

## ON THE ORNSTEIN-UHLENBECK OPERATOR IN $L^2$ SPACES WITH RESPECT TO INVARIANT MEASURES

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ABSTRACT. We consider a class of elliptic and parabolic differential operators with unbounded coefficients in  $\mathbb{R}^n$ , and we study the properties of the realization of such operators in suitable weighted  $L^2$  spaces.

### 1. INTRODUCTION

There is a very wide literature on boundary value problems for linear elliptic and parabolic equations in bounded domains in  $\mathbb{R}^n$ . A big part of the results can be extended easily to unbounded domains, provided the coefficients of the differential operators are bounded.

A comprehensive approach to the case of unbounded coefficients in  $\mathbb{R}^n$  may be found in [1], [2] and [3]. Under appropriate hypotheses, they are able to work in suitably weighted spaces. A typical simple example which is not in general covered by their results is the Ornstein-Uhlenbeck operator

$$(1.1) \quad Au = \frac{1}{2} \sum_{i,j=1}^n q_{ij} D_{ij} u + \sum_{i,j=1}^n b_{ij} x_i D_j u = \frac{1}{2} \operatorname{Tr}(QD^2 u) + \langle Bx, Du \rangle, \quad x \in \mathbb{R}^n,$$

and the associated semigroup

$$(1.2) \quad \begin{cases} (T(t)\varphi)(x) &= \frac{1}{(2\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^n} e^{-\langle Q_t^{-1}y, y \rangle/2} \varphi(e^{tB}x - y) dy, \quad t > 0, \\ T(0)\varphi &= \varphi. \end{cases}$$

Here  $Q = [q_{ij}]_{i,j=1,\dots,n}$  is any symmetric positive definite matrix, and  $B$  is any nonzero matrix.  $Q_t$  is the matrix defined by

$$(1.3) \quad Q_t = \int_0^t e^{sB} Q e^{sB^*} ds, \quad t \geq 0,$$

where  $e^{sB^*}$  is the exponential of the transpose matrix  $B^*$ .

Besides their own mathematical interest, operators with unbounded coefficients arise in stochastic perturbations of ODE's. Consider for instance the linear equation

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in  $\mathbb{R}^n$

$$u'(t) = Bu(t),$$

perturbed by the noise  $\sqrt{Q} dW_t$ , where  $W_t$  is a standard  $n$  dimensional Brownian motion. Then the semigroup  $T(t)$  defined in (1.2) is the Markov semigroup associated to the stochastic differential equation

$$dX = BX dt + \sqrt{Q} dW_t,$$

since we have

$$(T(t)\varphi)(x) = \mathbb{E}[\varphi(X(t, x))]$$

for a large class of initial data  $\varphi$ . See e.g. [6, Ch. 5]. It would be obviously important (see again [6]) to study also stochastic perturbations of nonlinear systems

$$u'(t) = Bu(t) + F(u(t)),$$

which give nonlinear coefficients to the operator  $\mathcal{A}$ . Surprisingly, the literature deals essentially with the case where  $F$  is bounded. Our work can be considered as a first step in the study of the more general case of a Lipschitz continuous  $F$ . Moreover, at the end of the paper we consider also an example in which  $\mathcal{A}$  has variable nonlinear coefficients.

In the paper [4] we have described the properties of the realizations of  $\mathcal{A}$  and  $T(t)$  in spaces of continuous and bounded functions in  $\mathbb{R}^n$ . Here we study the realizations of  $\mathcal{A}$  and  $T(t)$  in a  $L^2$  space with respect to a suitable measure. Besides the usual Lebesgue measure, which will be considered in a forthcoming paper [9], an appropriate measure in the study of a dynamical system is its invariant measure, which exists and is unique under suitable assumptions, see [6]. We assume that all the eigenvalues of the matrix  $B$  have negative real part, so that there exist  $C > 0$ ,  $\omega > 0$  such that

$$(1.4) \quad \|e^{tB}\| \leq Ce^{-\omega t}, \quad t \in \mathbb{R}.$$

Therefore, the matrix

$$(1.5) \quad Q_\infty = \int_0^\infty e^{sB} Q e^{sB^*} ds$$

is well defined. We consider the Gaussian weight associated to the matrix  $Q_\infty$ ,

$$(1.6) \quad \mu(x) = \frac{1}{(2\pi)^{n/2}(\det Q_\infty)^{1/2}} e^{-\langle Q_\infty^{-1}x, x \rangle/2}, \quad x \in \mathbb{R}^n.$$

The weighted space  $L_\mu^2$  is defined by

$$(1.7) \quad L_\mu^2 = \left\{ f : \mathbb{R}^n \mapsto \mathbb{C} \text{ measurable} \mid \|f\|_{L_\mu^2} = \left( \int_{\mathbb{R}^n} |f(x)|^2 \mu(x) dx \right)^{1/2} < \infty \right\}.$$

Similarly, the Sobolev weighted space  $H_\mu^s$ ,  $s > 0$ , is the subspace of  $L_\mu^2$  consisting of all the functions  $f$  such that  $x \mapsto f(x)\mu(x)^{1/2}$  belongs to  $H^s(\mathbb{R}^n)$ . If  $s$  is integer, it coincides with the space of all the functions  $f \in H_{loc}^s(\mathbb{R}^n)$  such that  $D^\beta f$  belongs to  $L_\mu^2$  for every multi-index  $\beta$  such that  $0 \leq |\beta| \leq s$ .

We shall see that the semigroup defined by (1.2) is analytic, strongly continuous, positivity preserving, and it is a contraction semigroup in  $L_\mu^2$ . Moreover, the

measure  $\mu(x)dx$  is invariant for  $T(t)$ , in the sense that

$$(1.8) \quad \int_{\mathbb{R}^n} (T(t)\varphi)(x)\mu(x)dx = \int_{\mathbb{R}^n} \varphi(x)\mu(x)dx \quad \forall t \geq 0, \varphi \in L^2_\mu.$$

The main result of the paper is the characterization of the domain of the realization  $A$  of  $\mathcal{A}$  in  $L^2_\mu$ . We prove that  $D(A) = H^2_\mu$ . The embedding  $H^2_\mu \subset D(A)$  is easy, while proving that  $D(A) \subset H^2_\mu$  is more delicate. Indeed, due to the strong decay of the weight  $\mu(x)$  as  $|x| \rightarrow \infty$ , there are difficulties in treating differential operators in  $L^2_\mu$  by the usual methods. We use a technique similar to the one employed in [8] to get optimal Schauder type estimates: we show that for every  $\alpha \in (0, 1)$ ,  $D(A)$  is continuously embedded in the interpolation space

$$(H^\alpha_\mu, H^{2+\alpha}_\mu)_{1-\alpha/2, 2} = H^2_\mu.$$

This is done by using the representation formula for the resolvent  $R(\lambda, A)$ ,

$$R(\lambda, A)\varphi = \int_0^\infty e^{-\lambda t} T(t)\varphi dt, \quad \lambda > 0,$$

and optimal estimates for  $\|T(t)\|_{L(L^2_\mu, H^\alpha_\mu)}$ ,  $\|T(t)\|_{L(L^2_\mu, H^{2+\alpha}_\mu)}$ , which are obtained with the aid of the explicit representation formula (1.2) and interpolation arguments.

