SIGNÉ QUASI-MEASURES

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Abstract. Let $X$ be a compact Hausdorff space and let $A$ denote the subsets of $X$ which are either open or closed. A quasi-linear functional is a map $\rho : C(X) \to \mathbb{R}$ which is linear on singly generated subalgebras and such that $|\rho (f)| \leq M\|f\|$ for some $M < \infty$. There is a one-to-one correspondence between the quasi-linear functional on $C(X)$ and the set functions $\mu : A \to \mathbb{R}$ such that i) $\mu (\emptyset) = 0$, ii) If $A, B, A \cup B \in A$ with $A$ and $B$ disjoint, then $\mu (A \cup B) = \mu (A) + \mu (B)$, iii) There is an $M < \infty$ such that whenever $\{U_\alpha \}$ are disjoint open sets, $\sum |\mu (U_\alpha)| \leq M$, and iv) if $U$ is open and $\varepsilon > 0$, there is a compact $K \subseteq U$ such that whenever $V \subseteq U \setminus K$ is open, then $|\mu (V)| < \varepsilon$. The space of quasi-linear functionals is investigated and quasi-linear maps between two $C(X)$ spaces are studied.

Let $X$ be a compact Hausdorff space and $C(X)$ the space of real-valued continuous functions on $X$. A map $\rho : C(X) \to \mathbb{R}$ is said to be a quasi-linear functional if $\rho$ is linear on singly generated subalgebras and bounded in the sense that there exists an $M < \infty$ such that $|\rho (f)| \leq M\|f\|$ for all $f \in C(X)$. Let $\|\rho\|$ be the minimal such $M$. If $\rho$ and $\eta$ are quasi-linear functionals, we define $\rho + \eta$ by pointwise action on functions. In this fashion, the collection of all quasi-linear functionals becomes a normed linear space. Call this space $QL(X)$.

Notice that if $\rho$ is quasi-linear, and $fg = 0$, then $\rho (f + g) = \rho (f) + \rho (g)$. In fact, if $f$ and $g$ are also positive, we have that the subalgebra generated by $f - g$ contains both $f$ and $g$. In general, we can break $f$ and $g$ into positive and negative parts to get the result. Also notice that if $c$ is a constant, $\rho (c + f) = \rho (c) + \rho (f)$. Thus, if $f$ is constant on the support of $g$, we still have that $\rho (f + g) = \rho (f) + \rho (g)$.

Our goal is to find set functions that produce all quasi-linear functionals on $C(X)$. We will use an approach inspired by the techniques in [1] where the theory of positive quasi-linear functionals is presented. We use the notation $f \prec U$ when $U$ is open to state that $0 \leq f \leq 1$ and $f$ has support contained in $U$. We also use the notation $sp f$ for the image of $f$.

Let $O$ be the collection of open sets in $X$ and $C$ the collection of closed sets. Also, let $A = O \cup C$. Thus $A$ is the collection of subsets of $X$ which are either open or closed.

Definition 1. A function $\mu : A \to \mathbb{R}$ is called a signed quasi-measure if the following hold:

(i) $\mu (\emptyset) = 0$,
(ii) If $A, B \in A$ are disjoint with $A \cup B \in A$, then $\mu (A \cup B) = \mu (A) + \mu (B)$.
(iii) There is a constant $M < \infty$ such that whenever $\{U_n\}$ is a finite disjoint collection of open sets, then $\sum |\mu(U_n)| \leq M$.

(iv) If an open set $U$ and $\varepsilon > 0$ are given, there exists a closed set $K \subseteq U$ such that if $V$ is an open set with $V \subseteq U \setminus K$, we have $|\mu(V)| < \varepsilon$.

We define $\|\mu\|$ to be the minimal $M$ such that (iii) holds.

For future reference, we note that property (ii) above is equivalent to the following three statements.

a) If $U$ and $V$ are disjoint open sets, then $\mu(U \cup V) = \mu(U) + \mu(V)$.

b) If $U$ and $V$ are open with $X = U \cup V$, then $\mu(U) + \mu(V) = \mu(X) + \mu(U \cap V)$.

c) If $U$ is open, then $\mu(X \setminus U) = \mu(X) - \mu(U)$.

This will allow us to define a quasi-measure by its action on only open sets.

Let $QM(X)$ denote the collection of all signed quasi-measures on $X$. If we define $\mu + \nu$ by action on sets, we see that $QM(X)$ is a normed linear space. We wish to show that in a natural way $QL(X)$ and $QM(X)$ are isomorphic as normed linear spaces, and are, in fact, Banach spaces.

