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STRASSEN THEOREMS FOR A CLASS OF ITERATED PROCESSES

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Abstract. A general direct Strassen theorem is proved for a class of stochastic processes and applied for iterated processes such as $W(L_t)$, where $W(\cdot)$ is a standard Wiener process and $L_{\cdot}$ is a local time of a Lévy process independent from $W(\cdot)$.

1. Introduction

Since the landmark paper of Burdzy [3] on iterated Brownian motion (IBM) the investigation of various IBM type processes became increasingly popular. To name a few: Arcones [1], Khoshnevisan and Lewis [13, 14], Hu and Shi [11], Khoshnevisan et al. [15], Shi [17, 18]. In [7] Csáki, Csörgő, Földes and Révész proved a Strassen type result for a class of iterated processes. In this paper some strongly related but at the same time markedly different Strassen type results will be proved.

2. Main results

Let $\mathcal{S}$ be the Strassen class of functions, i.e., $\mathcal{S} \subset C[0,1]$ is the class of absolutely continuous functions (with respect to the Lebesgue measure) on $[0,1]$ for which

$$f(0) = 0 \qquad \text{and} \qquad \int_0^1 \dot{f}^2(x)dx \le 1. \tag{2.1}$$

The set of $\mathbf{R}^2$ valued, absolutely continuous functions

$$\{(g(y), h(x)),\ 0 \le y \le 1, 0 \le x \le 1\} \tag{2.2}$$

for which $g(0) = h(0) = 0$ and

$$\int_0^1 \dot{g}^2(y)dy + \int_0^1 \dot{h}^2(x)dx \le 1 \tag{2.3}$$

will be called the *Strassen class* $\mathcal{S}^2$. In [7] the following Strassen-type theorem was proved.

Let $C_0[0,1] \subset C[0,1]$ be the set of the continuous functions $f(\cdot)$ on $[0,1]$ for which $f(0) = 0$. Let A be an operator on $C_0[0,1]$, satisfying

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(C.1) $Acf = c^\rho Af \qquad (\rho \geq 1,\ c > 0)$,
(C.2) $Af \geq 0$,
(C.3) $Af \in C_0[0,1]$,
(C.4) A is uniformly continuous on bounded subsets of $C_0[0,1]$, i.e., $\forall \varepsilon > 0,\ K > 0,\ \exists \delta = \delta(\varepsilon, K) > 0$ such that if $f, g \in C_0[0,1]$, $\sup_{0\leq x\leq 1} |f(x)| \leq K$, $\sup_{0\leq x\leq 1} |g(x)| \leq K$ and $\sup_{0\leq x\leq 1} |f(x) - g(x)| < \delta$, then $\sup_{0\leq x\leq 1} |Af(x) - Ag(x)| \leq \varepsilon$,
(C.5) $\sup_{f\in\mathcal{S}} Af(x) = \lambda(A, x) = \lambda_x \qquad 0 < \lambda_x \leq 1$.

Theorem A ([7]). *Let $W_1(\cdot)$ and $W_2(\cdot)$ be two independent standard Wiener processes starting from zero, and A be an operator satisfying conditions* (C.1)–(C.5). *Then for $0 \leq x \leq 1$, $0 \leq y \leq 1$, the limit set of the vector*

$$\left(\frac{W_1(yAW_2(xT))}{T^{\rho/4}(2\log\log T)^{(\rho+2)/4}}, \frac{W_2(xT)}{(2T\log\log T)^{1/2}}\right) \tag{2.4}$$

is $(g(yAh(x)), h(x))$, where $(g, h) \in \mathcal{S}^2$.

Remark 2.1. We will call this a *composite* (see Remark 2.2) type Strassen theorem, because of the composite structure of $g(yAh(x))$.

A result having this composite feature for processes of $W(L(t))$, where $L(\cdot)$ is the local time of a (not necessarily symmetric) Lévy process was claimed in Khoshnevisan [12]. For other composite type Strassen theorem for iterated processes see also Arcones [1] and Hu et al. [10].

The main source of inspiration of this work is a result of Marcus and Rosen [16] combined with a recent paper of Bertoin [2]. The following *direct* (as opposed to composite, see Remark 2.2) Strassen theorem was proved in [16]. Let $Z = \{Z(t),\ t \in \mathbf{R}^+\}$ be a symmetric Lévy process and set

$$E(e^{i\lambda Z(t)}) = exp(-t\psi(\lambda)). \tag{2.5}$$

Assume

$$\int_0^1 \frac{d\lambda}{\psi(\lambda)} = \infty \tag{2.6}$$

and

$$\int_0^\infty \frac{\log(1+\lambda)}{1+\psi(\lambda)}\, d\lambda < \infty. \tag{2.7}$$

Under these conditions the Lévy process $Z(t)$ is recurrent and possesses local times $\{L_t^y,\ (t,y) \in R^+ \times R\}$.

Moreover, define, for $\alpha > 0$,

$$\kappa(\alpha) = \int_0^\infty \frac{d\lambda}{\alpha + \psi(\lambda)}. \tag{2.8}$$

Let $\mathcal{A} \subset C[0,1]$ denote the set of functions $f(x)$, $\quad 0 \leq x \leq 1, \quad f(0) = 0$ and absolutely continuous with respect to the Lebesgue measure.