By a similar procedure it is possible to prove that for every  $\theta \in (0, 1)$  the domain of the realization of  $\mathcal{A}$  in  $H^\theta_\mu$  is  $H^{2+\theta}_\mu$ .

We consider also the case of matrices  $Q, B$  depending on  $x$ , with continuous coefficients, such that the limits  $\lim_{|x| \rightarrow \infty} Q(x) = Q$ ,  $\lim_{|x| \rightarrow \infty} B(x) = B$  exist, and  $Q, B$  satisfy the above assumptions.  $\mu$  is again the Gaussian weight associated to the matrix  $Q_\infty$  defined in (1.5). By using a suitable localization procedure we show that also in this case the domain of the realization  $A$  of  $\mathcal{A} = \text{Tr}(Q(\cdot)D^2) + \langle B(\cdot), D \rangle$  in  $L^2_\mu$  is  $H^2_\mu$ , and that  $A$  generates an analytic semigroup in  $L^2_\mu$ .

Once optimal regularity results for elliptic equations have been established, from the general theory of analytic semigroups one gets easily optimal regularity results for parabolic equations,

$$(1.9) \quad \begin{cases} u_t = \mathcal{A}u + f, & 0 < t < T, \quad x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

Precisely, one shows that for every  $u_0 \in H^1_\mu$  and for every  $f \in L^2((0, T) \times \mathbb{R}^n)$  with respect to the measure  $dt \times \mu(x)dx$  the solution  $u$  of (1.9) is such that  $u, u_t, D_i u, D_{ij} u$  ( $i, j = 1, \dots, n$ ) belong to  $L^2((0, T) \times \mathbb{R}^n)$  with respect to the measure  $dt \times \mu(x)dx$ .

## 2. THE SPACES $L^2_\mu$ AND $H^s_\mu$

The space  $L^2_\mu$  has been defined in (1.7). If there is no danger of confusion we shall write  $\|f\|$  instead of  $\|f\|_{L^2_\mu}$  for every  $f \in L^2_\mu$ .

We may assume, possibly changing coordinates, that

$$Q_\infty = \text{diag}(\lambda_1, \dots, \lambda_n),$$

with  $\lambda_i > 0$  for  $i = 1, \dots, n$ . Let  $\Gamma$  be the set of all multi-indexes  $\gamma = (\gamma_1, \dots, \gamma_n)$  with  $\gamma_i \in \mathbb{N} \cup \{0\}$ . For every  $\gamma \in \Gamma$  let  $H_\gamma$  be the Hermite polynomial in  $\mathbb{R}^n$  associated to the matrix  $Q_\infty$ , defined by

$$H_\gamma(x) = \prod_{k=1}^n H_{\gamma_k} \left( \frac{x_k}{\sqrt{\lambda_k}} \right), \quad x \in \mathbb{R}^n,$$

where for every nonnegative integer  $r$ ,  $H_r$  is the one dimensional  $r$ -th Hermite polynomial

$$H_r(x) = \frac{(-1)^r}{\sqrt{r!}} e^{x^2/2} \frac{d^r}{dx^r} e^{-x^2/2}, \quad x \in \mathbb{R}.$$

It is not hard to see that the set  $\{H_\gamma : \gamma \in \Gamma\}$  is an orthonormal basis in  $L_\mu^2$ .

For  $s \in \mathbb{N}$  we define the spaces  $H_\mu^s$  as

$$H_\mu^s = \{f \in L_\mu^2 : \forall |\beta| \leq s \exists D^\beta f \in L_\mu^2\}, \quad \|f\|_{H_\mu^s} = \sum_{|\beta| \leq s} \|D^\beta f\|_{L_\mu^2}.$$

For notational convenience we set also

$$H_\mu^0 = L_\mu^2.$$

Using the equalities

$$D_h H_\gamma(x) = \sqrt{\frac{\gamma_h}{\lambda_h}} H_{\gamma_h-1} \left( \frac{x_h}{\sqrt{\lambda_h}} \right) \prod_{k \neq h} H_{\gamma_k} \left( \frac{x_k}{\sqrt{\lambda_k}} \right), \quad \text{if } \gamma \in \Gamma, \gamma_h > 0$$

$$D_h H_\gamma(x) = 0, \quad \text{if } \gamma \in \Gamma, \gamma_h = 0,$$

( $h = 1, \dots, n$ ) one checks that a function  $\varphi \in L_\mu^2$ ,  $\varphi = \sum_{\gamma \in \Gamma} \varphi_\gamma H_\gamma$ , is differentiable with respect to  $x_h$  with derivative  $D_h \varphi \in L_\mu^2$  if and only if

$$\sum_{\gamma \in \Gamma} |\varphi_\gamma|^2 \gamma_h < \infty,$$

in which case we have

$$D_h \varphi(x) = \sum_{\gamma \in \Gamma, \gamma_h > 0} \varphi_\gamma \sqrt{\frac{\gamma_h}{\lambda_h}} H_{\gamma_h-1} \left( \frac{x_h}{\sqrt{\lambda_h}} \right) \prod_{k \neq h} H_{\gamma_k} \left( \frac{x_k}{\sqrt{\lambda_k}} \right),$$

and

$$\|D_h \varphi\|_{L_\mu^2}^2 = \frac{1}{\lambda_h} \sum_{\gamma \in \Gamma} |\varphi_\gamma|^2 \gamma_h.$$

Moreover one can see that for every  $s \in \mathbb{N}$  a function  $\varphi$  belongs to  $H_\mu^s$  if and only if

$$\sum_{\gamma \in \Gamma} |\varphi_\gamma|^2 \left( \sum_{h=1}^n \gamma_h \right)^s < \infty,$$

and that the norm

$$\varphi \mapsto \left( \sum_{\gamma \in \Gamma} |\varphi_\gamma|^2 \left( \sum_{h=1}^n \gamma_h \right)^s \right)^{1/2}$$

is equivalent to the norm of  $H_\mu^s$ .

A useful property of the space  $H_\mu^1$  is the following.