Given a signed quasi-measure $\mu$, we may define a new set function $|\mu|$ on open sets by

$$|\mu|(U) = \sup \left\{ \sum |\mu(U_n)| : U_n \subseteq U \text{ are disjoint open sets} \right\}.$$ 

Then we see that $\|\mu\| = |\mu|(X)$. It is important to note here that $|\mu|$ need not yield a quasi-measure. In particular, it is impossible to define $|\mu|$ on closed sets so that (ii) holds. An example of this will be seen later. It is clear, however, that (i) and (iii) hold for $|\mu|$. We will see later that (iv) does also.

**Proposition 2.** We have the following:

a) If $\{A_\alpha\}_{\alpha \in \mathcal{A}} \subseteq \mathcal{A}$ is a collection (possibly infinite) of disjoint subsets of $U \in \mathcal{C}$, then $\sum |\mu(A_\alpha)| \leq |\mu|(U)$.

b) If $U_1 \subseteq U_2 \subseteq \cdots$ are open, then $\mu \left( \bigcup_{i=1}^{\infty} U_i \right) = \lim_{i \to \infty} \mu(U_i)$.

c) If $\{U_\alpha\}$ is a collection (possibly infinite) of disjoint open sets, then $\mu(\bigcup U_\alpha) = \sum \mu(U_\alpha)$.

**Proof.** Both b) and c) are results of the regularity assumption (iv). For a), notice that there is a similar outer regularity for closed sets, so if $\{F_n\}_{n=1}^N$ are finitely many closed sets contained in $U$, and $\varepsilon > 0$, we may find disjoint open sets $U_n$ with $F_n \subseteq U_n \subseteq U$ and $|\mu(U_n \setminus F_n)| < \varepsilon / N$. Then $\sum |\mu(F_n)| \leq \sum |\mu(U_n)| + \varepsilon \leq |\mu|(U) + \varepsilon$.

For the general case, we may restrict to finitely many $A_\alpha$, and approximate by closed sets using inner regularity for open sets.

Now, let $f \in C(X)$ and $\alpha \in \mathbb{R}$, and define $\hat{f}(\alpha) = \mu(f^{-1}(\alpha, +\infty))$ and $\check{f}(\alpha) = \mu(f^{-1}[\alpha, +\infty))$. Notice that $\mu(f^{-1}(\alpha, \beta)) = \hat{f}(\alpha) - \hat{f}(\beta)$ and $\mu(f^{-1}[\alpha, \beta]) = \check{f}(\alpha) - \check{f}(\beta)$.

**Proposition 3.** Let $f \in C(X)$. Then

a) $\check{f}$ is continuous from the right and $\hat{f}$ is continuous from the left.

b) $\hat{f}(\alpha^-) = \check{f}(\alpha)$ and $\hat{f}(\alpha^+) = \check{f}(\alpha)$.

c) $\hat{f}$ and $\check{f}$ agree except at countably many $\alpha \in \mathbb{R}$.

d) If $\check{f}(\alpha) = \hat{f}(\alpha)$, then $f$ is continuous at $\alpha$.

e) $\check{f}$ is of bounded variation with variation less than $\|\mu\|$.
We have that Lemma 4.

\[ \mu \text{ the map } \]

Proof. a) If \( \alpha_n \) decreases to \( \alpha \), then

\[
\hat{f}(\alpha) = \mu \left( f^{-1}(\alpha, +\infty) \right)
\]

\[
= \lim_{n \to \infty} \mu \left( f^{-1}(\alpha_n, +\infty) \right)
\]

\[
= \lim_{n \to \infty} \hat{f}(\alpha_n).
\]

Thus \( \hat{f} \) is continuous from the right. Since \( \hat{f}(\alpha) = \mu(X) - (-f)^\vee(-\alpha) \), \( \hat{f} \) is continuous from the left.

b) Let \( \alpha_0 \) and \( \varepsilon > 0 \) be given and let \( U = f^{-1}(\alpha_0, +\infty) \). Pick \( K \) as in iv) of the definition of quasi-measure. Let \( \beta \) be the minimum value of \( f \) on \( K \). Then \( \alpha_0 < \beta \). If \( \alpha_0 < \alpha < \beta \), we have that \( f^{-1}(\alpha_0, \alpha) \subseteq U \setminus K \), so

\[
|\hat{f}(\alpha_0) - \hat{f}(\alpha)| = |\mu \left( f^{-1}(\alpha_0, \alpha) \right)| < \varepsilon.
\]

This shows that \( \hat{f}(\alpha_0^+) = \hat{f}(\alpha_0) \).

