epsilon$. If $X \in \mathcal{L}(\mathcal{H})$ satisfies $AX - XB' = K'$, then $(A - \beta)Xe_1 = X(B' - \beta)e_1 + K'e_1 \in \bigvee\{e_1\}$. Since $A - \beta$ is injective and upper triangular, $Xe_1 = x_{11}e_1$ for some number x_{11}. Similarly $(A - \beta)Xe_2 = X(B' - \beta)e_2 + K'e_2 \in \bigvee\{e_1, e_2\}$, and thus $Xe_2 = x_{12}e_1 + x_{22}e_2$. In general, $Xe_n = x_{1n}e_1 + \cdots + x_{nn}e_n$ and X admits an upper triangular representation with respect to the ONB $\{e_n\}_{n=1}^{\infty}$. Computation shows that $x_{nn} = \frac{\lambda_n}{a_n - \beta}$ $(n = 1, 2, \cdots)$. Thus $|x_{n_k n_k}| \to \infty$ $(k \to \infty)$. Therefore $K = K'U \notin \operatorname{ran} \tau_{AB}$. □

Lemma 2.7. *Let* $A, K \in \mathcal{L}(\mathcal{H})$ *be given by* $Ae_1 = 0$, $Ae_{n+1} = \frac{1}{n}e_n$ *and* $Ke_n = \frac{\epsilon}{2\sqrt{n}}e_n$ $(n = 1, 2, \cdots)$, *where* $\{e_n\}_{n=1}^{\infty}$ *is an ONB of* $\mathcal{H}$. *Then* K *is compact with* $||K|| < \epsilon$ *and* $K \notin \operatorname{ran}\tau_{AA^*}$.

Proof. If $AX - XA^* = K$ for some $X \in \mathcal{L}(\mathcal{H})$ and assuming that

$$X = \begin{pmatrix} x_{11} & x_{12} & \dots \\ x_{21} & x_{22} & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \\ \vdots \end{matrix},$$

then $\frac{1}{n}(x_{n+1,n} - x_{n,n+1}) = \frac{\epsilon}{\sqrt{n}}$ and $|x_{n+1,n} - x_{n,n+1}| = \sqrt{n}\epsilon \to \infty$ $(n \to \infty)$. Thus $K \notin \operatorname{ran} \tau_{AA^*}$. □

Lemma 2.8. *Given a domain* Ω*, there exists an essentially normal and co-subnormal operator* N *such that for each* $\epsilon > 0$*, there is* K *compact with* $||K|| < \epsilon$ *satisfying*

(i) $A(\Omega) = N + K \in \mathcal{B}_1(\Omega)$*;*
(ii) $\sigma(A(\Omega)) = \bar{\Omega}$ *and* $\sigma_e(A) = \partial\Omega$*;*
(iii) $\sigma_p(A(\Omega)) \cap \partial\bar{\Omega} = \varnothing$*.*

Proof. Let $N_+(\Omega_1^*)$ be the Bergman operator on $\mathcal{H}_0 = L_a^2(\Omega_1^*)$ and $S_0 = N_+(\Omega_1^*)^*$, where $\Omega_1 = \bar{\Omega}^0$. Denote $\sigma = \Omega_1 - \Omega$. Let $\{\mu_n\}_{n=1}^l$ $(0 \le l \le \infty)$ be a dense subset of σ and let $\{\mathcal{H}_n\}_{n=1}^\infty$ be a sequence of Hilbert spaces and $\{e_k^n\}_{k=1}^\infty$ be an ONB of Hilbert space $\mathcal{H}_n$ $(n = 0, 1, 2, \cdots)$. Define $A_n e_k^n = \frac{\epsilon}{(k!)^n} e_{k+1}^n = a_k^n e_{k+1}^n$ $(k = 1, 2, \cdots)$ and $S_n = \mu_n I_n + A_n$, where I_n is the identity on $\mathcal{H}_n$ $(n = 1, 2, \cdots)$. From a result of M. Cowen and R. G. Douglas [6], there is an analytic function $f : \Omega_1 \to \mathcal{H}_0$ such that $f(\lambda) \neq 0$ and

$$(S_0 - \lambda) f(\lambda) \equiv 0 \quad (\lambda \in \Omega_1).$$

Define

$$A(\Omega) = \begin{pmatrix} S_0 & & & 0 \\ C_1 & S_1 & & \\ C_2 & 0 & S_2 & \\ \vdots & & & \ddots \end{pmatrix} \begin{matrix} \mathcal{H}_0 \\ \mathcal{H}_1 \\ \mathcal{H}_2 \\ \vdots \end{matrix}$$

with respect to $\mathcal{H} = \bigoplus_{i=0}^l \mathcal{H}_i$, where

$$C_n = \frac{\epsilon}{2^n ||f(\mu_n)||} e_1^n \otimes f(\mu_n)^* \quad (n = 1, 2, \cdots).$$

Then $A(\Omega) = N + K$, where

$$N = S_0 \oplus \left(\bigoplus_{n=1}^l \mu_n I_n \right)$$

is essentially normal and co-subnormal and

$$K = \begin{pmatrix} 0 & & & 0 \\ C_1 & A_1 & & \\ C_2 & 0 & A_2 & \\ \vdots & & & \ddots \end{pmatrix}$$

is compact with $||K|| < \epsilon$.

