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ANOTHER NOTE ON WEYL'S THEOREM

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Abstract. "Weyl's theorem holds" for an operator T on a Banach space X when the complement in the spectrum of the "Weyl spectrum" coincides with the isolated points of spectrum which are eigenvalues of finite multiplicity. This is close to, but not quite the same as, equality between the Weyl spectrum and the "Browder spectrum", which in turn ought to, but does not, guarantee the spectral mapping theorem for the Weyl spectrum of polynomials in T. In this note we try to explore these distinctions.

Recall [2, 4, 6] that a bounded linear operator $T \in BL(X, X)$ on a Banach space X is *Fredholm* if $T(X)$ is closed and both $T^{-1}(0)$ and $X/\mathrm{cl}\,(TX)$ are finite dimensional: in this case, we define the *index* of T by index $(T) = \dim T^{-1}(0) - \dim X/T(X)$. An operator $T \in BL(X, X)$ is called *Weyl* if it is Fredholm of index zero, and is called *Browder* if it is Fredholm "of finite ascent and descent": equivalently ([6], Theorem 7.9.3) if T is Fredholm and $T - \lambda I$ is invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The (Fredholm) essential spectrum $\sigma_{\mathrm{ess}}(T)$, the Weyl spectrum $\omega_{\mathrm{ess}}(T)$ and the Browder spectrum $\omega_{\mathrm{ess}}^{\mathrm{comm}}(T)$ of T are defined by

$$\sigma_{\mathrm{ess}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}, \tag{0.1}$$

$$\omega_{\mathrm{ess}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl}\} \tag{0.2}$$

and

$$\omega_{\mathrm{ess}}^{\mathrm{comm}}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}; \tag{0.3}$$

evidently

$$\sigma_{\mathrm{ess}}(T) \subseteq \omega_{\mathrm{ess}}(T) \subseteq \omega_{\mathrm{ess}}^{\mathrm{comm}}(T) = \sigma_{\mathrm{ess}}(T) \cup \mathrm{acc}\ \sigma(T), \tag{0.4}$$

where we write acc K for the *accumulation points* of $K \subseteq \mathbb{C}$ and $\sigma(T)$ for the usual spectrum of T. If we write $\mathrm{iso}(K) = K \setminus \mathrm{acc}(K)$ and

$$\pi_0^{\mathrm{left}}(T) = \{\lambda \in \mathrm{iso}\ \sigma(T) : 0 < \dim\ (T - \lambda I)^{-1}(0) < \infty\} \tag{0.5}$$

for the isolated eigenvalues of finite multiplicity, and ([6], (9.8.3.4))

$$\pi_{00}(T) = \sigma(T) \setminus \omega_{\mathrm{ess}}^{\mathrm{comm}}(T) \tag{0.6}$$

for the *Riesz points* of T, then ([6], Theorem 9.8.4) with the help of the "punctured neighbourhood theorem"

$$\mathrm{iso}\ \sigma(T) \setminus \sigma_{\mathrm{ess}}(T) = \mathrm{iso}\ \sigma(T) \setminus \omega_{\mathrm{ess}}(T) = \pi_{00}(T) \subseteq \pi_0^{\mathrm{left}}(T). \tag{0.7}$$

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Note that some authors use the notation of (0.6) for the concept of (0.5).

1. Definition. We say that *Weyl's theorem holds for* $T \in BL(X,X)$ if

$$\sigma(T) \setminus \omega_{\text{ess}}(T) = \pi_0^{\text{left}}(T), \tag{1.1}$$

and we shall say that *Browder's theorem holds for* T if

$$\sigma(T) \setminus \omega_{\text{ess}}(T) = \pi_{00}(T). \tag{1.2}$$

Evidently "Weyl's theorem" implies "Browder's theorem":