**Lemma 2.1.** *If  $\varphi$  is differentiable with respect to  $x_h$  and  $D_h\varphi \in L^2_\mu$  then  $x \mapsto \psi(x) = x_h\varphi(x)$  belongs to  $L^2_\mu$ , with norm less than  $\text{const} \cdot (\|\varphi\|_{L^2_\mu} + \|D_h\varphi\|_{L^2_\mu})$ . Consequently, if  $a : \mathbb{R}^n \mapsto \mathbb{R}$  is any linear function, the mapping  $\varphi \mapsto a\varphi$  is bounded from  $H^1_\mu$  to  $L^2_\mu$ .*

*Proof.* It is sufficient to show that for every polynomial  $\varphi$  we have

$$\int_{\mathbb{R}^n} (\psi(x))^2 \mu(x) dx \leq C(\|\varphi\|^2 + \|D_h\varphi\|^2).$$

If  $\varphi$  is a polynomial, then

$$\begin{aligned} \int_{\mathbb{R}^n} (x_h\varphi(x))^2 \mu(x) dx &= \int_{\mathbb{R}^n} x_h\varphi(x)^2 (-\lambda_h/2)(D_h\mu(x)) dx \\ &= \frac{\lambda_h}{2} \int_{\mathbb{R}^n} (2x_h\varphi(x)D_h\varphi(x) + \varphi(x)^2)\mu(x) dx \\ &\leq \frac{\lambda_h}{2} \|\varphi\|^2 + \lambda_h \left( \int_{\mathbb{R}^n} (x_h\varphi(x))^2 \mu(x) dx \right)^{1/2} \|D_h\varphi\| \\ &\leq \frac{\lambda_h}{2} \|\varphi\|^2 + \frac{1}{2} \int_{\mathbb{R}^n} (x_h\varphi(x))^2 \mu(x) dx + \frac{\lambda_h^2}{2} \|D_h\varphi\|^2. \end{aligned}$$

Therefore,

$$\|\psi\|^2 \leq \lambda_h(\|\varphi\|^2 + \lambda_h\|D_h\varphi\|^2),$$

and the statement follows. □

In the next sections we shall use an explicit characterization of the interpolation spaces  $(L^2_\mu, H^s_\mu)_{\theta,2}$ . To this aim we define the spaces  $H^s_\mu$  for  $s > 0$  not integer. We set

$$\begin{cases} H^s_\mu = \{f \in L^2_\mu : x \mapsto f(x) \exp(-\langle Q_\infty x, x \rangle/4) \in H^s(\mathbb{R}^n)\}, \\ \|f\|_{H^s_\mu} = \|f \exp(-\langle Q_\infty^2 \cdot, \cdot \rangle/4)\|_{H^s(\mathbb{R}^n)}. \end{cases}$$

We define the strongly continuous semigroups  $T_h(t)$ ,  $h = 1, \dots, n$ , in  $L^2_\mu$  by

$$T_h(t)\varphi(x) = \varphi(x + te_h) \exp(-(t^2 + 2tx_h)/4\lambda_h), \quad x \in \mathbb{R}^n, t \geq 0.$$

The infinitesimal generator  $A_h$  of  $T_h(t)$  is the operator defined by

$$\begin{cases} D(A_h) = \{\varphi \in L^2_\mu : \exists D_h(\varphi e^{-x_h^2/4\lambda_h}), \quad x \mapsto e^{x_h^2/4\lambda_h} D_h(\varphi e^{-x_h^2/4\lambda_h}) \in L^2_\mu\}, \\ A_h\varphi(x) = e^{x_h^2/4\lambda_h} D_h(\varphi(x) e^{-x_h^2/4\lambda_h}) = D_h\varphi - \frac{x_h}{2\lambda_h}\varphi. \end{cases}$$

**Lemma 2.2.** *For every  $h = 1, \dots, n$  and  $m \in \mathbb{N}$  we have*

$$D(A_h^m) = \{\varphi \in L^2_\mu : \exists \partial^m / \partial x_h^m \varphi \in L^2_\mu\},$$

*and the graph norm of  $A_h^m$  is equivalent to*

$$\varphi \mapsto \sum_{k=0}^m \|\partial^k / \partial x_h^k \varphi\|_{L^2_\mu}.$$

*Proof.* Let us prove that the statement holds for  $m = 1$ . If  $\varphi \in L_\mu^2$  is differentiable with respect to  $x_h$  with derivative in  $L_\mu^2$ , by Lemma 2.1  $\varphi$  belongs to  $D(A_h)$ . Conversely, if  $\varphi$  is a polynomial then

$$\begin{aligned} A_h \varphi(x) &= \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \left( \sum_{m \geq 0} \varphi_{\tilde{\gamma}, m+1} \sqrt{m+1} (1/\sqrt{\lambda_h} + 1/\lambda_h) - \sum_{m \geq 2} \varphi_{\tilde{\gamma}, m-1} \sqrt{m}/\lambda_h \right) \\ &\quad \cdot H_m \left( \frac{x_h}{\sqrt{\lambda_h}} \right) \prod_{k=1}^{n-1} H_{\gamma_k} \left( \frac{x_k}{\sqrt{\lambda_k}} \right), \end{aligned}$$

where  $\tilde{\Gamma}$  is the set of all multi-indexes in  $(\mathbb{N} \cup \{0\})^{n-1}$  and  $\varphi_{\tilde{\gamma}, m+1}$  is the coefficient corresponding to the multi-index  $(\tilde{\gamma}_1, \dots, \tilde{\gamma}_{h-1}, m, \tilde{\gamma}_h, \dots, \tilde{\gamma}_{n-1})$ . Therefore,

$$\begin{aligned} \|A_h \varphi(x)\|^2 &= \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \sum_{m \geq 2} \left( \varphi_{\tilde{\gamma}, m+1} \sqrt{m+1} (1/\sqrt{\lambda_h} + 1/\lambda_h) - \varphi_{\tilde{\gamma}, m-1} \sqrt{m}/\lambda_h \right)^2 \\ &\quad + (1/\sqrt{\lambda_h} + 1/\lambda_h)^2 \sum_{\tilde{\gamma} \in \tilde{\Gamma}} (\varphi_{\tilde{\gamma}, 1}^2 + \sqrt{2} \varphi_{\tilde{\gamma}, 2}^2) \\ &= \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \sum_{m \geq 2} a_{\tilde{\gamma}, m}^2 + (1/\sqrt{\lambda_h} + 1/\lambda_h)^2 \sum_{\tilde{\gamma} \in \tilde{\Gamma}} (\varphi_{\tilde{\gamma}, 1}^2 + \sqrt{2} \varphi_{\tilde{\gamma}, 2}^2). \end{aligned}$$

For every  $m \geq 2$  and  $\varepsilon > 0$  we have

$$\begin{aligned} \left( (1/\sqrt{\lambda_h} + 1/\lambda_h) \varphi_{\tilde{\gamma}, m+1} \sqrt{m+1} \right)^2 &= (a_{\tilde{\gamma}, m} + \varphi_{\tilde{\gamma}, m-1} \sqrt{m}/\lambda_h)^2 \\ &\leq (1 + 1/\varepsilon) a_{\tilde{\gamma}, m}^2 + (1 + \varepsilon) \varphi_{\tilde{\gamma}, m-1}^2 \frac{m}{\lambda_h^2}, \end{aligned}$$

so that for every  $M \geq 2$

$$\begin{aligned} &(1/\sqrt{\lambda_h} + 1/\lambda_h)^2 \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \sum_{m \geq M} \varphi_{\tilde{\gamma}, m+1}^2 (m+1) \\ &\leq (1 + 1/\varepsilon) \|A_h \varphi\|^2 + C(M) \|\varphi\|^2 + (1 + \varepsilon) \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \sum_{m \geq M} \varphi_{\tilde{\gamma}, m-1}^2 \frac{m}{\lambda_h^2} \\ &\leq (1 + 1/\varepsilon) \|A_h \varphi\|^2 + C(M) \|\varphi\|^2 \\ &\quad + (1 + \varepsilon) \sum_{\tilde{\gamma} \in \tilde{\Gamma}} \lambda_h^{-2} \sum_{m \geq M-2} \varphi_{\tilde{\gamma}, m+1}^2 (m+1) \frac{m+2}{m+1}, \end{aligned}$$

which implies

$$\begin{aligned} &\sum_{\tilde{\gamma} \in \tilde{\Gamma}} \sum_{m \geq M} \varphi_{\tilde{\gamma}, m+1}^2 (m+1) \\ &\leq C(\varepsilon, M) (\|A_h \varphi\|^2 + \|\varphi\|^2) + \frac{1 + \varepsilon}{(\sqrt{\lambda_h} + 1)^2} \frac{M+2}{M+1} \sum_{m \geq M-2} \varphi_{\tilde{\gamma}, m+1}^2 (m+1). \end{aligned}$$