c) Notice that

\[
\sum |\hat{f}(\alpha) - \hat{f}(\beta)| = \sum |\mu(f^{-1}(\{\alpha\}))| \leq \|\mu\|, \text{ by Proposition 2.}
\]

Thus the set of \( \alpha \) where \( \hat{f}(\alpha) \neq \hat{f}(\alpha) \) is at most countable.

d) This follows from parts a) and b).

e) If \( \{(\alpha_n, \beta_n)\} \) is a disjoint collection of intervals, then

\[
\sum |\hat{f}(\alpha_n) - \hat{f}(\beta_n)| \leq \sum |\hat{f}(\alpha_n) - \hat{f}(\beta_n)| + \sum |\hat{f}(\beta_n) - \hat{f}(\beta_n)|
\]

\[
= \sum |\mu(f^{-1}(\alpha_n, \beta_n))| + \sum |\mu(f^{-1}(\beta_n))|\]

\[
\leq \|\mu\|
\]

by a) of Proposition 2. Thus \( \hat{f} \) is of bounded variation with variation at most \( \|\mu\| \).

\[
\square
\]

Since \( \hat{f} \) is of bounded variation, there is a signed measure \( \mu_f \) on \( \mathbb{R} \) such that

\[
\mu_f(\alpha, \beta) = \hat{f}(\alpha) - \hat{f}(\beta) = \hat{f}(\alpha) - \hat{f}(\beta) = \mu(f^{-1}(\alpha, \beta)) \text{ and } \|\mu_f\| = |\mu_f| = \|\mu\|.
\]

If \( O \) is any open set in \( \mathbb{R} \), we may write \( O \) as a disjoint union of open intervals to see that \( \mu_f(O) = \mu \left( f^{-1}(O) \right) \). It follows that \( \mu_f \) is concentrated on \( \text{sp } f \).

If \( f \in C(X) \) and \( \varphi \in C(\text{sp } f) \), we let \( \varphi^* \mu_f \) denote the image measure of \( \mu_f \) under the map \( \varphi \). The following lemma simplifies the proof of Proposition 3.2 of [1].

**Lemma 4.** We have that \( \mu_{\varphi \circ f} = \varphi^* \mu_f \).

**Proof.** Let \( O \subseteq \mathbb{R} \) be open. Then

\[
\mu_{\varphi \circ f}(O) = \mu \left( (\varphi \circ f)^{-1}(O) \right)
\]

\[
= \mu \left( f^{-1}(\varphi^{-1}(O)) \right)
\]

\[
= \mu_f(\varphi^{-1}(O))
\]

\[
= (\varphi^* \mu_f)(O).
\]

\[
\square
\]

Now we may define the functional \( \rho_\mu(f) = \int_\mathbb{R} i \, d\mu_f \) where \( i : \mathbb{R} \to \mathbb{R} \) is the function \( i(x) = x \). Since \( \mu_f \) is concentrated on \( \text{sp } f \), we have \( |\rho_\mu(f)| \leq \int_\mathbb{R} |i| \, d|\mu_f| \leq \)
Claim 3: If $\mu$ is bounded with $\|\mu\| \leq \|\rho\|$. Also

$$\rho_\mu(f) = \int_R i \ d\mu_{\varphi \circ f}$$

$$= \int_R i \ d\alpha_{\varphi} \mu_f$$

$$= \int_R \varphi \ d\mu_f,$$

so $\rho_\mu(\varphi \circ f + \psi \circ f) = \int_R \varphi + \psi \ d\mu_f = \rho_\mu(\varphi \circ f) + \rho_\mu(\psi \circ f)$. Thus $\rho_\mu$ is a quasi-linear functional on $C(X)$.

Theorem 5. The map $\mu \rightarrow \rho_\mu$ is an isometric isomorphism of the normed linear space $QM(X)$ onto $QL(X)$.

Proof. It is easy to see that this map is linear. We show that it is onto $QL(X)$.

Suppose $\rho$ is a quasi-linear functional on $C(X)$.

Claim 1: If $U$ is open in $X$ and $\varepsilon > 0$, there is a closed $K \subseteq U$ such that if $f \in C(X)$, $\|f\|_u \leq 1$, $\text{supp } f \subseteq U$, and $f = 0$ on $K$, then $|\rho(f)| < \varepsilon$.