2. Theorem. *Each of the following conditions is equivalent to Browder's theorem for* $T \in BL(X,X)$*:*

$$\sigma(T) = \omega_{\text{ess}}(T) \cup \pi_0^{\text{left}}(T); \tag{2.1}$$

$$\omega_{\text{ess}}(T) = \omega_{\text{ess}}^{\text{comm}}(T); \tag{2.2}$$

Necessary and sufficient for Weyl's theorem is Browder's theorem together with either of the following:

$$\omega_{\text{ess}}(T) \cap \pi_0^{\text{left}}(T) = \emptyset; \tag{2.3}$$

$$\pi_0^{\text{left}}(T) \subseteq \pi_{00}(T). \tag{2.4}$$

Proof. Implication (1.2) $\Longrightarrow$ (2.1) is the last part of (0.7). Conversely if (2.1) holds then $\sigma(T) \setminus \omega_{\text{ess}}(T) = \pi_0^{\text{left}}(T) \setminus \omega_{\text{ess}}(T) \subseteq \pi_{00}(T)$, giving (1.2). Equivalence (1.2) $\Longleftrightarrow$ (2.2) is (0.6). Implication (2.2) $\Longrightarrow$ (1.2) is the middle part of (0.7). Towards the second part of the theorem notice that (2.4) always implies (2.3): we claim that Browder's theorem together with (2.3) implies Weyl's theorem, and that Weyl's theorem implies (2.4). Indeed, using the last part of (0.7), Browder's theorem says that the complement in $\sigma(T)$ of the Weyl spectrum is a subset of ${\pi_0}^{\text{left}}(T)$, while (2.3) ensures that $\pi_0^{\text{left}}(T)$ is a subset of this complement. On the other hand, the second part of (0.7) together the inclusion $\pi_0^{\text{left}}(T) \subseteq \text{iso}\,\sigma(T)$ and Weyl's theorem gives (2.4). □

The disjointness condition (2.3) can fail whether or not Browder's theorem holds [9]:

3. Example. *If* $X = \ell_p$ *or* $X = c_0$ *and*

$$T = vw : (x_1, x_2, x_3, \cdots) \mapsto (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \cdots) \text{ on } X \tag{3.1}$$

is the product of the backward shift v *and the standard weight* w*, then*

$$\sigma(T) = \sigma_{\text{ess}}(T) = \omega_{\text{ess}}(T) = \omega_{\text{ess}}^{\text{comm}}(T) = \{0\} \tag{3.2}$$

and

$$\pi_0^{\text{left}}(T) = \{0\}. \tag{3.3}$$

Proof. T is quasinilpotent and compact, so not Fredholm, giving (3.2), while

$$T^{-1}(0) = \mathbb{C}\delta_1 = \{(\lambda, 0, 0, \cdots) : \lambda \in \mathbb{C}\}$$

is of dimension 1. □

In general the spectral mapping theorem is liable to fail for the Weyl spectrum ([2], Example 3.3): there is only ([2], Theorem 3.2) inclusion, since the product of Weyl operators is Weyl,

$$\omega_{\text{ess}} p(T) \subseteq p\omega_{\text{ess}}(T). \tag{3.4}$$

Similarly the Weyl spectrum of a direct sum need not be the union of the Weyl spectra of the components: we only have in general, since the direct sum of Weyl operators is Weyl and the index is additive on direct sums,

$$\omega_{\text{ess}}(T) \setminus \omega_{\text{ess}}(S) \subseteq \omega_{\text{ess}}(S \oplus T) \subseteq \omega_{\text{ess}}(S) \cup \omega_{\text{ess}}(T). \tag{3.5}$$

By contrast ([6], Theorem 9.8.2) the spectral mapping theorem holds for the Browder spectrum, and the Browder spectrum of a direct sum is the union of the Browder spectrum of the components. This might suggest that Browder's theorem for S and T is sufficient for equality in (3.4) and the second part of (3.5):

4. Theorem. *If Browder's theorem holds for $T \in BL(X,X)$ and $S \in BL(Y,Y)$ and if p is a polynomial, then*

$$\textit{Browder's theorem holds for } p(T) \iff p\omega_{\text{ess}}(T) \subseteq \omega_{\text{ess}} p(T), \tag{4.1}$$

and

$$\textit{Browder's theorem holds for } S \oplus T \iff \omega_{\text{ess}}(S) \cup \omega_{\text{ess}}(T) \subseteq \omega_{\text{ess}}(S \oplus T). \tag{4.2}$$