Thus $\sigma(A(\Omega)) \supset \Omega_1$, $\operatorname{ind}(\lambda - A(\Omega)) = 1$ $(\lambda \in \Omega_1)$ and $\sigma_e(A(\Omega)) = \sigma_e(N) = \partial\Omega$. Since $\sigma_p(S_n^*) \cap \Omega = \varnothing$ $(n = 0, 1, 2, \cdots)$, $\ker(\lambda - A(\Omega))^* = 0$ $(\lambda \in \Omega_1)$. Set

$$x(\lambda) = f(\lambda) \oplus \left[-\frac{\epsilon(f(\lambda), f(\mu_1))}{2||f(\mu_1)||} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} a_1^1 \cdots a_{k-1}^1}{(\mu_1 - \lambda)^k} e_k^1 \right] \oplus \cdots$$
$$\oplus \left[-\frac{\epsilon(f(\lambda), f(\mu_n))}{2^n||f(\mu_n)||} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} a_1^n \cdots a_{k-1}^n}{(\mu_n - \lambda)^k} e_k^n \right] \oplus \cdots .$$

Then computation shows that $x(\lambda) \in \ker(A(\Omega) - \lambda)$ $(\lambda \in \Omega_1)$.

Set $\bigvee\{x(\lambda) : \lambda \in \Omega\} = \mathcal{H}'$ and assume that $\bigoplus_{n=0}^{l} y_n \in \mathcal{H}'^{\perp}$, where $y_n \in \mathcal{H}_n$ and $y_n = \sum_{k=1}^{\infty} \bar{b}_k^n e_k^n$ $(n = 0, 1, 2, \cdots)$; then

$$(f(\lambda), y_0) = \frac{\epsilon(f(\lambda), f(\mu_1))}{2||f(\mu_1)||} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} a^1 \cdots a_{k-1}^1 b_k^1}{(\mu_1 - \lambda)^k} + \cdots$$
$$+ \frac{\epsilon(f(\lambda), f(\mu_n))}{2^n||f(\mu_n)||} \sum_{k=1}^{\infty} \frac{a_1^n \cdots a_{k-1}^n b_k^n}{(\mu_n - \lambda)^k} + \cdots .$$

Since $(f(\lambda), y_0)$ is analytic in Ω_1 (including μ_k $(k = 1, 2, \cdots)$) and since $f(\mu_k) \neq 0$, $b_k^n = 0$ $(k = 1, 2, \cdots;\ n = 1, 2, \cdots)$. Therefore $\bigvee\{x(\lambda) : \lambda \in \Omega\} = \mathcal{H}$ and $A(\Omega) \in \mathcal{B}_1(\Omega)$.

If $\lambda \notin \bar{\Omega}$, then $\lambda \in \rho(N)$. Since $\lambda \notin \sigma_p(A(\Omega)^*)$, $\lambda \in \rho(A(\Omega))$ and $\sigma(A(\Omega)) = \overline{\Omega}$. Since $\sigma_p(S_0) \cap \partial\Omega_1 = \varnothing$ and since $\partial\Omega_1 \subset \rho(S_n)$ $(n = 1, 2, \cdots)$, $\sigma_p(A(\Omega)) \cap \partial\Omega_1 = \varnothing$. □

Given a compact subset σ of C, consider the Sobolev space

$$W^{22}(\mathcal{D}) = \left\{ f \in L^2(\mathcal{D}, dA) : \begin{matrix} \text{the distributional partial derivatives of first} \\ \text{and second order of } f \text{ belong to } L^2(\mathcal{D}, dA) \end{matrix} \right\},$$

where $\mathcal{D} = (a, b)^2 \supset \sigma$ and dA denotes the planar Lebesgue measure. It is well known that $W^{22}(\mathcal{D})$ is a Hilbert space of continuous functions (by Sobolev's embedding theorem) under the norm

$$||f||_{W^{22}(\mathcal{D})} = \left(\int_{\mathcal{D}} \sum_{|\alpha| \leq 2} |D^\alpha f|^2 dA \right)^{1/2},$$

$W^{22}(D)$ is a regular Banach algebra with identity under pointwise multiplication and an equivalent norm whose maximal ideal space can be naturally identified with $\bar{\mathcal{D}}$ via "point evaluations".

Consider the subalgebra $W_0^{22}(\mathcal{D})$ of $W^{22}(\mathcal{D})$

$$W_0^{22}(\mathcal{D}) = \{f \in W^{22}(\mathcal{D}),\ f|_{\partial\mathcal{D}} \equiv 0\}.$$

For each $f \in W^{22}(\mathcal{D})$, the operator $M_f^0 \in \mathcal{L}(W_0^{22}(\mathcal{D}))$ is given by

$$M_f^0 g(\lambda) = f(\lambda) g(\lambda),\ g \in W_0^{22}(\mathcal{D}).$$

Set $m_0(\sigma) = \{f \in W_0^{22}(\mathcal{D}) : f|_\sigma \equiv 0\}$. Then it is clear that $m_0(\sigma)$ leaves invariant under M_λ^0, i.e.

$$M_\lambda^0 = \begin{pmatrix} M_\lambda^0|_{m_0(\sigma)} & * \\ 0 & M_\lambda^0(\sigma) \end{pmatrix}.$$

with respect to the decomposition $W_0^{22}(\mathcal{D}) = m_0(\sigma) \oplus [W_0^{22}(\mathcal{D}) \ominus m_0(\sigma)]$, where M_λ^0 is the operator "multiplication by λ" and $M_\lambda^0(\sigma)$ is the compression of M_λ^0 to $W_0^{22}(\mathcal{D}) \ominus m_0(\sigma)$.