Suppose no such $K$ exists for some $\varepsilon > 0$. Pick $f_1 \in C(X)$ such that $\|f_1\|_u \leq 1$, $\text{supp } f_1 \subseteq U$, and $|\rho(f_1)| \geq \varepsilon$. Pick $|a_1| = 1$ such that $\rho(a_1 f_1) = |\rho(f_1)| \geq \varepsilon$. Now $K_1 = \text{supp } f_1$ fails the conditions of the claim, so there is an $f_2 \in C(X)$ supported in $U$ such that $\|f_2\| \leq 1$, $f_2 = 0$ on $K_1$, and $|\rho(f_2)| \geq \varepsilon$. Pick $|a_2| = 1$ such that $\rho(a_2 f_2) = |\rho(f_2)| \geq \varepsilon$. Since $f_2 = 0$ on the support of $f_1$, we have that $\rho(a_1 f_1 + a_2 f_2) = \rho(a_1 f_1) + \rho(a_2 f_2) \geq 2\varepsilon$ and $\|a_1 f_1 + a_2 f_2\|_u \leq 1$. Continuing by induction, we may find $f_n \in C(X)$ supported in $U$ that vanishes on the support of $a_1 f_1 + a_2 f_2 + \cdots + a_n f_n$ and $\|f_n\| \leq 1$, while $\rho(a_n f_n) = |\rho(f_n)| \geq \varepsilon$. But then $\rho(\sum_{i=1}^n a_i f_i) \geq n\varepsilon$, which violates the boundedness of $\rho$ for large $n$.

Claim 2: For $U$ open, $\lim_{f \in U} \rho(f)$ exists where the $f$ are ordered pointwise.

We show that this net is a Cauchy net. In fact, let $\varepsilon > 0$ and let $K \subseteq U$ be the closed set of Claim 1. Let $f$ be any function such that $f = 1$ on $K$ and $f \ll U$. If $f \leq g, h \ll U$, then pick $k$ with $k = 1$ on $g \cup \text{supp } h$, and $k \ll U$. Then $g - k$ and $h - k$ vanish on $K$, so we have that $|\rho(g) - \rho(h)| \leq |\rho(g) - \rho(k)| + |\rho(h) - \rho(k)| = |\rho(g - k)| + |\rho(h - k)| \leq 2\varepsilon$.

Define $\mu(U) = \lim_{f \in U} \rho(f)$ for $U$ open in $X$.

Claim 3: $\mu$ is a signed quasi-measure on $X$.

Easily, $\mu(U \cup V) = \mu(U) + \mu(V)$ if $U$ and $V$ are disjoint. Also $\mu(\emptyset) = 0$. Notice also that $\mu(X) = \rho(1)$. We next show property b) after the definition of a signed quasi-measure.

Suppose that $U \cup V = X$. Pick $C \subseteq U$ and $K \subseteq V$, closed such that $C \cup K = X$. Pick $f_0 \ll U$, $g_0 \ll V$ with $f_0 = 1$ on $C$, $g_0 = 1$ on $K$, and such that $f_0 \ll f \ll U$ implies $|\rho(f) - \mu(U)| < \varepsilon$ and $g_0 \ll g \ll V$ implies $|\rho(g) - \mu(V)| < \varepsilon$. Let $h_0 \ll U \cap V$ with $h_0^2 = 1$ on $C \cap K$ and such that $h_0^2 \ll h \ll U \cap V$ implies that $|\rho(h) - \mu(U \cap V)| < \varepsilon$. Now, set $f = \max\{f_0, h_0\}$ and $g = \max\{g_0, h_0\}$. Then $f \ll U$, $g \ll V$, and $fg \ll U \cap V$, so $|\rho(f) - \mu(U)| < \varepsilon$, $|\rho(g) - \mu(V)| < \varepsilon$ and $|\rho(fg) - \mu(U \cap V)| < \varepsilon$. Also, since $f = 1$ on $C$ and $g = 1$ on $K$, and $C \cup K = X$, we have that $(1 - f)(1 - g) = 0$, so $\rho((1 - f) + (1 - g)) = \rho(1 - f) + \rho(1 - g)$ and
\[ \rho(f + g) = \rho(1 + fg). \] This gives that \( \rho(f) + \rho(g) = \rho(1 + fg) \), which shows that \( |\mu(U) + \mu(V) - \mu(X) - \mu(U \cap V)| < 3\varepsilon. \)

Now suppose that \( \{U_n\}_{n=1}^N \) is a finite, disjoint collection of open sets. Let \( \varepsilon > 0 \) be given and choose \( f_n \prec U_n \) such that \( |\rho(f_n) - \mu(U_n)| < \varepsilon \). Now choose \( |a_n| = 1 \) such that \( |\rho(f_n)| = \rho(a_n f_n) \). Then \( \rho \left( \sum a_n f_n \right) \leq ||\rho||, \) so \( \sum |\mu(U_n)| \leq ||\rho|| + N\varepsilon. \) Now let \( \varepsilon \to 0 \).