Define

$$\mathcal{D}^{(\gamma)} = \{f:\ f \in \mathcal{A}, \quad \int_0^1 |\dot{f}(x)|^\gamma\, dx \leq 1\}. \tag{2.9}$$

(Observe that $\mathcal{D}^{(\gamma)}$ is a generalization of the usual Strassen class as $\mathcal{D}^{(2)} = \mathcal{S}$.) Denote by $\mathcal{D}_M^{(\gamma)}$ the class of nondecreasing functions of $\mathcal{D}^{(\gamma)}$.

Theorem B ([16])**.** *Let Z be a symmetric Lévy process satisfying* (2.5)–(2.7)*, and for which ψ is regularly varying at zero with index $1 < \beta \leq 2$. Then the set of functions*

$$L_t(x) = \frac{L^0_{xt}}{\gamma(\bar{\beta})(\log\log t)\kappa(\frac{\log\log t}{t})}, \qquad 0 \leq x \leq 1, \tag{2.10}$$

where

$$\gamma(\bar{\beta}) = \beta^{1/\beta}\bar{\beta}^{1/\bar{\beta}} \tag{2.11}$$

and

$$\frac{1}{\beta} + \frac{1}{\bar{\beta}} = 1 \tag{2.12}$$

is relatively compact in $C[0,1]$ and the set of its limit points is $\mathcal{D}_M^{(\beta)}$ almost surely.

In an elegant paper Bertoin [2] points out that the inverse process of the maximum of the iterated Brownian motion is a stable subordinator with index $1/4$. On the other hand the above-mentioned theorem of Marcus and Rosen, combined with the fact (see Dellacherie and Meyer [9, p. 275]) that the inverse local time process of a stable process with index α is also a stable process with index $1 - 1/\alpha$, tells us that if $Z(t)$ is stable with index $4/3$ then its local time has an inverse process which is a stable subordinator with index $1/4$. Combining this observation and Bertoin's result, it suggests that the Strassen class of the supremum of IBM and the Strassen class of the local time of a stable process of index $4/3$ should be the same. However the two Strassen type theorems quoted above do not yield this conjecture. Our first result shows that the above argument is correct, the composite and the direct Strassen theorems are equivalent, subject to a natural condition, namely the monotonicity of the inner process.

Theorem 2.1. *Let W_1 and W_2 be two independent standard Wiener processes. Consider*

$$u_t(x) = \frac{W_1(\max_{0\leq s\leq xt} W_2(s))}{2^{5/4}3^{-3/4}t^{1/4}(\log\log t)^{3/4}}, \quad 0 \leq x \leq 1. \tag{2.13}$$

Then the set of functions $\{u_t(\cdot) : 1 \leq t < \infty\}$ is relatively compact in $C[0,1]$ and the set of its limit points as $t \to \infty$ is $\mathcal{D}^{(4/3)}$ a.s.

Clearly Theorem 2.1 states that for the operator $Af(x) = \max\limits_{0\leq y\leq x} f(y)$ the composite Strassen class of Theorem A and the direct Strassen class of Theorem 2.1 are equivalent. However we can't expect a direct Strassen class for all the operators satisfying Theorem A. To see this observe that for instance the operator $Af(x) = |f(x)|$ is not monotone, and we will show in Example 1 that there is no direct Strassen class for $W_1(|W_2(\cdot)|)$.

However it is easy to see that the Strassen class of $\max\limits_{0\leq s\leq x} W(s)$ and that of $\max\limits_{0\leq s\leq x}|W(s)|$ are identical, both being $\mathcal{D}_M^{(2)}(= \mathcal{S}_M)$. This fact and Lemma 2.1 implies that Theorem 2.1 remains true if $W_2(s)$ is replaced by $|W_2(s)|$.

Applying the well-known result of P. Lévy

$$\{L_2(t),\ t \geq 0\} \stackrel{\mathcal{D}}{=} \left\{\max_{0\leq s\leq t} W_2(s),\ t \geq 0\right\}, \tag{2.14}$$

where $L_2(\cdot)$ is the local time at zero of $W_2(\cdot)$, we can rephrase Theorem 2.1 as

Theorem 2.2. *Let W_1 be a standard Wiener process and let L_2 be the local time at zero of another Wiener process W_2 independent from W_1. Consider*

$$s_t(x)=\frac{W_1(L_2(xt))}{2^{5/4}3^{-3/4}t^{1/4}(\log\log t)^{3/4}},\quad 0\le x\le 1. \tag{2.15}$$

Then the set of functions $\{s_t(\cdot):\ 1\le t<\infty\}$ is relatively compact in $C[0,1]$ and the set of its limit points as $t\to\infty$ is $\mathcal{D}^{(4/3)}$ a.s.