Taking  $\varepsilon$  small and  $M$  large in such a way that

$$\frac{1 + \varepsilon}{(\sqrt{\lambda_h} + 1)^2} \frac{M + 2}{M + 1} < 1$$

we get

$$\sum_{\tilde{\gamma} \in \tilde{\Gamma}} \sum_{m \geq 0} \varphi_{\tilde{\gamma}, m+1}^2 (m + 1) \leq K(\|A_h \varphi\|^2 + \|\varphi\|^2).$$

Since the set of all polynomials is dense in  $D(A_h)$ , the statement is proved for  $m = 1$ . Arguing by recurrence, one can prove that the statement holds for every  $m$ .  $\square$

The characterization of the domain of  $A_h^m$  allows us to characterize the interpolation spaces  $(L_\mu^2, H_\mu^m)_{\theta, 2}$ .

**Proposition 2.3.** *For every  $m \in \mathbb{N}$  and  $\theta \in (0, 1)$  we have*

$$(L_\mu^2, H_\mu^m)_{\theta, 2} = H_\mu^{\theta m},$$

with equivalence of the respective norms.

*Proof.* Following [11, §1.13.3] we set

$$K^m = \bigcap_{0 \leq k \leq m, 1 \leq h \leq n} D(A_h^k).$$

By Lemma 2.2 we have  $K^m = H_\mu^m$ . By [11, Thm. 1.13.6.1] we have

$$(L_\mu^2, H_\mu^m)_{\theta, 2} = \{f \in L_\mu^2 : \|f\| < \infty\},$$

where

$$\|f\| = \|f\|_{L_\mu^2} + \left( \int_{[0,1]^n} |y|^{-2\theta m - n} \left\| \left( \prod_{s=1}^n T_s(y_s) - I \right)^m f \right\|_{L_\mu^2}^2 dt \right)^{1/2}.$$

For every  $y \in \mathbb{R}^n$  we have

$$\left( \prod_{s=1}^n T_s(y_s) f \right)(x) = f(x + y) e^{-\langle Q_\infty^{-1} y, y \rangle + 2\langle Q_\infty^{-1} y, x \rangle / 4}, \quad x \in \mathbb{R}^n,$$

so that

$$\left[ \left( \prod_{s=1}^n T_s(y_s) - I \right)^m f \right](x) e^{-\langle Q_\infty^{-1} x, x \rangle / 4} = \sum_{k=0}^m \binom{m}{k} (-1)^k \varphi(x + ky),$$

where

$$\varphi(x) = f(x) e^{-\langle Q_\infty^{-1} x, x \rangle / 4}.$$

By [11, §2.5.1] we get  $\|f\| < \infty$  iff  $\varphi \in H^{\theta m}(\mathbb{R}^n)$ , and the statement follows.  $\square$

### 3. PROPERTIES OF $T(t)$

The measure  $\mu(x)dx$  is invariant for the semigroup  $T(t)$ , in the sense specified by the following lemma.

**Lemma 3.1.** *For every  $f \in L_\mu^1$  and  $t > 0$  we have*

$$(3.1) \quad \int_{\mathbb{R}^n} (T(t)f)(x) \mu(x) dx = \int_{\mathbb{R}^n} f(x) \mu(x) dx.$$

*Proof.* Since the set of the functions  $x \mapsto e^{i\langle h, x \rangle}$  is dense in  $L^1_\mu$ , it is sufficient to prove that for every  $h \in \mathbb{R}^n$  we have

$$(3.2) \quad \int_{\mathbb{R}^n} (T(t)e^{i\langle h, \cdot \rangle})(x)\mu(x)dx = \int_{\mathbb{R}^n} e^{i\langle h, x \rangle} \mu(x)dx.$$

To this aim we remark that

$$(T(t)e^{i\langle h, \cdot \rangle})(x) = e^{i\langle h, e^{tB}x \rangle - \langle Q_t h, h \rangle / 2}, \quad t > 0,$$

and

$$\int_{\mathbb{R}^n} e^{i\langle h, x \rangle} \mu(x)dx = e^{-\langle Q_\infty h, h \rangle / 2}.$$

So, for  $t > 0$  we have

$$\int_{\mathbb{R}^n} (T(t)e^{i\langle h, \cdot \rangle})(x)\mu(x)dx = e^{-\langle (Q_t h, h) + (e^{tB} Q_\infty e^{tB^*} h, h) \rangle / 2}.$$

Taking into account that

$$\begin{aligned} Q_t + e^{tB} Q_\infty e^{tB^*} &= \int_0^t e^{sB} Q e^{sB^*} ds + \int_0^\infty e^{(t+s)B} Q e^{(t+s)B^*} ds \\ &= \int_0^t e^{sB} Q e^{sB^*} ds + \int_t^\infty e^{sB} Q e^{sB^*} ds = Q_\infty, \end{aligned}$$

(3.2) follows. □

Estimates for  $T(t)f$  and its derivatives are provided by the following lemma.

**Lemma 3.2.** *For every  $f \in L^2_\mu$  and  $t > 0$  we have*

$$(3.3) \quad \|T(t)f\|_{L^2_\mu} \leq \|f\|_{L^2_\mu},$$

$$(3.4) \quad \|D^\beta T(t)f\|_{L^2_\mu} \leq \frac{C}{t^{|\beta|/2}} \|f\|_{L^2_\mu}, \quad |\beta| \leq 3.$$

*Proof.* Using the Hölder inequality in formula (1.2) we get

$$|(T(t)f)(x)|^2 \leq (T(t)f^2)(x), \quad x \in \mathbb{R}^n,$$

and (3.3) follows from (3.1).