Lemma 2.9 ([20, Propositions 3.1, 3.2])**.** (i) $\sigma(M_\lambda^0(\sigma)) = \sigma_e(M_\lambda^0(\sigma)) = \sigma$*;*

(ii) $\mathrm{nul}\,(\lambda - M_\lambda^0(\sigma)) = 0$ *and* $\mathrm{nul}\,(\lambda - M_\lambda^0(\sigma))^* = 1$ $(\lambda \in \sigma)$*;*

(iii) *For each dense subset* $\{\lambda_i\}_{i=1}^\infty$ *of* σ, $\bigvee_{i=1}^\infty \ker(\lambda_i - M_\lambda^0(\sigma))^* = W_0^{22}(\mathcal{D}) \ominus m_0(\sigma)$*;*

(iv) $M_\lambda^0(\sigma) \simeq$ *normal+compact;*

(v) *If* σ *is connected, then* $M_\lambda^0(\sigma) \in$ *(SI).*

Denote $W_0(\sigma) = \{M_f^0(\sigma) : f \in W^{22}(\mathcal{D})\}$; then it is easy to see that $W_0(\sigma)$ is a strictly cyclic operator algebra, where $M_f^0(\sigma)$ is the compression of M_f^0, the multiplication by function f, to the subspace $W_0^{22}(\mathcal{D}) \ominus m_0(\sigma)$. It was proved in [20] that $\mathcal{A}'(M_\lambda^0(\sigma)) = W_0(\sigma)$ and there is a vector $f_\mu \in \ker(M_\lambda^0(\sigma) - \mu)^*$ such that $(f, f_\mu) = f(\mu)$ for each $f \in W_0^{22}(\mathcal{D}) \ominus m_0(\sigma)$ and $\mu \in \sigma$. Thus, if P is an idempotent and $PM_\lambda^0(\sigma)^* = M_\lambda^0(\sigma)^* P$, then $Pf_\mu \in \ker(M_\lambda^0(\sigma) - \mu)^*$. Since $\dim\ker(M_\lambda^0(\sigma) - \mu)^* = 1$, $Pf_\mu = a(\mu) f_\mu$ for some number $a(\mu)$. Since $P^2 = P$, $a(\mu) = 0$ or 1 $(\mu \in \sigma)$.

Lemma 2.10. *Let* Ω *be a bounded domain, and let* A *be an operator on* $\mathcal{H}$ *in* $\mathcal{B}_1(\Omega)$ *such that* $\sigma(A) = \overline{\Omega}$ *and* $\sigma_p(A) \cap \partial\overline{\Omega} = \varnothing$*. Also let* σ *be a compact subset of the plane such that the components* $\{\sigma_n\}_{n\in\Lambda}$ *of* σ *each have more than one point. Let* σ' *denote the union of those components of* σ *which intersect* $\partial\bar{\Omega}$*. Let* $\mathcal{D}$ *be an open square containing* σ^**, and set* $B = M_\lambda^0(\sigma^*)^*$ *acting on* $\mathcal{H}_1 = W^{22}(\mathcal{D}) \ominus m_0(\sigma^*)$*. Then given* $\epsilon > 0$*, there is a compact operator* $D \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ *with* $||D|| < \epsilon$ *such that* $D \notin \mathrm{ran}\,\tau_{AB}$ *and having the property: if there are an idempotent* P *in* $\mathcal{A}'(B)$ *and an operator* $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$ *such that* $AXP - XPB = DP$*, then* $a(\lambda)|_{\sigma'^*} = 0$ *and* $Pf_\mu = a(\mu) f_\mu$ *for* $\mu \in \sigma^*$*.*

Similarly, there is a compact operator $D' \in \mathcal{L}(\mathcal{H}, \mathcal{H}_1)$ *with* $||D'|| < \epsilon$ *such that* $D' \notin \mathrm{ran}\tau_{BA}$ *and having the property: if there are an idempotent* P *in* $\mathcal{A}'(B)$ *and an operator* $X \in \mathcal{L}(\mathcal{H}, \mathcal{H}_1)$ *such that* $BPX - PXA = PD$*, then* $a(\lambda)|_{\sigma'^*} = 0$ *and* $Pf_\mu = a(\mu) f_\mu$ *for* $\mu \in \sigma^*$*.*

Proof. Choose two sets $\{\lambda_i^1\}_{i=1}^\infty$ and $\{\mu_i^1\}_{i=1}^\infty$ of distinct complex numbers satisfying

(i) $\{\lambda_i^1\}_{i=1}^\infty \subset \Omega$ and $\{\mu_i^1\}_{i=1}^\infty \subset \sigma'$;

(ii) $\mathrm{card}\{i,\ \mu_i^1 \in \sigma_n\} = \infty$ for each σ_n, $\sigma_n \cap \partial\bar{\Omega} \neq \varnothing$;

(iii) $|\mu_i^1 - \lambda_i^1| < \frac{1}{2^i}$ $(i = 1, 2, \cdots)$.

Choose dense subsets $\{\lambda_k\}_{k=1}^\infty$ of Ω and $\{\mu_k\}_{k=1}^\infty$ of σ such that $\lambda_i^1 = \lambda_{2i}$, $\mu_i^1 = \mu_{2i}$ $(i = 1, 2, \cdots)$ and $\lambda_i \neq \lambda_j$, $\mu_i \neq \mu_j$ $(i \neq j)$. Since $\bigvee\{\ker(\lambda_i - A) : i = 1, 2, \cdots\} = \mathcal{H}$ and $\bigvee\{\ker(\mu_i - B) : i = 1, 2, \cdots\} = \mathcal{H}_1$, it is completely apparent that A and B have upper triangular matrices with respect to ONB $\{e_n\}_{n=1}^\infty$ of $\mathcal{H}$ and respectively, $\{f_n\}_{n=1}^\infty$ of $\mathcal{H}_1$ obtained by Gram-Schmidt orthnormalization of unit vectors in $\bigcup_{i=1}^\infty \{\ker(\lambda_i - A)\}$ and $\bigcup_{i=1}^\infty \{\ker(\mu_i - B)\}$ respectively

$$A = \begin{pmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ 0 & & \ddots \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \end{matrix},$$

$$B = \begin{pmatrix} \mu_1 & & * \\ & \mu_2 & \\ 0 & & \ddots \end{pmatrix} \begin{matrix} f_1 \\ f_2 \\ \vdots \end{matrix}$$

and

$$UBU^* = B' = \begin{pmatrix} \mu_1 & & * \\ & \mu_2 & \\ 0 & & \ddots \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \end{matrix}$$

where U is an unitary, $Uf_i = e_i$ $(i = 1, 2, \cdots)$.