Corollary 2.1. *Let $\{X_n\}$ be a sequence of i.i.d. integer valued random variables such that $Ee^{itX_1}=1$ if and only if t is an integer multiple of 2π. Assume that $E|X_1|^3<\infty$ and $EX_1=0$. Let $\rho(x)$ be a real valued function on integers such that*

$$\sum_{x=-\infty}^{\infty}|x|^{1+\delta}|\rho(x)|<\infty \text{ for some } \delta>0.$$

Set $S_n=X_1+...+X_n$ and

$$A_n=\frac{\sigma^{1/2}}{d}\sum_{i=1}^{n}\rho(S_i),$$

where

$$\sigma^2=EX_1^2,$$

$$d^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|\sum_{x=-\infty}^{\infty}\rho(x)e^{ivx}|^2\frac{1+E(e^{ivX_1})}{1-E(e^{ivX_1})}dv,$$

and let $\bar\rho=\sum_{x=-\infty}^{\infty}\rho(x)$ and $\xi(n)=\#\{i:\ 1\le i\le n,\ S_i=0\}$,
Consider

$$\tilde f_n(x)=\frac{A_{[xn]}-\bar\rho\xi([xn])}{2^{5/4}3^{-3/4}n^{1/4}(\log\log n)^{3/4}},\qquad 0\le x\le 1. \tag{2.16}$$

Then the set of functions $\{\tilde f_n(\cdot):1\le n<\infty\}$ is relatively compact in $C[0,1]$ and the set of its limit points as $n\to\infty$ is $\mathcal{D}^{(4/3)}$ a.s.

Corollary 2.2. *Let B be a standard Wiener process and let ℓ be its local time at zero. Assume that $\rho(x)$ is an integrable function on R such that*

$$\int_{-\infty}^{\infty}|x|^{1+\delta}|\rho(x)|\,dx<\infty \text{ for some } \delta>0.$$

Let

$$\bar\rho=\int_{-\infty}^{\infty}\rho(x)\,dx,$$

$$\sigma^2=4\int_{-\infty}^{0}(\int_{-\infty}^{x}\rho(y)\,dy)^2+4\int_{0}^{\infty}(\int_{x}^{\infty}\rho(y)\,dy)^2\,dx,$$

$$A(t)=\int_0^t\rho(B(s))\,ds$$

and

$$\hat f_t(x)=\frac{A(xt)-\bar\rho\ell(xt)}{2^{5/4}3^{-3/4}\sigma t^{1/4}(\log\log t)^{3/4}},\qquad 0\le x\le 1. \tag{2.17}$$

Then the set of functions $\{\hat{f}_t(\cdot) : 1 \le t < \infty\}$ *is relatively compact in* $C[0,1]$ *and the set of its limit points as* $t \to \infty$ *is* $\mathcal{D}^{(4/3)}$ *a.s.*

Corollary 2.3. *Let* B *be a standard Wiener process and let* $\ell(z,t)$ *be its local time. Put*

$$\tilde{f}_t(x) = \frac{\ell(z,xt) - \ell(0,xt)}{2^{9/4}3^{-3/4}|z|^{1/2}t^{1/4}(\log\log t)^{3/4}}. \tag{2.18}$$

Then for fixed z *the set* $\{\tilde{f}_t(\cdot) : 1 \le t < \infty\}$ *is relatively compact in* $C[0,1]$ *and its set of limit points as* $t \to \infty$ *is* $\mathcal{D}^{(4/3)}$ *a.s.*

The proof of Theorem 2.1 is based on the following lemma, which is interesting for its own sake as well. Put

$$\mathcal{F}^{(\beta)} = \left\{ f : \; f \in \mathcal{A}, \quad \int_0^1 |\dot{f}(x)|^{2\beta/(1+\beta)}\, dx \le \frac{\beta^{\beta/(1+\beta)}}{1+\beta} \right\} \tag{2.19}$$

and

$$\begin{aligned}\mathcal{G}^{(\beta)} = \{f :\; & f = g \circ h,\; g, h \in \mathcal{A},\; h \text{ is nondecreasing} \\ & \text{and } \int_0^1 (|\dot{g}(x)|^2 + |\dot{h}(x)|^\beta)\, dx \le 1\}.\end{aligned} \tag{2.20}$$

Observe that $h(0) = 0$ and Lemma C (see Section 3) implies that $h(1) \le 1$.

Remark 2.2. Strassen classes described in terms of one function and $\mathcal{D}^{(\gamma)}$ (or $\mathcal{F}^{(\beta)}$) are called *direct*, and Strassen classes described in terms of a pair of functions, their composition (see $\mathcal{G}^{(\beta)}$, or Theorem A) are called *composite*.

The next lemma reveals an interesting connection between direct and composite Strassen classes.

Lemma 2.1. *For* $\beta \ge 1$, *the classes* $\mathcal{F}^{(\beta)}$ *and* $\mathcal{G}^{(\beta)}$ *are identical.*

Remark 2.3. We will show in an example that the monotonicity of $h(\cdot)$ is essential in the lemma.

Theorem 2.2 suggests that we might try to prove a direct Strassen theorem for the process $W_1(\hat{L}_2(t))$ where $\hat{L}_2(t)$ is the local time process of a Lévy process independent from $W_1(\cdot)$. In what follows we will go one step further in generalization and formulate a direct Strassen theorem for a class of stochastic processes satisfying two conditions. The first condition requires that the ordinary LIL should hold for certain linear combinations and the second condition controls the increment behaviour of the process.

Theorem 2.3. *Let* $\{X(t), t \ge 0\}$ *be a stochastic process with continuous sample paths and the following two properties*
Property 1.

$$\limsup_{t\to\infty} \frac{\sum_{i=1}^d c_i(X(it) - X((i-1)t))}{\chi(t)} = 1 \quad a.s., \tag{2.21}$$

$$\liminf_{t\to\infty} \frac{\sum_{i=1}^d c_i(X(it) - X((i-1)t))}{\chi(t)} = -1 \quad a.s., \tag{2.22}$$

with some $\chi(t) \nearrow \infty$ provided that $\sum_{i=1}^{d} |c_i|^q = 1,\ q > 1,\ d = 1, 2, \ldots$.
Property 2. *For $0 < c \leq 1$*

$$\limsup_{T\to\infty} \sup_{0\le t\le T-cT} \sup_{0\le s\le cT} \frac{|X(t+s)-X(t)|}{\chi(T)} \le A = A(c) \quad a.s. \tag{2.23}$$

where $\lim_{c\searrow 0} A(c) = 0$.