Finally, if \( U \) is an open set and \( \varepsilon > 0 \) are given, choose \( K \subseteq U \) as in Claim 1, and argue as in the previous paragraph to show that if \( \{U_n\} \) are disjoint and open with \( U_n \subseteq U \setminus K, \) then \( \sum |\mu(U_n)| < \varepsilon. \) In particular, if \( V \subseteq U \setminus K \) is open, \( |\mu(V)| < \varepsilon. \)

Notice that \( ||\mu|| \leq ||\rho||. \)

**Claim 4:** We have that \( \rho = \rho_{\mu}. \)

For each \( f \in C(X), \) the map \( \varphi \to \rho(\varphi \circ f) \) is bounded and linear on \( C(sp \ f). \) Thus, there is a signed measure \( \nu_f \) on \( sp \ f \) such that

\[ \rho(\varphi \circ f) = \int_{\mathbb{R}} \varphi \ d\nu_f \]

for all \( \varphi \in C(sp \ f). \) Since \( \rho(f) = \int_{\mathbb{R}} i \ d\nu_f, \) we need only show that \( \nu_f = \mu_f \) for each \( f \in C(X) \). Notice that both measures are measures on \( \mathbb{R}. \)

Suppose that \( O \) is an open set in \( \mathbb{R}. \) Pick closed sets \( C_n \subseteq O \) such that \( C_n \subseteq \text{int}(C_{n+1}) \) and \( O = \bigcup C_n. \) Choose \( \varphi_n \prec O \) such that \( \varphi_n = 1 \) on \( C_n. \) Since \( X \) is compact, the sequence \( \varphi_n \circ f \) is cofinal in the collection of functions \( g \) such that \( g \prec f^{-1}(O). \) Thus

\[ \nu_f(O) = \lim_{n \to \infty} \int \varphi_n \ d\nu_f = \lim_{n \to \infty} \rho(\varphi_n \circ f) = \mu \left( f^{-1}(O) \right) = \mu_f(O) \]

giving the required equality of measures. Thus \( ||\mu|| \leq ||\rho|| = ||\rho_{\mu}|| \leq ||\mu||. \)

This shows that the map \( \mu \to \rho_{\mu} \) is onto \( QL(X), \) and in fact, that any \( \rho \in QL(X) \) is the image of some \( \mu \in QM(X) \) of the same norm. If we show that our map is one-to-one, we will be finished.

Assume \( \rho_{\mu} = 0. \) Then \( \mu_f = 0 \) for all \( f \in C(X). \) Thus \( \hat{f}(\alpha) = \mu_f(\alpha, +\infty) = 0 \) for all \( \alpha \in \mathbb{R}. \) If, now, \( U \subseteq X \) is open and \( \varepsilon > 0, \) pick \( K \subseteq U \) as in part (iv) of the definition of a signed quasi-measure. Choose any \( f \in C(X) \) with \( K \prec f \prec U. \) Then

\[ |\mu(U)| \leq |\mu(U) - \mu(K)| + |\hat{f}(1) - \mu(K)| + |\hat{f}(1)| \]
\[ = |\mu(U \setminus K)| + |\mu((f^{-1}(1/2, +\infty)) \setminus K)| + |\hat{f}(1)| \]
\[ \leq 2\varepsilon. \]

Thus \( \mu(U) = 0 \) for all open sets, so \( \mu = 0. \)

It should be noted that in Claim 2 we actually showed a stronger form of regularity. If \( U \) is open and \( \varepsilon > 0, \) then there is a closed set \( K \subseteq U \) with \( |\mu|(U \setminus K) < \varepsilon. \) Thus \( |\mu| \) obeys part (iv) of the definition of a quasi-measure.
There is yet another representation of $QL(X)$ that is sometimes useful. For each $f \in C(X)$, let $M(\text{sp} f)$ denote the collection of regular Borel measures on the compact set $\text{sp} f$ with the usual measure norm. Define $PM(X)$ to be

$$\left\{ (\nu_f) \in \prod_{f \in C(X)} M(\text{sp} f) : \nu_{\varphi f} = \varphi^* \nu_f \text{ for } \varphi \in C(\text{sp} f) \text{ and } \sup \|\nu_f\| < \infty \right\}.$$  

Define a norm on $PM(X)$ by $\| (\nu_f) \| = \sup \|\nu_f\|$. Then it is easy to see that $PM(X)$ is a Banach space since this is true for each $M(\text{sp} f)$.