Define $D'e_k = a_k e_k$ where

$$a_k = \begin{cases} \dfrac{\epsilon}{2i}, & \text{if } k = 2i, \\ 0, & \text{otherwise;} \end{cases}$$

then D' is compact with $||D'|| < \epsilon$.

If $AX - XB' = D'$ for some X, then calculations show that X admits an upper triangular matrix with respect to $\{e_n\}_{n=1}^{\infty}$. Thus $|x_{2i,2i}| = \frac{a_{2i}}{|\lambda_{2i} - \mu_{2i}|} \geq \frac{2^{i-1}\epsilon}{i} \to \infty$ $(i \to \infty)$, where $x_{2i,2i} = (Xe_{2i}, e_{2i})$ $(i = 1, 2, \cdots)$.

Therefore $D' \notin \operatorname{ran} \tau_{AB'}$ and $D = D'U \notin \operatorname{ran} \tau_{AB}$. Suppose that $P \in \mathcal{A}'(B)$ is an idempotent and $Pf_\mu = a(\mu) f_\mu$ $(\mu \in \sigma^*)$. If the function $a(\mu)$ takes two values 1 and 0 on a σ_n^*, then there is a sequence $\{h_k\}_{k=0}^{\infty} \subset \sigma_n$ such that $h_k \to h_0$ $(k \to \infty)$, $a(\bar{h}_k) = 0$ $(k \geq 1)$ and $a(\bar{h}_0) = 1$ (if $a(\bar{h}_k) = 1$ $(k \geq 1)$ and $a(\bar{h}_0) = 0$, consider the idempotent $I_1 - P$, where I_1 is the identity on $\mathcal{H}_1$).

Set $\mathcal{M} = \bigvee\{f_{\bar{\mu}} : a(\bar{\mu}) = 1, \ \mu \in \sigma\}$, $\mathcal{N} = \bigvee\{f_{\bar{\mu}} : a(\bar{\mu}) = 0, \ \mu \in \sigma\}$; then $f_{\bar{h}_k} \in \mathcal{M}$ $(k \geq 1)$ and $f_{\bar{h}_0} \in \mathcal{N}$. Since $\mathcal{M} \subset \operatorname{ran} P$ and $\mathcal{N} \subset \ker P$, $\mathcal{M} \cap \mathcal{N} = \{0\}$.

Since $f_{\bar{h}_k} \to f_{\bar{h}_0}$ weakly, $f_{\bar{h}_0} \in \mathcal{M} \cap \mathcal{N}$. This contradiction implies that $a(\mu)$ takes the same value (1 or 0) on each σ_n^*. Similar arguments show that $a(\mu)$ is continuous on σ_n^*.

Assume that P satisfies $AXP - XPB = DP$ for some $X \in \mathcal{L}(\mathcal{H}_1, \mathcal{H})$. Note that P has an upper triangular matrix representation

$$P = \begin{pmatrix} p_1 & & * \\ & p_2 & \\ 0 & & \ddots \end{pmatrix}$$

with respect to the ONB $\{f_i\}_{i=1}^{\infty}$. Thus XP admits an upper triangular matrix form

$$XP = \begin{pmatrix} y_1 & & * \\ & y_2 & \\ 0 & & \ddots \end{pmatrix}$$

with respect to the ONB's $\{e_i\}_{i=1}^{\infty}$ and $\{f_i\}_{i=1}^{\infty}$.

Suppose that $\sigma_n \cap \partial\overline{\Omega} \neq 0$ and $a(\bar{\mu}) = 1$ $(\mu \in \sigma_n)$. Also, suppose that $\mu_{i_j}^1 \in \sigma_n$ $(j = 1, 2, \cdots)$. Thus $AXP - XPB = DP$ indicates that $\lambda_{2i_j} y_{2i_j} - y_{2i_j} \mu_{2i_j} = a_{2i_j} p_{2i_j}$.

Since $a(\bar{\mu}_{2i_j}) = 1$, $p_{2i_j} = 1$. Thus $y_{2i_j} = \frac{2^{i_j - 1}\epsilon}{i_j} \to \infty$ $(j \to \infty)$. This contradiction implies that $a(\mu)|_{\sigma'^*} = 0$. Same arguments work in the case that P satisfies $BPX - PXA = PD$ for some $X \in \mathcal{L}(\mathcal{H}, \mathcal{H}_1)$. $\square$

3. The proof of the Main Theorem

Let T be an essentially normal operator with connected spectrum. The spectral picture of T is determined by the sets

$$\rho_F^0(T)\cap\sigma(T):=\{\lambda\in\sigma(T)\,:\,\mathrm{ind}\,(T-\lambda I)=0\}=\bigcup_{i\in\Lambda_1}\Omega_{1i},$$
$$\rho_F^+(T):=\{\lambda\in\sigma(T)\,:\,\mathrm{ind}\,(T-\lambda I)>0\}=\bigcup_{i\in\Lambda_2}\Omega_{2i},$$
$$\rho_F^-(T):=\{\lambda\in\sigma(T)\,:\,\mathrm{ind}\,(T-\lambda I)<0\}=\bigcup_{i\in\Lambda_3}\Omega_{3i},$$
$$\sigma=\sigma(T)\setminus\mathrm{int}\,[\rho_F(T)^-],$$

where $\Omega_{1i},\ \Omega_{2i},\ \Omega_{3i}$ are connected components.