Moreover, setting  $\mu_t(y) = (2\pi)^{-n/2} (\det Q_t)^{-1/2} e^{-\langle Q_t^{-1}y, y \rangle / 2}$ , for every  $t > 0$  we have

$$(DT(t)f)(x) = - \int_{\mathbb{R}^n} e^{tB^*} Q_t^{-1} y f(e^{tB}x + y) \mu_t(y) dy.$$

By the Hölder inequality,

$$\begin{aligned} \|D_i T(t)f\|^2 &\leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |\langle Q_t^{-1/2} e^{tB} e_i, Q_t^{-1/2} y \rangle|^2 \mu_t(y) dy \right. \\ &\quad \left. \cdot \int_{\mathbb{R}^n} (f(e^{tB}x - y))^2 \mu_t(y) dy \right) \mu(x) dx \\ &\leq |Q_t^{-1/2} e^{tB} e_i|^2 \int_{\mathbb{R}^n} (T(t)f^2)(x) \mu(x) dx = |Q_t^{-1/2} e^{tB} e_i|^2 \|f\|_{L^2_\mu}^2 \leq Ct^{-1} \|f\|_{L^2_\mu}^2, \end{aligned}$$

so that (3.4) holds for  $|\beta| = 1$ .



To estimate the second order derivatives we remark that for every regular  $\varphi \in L^2_\mu$  we have

$$DT(t)\varphi = e^{tB^*}T(t)D\varphi, \quad t > 0.$$

It follows that for  $i, j = 1, \dots, n$  we have

$$\|D_{ij}T(t)f\| = \|D_i(D_jT(t/2)T(t/2)f)\| = \|D_i(e^{tB^*/2}T(t/2)DT(t/2)f)_j\|$$

$$\leq C\|DT(t/2)f\|_{(L^2_\mu)^n} \|DT(t/2)\|_{L(L^2_\mu, (L^2_\mu)^n)} \leq Ct^{-1}\|f\|,$$

and (3.4) follows for  $|\beta| = 2$ . The proof for  $|\beta| = 3$  is similar. □

From (3.3) and (3.4) it follows by interpolation that for  $0 < t < 1, 0 < \alpha \leq 3$

$$(3.5) \quad \|T(t)\|_{L(L^2_\mu, H^\alpha_\mu)} \leq Ct^{-\alpha/2}.$$

This estimate is not optimal for  $t$  near 0, and it will be improved later. However, we are going to use it in the next proposition to characterize the interpolation spaces  $D_A(\theta, 2)$ ,  $A$  being the infinitesimal generator of  $T(t)$ .

It is not difficult to see that the semigroup  $T(t)$  is analytic in  $L^2_\mu$ . In [4, §2.2] one can find a proof which is an adaptation to the finite dimensional case of a result for equations in infinitely many variables due to [7]. Here we give a simple direct proof.

**Proposition 3.3.** *The semigroup  $T(t)$  defined in (1.2) is analytic in  $L^2_\mu$ .*

*Proof.* By estimates (3.4) with  $|\beta| = 2$ , for every  $f \in L^2_\mu$  and  $t > 0$  we have

$$\left\| \sum_{i,j=1}^n q_{ij}D_{ij}T(t)f \right\|_{L^2_\mu} \leq \frac{C}{t} \|f\|_{L^2_\mu}.$$

Moreover, due to Lemma 2.1,  $x \mapsto \langle Bx, DT(t)f \rangle$  belongs to  $L^2_\mu$  and by estimates (3.4) we have

$$\|\langle B \cdot, DT(t)f \rangle\|_{L^2_\mu} \leq C\|T(t)f\|_{H^1_\mu} \leq C(1 + t^{-1/2} + t^{-1})\|f\|_{L^2_\mu}.$$

Therefore, for every  $f \in L^2_\mu$  and  $t > 0$

$$\frac{\partial}{\partial t} T(t)f = \mathcal{A}T(t)f \in L^2_\mu, \quad \left\| \frac{\partial}{\partial t} T(t)f \right\|_{L^2_\mu} \leq C(1 + t^{-1/2} + t^{-1})\|f\|_{L^2_\mu}.$$

It follows that  $t \mapsto T(t)f$  is differentiable for  $t > 0$  with values in  $L^2_\mu$ , and

$$\left\| \frac{d}{dt} T(t)f \right\|_{L^2_\mu} = \|\mathcal{A}T(t)f\|_{L^2_\mu} \leq C(1 + t^{-1/2} + t^{-1})\|f\|_{L^2_\mu}, \quad t > 0.$$

Since  $\|T(t)f\|_{L^2_\mu} \leq \|f\|_{L^2_\mu}$  by (3.3), then  $T(t)$  is an analytic semigroup. □

We recall that if  $X$  is any Banach space and  $A : D(A) \subset X \mapsto X$  generates an analytic semigroup  $T(t)$  in  $X$ , for  $0 \leq \theta < 1$  the space  $D_A(\theta, 2)$  is defined by

$$\begin{cases} D_A(\theta, 2) = \{f \in X : [f]_{\theta,2} = \int_0^1 t^{1-2\theta} \|\mathcal{A}T(t)f\|_X^2 < \infty\}, \\ \|f\|_{D_A(\theta,2)} = \|f\|_X + [f]_{\theta,2}. \end{cases}$$

It is well known that for  $\theta \in (0, 1)$  the space  $D_A(\theta, 2)$  coincides with the interpolation space  $(X, D(A))_{\theta,2}$ , with equivalence of the respective norms. In the next proposition we characterize such spaces.

**Proposition 3.4.** *We have*

$$D_A(\theta, 2) = \begin{cases} L_\mu^2, & \theta = 0, \\ H_\mu^{2\theta}, & 0 < \theta < 1, \end{cases}$$

*with equivalence of the respective norms.*

*Proof.* Let  $0 < \theta < 1$ . Since  $H_\mu^2$  is continuously embedded in  $D(A)$  by Lemma 2.1, then  $H_\mu^{2\theta} = (L_\mu^2, H_\mu^2)_{\theta,2}$  is continuously embedded in  $D_A(\theta, 2)$ . We are going to show that for every  $\alpha \in (0, 1)$ ,  $H_\mu^{2\alpha}$  belongs to the class  $J_\alpha$  between  $L_\mu^2$  and  $D(A)$ , i.e.

$$(3.6) \quad \|f\|_{H_\mu^{2\alpha}} \leq C \|f\|_{L_\mu^2}^{1-\alpha} \|f\|_{D(A)}^\alpha, \quad \forall f \in D(A).$$

Indeed, given any  $f \in D(A)$ , for  $\lambda > 0$  we have, due to (3.5),

$$\begin{aligned} \|f\|_{H_\mu^{2\alpha}} &= \left\| \int_0^\infty e^{-\lambda t} T(t) (\lambda f - Af) dt \right\|_{H_\mu^{2\alpha}} \leq C \int_0^\infty e^{-\lambda t} t^{-\alpha} dt \|\lambda f - Af\|_{L_\mu^2} \\ &\leq C' (\lambda^\alpha \|f\|_{L_\mu^2} + \lambda^{\alpha-1} \|Af\|_{L_\mu^2}). \end{aligned}$$

Taking the minimum for  $\lambda > 0$  we get (3.6). Then we may apply the Reiteration Theorem ([11, §1.10]) to get, for  $0 < \theta < \alpha < 1$ ,

$$D_A(\theta, 2) = (L_\mu^2, D(A))_{\theta,2} \subset (L_\mu^2, H_\mu^{2\alpha})_{\theta/\alpha,2} = H_\mu^{2\theta},$$

and the statement follows for  $0 < \theta < 1$ .