Proof. If $\omega_{\text{ess}}^{\text{comm}} p(T) \subseteq \omega_{\text{ess}} p(T)$, then, with no other restriction on T,

$$p\omega_{\text{ess}}(T) \subseteq p\omega_{\text{ess}}^{\text{comm}}(T) = \omega_{\text{ess}}^{\text{comm}} p(T) \subseteq \omega_{\text{ess}} p(T),$$

which is the right hand side of (4.1); conversely if Browder's theorem holds for T as well as this inclusion, then $\omega_{\text{ess}}^{\text{comm}} p(T) = p\omega_{\text{ess}}^{\text{comm}}(T) \subseteq p\omega_{\text{ess}}(T) \subseteq \omega_{\text{ess}} p(T)$. Similarly, if Browder's theorem holds for $S \oplus T$, then, with no other restriction on either S or T,

$$\omega_{\text{ess}}(S) \cup \omega_{\text{ess}}(T) \subseteq \omega_{\text{ess}}^{\text{comm}}(S) \cup \omega_{\text{ess}}^{\text{comm}}(T) = \omega_{\text{ess}}^{\text{comm}}(S \oplus T) \subseteq \omega_{\text{ess}}(S \oplus T),$$

which is the right hand side of (4.2); conversely if Browder's theorem holds for S and for T as well as this inclusion, then $\omega_{\text{ess}}^{\text{comm}}(S \oplus T) = \omega_{\text{ess}}^{\text{comm}}(S) \cup \omega_{\text{ess}}^{\text{comm}}(T) \subseteq \omega_{\text{ess}}(S) \cup \omega_{\text{ess}}(T) \subseteq \omega_{\text{ess}}(S \oplus T)$. □

5. Theorem. *If $T \in BL(X,X)$, then the following are equivalent:*

$$\operatorname{Index}(T - \lambda I) \operatorname{Index}(T - \mu I) \geq 0 \textit{ for each pair } \lambda, \mu \in \mathbb{C} \setminus \sigma_{\text{ess}}(T); \tag{5.1}$$

$$p\omega_{\text{ess}}(T) \subseteq \omega_{\text{ess}} p(T) \textit{ for each polynomial } p. \tag{5.2}$$

Also if

$$\omega_{\text{ess}}(T) = \sigma_{\text{ess}}(T), \tag{5.3}$$

then

$$\omega_{\text{ess}}(S) \cup \omega_{\text{ess}}(T) \subseteq \omega_{\text{ess}}(S \oplus T) \textit{ for each } Y \textit{ and } S \in BL(Y,Y), \tag{5.4}$$

which in turn implies the condition (5.1).

Proof. The spectral mapping theorem for the Weyl spectrum may be rewritten as the implication, for arbitrary $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}^n$,

$$\begin{aligned}(T-\lambda_1 I)(T-\lambda_2 I)\cdots(T-\lambda_n I) \text{ Weyl}\\ \implies T-\lambda_j I \text{ Weyl for each } j=1,2,\cdots,n.\end{aligned} \tag{5.5}$$

Now if $\mathrm{Index}(T-zI) \geq 0$ on $\mathbb{C}\setminus\sigma_{\mathrm{ess}}(T)$, then we have

$$\begin{aligned}\sum_{j=1}^{n}\mathrm{Index}(T-\lambda_j I) = \mathrm{Index}\prod_{j=1}^{n}(T-\lambda_j I) = 0\\ \implies \mathrm{Index}(T-\lambda_j I) = 0 \ (j=1,2,\cdots,n),\end{aligned}$$

and similarly if $\mathrm{Index}(T-zI) \leq 0$ off $\sigma_{\mathrm{ess}}(T)$. If conversely there exist λ, μ for which

$$\mathrm{Index}(T-\lambda I) = -m < 0 < k = \mathrm{Index}(T-\mu I), \tag{5.6}$$