If $\mu$ is a signed quasi-measure, then the collection $(\mu_f)$ in the definition of $\rho_\mu$ is an element of $PM(X)$ with $\| (\mu_f) \| \leq \| \mu \|$. The induced map from $QM(X)$ to $PM(X)$ is evidently linear.

On the other hand, if $(\nu_f) \in PM(X)$, we may define $\rho(f) = \int f \, d\nu_f$. Then the argument just before the statement of the theorem shows that $\rho \in QL(X)$ with $\|\rho\| \leq \| (\nu_f)\|$. This, with the last paragraph shows that $PM(X)$ is isometrically isomorphic to both $QM(X)$ and $QL(X)$.

In particular $QM(X)$ is a Banach space. We can make it an ordered Banach space by taking the positive cone to be the collection of positive quasi-measures. The norm on this space is additive on the positive cone, but $QM(X)$ does not have to be a lattice. Thus, $QM(X)$ need not be an $L$-space. For example, in [3], Aarnes finds positive $[0, 1]$-valued quasi-measures $\mu_1, \mu_2, \mu_3, \mu_4$ with $\mu_1 + \mu_3 = \mu_2 + \mu_4$. Since $[0, 1]$-valued quasi-measures are extremal, there will then be no supremum of $\{\mu_1, \mu_2\}$. 

For future convenience, we define the notation $\langle \mu, f \rangle = \langle \rho_\mu, f \rangle = \rho_\mu(f)$.

Another aspect of the failure of the lattice property is that the set function $|\mu|$ need not have an extension to a positive quasi-measure on $X$. An example of this is given next.

**Example.** Let $X = [0, 1] \times [0, 1]$. In [2] and [6], it is shown how to construct the so-called three point quasi-measures. This is done as follows. A subset $A$ of $X$ is said to be solid if both $A$ and $X \setminus A$ are connected. If $C = \{x_1, x_2, x_3\}$ is a set with three elements, we define $\mu_C$ on solid sets by

$$\mu_C(A) = \begin{cases} 0 & \text{if card}(A \cap C) \leq 1, \\ 1 & \text{if card}(A \cap C) \geq 2. \end{cases}$$

There is then a unique extension of $\mu_C$ to a $[0, 1]$-valued quasi-measure on $X$.

Now let $x_1 = (0, 0), x_2 = (1, 0), x_3 = (1, 1),$ and $x_4 = (0, 1)$. Let $C = \{x_1, x_2, x_3\}$, $D = \{x_2, x_3, x_4\}$, and $\mu = \mu_C - \mu_D$. If we let $U_1 = [0, 1] \times [0, \frac{1}{2})$ and $U_2 = [0, 1] \times (\frac{1}{2}, 1]$, we see that $2 = |\mu(U_1)| + |\mu(U_2)| \leq |\mu|(X) = \|\mu\| \leq 2$. Hence, $|\mu|(X) = 2$. We show that $|\mu|$ cannot be extended to closed sets in such a way that it is a positive quasi-measure. In particular, (ii) does not hold in the definition of a quasi-measure.

Assume such an extension exists. Let $V_1 = [0, \frac{3}{4}) \times [0, 1]$ and $V_2 = (\frac{1}{4}, 1] \times [0, 1]$. Write $K_1 = X \setminus V_1$ and $K_2 = X \setminus V_2$. Since $\mu_C(V_1) = \mu_D(V_1) = 0$, we see that $|\mu|(V_1) = 0 = |\mu|(V_1 \cap V_2)$. Hence $|\mu|(K_1) = |\mu|(K_1 \cup K_2) = 2$. Since $K_1$ and $K_2$ are disjoint, we must have that $|\mu|(K_2) = 0$, in other words that $|\mu|(V_2) = 2$. However this is not the case. In fact, $|\mu|(V_2) = 0$.

To see this we show that if $U \subseteq V_2$ is open, then $\mu(U) = 0$; that is $\mu_C(U) = \mu_D(U)$. Using symmetry and the fact that both $\mu_C$ and $\mu_D$ take on only 0 and
1 as values, we only show that \( \mu_C(U) = 0 \) implies \( \mu_D(U) = 0 \). Furthermore, by considering components, it is enough to show this for connected open sets.