Choose $\bigcup_{l\in\Lambda_4}\sigma_l\subset\sigma$, where σ_l is a connected compact set consisting of more than one point, $\sigma_l\cap\sigma_{l'}=\varnothing$ $(l\neq l')$ and for each l, there exists at least one connected component Ω of $\rho_F(T)\cap\sigma(T)$ such that $\sigma_l\cap\partial\Omega\neq\varnothing$ and $\sigma=\overline{\bigcup_{l\in\Lambda_4}\sigma_l}$.

Construct operators $A(\Omega_{1i})\in\mathcal{B}_1(\Omega_{1i})$ and $A(\Omega_{1i}^*)\in\mathcal{B}_1(\Omega_{1i}^*)$ (Lemma 2.8) with respect to Ω_{1i} and, respectively, Ω_{1i}^* $(i\in\Lambda_1)$ and let the norm of the compact operator K (in Lemma 2.8) be so small that $\bigoplus_{i\in\Lambda_1}A(\Omega_{1i})$ and $\bigoplus_{i\in\Lambda_1}A(\Omega_{1i}^*)$ are sums of an essentially normal and co-subnormal operator and a compact operator. Set $C(\Omega_{1i})=A(\Omega_{1i}^*)^*$ $(i\in\Lambda_1)$.

For each domain Ω_{2j}, according to Lemma 2.5 and Lemma 2.8 construct operator $A(\Omega_{2j})\in B_1(\Omega_{2j})$ and compact operator $Q(\Omega_{2j})$. Let

$$n_j=\mathrm{ind}\,(T-\lambda)\text{ for }\lambda\in\Omega_{2j}.$$

Using Lemma 2.8 construct operator

$$\bar{A}(\Omega_{2j})\in B_{n_j-1}(\Omega_{2j})$$

if $n_j>1$ such that $Q(\Omega_{2j})\notin\tau_{\bar{A}(\Omega_{2j})A(\Omega_{2j})}$ and $\ker\tau_{A(\Omega_{2j})\bar{A}(\Omega_{2j})}=\{0\}$ and the norm of $Q(\Omega_{2j})$ is so small that $\bigoplus Q(\Omega_{2j})$ is compact. Let the norm of the compact operator K in Lemma 2.8 and Lemma 2.5 be so small that $\bigoplus_{j\in\Lambda_2}A(\Omega_{2j})$ and $\bigoplus\bar{A}(\Omega_{2j})$ are sums of an essentially normal co-subnormal operator and a compact operator.

For each domain Ω_{3k}, according to Lemma 2.8 construct operators $A(\Omega_{3k})\in\mathcal{B}_1(\Omega_{3k})$ and $A(\Omega_{3k}^*)\in\mathcal{B}_1(\Omega_{3k}^*)$. Let $n_k=$ind$(T-\lambda)$ for $\lambda\in\Omega_{3k}$. Using Lemma 2.5 and Lemma 2.8, construct operators $\bar{A}(\Omega_{3k}^*)\in\mathcal{B}_{|n_k|}(\Omega_{3k}^*)$ and $Q(\Omega_{3k}^*)$ compact if $|n_k|>1$ such that

$$Q(\Omega_{3k}^*)\notin\mathrm{ran}\tau_{\bar{A}(\Omega_{3k}^*)A(\Omega_{3k}^*)},\qquad\ker\tau_{A(\Omega_{3k}^*)\bar{A}(\Omega_{3k}^*)}=\{0\}$$

and the norm of $Q(\Omega_{3k}^*)$ is so small that $\bigoplus Q(\Omega_{3k}^*)$ is compact. Let the norm of the compact operator K in Lemma 2.5 and Lemma 2.8 is so small that $\bigoplus_{k\in\Lambda_3}A(\Omega_{3k})$, $\bigoplus_{k\in\Lambda_3}A(\Omega_{3k}^*)$ and $\bigoplus\bar{A}(\Omega_{3k}^*)$ are sums of an essentially normal co-subnormal operator and a compact operator.

Set $C(\Omega_{3k})=A(\Omega_{3k}^*)^*$ and $\bar{C}(\Omega_{3k})=\bar{A}(\Omega_{3k}^*)^*$.

Construct $B=[M_\lambda^0(\sigma^*)]^*$ on Hilbert space $W_0^{22}(\mathcal{D})\ominus m_0(\sigma^*)$ where $\mathcal{D}=(a,b)^2\supset\sigma^*$.

The first step:

Rearrange

$$\{A(\Omega_{2j})\,:\,j\in\Lambda_2\}=\{A_k\}_{k=1}^{l_1}\qquad(0\le l_1\le\infty)$$

and

$$\{A(\Omega_{1j})\,:\,j\in\Lambda_1,\ A(\Omega_{3k});\ k\in\Lambda_3\}=\{A_k^a\}_{k=1}^{l_2}\qquad(0\le l_2\le\infty).$$

Thus $\bigoplus_{k=1}^{l_1}A_k$ and $\bigoplus_{k=1}^{l_2}A_k^a$ are essentially normal and co-subnormal + compact type. Therefore,

$$\sigma\left[\bigoplus_{k=1}^{l_1}A_k\right]=\left[\bigcup_{k=1}^{l_1}\sigma(A_k)\right]^-,\qquad\sigma\left[\bigoplus_{k=1}^{l_2}A_k^a\right]=\left[\bigcup_{k=1}^{l_2}\sigma(A_k^a)\right]^-.$$