then

$$(T-\lambda I)^k (T-\mu I)^m \tag{5.7}$$

is a Weyl operator whose factors are not Weyl. This proves the equivalence of the conditions (5.1) and (5.2). To see that (5.3) $\implies$ (5.4), recall that the index of a direct sum is the sum of the indices:

$$\mathrm{Index}(S\oplus T-\lambda(I\oplus I)) = \mathrm{Index}(S-\lambda I) + \mathrm{Index}(T-\lambda I) \tag{5.8}$$

whenever $\lambda \notin \sigma_{\mathrm{ess}}(S\oplus T) = \sigma_{\mathrm{ess}}(S)\cup\sigma_{\mathrm{ess}}(T)$. Conversely if (5.1) fails, so that (5.6) holds, then also the direct sum of $(T-\lambda I)^k$ and $(T-\mu I)^m$ is a Weyl operator whose factors are not Weyl; thus in particular $0 \in \mathbb{C}$ is in the Weyl spectrum of the operator $T-\lambda I$ but not in that of the direct sum

$$T-\lambda I \oplus S-\lambda I \quad \text{with } S-\lambda I = (T-\lambda I)^{k-1}(T-\mu I)^m.$$

□

Of course the condition (5.3) implies the condition (5.1). We can rewrite (5.1) in terms of the "spectral picture" ([11], Definition 1.22) of the operator T, which consists of the essential spectrum of T together with the mapping which associates with each "hole" H (bounded component of the complement) of $\sigma_{\mathrm{ess}}(T)$ the integer $J_T(H) = \mathrm{Index}(T-\lambda I)$ with $\lambda \in H$ (independent of the choice of $\lambda \in H$). When X is a Hilbert space then (5.1) is ([11], Definition 4.8; Theorem 1.31) the condition that $T \in BL(X,X)$ be "semi-quasitriangular", in the sense that either T or T^* is quasitriangular.

We have a familiar example of an operator for which the spectral mapping theorem holds for the Weyl spectrum, which does not coincide with the Browder spectrum:

6. Example. *If*

$$T = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : \begin{pmatrix} Y \\ Y \end{pmatrix} \to \begin{pmatrix} Y \\ Y \end{pmatrix} \tag{6.1}$$

with $Y = \ell_p$ or $Y = c_0$ and the forward and backward shifts u and v, and if $|\lambda| < 1$, then

$$T-\lambda I \text{ is Weyl and not Browder}, \tag{6.2}$$

but the inclusion (5.2) does hold. At the same time Browder's theorem holds for each of u and v, but not for $T = u \oplus v$.

Proof. It is clear that $T - \lambda I$ is Fredholm and has index zero, therefore is Weyl; alternatively (cf. [4], Example 4.3; [6], (7.7.6.6)-(7.7.6.9)) if

(6.3)
$$S = \begin{pmatrix} u-\lambda & 1-uv \\ 0 & v-\lambda \end{pmatrix} \text{ and } S' = \begin{pmatrix} v(1-\lambda v)^{-1} & 0 \\ (1-\lambda u)^{-1}(1-uv)(1-\lambda v)^{-1} & (1-\lambda u)^{-1}u \end{pmatrix},$$

then

$$S'S = I = SS' \text{ and } T - \lambda I - S \text{ is finite rank;} \tag{6.4}$$

make the calculations

$$(1-\lambda v)^{-1}(u-\lambda) = u\ ; \qquad (v-\lambda)(1-\lambda u)^{-1} = v\ ;$$
$$(1-\lambda v)^{-1}(1-uv) = 1-uv = (1-uv)(1-\lambda u)^{-1}.$$

To see that $T - \lambda I$ is not Browder, recall the eigenvector

$$\delta_1 = (1, 0, 0, \cdots) \in v^{-1}(0);$$

we claim that

$$y = u^n(1-\lambda u)^{-(n+1)}\delta_1 \Longrightarrow (v-\lambda)^{n+1}y = 0 \neq (v-\lambda)^n y,$$

noting that

$$(v-\lambda)^n(1-\lambda u)^{-n} = v^n,$$

and hence

$$x = \begin{pmatrix} 0 \\ y \end{pmatrix} \Longrightarrow (T-\lambda I)^{n+1}x = 0 \neq (T-\lambda I)^n x.$$