Let now  $\theta = 0$ . We remark preliminarily that if  $X$  is a Banach space and  $A$  generates a bounded analytic semigroup  $T(t)$  in  $X$ , then  $X = D_A(0, 2)$  if and only if  $D(A) = (X, D(A^2))_{1/2,2}$ . Indeed, we may replace  $A$  by  $\tilde{A} = A - I$ ,  $T(t)$  by  $\tilde{T}(t) = T(t)e^{-t}$  and we get  $D_A(0, 2) = D_{\tilde{A}}(0, 2)$ ,  $(X, D(\tilde{A}^2))_{1/2,2} = (X, D(A^2))_{1/2,2}$ . By [11, §1.14.5],

$$(X, D(\tilde{A}^2))_{1/2,2} = \{f \in X : \| \|f\| \| = \int_0^1 t \|\tilde{A}^2 T(t) f\|^2 dt < \infty\},$$

and the norm  $\| \|f\| \|$  is equivalent to the norm of  $(X, D(A^2))_{1/2,2}$ . Therefore,  $(X, D(\tilde{A}^2))_{1/2,2} = D(\tilde{A})$  means that a function  $f$  belongs to  $D(\tilde{A})$  if and only if  $\| \|f\| \|$  is finite. Since  $\tilde{A}$  is invertible, applying  $\tilde{A}^{-1}$  we get that a function  $g$  belongs to  $X$  if and only if

$$\int_0^1 t \|\tilde{A} T(t) g\|^2 dt < \infty,$$

which means that  $g \in D_{\tilde{A}}(0, 2)$ .

So, it is sufficient to prove that  $(X, D(A^2))_{1/2,2} = D(A)$ . To this aim, we remark that  $T(t)$  is a contraction semigroup so that  $A$  is m-accretive and it admits bounded imaginary powers (see e.g. [10, §2]). Consequently, by [11, §1.15.3] we get  $D(A) = [X, D(A^2)]_{1/2}$  (complex interpolation). Since in our case  $X$  and  $D(A^2)$  are Hilbert spaces, then  $[X, D(A^2)]_{1/2} = (X, D(A^2))_{1/2,2}$  with equivalence of the norms. The statement follows.  $\square$

Once the spaces  $H_\mu^\alpha$  have been characterized as interpolation spaces we may improve estimates (3.5). We shall state just the estimates we need for the sequel. We shall use the following lemma.

**Lemma 3.5.** *Let  $A$  be the generator of an analytic semigroup  $T(t)$  in a Banach space  $X$ , and let  $0 \leq \theta < 1$ . Then for every  $f \in D_A(\theta, 2)$  and  $\theta < \alpha < 1$*

$$\int_0^1 t^{2(\alpha-\theta)-1} \|T(t)f\|_{D_A(\alpha,2)}^2 dt \leq C \|f\|_{D_A(\theta,2)}.$$

*Proof.* Since  $\|T(t)\|_{L(D_A(\theta,2),X)}$  is bounded in  $(0, 1)$ , it is sufficient to prove that  $\int_0^1 t^{2(\alpha-\theta)-1} [T(t)f]_{D_A(\alpha,2)}^2 dt$  is bounded by  $C \|f\|_{D_A(\theta,2)}^2$ . Indeed,

$$\begin{aligned} & \int_0^1 t^{2(\alpha-\theta)-1} [T(t)f]_{D_A(\alpha,2)}^2 dt = \int_0^1 t^{2(\alpha-\theta)-1} \int_0^1 \xi^{1-2\alpha} \|AT(t+\xi)f\|^2 d\xi \\ &= \int_0^1 t^{2(\alpha-\theta)-1} \int_t^{t+1} (s-t)^{1-2\alpha} \|AT(s)f\|^2 ds \\ &\leq \int_0^2 \|AT(s)f\|^2 \int_0^s t^{2(\alpha-\theta)-1} (s-t)^{1-2\alpha} dt ds \\ &= \int_0^2 s^{1-2\theta} \|AT(s)f\|^2 \int_0^1 \sigma^{2(\alpha-\theta)-1} (1-\sigma)^{1-2\alpha} d\sigma ds \\ &= C [f]_{D_A(\theta,2)}^2. \quad \square \end{aligned}$$

**Corollary 3.6.** *Let  $0 \leq \theta < \alpha < 1$ , and let  $T(t)$  be the semigroup defined in (1.2). There exists  $C > 0$  such that for every  $f \in H_\mu^\theta$  we have*

$$(3.7) \quad \int_0^1 t^{\alpha-\theta-1} \|T(t)f\|_{H_\mu^\alpha}^2 dt \leq C \|f\|_{H_\mu^\theta}^2,$$

$$(3.8) \quad \int_0^1 t^{\alpha-\theta+1} \|T(t)f\|_{H_\mu^{2+\alpha}}^2 dt \leq C \|f\|_{H_\mu^\theta}^2.$$

*Proof.* Estimate (3.7) is an immediate consequence of Proposition 3.4 and Lemma 3.5.

To prove (3.8) we remark that Proposition 3.4 and estimate (3.5) imply that  $\|T(t)\|_{L(H_\mu^\alpha, H_\mu^{2+\alpha})} \leq Ct^{-1}$  for  $0 < \alpha < 1$ ,  $0 < t < 1$ , so that

$$\begin{aligned} & \int_0^1 t^{\alpha-\theta+1} \|T(t)f\|_{H_\mu^{2+\alpha}}^2 dt \\ &\leq \int_0^1 t^{\alpha-\theta+1} \|T(t/2)\|_{L(H_\mu^\alpha, H_\mu^{2+\alpha})}^2 \|T(t/2)f\|_{H_\mu^\alpha}^2 dt \\ &\leq C \int_0^1 t^{\alpha-\theta-1} \|T(t/2)f\|_{H_\mu^\alpha}^2 dt \leq C \|f\|_{H_\mu^\theta}^2. \quad \square \end{aligned}$$

#### 4. CHARACTERIZATION OF THE DOMAIN OF $A$

The main result of the paper is the following theorem.