Define

$$G=\begin{pmatrix}\bigoplus A_k & D_1 & 0\\ 0 & B & D_2\\ 0 & 0 & \bigoplus A_k^a\end{pmatrix}\qquad\text{on }\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3,$$

where $\mathcal{H}_1=\bigoplus_{k=1}^{l_1}\mathcal{H}_k^1$, $\mathcal{H}_2=W_0^{22}(D)\ominus m_0(\sigma^*)$, $\mathcal{H}_3=\bigoplus_{k=1}^{l_2}\mathcal{H}_k^3$, and for each k, $A_k\in\mathcal{L}(\mathcal{H}_k^1)$, $A_k^a\in\mathcal{L}(\mathcal{H}_k^3)$. Assume that

$$D_1=\begin{pmatrix}D_1^1\\ D_2^1\\ \vdots\end{pmatrix}\qquad\text{and}\qquad D_2=(D_1^2,D_2^2,\cdots).$$

Choose $D_k^1\notin\operatorname{ran}\tau_{A_kB}$ and $D_k^2\notin\operatorname{ran}\tau_{BA_k^a}$, D_k^1, D_k^2 are compact and the norms of them are so small that D_1, D_2 are compact (Lemma 2.10).

Suppose that $P\in\mathcal{A}'(G)$ is an idempotent and

$$P=\begin{pmatrix}P_{11} & P_{12} & P_{13}\\ P_{21} & P_{22} & P_{23}\\ P_{31} & P_{32} & P_{33}\end{pmatrix}\begin{matrix}\mathcal{H}_1\\ \mathcal{H}_2\\ \mathcal{H}_3\end{matrix}$$

Then $P_{31}=0$, $P_{21}=0$ and $P_{32}=0$ by Lemma 2.1. Since $A_k\in$ (SI) and $A_k^a\in$ (SI) $(k=1,2,\cdots)$, from Lemma 2.1, $P_{11}=\bigoplus_{i=1}^{l_1}\delta_i^1I_i^1$ and $P_{33}=\bigoplus_{i=1}^{l_2}\delta_i^3I_i^3$, where $\delta_i^1(\delta_i^3)$ is 0 or 1, and $I_i^1(I_i^3)$ is the identity operator on $\mathcal{H}_i^1(\mathcal{H}_i^3)$, $i=1,2,\cdots$. Since $P_{22}\in\mathcal{A}'(B)$ is an idempotent, $P_{22}f_\mu=a(\bar\mu)f_\mu(\mu\in\sigma)$ and $a(\mu)$ takes constant (1 or 0) on each σ_l (Lemma 2.10). Set $\Delta_1=\{\sigma_l\,:\,a(\mu)|_{\sigma_l^*}\equiv0\}$, $\Delta_2=\{\sigma_l\,:\,a(\mu)|_{\sigma_l^*}\equiv1\}$ and denote $\Gamma_1=\bigcup_{\Delta_1}\sigma_l$, $\Gamma_2=\bigcup_{\Delta_2}\sigma_l$. Thus $\Gamma_1\cup\Gamma_2=\sigma$ and $\Gamma_1\cap\Gamma_2=\varnothing$. Rearrange $\{\Omega_{1i}\ (i\in\Lambda_1),\ \Omega_{2i}\ (i\in\Lambda_2),\ \Omega_{3i}\ (i\in\Lambda_3)\}=\{\Omega_k\}_{k=1}^l$, set $\Delta_3=\{\overline{\Omega}\,:\,a(\bar\mu)\equiv0,\ \mu\in\partial\overline{\Omega}_k\}$, $\Delta_4=\{\overline{\Omega}_k\,:\,a(\bar\mu)\equiv1,\ \mu\in\partial\overline{\Omega}_k\}$. Denote $\Sigma_1=\Gamma_1\cup[\bigcup_{\Delta_3}\overline{\Omega}_k]$ and $\Sigma_2=\Gamma_2\cup[\bigcup_{\Delta_4}\overline{\Omega}_k]$. Thus $\sigma(T)=\overline{\Sigma}_1\cup\overline{\Sigma}_2$. If $\mu_0\in\overline{\Sigma}_1\cup\overline{\Sigma}_2$, $\mu_0\in\sigma$, since μ_0 is not in any $(\overline{\Omega}_k)^0$. Suppose that $\mu_0\in\Gamma_1$. Since $\mu_0\in\overline{\Sigma}_2$, it is always possible to find a sequence $\{\mu_k\}_{k=1}^\infty\subset\Gamma_2$ such that $\mu_k\to\mu_0$. Since $a(\bar\mu)$ is continuous, $a(\bar\mu_0)=1$. Similarly, if $\mu_0\in\Gamma_2$, we get $a(\bar\mu_0)=0$. The contradiction implies that $\overline{\Sigma}_1\cap\overline{\Sigma}_2=\varnothing$. Since $\sigma(T)=\sigma(G)$ is connected, one of $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ must be empty. Thus one of Γ_1 and Γ_2 must be empty and $a(\bar\mu)\equiv0$ or 1 $(\mu\in\sigma)$, i.e. $P_{22}=\delta_2I_2$, $\delta_2=0$ or 1, where I_2 is the identity operator on $\mathcal{H}_2$.