Suppose, then, that \( U \) is connected and let \( K \) be any connected closed subset of \( U \). Let \( \hat{K} \) be the solid hull of \( K \) as a subset of \( V_2 \) as in [2]. Then \( \hat{K} \) is a solid closed set in \( V_2 \) containing \( K \). If \( \mu_C(\hat{K}) = 0 \), then \( \{ x_2, x_3 \} \cap \hat{K} \) has at most one element, so \( \mu_D(K) \leq \mu_D(\hat{K}) = 0 \). On the other hand, if \( \mu_C(\hat{K}) = 1 \), there is some interior component \( W \) of \( X \setminus K \) with \( \mu_C(W) = 1 \). This follows since \( \mu_C(K) = 0 \). Since \( W \subseteq \hat{K} \subseteq V_2 \), we have that \( x_2, x_3 \in W \). Since \( W \) is solid, we then have that \( \mu_D(W) = 1 \), so \( \mu_D(K) \leq \mu_D(X \setminus W) = 0 \). Finally, \( \mu_D(U) \) is the supremum of \( \mu_D(K) \) as above, so \( \mu_D(U) = 0 \).

It would be nice to know that every signed quasi-measure is the difference of two positive quasi-measures. However, the failure of the lattice property and the fact that \( |\mu| \) need not be a quasi-measure brings this into question. At this point the issue remains open.

We now turn to another topology on \( QM(X) \) that is very useful.

**Definition 6.** The weak-* topology on \( QM(X) \) is the weakest topology making each map \( \mu \to \langle \mu, f \rangle \) continuous where \( f \) ranges over \( C(X) \).

Since each such map is linear and the collection of these maps separates points of \( QM(X) \), we obtain a locally convex topology on \( QM(X) \) where a net \( \mu_\alpha \) converges to \( \mu \) if and only if \( \langle \mu_\alpha, f \rangle \) converges to \( \langle \mu, f \rangle \) for every \( f \in C(X) \). This topology has been studied on the space of positive quasi-measures in [3].

**Proposition 7.** Let \( \mu_\alpha \) be a net in \( QM(X) \) and \( \mu \in QM(X) \). Let \( (\mu_{\alpha, f}) \) and \( (\mu_f) \) be the corresponding elements of \( PM(X) \). Then \( \mu_\alpha \) converges to \( \mu \) in the weak-* topology if and only if \( \mu_{\alpha, f} \) converges to \( \mu_f \) in the weak-* topology of \( M(\text{sp } f) \) for each \( f \in C(X) \). Thus, the unit ball in \( QM(X) \) is weak-* compact.

**Proof.** If \( \mu_{\alpha, f} \) converges to \( \mu_f \) for each \( f \in C(X) \), then \( \langle \mu_\alpha, f \rangle = \int_{\mathbb{R}} i \, d\mu_{\alpha, f} \) converges to \( \int_{\mathbb{R}} i \, d\mu_f = \langle \mu, f \rangle \).

Conversely, if \( \mu_\alpha \) converges to \( \mu \) weak-* , then for each \( \varphi \in C(\text{sp } f) \), we have

\[
\lim_{\alpha} \int_{\mathbb{R}} \varphi \, d\mu_{\alpha, f} = \lim_{\alpha} \langle \mu_\alpha, \varphi \circ f \rangle = \langle \mu, \varphi \circ f \rangle = \int_{\mathbb{R}} \varphi \, d\mu_f.
\]

Thus \( \mu_{\alpha, f} \) converges to \( \mu_f \) in the weak-* topology.

Since the map \( \mu_f \to \varphi^* \mu_f \) is weak-* continuous, the compactness of the unit ball of \( QM(X) \) follows from the compactness of the unit balls of \( M(\text{sp } f) \). \( \Box \)

**Definition 8.** Let \( X \) and \( Y \) be compact Hausdorff spaces. A quasi-linear map from \( C(X) \) to \( C(Y) \) is a map, \( T \), which is linear on each singly generated subalgebra of \( C(X) \) and which is bounded in the sense that there is an \( M < \infty \) with \( \|T(f)\| \leq M\|f\| \). If, in addition, \( T \) is multiplicative on each singly generated subalgebra, we say that \( T \) is a quasi-homomorphism.

For example, let \( \rho \) be a positive quasi-linear functional on \( X \) and let \( Y \) be any compact Hausdorff space. Define \( T_\rho : C(X \times Y) \to C(Y) \) by \( T_\rho(f)(y) = \rho(f^y) \) where \( f^y(x) = f(x, y) \). It is noted in [5] that \( T_\rho \) is a quasi-linear map. If \( \eta : C(Y) \to \mathbb{R} \)
is an additional quasi-linear functional, it is also noted there that the composition of $T_\rho$ and $\eta$ need not be quasi-linear.

**Proposition 9.** There is a one to one correspondence between quasi-linear maps from $C(X)$ to $C(Y)$ and norm-bounded functions $Y \to QM(X)$ which are weak-* continuous. Specifically, if $y \to \mu_y$ is a bounded, weak-* continuous map of $Y$ into $QM(X)$, the corresponding quasi-linear map is $T(f)(y) = \langle \mu_y, f \rangle$.