Let

$$\eta(x) = \eta_t(x) = \frac{X(xt)}{\chi(t)} \quad (0 \le x \le 1). \tag{2.24}$$

Then the set

$$\{\eta_t(x), \quad 0 \le x \le 1\} \quad (t \to \infty) \tag{2.25}$$

is relatively compact in $C[0,1]$ and its set of limit points is $\mathcal{D}^{(p)}$ almost surely, where $1/p + 1/q = 1$.

As an application of Theorem 2.3 we prove the following generalization of our Theorem 2.2

Theorem 2.4. *Consider a symmetric Lévy process $\{Z(t),\ t \in \mathbf{R}^+\}$ for which the conditions of Theorem* B *hold. Denote its local time process at zero by L_t. Let $Y(t) = W(L_t)$ where $W(\cdot)$ is a standard Wiener process, independent from $Z(\cdot)$ (and hence also from $L_.$). The set of functions*

$$f_t(x) = \frac{Y(xt)}{G(t)}, \qquad 0 \le x \le 1, \tag{2.26}$$

with

$$G(t) = K_\beta \log\log t \left(\kappa\left(\frac{\log\log t}{t}\right)\right)^{1/2}, \tag{2.27}$$

$$K_\beta = \frac{2^{1/2}\beta}{(\beta+1)^{(\beta+1)/(2\beta)}(\beta-1)^{(\beta-1)/(2\beta)}} \tag{2.28}$$

is relatively compact in $C[0,1]$ and the set of its limit points as $t \to \infty$ is $\mathcal{D}^{(\frac{2\beta}{\beta+1})}$ almost surely.

Remark 2.4. The process $W(L_t)$ appears as the limit process of additive functionals (see Khoshnevisan [12]).

3. Proofs of Theorems 2.1–2.3

Proof of Lemma 2.1. First assume that $f = g \circ h \in \mathcal{G}^{(\beta)}$. Then obviously $f \in \mathcal{A}$. Moreover, by Hölder's inequality,

$$\begin{aligned}
\int_0^1 |\dot f(x)|^{2\beta/(1+\beta)}\,dx &= \int_0^1 |\dot g(h(x))\dot h(x)|^{2\beta/(1+\beta)}\,dx \\
&= \int_0^1 \left(|\dot g(h(x))|^{2\beta/(1+\beta)}|\dot h(x)|^{\beta/(1+\beta)}\right)|\dot h(x)|^{\beta/(1+\beta)}\,dx \\
&\le \left(\int_0^1 |\dot g(h(x))|^2 \dot h(x)\,dx\right)^{\beta/(1+\beta)}\left(\int_0^1 |\dot h(x)|^\beta\,dx\right)^{1/(1+\beta)} \\
&\le \left(\int_0^1 |\dot g(u)|^2\,du\right)^{\beta/(1+\beta)}\left(\int_0^1 |\dot h(x)|^\beta\,dx\right)^{1/(1+\beta)} = A^{\beta/(1+\beta)}B^{1/(1+\beta)}.
\end{aligned}$$

It is easy to see that under the condition $A+B \le 1$, we have $\max A^{\beta/(1+\beta)}B^{1/(1+\beta)} = \beta^{\beta/(1+\beta)}/(1+\beta)$, i.e. $f \in \mathcal{F}^{(\beta)}$.

Now assume that $f \in \mathcal{F}^{(\beta)}$. Define

$$h(x) = \frac{1}{\beta^{1/(1+\beta)}}\int_0^x |\dot f(u)|^{2/(1+\beta)}\,du,$$

$$g(u) = \begin{cases} f(h^{-1}(u)) & \text{for} \quad 0 \le u \le h(1), \\ f(1) & \text{for} \quad h(1) \le u \le 1. \end{cases}$$

Observe that $h(1) \le (1+\beta)^{-1/\beta} \le 1$. Indeed, by Jensen's inequality,

$$h(1) = \int_0^1 |\dot f(x)|^{2/(1+\beta)}\,dx \le \left(\int_0^1 |\dot f(x)|^{2\beta/(1+\beta)}\,dx\right)^{1/\beta} \le \beta^{1/(1+\beta)}(1+\beta)^{-1/\beta},$$

since $\beta \ge 1$. Then

$$\begin{aligned}
\int_0^1 |\dot g(u)|^2\,du + \int_0^1 |\dot h(x)|^\beta\,dx &= \int_0^1 |\dot g(h(x))|^2 \dot h(x)\,dx + \int_0^1 |\dot h(x)|^\beta\,dx \\
&= \beta^{1/(1+\beta)}\int_0^1 |\dot f(x)|^{2\beta/(1+\beta)}\,dx + \beta^{-\beta/(1+\beta)}\int_0^1 |\dot f(x)|^{2\beta/(1+\beta)}\,dx \\
&\le (\beta^{1/(1+\beta)} + \beta^{-\beta/(1+\beta)})\frac{\beta^{\beta/(1+\beta)}}{1+\beta} = 1.
\end{aligned}$$

Hence $f \in \mathcal{G}^{(\beta)}$, and Lemma 2.1 is proved. □

Remark 3.1. If $h(\cdot)$ is not restricted to the class of nondecreasing functions, then $\mathcal{F}^{(\beta)}$ and $\mathcal{G}^{(\beta)}$ are no longer equivalent.