Assume that

$$P_{12} = \begin{pmatrix} P_1^1 \\ P_2^1 \\ \vdots \end{pmatrix}, \qquad P_{23} = (P_1^2, P_2^2, \cdots).$$

Then

$$\big(\bigoplus \delta_k^1 I_k^1\big) D_1 + P_{12} B = \big(\bigoplus A_k\big) P_{12} + D_1 P_{22}$$

and

$$P_{22} D_2 + P_{23} \big(\bigoplus A_k^a\big) = B P_{23} + D_2 \big(\bigoplus \delta_k^3 I_k^3\big)$$

imply

$$A_k P_k^1 - P_k^1 B = D_k^1 (\delta_k^1 - \delta_2) I_2 \ (k = 1, 2, \cdots, l_1)$$

and

$$B P_k^2 - P_k^2 A_k^2 = (\delta_2 - \delta_k^3) I_2 D_k^2 \ (k = 1, 2, \cdots, l_2).$$

Using Lemma 2.10 with respect to the idempotents $(\delta_k^1 - \delta_2)I_2$ $(k = 1, 2, \cdots, l_1)$ and $(\delta_2 - \delta_k^3)I_2$ $(k = 1, 2, \cdots, l_2)$, we know that $\delta_k^1 = \delta_k^2 = \delta_2$ for all k. i.e. $P = 0$ or I, where I is the identity operator on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. Thus $G \in$ (SI).

The second step: Recovery of the positive indices.

Rearrange and reexpress $\{\bar{A}(\Omega_{2j})\} = \{\bar{A}_k\}$. Then $\sigma\big(\bigoplus \bar{A}_k\big) = \big[\bigcup \sigma(\bar{A}_k)\big]^-$.

Define

$$M = \begin{pmatrix} \bigoplus \bar{A}_k & D_3 & & 0 \\ & \bigoplus A_k & D_1 & \\ & & B & D_2 \\ 0 & & & \bigoplus A_k^a \end{pmatrix} \text{ on } \mathcal{H}_4 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3,$$

where $\mathcal{H}_4 = \oplus \mathcal{H}_k^4$, $\bar{A}_k \in \mathcal{L}(\bar{\mathcal{H}}_k)$ $(k = 1, 2, \cdots)$, and

$$D_3 = \begin{pmatrix} D_{11}^3 & D_{12}^3 & \cdots \\ D_{21}^3 & D_{22}^3 & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix},$$

$$D_{ki}^3 = \begin{cases} Q(\Omega_{2j}), & \text{if } \sigma(\bar{A}_k) = \sigma(A_i) = \overline{\Omega}_{2j}; \\ 0, & \text{otherwise.} \end{cases}$$

Thus D_3 is compact.

Suppose that $E \in \mathcal{A}'(M)$ is an idempotent and

$$E = \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ E_{21} & E_{22} & E_{23} & E_{24} \\ E_{31} & E_{32} & E_{33} & E_{34} \\ E_{41} & E_{42} & E_{43} & E_{44} \end{pmatrix} \begin{matrix} \mathcal{H}_4 \\ \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \end{matrix} = \begin{pmatrix} E_{11} & E_{12}' \\ E_{21}' & E_{22}' \end{pmatrix} \begin{matrix} \mathcal{H}_4 \\ \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \end{matrix}.$$

Then $E_{41} = 0$ and $E_{31} = 0$, since Lemma 2.1. From Lemma 2.1 and Lemma 2.5, $E_{21} = 0$. Since $\sigma(\bar{A}_k)^0 \cap \sigma(\bar{A}_j)^0 = \varnothing$ $(k \neq j)$ and since $\bar{A}_k \in$ (SI), $E_{11} = \bigoplus \delta_k^4 I_k^4$, $\delta_k^4 = 0$ or 1, where I_k^4 is the identity operator on $\mathcal{H}_k^4$ $(k = 1, 2, \cdots)$. Since $G \in$ (SI), $E_{22}' = 0$ or $I_1 \oplus I_2 \oplus I_3$. Assume that $E_{22}' = 0$ (if $E_{22}' = I_1 \oplus I_2 \oplus I_3$, consider

$I_4 \oplus I_1 \oplus I_2 \oplus I_3 - E)$. Thus we have $E_{11}D_3 + E_{12}(\bigoplus A_k) = (\bigoplus \bar{A}_k)E_{12}$. Suppose that

$$E_{12} = \begin{pmatrix} L_{11} & L_{12} & \dots \\ L_{21} & L_{22} & \dots \\ \dots & \dots & \dots \end{pmatrix},$$

For each k, find i such that $\sigma(\bar{A}_k) = \sigma(A_i)$; then $\bar{A}_k L_{ki} - L_{ki}A_i = \delta_k^4 D_{ki}^4$ implies $\delta_k^4 = 0$. Therefore $E = 0$ and $M \in$ (SI).

The third step. The recovery of the zero indices.

Take $\lambda_{1k} \in \partial\overline{\Omega}_{1k}$ $(k = 1, 2, \cdots, l_1)$, $\lambda_{3k} \in \partial\Omega_{3k}$ $(k = 1, 2, \cdots, l_2)$. Rearrange

$$\{\lambda_{1k}\ (k = 1, 2, \cdots, l_1),\ \lambda_{3k}\ (k = 1, 2, \cdots, l_2)\} = \{\lambda_k\}_{k=1}^{l_1+l_2}.$$

Define $B_k \in \mathcal{L}(\mathcal{H}_k)$ by $B_k e_n = \frac{1}{nk}e_{n-1}$ $(n = 2, 3, \cdots)$, $B_k e_1 = 0$, where $\{e_n\}_{n=1}^{\infty}$ is an ONB of Hilbert space $R_k = \mathcal{H}$ $(k = 1, 2, \cdots, l_1 + l_2)$. Set $\bar{B} = \bigoplus B_k \in \mathcal{L}(\mathcal{H}_5)$, where $\mathcal{H}_5 = \bigoplus R_k$.