**Theorem 4.1.** *Let  $\mathcal{A}$  be the operator defined by (1.1) and let  $A, A_\theta$  be the realizations of  $\mathcal{A}$  in  $L_\mu^2$  and in  $H_\mu^\theta$ , respectively ( $0 < \theta < 1$ ). Then*

$$D(A) = H_\mu^2, \quad D(A_\theta) = H_\mu^{2+\theta},$$

*with equivalence of the respective norms.*

*Proof.* Let us prove the embeddings  $\subset$ . By Proposition 3.4,  $L_\mu^2 = D_A(0, 2)$  and  $H_\mu^\theta = D_A(\theta/2, 2)$ . So, it is sufficient to show that if  $f \in D(A)$  is such that  $Af \in D_A(\theta/2, 2)$  with  $0 \leq \theta < 1$ , then  $f \in H_\mu^{2+\theta}$ . By Proposition 2.3 and the Reiteration Theorem ([11, §1.10]) we have

$$H_\mu^{2+\theta} = (H_\mu^\alpha, H_\mu^{2+\alpha})_{1-(\alpha-\theta)/2, 2}, \quad \forall \alpha \in (\theta, 1).$$

So, we fix once and for all  $\alpha \in (\theta, 1)$  and we prove that

$$(4.1) \quad f \in (H_\mu^\alpha, H_\mu^{2+\alpha})_{1-(\alpha-\theta)/2, 2}.$$

Let  $\lambda > 0$ , and set  $\varphi = \lambda f - Af$ . Then

$$f = \int_0^\infty e^{-\lambda t} T(t) f dt.$$

We recall that if  $X, Y$  are Banach spaces such that  $Y \subset X$ , the interpolation space  $(X, Y)_{1-(\alpha-\theta)/2, 2}$  is the set of all  $f \in X$  such that the function

$$\xi \mapsto k(\xi, f) = \xi^{(-3+\alpha-\theta)/2} \inf_{f=a+b, a \in X, b \in Y} (\|a\|_X + \xi \|b\|_Y)$$

belongs to  $L^2(0, 1)$ , and the  $(X, Y)_{1-(\alpha-\theta)/2, 2}$  norm is equivalent to  $\|k(\cdot, f)\|_{L^2(0,1)}$ . In our case it is convenient to split  $f$  as  $f = a(\xi) + b(\xi)$ , where

$$a(\xi) = \int_0^\xi e^{-\lambda t} T(t) f dt, \quad b(\xi) = \int_\xi^\infty e^{-\lambda t} T(t) f dt, \quad 0 \leq \xi \leq 1.$$

By the Hardy-Young inequality and estimate (3.7) we get

$$\begin{aligned} & \int_0^1 \xi^{\alpha-\theta-3} \|a(\xi)\|_{H_\mu^\alpha}^2 d\xi \leq \int_0^1 \xi^{\alpha-\theta-3} \left( \int_0^\xi e^{-\lambda t} \|T(t)f\|_{H_\mu^\alpha} dt \right)^2 d\xi \\ & \leq \left( 1 - \frac{\alpha-\theta}{2} \right)^{-2} \int_0^1 t^{\alpha-\theta-1} \|T(t)f\|_{H_\mu^\alpha}^2 dt \leq C \|f\|_{D_A(\theta/2, 2)}^2. \end{aligned}$$

By the Hardy-Young inequality and estimate (3.8) we get

$$\begin{aligned} & \int_0^1 \xi^{\alpha-\theta-1} \|b(\xi)\|_{H_\mu^{2+\alpha}}^2 \int_0^1 \xi^{\alpha-\theta-1} \left( \int_\xi^\infty e^{-\lambda t} \|T(t)f\|_{H_\mu^{2+\alpha}} dt \right)^2 d\xi \\ & \leq \left( \frac{\alpha-\theta}{2} \right)^{-2} \int_0^\infty t^{\alpha-\theta+1} \|T(t)f\|_{H_\mu^{2+\alpha}}^2 dt \leq C \|f\|_{D_A(\theta/2, 2)}^2. \end{aligned}$$

Since

$$k(\xi, f) \leq \xi^{(-3+\alpha-\theta)/2} (\|a(\xi)\|_{H_\mu^\alpha} + \xi \|b(\xi)\|_{H_\mu^{2+\alpha}}),$$

we get

$$\|k(\cdot, f)\|_{L^2(0,1)} \leq C \|f\|_{D_A(\theta/2, 2)},$$

and (4.1) follows.

The embedding  $H_\mu^2 \subset D(A)$  is an obvious consequence of Lemma 2.1. Moreover, by Lemma 2.1, for every linear function  $a$  the mapping  $\varphi \mapsto a\varphi$  is bounded from  $H_\mu^1$  to  $L_\mu^2$ . Consequently, it is bounded from  $H_\mu^2$  to  $H_\mu^1$ . By interpolation, it is bounded from  $H_\mu^{\theta+1}$  to  $H_\mu^\theta$  for every  $\theta \in (0, 1)$ . Therefore,  $\varphi \mapsto \langle B \cdot, D\varphi \rangle \in L(H_\mu^{\theta+2}, H_\mu^\theta)$ , so that  $H_\mu^{\theta+2}$  is continuously embedded in  $D(A_\theta)$ .  $\square$

5. THE CASE OF COEFFICIENTS DEPENDING ON  $x$

Let us consider now the case of coefficients  $q_{ij}, b_{ij}$  depending on  $x$ . The assumptions on the coefficients are the following:

$$(5.1) \quad \begin{cases} q_{ij}, b_{ij} \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), & q_{ij} = q_{ji}, \quad i, j = 1, \dots, n, \\ \sum_{i,j=1}^n q_{ij}(x) \xi_i \xi_j \geq \nu |\xi|^2 \quad \forall x, \xi \in \mathbb{R}^n, \end{cases}$$

for some  $\nu > 0$ . Moreover,

$$(5.2) \quad \exists \lim_{|x| \rightarrow \infty} q_{ij}(x) = q_{ij} \in \mathbb{R}, \quad \exists \lim_{|x| \rightarrow \infty} b_{ij}(x) = b_{ij} \in \mathbb{R}.$$

Setting  $Q = [q_{ij}]_{i,j=1, \dots, n}$ ,  $B = [b_{ij}]_{i,j=1, \dots, n}$  we assume that

$$(5.3) \quad B \text{ satisfies (1.4).}$$

Let  $\mathcal{A}$  be the differential operator defined by

$$(5.4) \quad \mathcal{A}f = \frac{1}{2} \sum_{i,j=1}^n q_{ij}(x) D_{ij} f(x) + \sum_{i,j=1}^n b_{ij}(x) x_i D_j f(x), \quad x \in \mathbb{R}^n.$$

Let  $\mu$  be the Gaussian weight associated to the matrix  $Q_\infty$  defined in (1.5). Define the operators  $A_\infty, A : H_\mu^2 \mapsto L_\mu^2$  by

$$(A_\infty f)(x) = \text{Tr}(Q D^2 f(x)) + \langle Bx, Df(x) \rangle,$$

$$(Af)(x) = \text{Tr}(Q(x) D^2 f(x)) + \langle B(x)x, Df(x) \rangle = \mathcal{A}f(x).$$

**Theorem 5.1.** *Under assumptions (5.1), (5.2), (5.3),  $A : D(A) = H_\mu^2 \mapsto L_\mu^2$  generates an analytic semigroup in  $L_\mu^2$ .*

*Proof.* As a first step we prove that there are  $K, \omega > 0$  such that for  $\text{Re } \lambda > \omega$  and for every  $f \in H_\mu^2$

$$(5.5) \quad |\lambda| \|f\|_{L_\mu^2} + \|f\|_{H_\mu^2} \leq K \|\lambda f - Af\|_{L_\mu^2}.$$

For every  $\varepsilon > 0$  let  $R > 1$  be such that

$$|q_{ij}(x) - q_{ij}| + |b_{ij}(x) - b_{ij}| \leq \varepsilon \quad \text{for } |x| \geq R - 1.$$