To see this consider the following example.

Example 1. Let $\beta = 2$ and consider for $n = 1, 2, \ldots$

$$g_n(u) = \begin{cases} \sqrt{3}nu/2 & \text{if} \quad 0 \le u \le 1/n^2, \\ \sqrt{3}/(2n) & \text{if} \quad u > 1/n^2, \end{cases}$$

$$h_n(u) = \begin{cases} u/2 & \text{if} \quad 0 \le u \le 1/(2n^2), \\ 1/(2n^2) - u/2 & \text{if} \quad 1/(2n^2) \le u \le 1/n^2, \end{cases}$$

$$h_n\left(u+\frac{1}{n^2}\right)=h_n(u).$$

It is easy to see that for these $g_n(\cdot)$, and $h_n(\cdot)$, the integral in (2.19) $\to\infty$, as $n\to\infty$, while the integral in (2.20) is always 1.

To see that there is no direct Strassen class for the process $W_1(|W_2(\cdot)|)$, observe that the above sequence of functions $(g_n,h_n)\in\mathcal{S}^2$, but for their composition $f_n=g_n(h_n(\cdot))$ we have $|\dot{f}_n(x)|=\frac{\sqrt{3}}{4}n$, so $\int|\dot{f}_n(x)|^\alpha dx\to\infty$ with any choice of $\alpha>0$.

Proof of Theorem 2.1. It was proved in [7] that Theorem A holds for $Af=\max f$, which amounts to saying that the limit set of

$$u_t^*(x)=\frac{W_1(\max_{0\le s\le xt}W_2(s))}{t^{1/4}(2\log\log t)^{3/4}},\quad 0\le x\le 1,$$

is $\mathcal{G}^{(2)}$. Applying our Lemma 2.1 we get Theorem 2.1.

Proof of Corollaries 2.1 and 2.2. These follow from the strong approximation results of [6]. □

Proof of Corollary 2.3. This follows from the strong approximation theorem proved in [5]. □

For the proof of Theorem 2.3 we will use the following well-known result.

Theorem C (Riesz and Sz. Nagy [19, p. 75])**.** *Let f be a real valued function on $[0,1]$. The following two conditions are equivalent:*

(i) f *is absolutely continuous and* $\int_0^1|\dot{f}(x)|^p dx\le 1$

(ii) f *is continuous on* $[0,1]$ *and* $\sum_{i=1}^r\left(\frac{f(\frac{i}{r})-f(\frac{i-1}{r})}{1/r}\right)^p\frac{1}{r}\le 1$ *for any* $r=1,2,\ldots$.

Now we prove two lemmas.

Lemma 3.1. *Denote the set of the limit points of the vectors*

$$\left(\frac{X(t)}{\chi(t)},\frac{X(2t)-X(t)}{\chi(t)}\right)$$

defined in Theorem 2.3 by $\mathcal{M}_2$.($\mathcal{M}_2\subset\mathbf{R}^2$). Then

(i) $\mathcal{M}_2\subset\{(\xi_1,\xi_2):\ |\xi_1|^p+|\xi_2|^p\le 1\}\quad(p>0)$,

(ii) $\{(\xi_1,\xi_2):\ |\xi_1|^p+|\xi_2|^p=1\}\subset\mathcal{M}_2$.

Proof of Lemma 3.1. Consider an element $(Y_1,Y_2)\in\mathcal{M}_2$. Let q be the conjugate exponent to p (that is $1/p+1/q=1$), and denote

$$a_1=\frac{|Y_1|^\alpha\operatorname{sgn}(Y_1)}{(|Y_1|^{\alpha q}+|Y_2|^{\alpha q})^{\frac{1}{q}}},\qquad a_2=\frac{|Y_2|^\alpha\operatorname{sgn}(Y_2)}{(|Y_1|^{\alpha q}+|Y_2|^{\alpha q})^{\frac{1}{q}}}$$

where $\alpha=p-1$. Then $|a_1|^q+|a_2|^q=1$. Since

$$\alpha+1=\alpha q=p$$

we have

$$a_1Y_1+a_2Y_2=\frac{|Y_1|^{\alpha+1}+|Y_2|^{\alpha+1}}{(|Y_1|^{\alpha q}+|Y_2|^{\alpha q})^{\frac{1}{q}}}=(|Y_1|^p+|Y_2|^p)^{\frac{1}{p}}\le 1\tag{3.1}$$

by Property 1. Hence we have (i).