Define

$$L = \begin{pmatrix} \bigoplus \bar{A}_k & D_3 & & & 0 \\ & \bigoplus A_k & D_1 & & \\ & & B & D_2 & \\ & & & \bigoplus A_k^a & D_4 \\ 0 & & & & \bar{B} \end{pmatrix} \begin{matrix} \mathcal{H}_4 \\ \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_5 \end{matrix}$$

where

$$D_4 = \begin{pmatrix} D_{11}^4 & D_{12}^4 & \dots \\ D_{21}^4 & D_{22}^4 & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

and $D_{kj}^4 \notin \operatorname{ran} \tau_{A_k^a B_j}$, D_{kj}^4 is compact if $\sigma(A_k^a) \cap \sigma(B_j) \neq \varnothing$ (Lemma 2.6), otherwise $D_{kj}^4 = 0$. Choose $||D_{kj}^4||$ so small that D_4 is compact. By the same arguments used above, $L \in$ (SI).

Set $B'_k = \lambda_k + B_k \in \mathcal{L}(R_k)$ and $B' = \bigoplus B'_k \in \mathcal{L}(\mathcal{H}_6)$, where $\mathcal{H}_6 = \bigoplus R_k$.

Define

$$Q = \begin{pmatrix} \bigoplus \bar{A}_k & D_3 & & & & 0 \\ & \bigoplus A_k & D_1 & & & \\ & & B & D_2 & & \\ & & & \bigoplus A_k^a & D_4 & \\ & & & & \bar{B} & D_5 \\ 0 & & & & & B' \end{pmatrix} \begin{matrix} \mathcal{H}_4 \\ \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_5 \\ \mathcal{H}_6 \end{matrix}$$

where

$$D_5 = \begin{pmatrix} D_1^5 & & 0 \\ & D_2^5 & \\ 0 & & \ddots \end{pmatrix}$$

and $D_k^5 \notin \operatorname{ran} \tau_{B_k B'_k}$, D_k^5 is compact $(k = 1, 2, \cdots)$ (Lemma 2.7) and choose $||D_k^5||$ so small that D_5 is compact. Then $Q \in$ (SI).

Rearrange $\{C(\Omega_{1i})\ (i \in \Lambda_1),\ C(\Omega_{3k})\ (k \in \Lambda_3)\} = \{C_k\}_{k=1}^{l}$.

Set $C = \bigoplus C_k$ on $\mathcal{H}_7$, where $H_7 = \bigoplus_{k=1}^{l} \mathcal{M}_k$, $C_k \in \mathcal{L}(\mathcal{M}_k)$; then $\sigma(C) = [\bigcup \sigma(C_k)]^{-}$.

Define

$$R = \begin{pmatrix} \bigoplus \bar{A}_k & D_3 & & & & & 0 \\ & \bigoplus A_k & D_1 & & & & \\ & & B & D_2 & & & \\ & & & \bigoplus A_k^a & D_4 & & \\ & & & & \bar{B} & D_5 & \\ & & & & & B' & D_6 \\ 0 & & & & & & C \end{pmatrix} \begin{matrix} \mathcal{H}_4 \\ \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_5 \\ \mathcal{H}_6 \\ \mathcal{H}_7 \end{matrix}$$

where

$$D_6 = \begin{pmatrix} D_1^6 & & 0 \\ & D_2^6 & \\ 0 & & \ddots \end{pmatrix}$$

and $D_k^{6*} \notin \operatorname{ran} \tau_{C_k^* B_K'^*}$, D_k^6 is compact $(k = 1, 2, \cdots)$ (Lemma 2.7) and choose $||D_k^6||$ so small that D_6 is compact. Then $R \in$ (SI).

The fourth step. The recovery of the negative indices.

Assume that $\{\bar{C}(\Omega_{3k})\} = \{\bar{C}_k\}$ and set $\bar{C} = \bigoplus \bar{C}_k$ on $\mathcal{H}_8 = \bigoplus \mathcal{N}_k$ where $\bar{C}_k \in \mathcal{L}(\mathcal{N}_k)$ $(k = 1, 2, \cdots)$; then $\sigma(\bar{C}) = [\cup \sigma(\bar{C}_k)]^-$.

Define

$$S = \begin{pmatrix} \bigoplus \bar{A}_k & D_3 & & & & & & 0 \\ & \bigoplus A_k & D_1 & & & & & \\ & & B & D_2 & & & & \\ & & & \bigoplus A_k^a & D_4 & & & \\ & & & & \bar{B} & D_5 & & \\ & & & & & B' & D_6 & \\ & & & & & & C & D_7 \\ 0 & & & & & & & \bar{C} \end{pmatrix} \begin{matrix} \mathcal{H}_4 \\ \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_5 \\ \mathcal{H}_6 \\ \mathcal{H}_7 \\ \mathcal{H}_8 \end{matrix}$$

where

$$D_7 = \begin{pmatrix} D_{11}^7 & D_{12}^7 & \cdots \\ D_{21}^7 & D_{22}^7 & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

and $D_{ik}^7 \notin \operatorname{ran} \tau_{C_i, \bar{C}_k}$, D_{ik}^7 is compact if $\sigma(C_i) = \sigma(\bar{C}_k)$ (Lemma 2.5), otherwise $D_{ik}^7 = 0$. Choose $||D_{ik}^7||$ so small that D_7 is compact. Then $S \in$ (SI). It is a routine exercise to check that S =essentially normal + compact, $\sigma(T) = \sigma(S)$, $\sigma_e(T) = \sigma_e(S)$ and $\operatorname{ind}(\lambda - T) = \operatorname{ind}(\lambda - S)$ $(\lambda \in \rho_F(T))$. Thus by the Brown-Douglas-Fillmore Theorem [3], there is a compact operator K such that $T + K$ is unitarily equivalent to S and the proof of the Main Theorem is now complete. □

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