The proof of this is evident. For comparison, there is a similar correspondence between the quasi-homomorphisms from $C(X)$ to $C(Y)$ and weak-* continuous functions $Y \to X^*$ where $X^*$ is the collection of $\{0,1\}$-quasi-measures on $X$. See [4] for details.

**Proposition 10.** Let $T : C(X) \to C(Y)$ be a quasi-homomorphism and $S : C(Y) \to C(Z)$ be a quasi-linear map. Then the composition $S \circ T : C(X) \to C(Z)$ is a quasi-linear map. If $y \to \mu_y$ is the map from $Y$ to $X^*$ corresponding to $T$, $z \to \nu_z$ the map from $Z$ into $QM(Y)$ corresponding to $S$, and $z \to \omega_z$ the map from $Z$ to $QM(X)$ corresponding to $S \circ T$, then for $U$ open in $X$,

$$\omega_z(U) = \nu_z(\{y \in Y : \mu_y(U) = 1\}).$$

**Proof.** That the composition is quasi-linear is evident.

Let $U \subseteq X$ be open and $\varepsilon > 0$. Let $W = \{y : \mu_y(U) = 1\}$.

**Claim:** For each $y \in W$ there is a compact $K_y \subseteq U$ and open set $y \in V_y$ such that $y' \in V_y$ implies $\mu_y(K_y) = 1$.

Since $\mu_y(U) = 1$, there is a compact $K_1 \subseteq U$ with $\mu_y(K_1) = 1$. Let $f \in C(X)$ with $K_1 \prec f \prec U$. Then $\langle \mu_y, f \rangle = 1$, so by weak-* continuity, there is an open set $y \in V$ such that $y' \in V$ implies that $\langle \mu_y', f \rangle > \frac{2}{3}$. Let $K_y = \{x : f(x) \geq \frac{1}{3}\}$. Since each $\mu_y'$ is a $\{0,1\}$-quasi-measure, $\mu_y'(K_y) = 1$ for $y' \in V$.

In particular, we see that $W$ is open in $Y$. This also shows that if $A \subseteq X$ is closed then $\{y : \mu_y(A) = 1\}$ is closed in $Y$. Now we can find compact sets $K \subseteq U$ and $C \subseteq W$ such that $K \prec f \prec U$ implies $|\langle \omega_z, f \rangle - \omega_z(U)| < \varepsilon$ and $C \prec g \prec W$ implies $|\langle \nu_z, g \rangle - \nu_z(W)| < \varepsilon$.

For each $y \in C$, pick $V_y$ and $K_y$ as in the claim. Choose finitely many $V_y$ to cover $C$, say $V_1, \ldots, V_n$. Let $K_1, \ldots, K_n$ be the corresponding $K_y$.

Set $L = K \cup K_1 \cup \cdots \cup K_n$ and find $L \prec f \prec U$. Recall $T(f)(y) = \langle \mu_y, f \rangle$ for $y \in Y$. If $y \in C$, say $y \in V_j$, $1 \leq \mu_y(K_j) \leq \mu_y(L) \leq \langle \mu_y, f \rangle \leq \mu_y(U) \leq 1$, so $C \prec T(f)$. Also, if $F$ is the support of $f$, then $E = \{y : \mu_y(F) = 1\}$ is closed and contained in $W$. If $y \notin E$, $\mu_y(F) = 0$, so $\langle \mu_y, f \rangle = 0$. Thus, the support of $T(f)$ is contained in $E \subseteq W$. Hence $C \prec T(f) \prec W$.

Thus, $|\omega_z(U) - \langle \omega_z, f \rangle| < \varepsilon$ and $|\nu_z(W) - \langle \nu_z, T(f) \rangle| < \varepsilon$. However, $\langle \omega_z, f \rangle = S \circ T(f)(z) = \langle \nu_z, T(f) \rangle$. Hence $|\omega_z(U) - \nu_z(W)| < 2\varepsilon$. Now let $\varepsilon$ go to 0. \qed

Another interpretation of this result may be obtained by noting that a quasi-homomorphism $T : C(X) \to C(Y)$ induces an image transformation $q : A(X) \to A(Y)$ (see [4] for details). In particular, for $A \in A(X)$, we have $q(A) = \{y : \mu_y(A) = 1\}$. For $\nu \in QM(Y)$, we can then define $q^*\nu \in QM(X)$ by $q^*\nu(A) = \nu(q(A))$. The above proposition then states that $\omega_z = q^*\nu_z$ for $z \in Z$.

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