Let $|\xi_1|^p + |\xi_2|^p = 1$ and introduce

$$a_1 = |\xi_1|^\alpha \operatorname{sgn}(\xi_1), \quad a_2 = |\xi_2|^\alpha \operatorname{sgn}(\xi_2). \tag{3.2}$$

Then

$$|a_1|^q + |a_2|^q = 1. \tag{3.3}$$

Hence by Hölder's inequality and (i)

$$\begin{aligned} a_1Y_1 + a_2Y_2 &\le |a_1Y_1| + |a_2Y_2| \le (|\xi_1|^{\alpha q} + |\xi_2|^{\alpha q})^{\frac{1}{q}} (|Y_1|^p + |Y_2|^p)^{\frac{1}{p}} \\ &= (|\xi_1|^p + |\xi_2|^p)^{\frac{1}{q}} (|Y_1|^p + |Y_2|^p)^{\frac{1}{p}} \le 1. \end{aligned} \tag{3.4}$$

By Property 1 there exists a sequence of time points such that for the corresponding limit vector (Y_1, Y_2)

$$a_1Y_1 + a_2Y_2 = 1. \tag{3.5}$$

Hence for such a (Y_1, Y_2) we have equality everywhere in (3.4). By Hölder's inequality if we have equality in the second place in (3.4), then $|a_1|^q = |\xi_1|^{\alpha q} = |Y_1|^p$ and $|a_2|^q = |\xi_2|^{\alpha q} = |Y_2|^p$. Since $\alpha q = p$, this implies $|\xi_1| = |Y_1|$, $|\xi_2| = |Y_2|$. As we have equality in the first place of (3.4) as well, and as (3.2) implies that $\operatorname{sgn}(\xi_1) = \operatorname{sgn}(a_1)$, $\operatorname{sgn}(\xi_2) = \operatorname{sgn}(a_2)$, we get that $\xi_1 = Y_1$, $\xi_2 = Y_2$, proving (ii). □

Clearly Lemma 3.1 can be proved for (Y_1, Y_2, Y_3), where

$$Y_3 = \frac{X(3t) - X(2t)}{\chi(t)}.$$

Lemma 3.2. *Denote the set of the limit points of the vectors*

$$\left(\frac{X(t)}{\chi(t)}, \frac{X(2t) - X(t)}{\chi(t)}, \frac{X(3t) - X(2t)}{\chi(t)} \right)$$

by $\mathcal{M}_3$ ($\mathcal{M}_3 \subset \mathbf{R}^3$). Then

(i) $\mathcal{M}_3 \subset \{(\xi_1, \xi_2, \xi_3) : \ |\xi_1|^p + |\xi_2|^p + |\xi_3|^p \le 1\} \quad (p > 0)$,

(ii) $\{(\xi_1, \xi_2, \xi_3) : \ |\xi_1|^p + |\xi_2|^p + |\xi_3|^p = 1\} \subset \mathcal{M}_3$.

Lemma 3.3. *In the notation of Lemma* 3.1*, we have*

$$\mathcal{M}_2 = \{(\xi_1, \xi_2) : \ |\xi_1|^p + |\xi_2|^p \le 1\}.$$

Proof of Lemma 3.3. Let

$$|\xi_1|^p + |\xi_2|^p \le 1$$

and select a ξ_3 satisfying

$$|\xi_1|^p + |\xi_2|^p + |\xi_3|^p = 1.$$

Then by Lemma 3.2 there exists a sequence of time points such that for the corresponding limit point (Y_1, Y_2, Y_3) we have

$$(Y_1, Y_2, Y_3) = (\xi_1, \xi_2, \xi_3).$$

Hence

$$(Y_1, Y_2) = (\xi_1, \xi_2),$$

and we have Lemma 3.3. □

Let $f \in C[0,1]$ and define

$$f^{(d)}(x) = f\left(\frac{i-1}{d}\right) + d\left(f\left(\frac{i}{d}\right) - f\left(\frac{i-1}{d}\right)\right)\left(x - \frac{i-1}{d}\right)$$

$$\text{if } \frac{i-1}{d} \le x \le \frac{i}{d}, \quad i = 1, 2, ..., d,$$

$$C_d = \{f^{(d)} : \ f \in C[0,1]\},$$

$$\mathcal{D}_d^{(p)} = \{f^{(d)} : \ f \in \mathcal{D}^{(p)}\},$$

where $\mathcal{D}^{(p)}$ was defined in (2.9).

Lemma 3.3 can be reformulated as follows: *The limit set of* $\eta_t^{(2)}(x)$ *is* $\mathcal{D}_2^{(p)}$. Similarly we have

Lemma 3.4. *For any* $d = 1, 2, \ldots$ *the limit set of* $\eta_t^{(d)}(x)$ *is* $\mathcal{D}_d^{(p)}$.

Proof of Theorem 2.3. Clearly

$$\sup_{0 \le x \le 1} |\eta_t(x) - \eta_t^{(d)}(x)| \le \sup_{0 \le x \le 1} \sup_{0 \le s \le \frac{1}{d}} |\eta_t(x+s) - \eta_t(x)|.$$

By Property 2

$$\limsup_{t \to \infty} \sup_{0 \le x \le 1} |\eta_t(x) - \eta_t^{(d)}(x)| \le A(1/d).$$

In order to complete the proof of Theorem 2.3 we need to show that

$$\lim_{d \to \infty} \mathcal{D}_d^{(p)} = \mathcal{D}^{(p)}. \tag{3.6}$$

But this clearly follows from Lemma C. □

4. Proof of Theorem 2.4

First we prove the following

Lemma 4.1. *In the notations of Theorem* 2.4*, let* $\alpha_i, \ i = 1, \ldots, k$*, be real numbers with* $\sum_{i=1}^k |\alpha_i|^{2\bar{\beta}} = 1$*, where* $1/\beta + 1/\bar{\beta} = 1$*. Then*

$$\limsup_{t \to \infty} \frac{\sum_{i=1}^k \alpha_i (Y(it) - Y((i-1)t))}{G(t)} = 1 \quad a.s. \tag{4.1}$$

and

$$\liminf_{t \to \infty} \frac{\sum_{i=1}^k \alpha_i (Y(it) - Y((i-1)t))}{G(t)} = -1 \quad a.s. \tag{4.2}$$