Let  $\theta$  be a smooth cutoff function such that

$$\begin{cases} \theta \equiv 0 \text{ in } B(0, R - 1), & \theta \equiv 1 \text{ outside } B(0, R), \\ |D_i \theta| \leq 1, & |D_{ij} \theta| \leq 1, \quad i, j = 1, \dots, n. \end{cases}$$

Let  $\text{Re } \lambda \geq 1$  and let  $f \in H_\mu^2$ . Then  $\theta f$  satisfies

$$\begin{aligned} & \lambda(\theta f)(x) - A_\infty(\theta f)(x) = \theta(x)(\lambda f(x) - Af(x)) \\ & - \theta \left( \sum_{i,j=1}^n (q_{ij} - q_{ij}(x)) D_{ij} f(x) + \sum_{i,j=1}^n (b_{ij} - b_{ij}(x)) x_j D_i f(x) \right) \\ & - \sum_{i,j=1}^n q_{ij}(f(x) D_{ij} \theta(x) + 2D_i \theta(x) D_j f(x)) - \sum_{i,j=1}^n b_{ij} x_j f(x) D_i \theta(x) \end{aligned}$$

so that by Theorem 4.1 and Proposition 3.3

$$\begin{aligned}
 (5.6) \quad & |\lambda| \|\theta f\|_{L^2_\mu} + \|\theta f\|_{H^2_\mu} \leq C(\|\theta(\lambda f - Af)\|_{L^2_\mu} + \varepsilon\|f\|_{H^2_\mu} + \|f\|_{H^1_\mu}) \\
 & \leq C'(\|\theta(\lambda f - Af)\|_{L^2_\mu} + \varepsilon\|f\|_{H^2_\mu} + C(\varepsilon)\|f\|_{L^2_\mu}).
 \end{aligned}$$

The function  $(1 - \theta)f$  vanishes outside  $B(0, R)$  and satisfies

$$\begin{aligned}
 & \lambda((1 - \theta)f)(x) - A((1 - \theta)f)(x) = (1 - \theta(x))(\lambda f(x) - Af(x)) \\
 & - \sum_{i,j=1}^n q_{ij}(x)(f(x)D_{ij}\theta(x) + D_i\theta(x)D_j f(x)) \\
 & - \sum_{i,j=1}^n b_{ij}(x)x_j f(x)D_i\theta(x).
 \end{aligned}$$

By the well known *a priori* estimates for elliptic equations with regular coefficients in bounded sets, if  $\text{Re } \lambda$  is large enough we have

$$\begin{aligned}
 & |\lambda| \|(1 - \theta)f\|_{L^2(B(0,R))} + \|(1 - \theta)f\|_{H^2(B(0,R))} \\
 & \leq C(\|(1 - \theta)(\lambda f - Af)\|_{L^2(B(0,R))} + \|f\|_{H^1(B(0,R))} + R\|f\|_{L^2(B(0,R))}),
 \end{aligned}$$

so that for every  $\delta > 0$

$$\begin{aligned}
 (5.7) \quad & |\lambda| \|(1 - \theta)f\|_{L^2(B(0,R))} + \|(1 - \theta)f\|_{H^2(B(0,R))} \\
 & \leq C(\|(1 - \theta)(\lambda f - Af)\|_{L^2(B(0,R))} + \delta\|f\|_{H^2(B(0,R))} + C(\delta, R)\|f\|_{L^2(B(0,R))}).
 \end{aligned}$$

Using (5.6) and (5.7) we get

$$\begin{aligned}
 & |\lambda| \|f\|_{L^2_\mu} + \|f\|_{H^2_\mu} \leq |\lambda|(\|\theta f\|_{L^2_\mu} + C(R)\|(1 - \theta)f\|_{L^2(B(0,R))}) \\
 & + \|\theta f\|_{H^2_\mu} + C(R)\|(1 - \theta)f\|_{H^2(B(0,R))} \\
 & \leq C'\varepsilon\|f\|_{H^2_\mu} + C_1(R)(\|\lambda f - Af\|_{L^2_\mu} + \delta\|f\|_{H^2_\mu} + C(\varepsilon, \delta, R)\|f\|_{L^2(B(0,R))}).
 \end{aligned}$$

Taking  $\varepsilon$  so small that  $C'\varepsilon \leq 1/4$  and then  $\delta$  so small that  $C_1(R)\delta \leq 1/4$  we get

$$|\lambda| \|f\|_{L^2_\mu} + \|f\|_{H^2_\mu} \leq \frac{1}{2}\|f\|_{H^2_\mu} + K(\|\lambda f - Af\|_{L^2_\mu} + \|f\|_{L^2_\mu}),$$

and (5.5) follows.

To conclude, we remark that for  $\text{Re } \lambda$  large and for every  $g \in L^2_\mu$ , the equation

$$\lambda f - Af = g$$

has a unique solution  $f \in H^2_\mu$ . This can be seen using the continuity method: for every  $\varepsilon \in [0, 1]$  consider the problem

$$(5.8) \quad \lambda f - (1 - \varepsilon)A_\infty f - \varepsilon Af = g.$$

Using the *a priori* estimate (5.5) and the fact that  $A_\infty$  generates an analytic semi-group it is not hard to see that the set of all  $\varepsilon$ 's such that (5.8) is uniquely solvable in  $H^2_\mu$  is open and closed in  $[0, 1]$ , so that it coincides with  $[0, 1]$ . Taking  $\varepsilon = 1$  the statement follows.  $\square$

## 6. OPTIMAL REGULARITY IN PARABOLIC PROBLEMS

We consider here the parabolic problem

$$(6.1) \quad \begin{cases} u_t = \mathcal{A}u + f, & 0 < t < T, \quad x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where  $\mathcal{A}$  is the operator defined in (5.4). We assume that (5.1), (5.2) hold.

We recall that if  $A$  is the generator of an analytic semigroup in a Hilbert space  $H$  and  $f \in L^2(0, T; H)$ ,  $u_0 \in D_A(1/2, 2)$ , then the problem

$$\begin{cases} u' = Au + f, & 0 < t < T, \\ u(0) = u_0, \end{cases}$$

has a unique solution  $u \in L^2(0, T; D(A)) \cap H^1(0, T; H)$ , and

$$\|u\|_{L^2(0, T; D(A))} + \|u\|_{H^1(0, T; H)} \leq C(\|u_0\|_{D_A(1/2, 2)} + \|f\|_{L^2(0, T; H)}).$$

Applying this result to problem (6.1) we get that if  $u_0 \in H_\mu^1$  and  $f$  belongs to  $L^2((0, T) \times \mathbb{R}^n)$  with respect to the measure  $dt \times \mu(x)dx$  then the solution  $u$  of (6.1) is such that  $u$  and the derivatives  $u_t$ ,  $D_i u$ ,  $D_{ij} u$  ( $i, j = 1, \dots, n$ ) belong to  $L^2((0, T) \times \mathbb{R}^n)$  with respect to the measure  $dt \times \mu(x)dx$ .

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