Proof of Lemma 4.1. Our proof relies on the results of Marcus and Rosen [16] (see Theorem B above). Under our conditions κ is regularly varying with index $-1/\bar{\beta}$ (see [16]). Put

$$S_t = \sum_{i=1}^k \alpha_i (Y(it) - Y((i-1)t)).$$

First we show that

$$\limsup_{t \to \infty} \frac{S_t}{G(t)} \le 1. \tag{4.3}$$

Given $\{L_{it},\ i = 1, \ldots, k\}$, we see that $\{Y(it) - Y((i-1)t) = W(L_{it}) - W(L_{(i-1)t}),\ i = 1, \ldots, k\}$ are independent normal random variables with mean zero and variances $\{L_{it} - L_{(i-1)t},\ i = 1, \ldots, k\}$; hence we have

$$E(\exp(\lambda S_t)) = E\left(\exp\left(\frac{\lambda^2}{2}\sum_{i=1}^{k}\alpha_i^2(L_{it} - L_{(i-1)t})\right)\right)$$

and by (2.9) and (4.2) of [16]

$$\begin{aligned} P(S_t \geq z) &\leq \exp(-\lambda z)\prod_{i=1}^{k} E\left(\exp\left(\frac{\lambda^2\alpha_i^2}{2}L_t\right)\right) \\ &\leq C^k \exp(-\lambda z)\prod_{i=1}^{k}\exp\left(\kappa^{-1}\left(\frac{2}{\lambda^2\alpha_i^2}\right)t\right) \\ &= C^k \exp\left(-\lambda z + \sum_{i=1}^{k}\kappa^{-1}\left(\frac{2}{\lambda^2\alpha_i^2}\right)t\right) \end{aligned}$$

with some constant C.

Now put $z = (1+\varepsilon)G(t)$ and

$$\lambda = \frac{\bar{K}_\beta}{(\kappa(\frac{\log\log t}{t}))^{1/2}}, \qquad \bar{K}_\beta = 2^{1/2}\left(\frac{\beta-1}{\beta+1}\right)^{(\beta-1)/(2\beta)}.$$

Using the regular variation of κ at zero, we have for large enough t,

$$\begin{aligned} &P(S_t \geq (1+\varepsilon)G(t)) \\ &\quad\leq C^k \exp\left(-(1+\varepsilon)K_\beta\bar{K}_\beta\log\log t + (1+\varepsilon)\kappa^{-1}\left(\frac{1}{\lambda^2}\right)2^{-\bar{\beta}}\sum_{i=1}^{k}|\alpha_i|^{2\bar{\beta}}t\right) \\ &\quad\leq C^k \exp(-(1+\varepsilon)(K_\beta\bar{K}_\beta - 2^{-\bar{\beta}}\bar{K}_\beta^{2\bar{\beta}})\log\log t) \\ &\quad= C^k \exp(-(1+\varepsilon)\log\log t). \end{aligned}$$

By taking $t = t_n = \theta^n$, $\theta > 1$, it follows that

$$\limsup_{n\to\infty}\frac{S_{t_n}}{G(t_n)} \leq 1 \quad \text{a.s.}$$

If $t_{n-1} < t \leq t_n$, then

$$\begin{aligned} |S_t - S_{t_{n-1}}| &= |\sum_{i=1}^{k}\alpha_i(Y(it) - Y(it_{n-1}) - Y((i-1)t) + Y((i-1)t_{n-1}))| \\ &\leq 2k\max_{1\leq i\leq k}\max_{t_{n-1}<t\leq t_n}|Y(it) - Y(it_{n-1})|. \end{aligned}$$

But since (see [16])

$$L_{it_n} - L_{it_{n-1}} = O\left((\log\log t_n)\kappa\left(\frac{\log\log t_n}{t_n - t_{n-1}}\right)\right) \quad \text{a.s.}$$

it follows from the increment results of Csörgő and Révész [8, Theorem 1.2.1, p.30], that

$$\max_{1\leq i\leq k}\max_{t_{n-1}<t\leq t_n}|Y(it) - Y(it_{n-1})| = O\left((\log\log t_n)\kappa^{1/2}\left(\frac{\log\log t_n}{t_n - t_{n-1}}\right)\right) \quad \text{a.s.}$$

which can be made arbitrary small compared to $G(t_n)$ by choosing θ close to 1. This proves (4.3).

Now we prove

$$\limsup_{t\to\infty} \frac{S_t}{G(t)} \geq 1 \qquad \text{a.s.} \tag{4.4}$$

Put

$$V_t = \sum_{i=1}^{k} \alpha_i^2 (L_{it} - L_{(i-1)t}).$$

Then $S_t/V_t^{1/2}$ and V_t are independent random variables, the first one being standard normal. Consider the events

$$A_t = \left\{ S_t \geq (c_1 V_t \log\log t)^{1/2},\ V_t \geq (1-\varepsilon) c_2 (\log\log t) \kappa\left(\frac{\log\log t}{t}\right) \right\}$$

with $\varepsilon > 0$ and

$$\frac{c_1}{2} + (\bar\beta - 1)\left(\frac{c_2}{\bar\beta}\right)^{\bar\beta/(\bar\beta-1)} \leq 1. \tag{4.5}$$

By using the lower estimation (4.10) in Marcus and Rosen [16], one can see similarly to the proof of their Lemma 3.2 that

$$P^x\left(V_t \geq (1-\varepsilon) c_2 (\log\log t)\kappa\left(\frac{\log\log t}{t}\right)\right)$$

$$\geq C a_\nu(x,t) \exp\left(-\lambda(1-\varepsilon)c_2(\log\log t)\kappa\left(\frac{\log\log t}{t}\right) + \sum_{i=1}^{k} \kappa^{-1}\left(\frac{1}{\lambda\alpha_i^2}\right) t\right)$$

with sufficiently small ν, where P^x denotes the conditional probability under $Z(0) = x$,

$$a_\nu(x,t) = \frac{\int_0^{\nu t} p_v(x)\, dv}{\kappa(\frac{\log\log t}{t})}$$

and

$$\frac{1}{\lambda} = \left(\frac{\bar\beta}{c_2}\right)^{1/(\bar\beta-1)} \kappa\left(\frac{\log\log t}{t}\right).$$

By the independence of $S_t/V_t^{1/2}$ and V_t we obtain

$$P^x(A_t) \geq C a_\nu(x,t) \exp\left(-\left(\frac{c_1}{2} + (\bar\beta-1)\left(\frac{c_2}{\bar\beta}\right)^{\bar\beta/(\bar\beta-1)}\right) \log\log t\right)$$

$$\geq C a_\nu(x,t) \exp(-\log\log t)$$

provided that (4.5) holds.

Now let θ be large enough, $t_n = \theta^n$, put

$$S_n^* = \alpha_1(Y(t_n) - Y(kt_{n-1})) + \sum_{i=2}^{k} \alpha_i (Y(it_n) - Y((i-1)t_n))$$

and

$$V_n^* = \alpha_1^2(L_{t_n} - L_{kt_{n-1}}) + \sum_{i=2}^{k} \alpha_i^2(L_{it_n} - L_{(i-1)t_n})$$

and consider the events

$$A_n^* = \left\{ S_n^* \geq (c_1 V_n^* \log\log t_n)^{1/2},\ V_n^* \geq (1-\varepsilon)c_2(\log\log t_n)\kappa\left(\frac{\log\log t_n}{t_n}\right)\right\}.$$

By using the above estimations and the argument of Marcus and Rosen [16] to prove their lower bound, similarly to their (3.32) one can see that

$$\sum_{n=1}^{\infty} P^{Z(t_{n-1})}(A_n^*) = \infty \quad \text{a.s.}$$

implying

$$P(A_n^* \text{ i.o.}) = 1.$$

Upon choosing

$$c_1 = \frac{2\beta}{\beta+1}, \qquad c_2 = \frac{\beta}{(\beta-1)^{(\beta-1)/\beta}(\beta+1)^{1/\beta}},$$

we can see that

$$\limsup_{n\to\infty} \frac{S_n^*}{G(t_n)} \geq 1 \quad \text{a.s.}$$

Since

$$Y(kt_{n-1}) = O(G(t_{n-1})) \quad \text{a.s}$$

and this can be made arbitrary small compared to $G(t_n)$ by choosing θ large enough, we have (4.4). This completes the proof of (4.1). The proof of (4.2) is similar.

Lemma 4.2. *Under the conditions of Theorem* 2.4

$$\limsup_{T\to\infty} \sup_{0\leq t\leq T-cT} \sup_{0\leq s\leq cT} \frac{|Y(t+s)-Y(t)|}{G(T)} = O\left(c^{1/(2\bar{\beta})}\right) \tag{4.6}$$

Proof. As $Y(t) = W(L_t)$, according to the previously mentioned increment result in Csörgő and Révész [8, Theorem 1.2.1, p.30]

$$\limsup_{T\to\infty} \sup_{0\leq t\leq T-cT} \sup_{0\leq s\leq cT} \frac{|W(L_{t+s})-W(L_t)|}{G(T)}$$

$$\leq \limsup_{T\to\infty} \sup_{0\leq u\leq L_T} \sup_{0\leq y\leq a_c(T)} \frac{|W(u)-W(u-y)|}{G(T)}$$

$$\leq O(1)\limsup_{T\to\infty} \frac{\left(a_c(T)\log\frac{L_T}{a_c(T)} + \log\log L_T\right)^{1/2}}{G(T)}, \tag{4.7}$$

$$a_c(T) = \sup_{0\leq t\leq T-cT} (L_{t+cT} - L_t).$$

From [16, Theorem 1.1 and (4.17)] we obtain

$$L_T = O\left((\log\log T)\kappa\left(\frac{\log\log T}{T}\right)\right), \tag{4.8}$$

$$a_c(T) = O\left(c^{1/\bar{\beta}} L_T\right). \tag{4.9}$$

Combining (4.6)–(4.8) gives the result of the lemma. □

Proof of Theorem 2.4. In Lemmas 4.1 and 4.2 it was proved that the process $Y(t)$ possesses the two properties required by Theorem 2.3. Hence the theorem follows. □

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