For (5.1), observe that trivially T satisfies the condition (5.2) (since there is only one hole in the essential spectrum). □

We have a very similar example in the opposite direction:

7. Example. *If $Y = \ell_p$ or $Y = c_0$ and*

$$T = \begin{pmatrix} u+1 & 0 \\ 0 & v-1 \end{pmatrix} : \begin{pmatrix} Y \\ Y \end{pmatrix} \to \begin{pmatrix} Y \\ Y \end{pmatrix} \tag{7.1}$$

with the forward and backward shifts u and v on Y, then Browder's theorem holds for T while the spectral mapping theorem for the Weyl spectrum fails.

Proof. We claim ([4], Example 4.4; [6], (7.6.4.9)), using the first part of (3.5), that

$$\sigma(T) = \omega_{\text{ess}}(T) = \{|1-z| \leq 1\} \cup \{|1+z| \leq 1\},$$

since both the spectrum and the Weyl spectrum of each of the shifts is the closed unit disc. Thus Browder's theorem certainly holds for T; to see the failure of the spectral mapping theorem with the polynomial $p = z^2$, observe that

$$1 \in p\omega_{\text{ess}}(T) \supseteq \sigma_{\text{ess}}p(T) \text{ and } 1 \notin \omega_{\text{ess}}p(T) = \omega_{\text{ess}}(T^2);$$

for this last part observe that $\text{Index}(T-I) = -1 = -\text{Index}(T+I)$, or alternatively make a direct calculation ([4], Example 4.4; [6], (7.6.4.13)) as for Example 6. □

Weyl's theorem may or may not hold for quasinilpotent operators, and is not transmitted to or from dual operators: for example it fails for the quasinilpotent $T = vw$ of Example 3, but holds for its adjoint $T^* = wu$: if $T = vw$ on ℓ_2 then

$$\sigma(T^*) = \omega_{\text{ess}}(T^*) = \{0\} \text{ and } \pi_0^{\text{left}}(T^*) = \emptyset. \tag{7.2}$$

Once again Browder's theorem performs better:

8. Theorem. *If $T \in BL(X, X)$, then*

$$\textit{Browder's theorem holds for } T \iff \textit{Browder's theorem holds for } T^\dagger. \tag{8.1}$$

Proof. Observe that

$$\omega_{\text{ess}}(T^\dagger) = \omega_{\text{ess}}(T) \text{ and iso } \sigma(T^\dagger) = \text{iso } \sigma(T), \tag{8.2}$$

which together with (0.7) and (2.2) gives (8.1). □

Combining (3.3) and (7.2) shows that the Riesz points need not coincide with the intersection of the isolated eigenvalues of finite multiplicity for the operator and its dual:

$$T = vw \oplus wu \Longrightarrow \pi_0^{\text{left}}(T) \cap \pi_0^{\text{left}}(T^\dagger) = \{0\} \neq \pi_{00}(T) = \emptyset. \tag{8.3}$$

9. Theorem. *Necessary and sufficient for Browder's theorem to hold for $T \in BL(X, X)$ is that*

$$\text{acc } \sigma(T) \subseteq \omega_{\text{ess}}(T). \tag{9.1}$$

Hence in particular Browder's theorem holds for quasinilpotent operators, compact operators and algebraic operators.

Proof. If (9.1) holds then

$$\sigma(T) \setminus \omega_{\text{ess}}(T) \subseteq \text{iso } \sigma(T),$$

giving Browder's theorem by (0.7); the converse is (0.4). If $\sigma(T)$ consists of isolated points then T satisfies (9.1); thus Browder's theorem holds for quasinilpotents, algebraic operators and compact operators with finite spectrum. For general compact operators (more generally, "Riesz operators"), we have (in infinite dimensions)

$$\text{acc } \sigma(T) \subset \{0\} \subseteq \sigma_{\text{ess}}(T), \tag{9.2}$$

giving again (9.1). □

An example of Berberian shows that on a Hilbert space X it is not sufficient, for Weyl's theorem for $T \in BL(X, X)$, that T is reduced by its finite dimensional eigenspaces ([1], Example 1): take $T = T_1 \oplus T_2$, where T_1 is the one-dimensional zero operator and T_2 is an injective quasinilpotent compact operator as in (7.2). This condition is, however, sufficient for Browder's theorem:

10. Theorem. *If X is a Hilbert space and $T \in BL(X, X)$ is reduced by its finite dimensional eigenspaces, then Browder's theorem holds for T.*

Proof. If T is reduced by its finite dimensional eigenspaces, then $T = T_1 \oplus T_2$ with

$$T_1 \text{ normal and } \omega_{\text{ess}}(T_2) = \sigma(T_2). \tag{10.1}$$

In fact, take ([1], Example 5)

$$X_2^\perp = X_1 = \sum_{\lambda \in \Lambda} (T - \lambda I)^{-1}(0)$$

to be the sum of the (not necessarily isolated) eigenvalues of finite multiplicity. Evidently both the condition (5.3) and Browder's theorem hold for each of T_1 and T_2. □

Theorem 10 shows (cf. [3], Theorem 3.1) that Browder's theorem holds for hyponormal operators, since hyponormal operators are reduced by their eigenspaces.

Weyl's theorem is transmitted ([10], Theorem 3) from $T \in BL(X,X)$ to $T-K$ for commuting nilpotents $K \in BL(X,X)$: notice $(T-K)^{-1}(0) \subseteq T^{-n}(0)$ if $K^n = 0$. This however does not extend to quasinilpotents: recall the quasinilpotent vw of Example 3 and set

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} Y \\ Y \end{pmatrix} \to \begin{pmatrix} Y \\ Y \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 & 0 \\ 0 & vw \end{pmatrix} : \begin{pmatrix} Y \\ Y \end{pmatrix} \to \begin{pmatrix} Y \\ Y \end{pmatrix}. \tag{10.2}$$

Evidently K is quasinilpotent and commutes with T; but Weyl's theorem holds for T because

$$\sigma(T) = \omega_{\text{ess}}(T) = \{0,1\} \text{ and } \pi_0^{\text{left}}(T) = \emptyset, \tag{10.3}$$

while Weyl's theorem does not hold for $T+K$ because

$$\sigma(T+K) = \omega_{\text{ess}}(T+K) = \{0,1\} \text{ and } \pi_0^{\text{left}}(T+K) = \{0\}. \tag{10.4}$$

Oberai [10] has examples which show that Weyl's theorem for T is not sufficient either for Weyl's theorem for $T+K$ with finite rank K ([10], Example 2) or for Weyl's theorem for $p(T)$ with a polynomial p ([10], Example 1). Browder's theorem behaves better, at least for commuting perturbations:

11. Theorem. *If Browder's theorem holds for $T \in BL(X,X)$, then Browder's theorem holds for $T+K$ if K commutes with T and is either quasinilpotent or compact.*

Proof. For the first part recall the argument of Oberai ([10], Lemma 2): if K is quasinilpotent and commutes with a Weyl operator T, then $0 \notin \sigma_{\text{ess}}(T+\lambda K)$ for arbitrary $\lambda \in \mathbb{C}$, which by index continuity forces $T+\lambda K$ to have index zero for all $\lambda \in \mathbb{C}$, in particular with $\lambda = 1$. Thus if K is quasinilpotent and commutes with T, then

$$\omega_{\text{ess}}(T+K) = \omega_{\text{ess}}(T). \tag{11.1}$$

It is also clear ([6], Theorem 7.4.3) that, for the same K,

$$\sigma(T+K) = \sigma(T) \text{ and } \sigma_{\text{ess}}(T+K) = \sigma_{\text{ess}}(T), \tag{11.2}$$

and hence also the accumulation points of the spectrum coincide. By (0.4) it follows that also

$$\omega_{\text{ess}}^{\text{comm}}(T+K) = \omega_{\text{ess}}^{\text{comm}}(T) \tag{11.3}$$

whenever K is quasinilpotent and commutes with T. If instead K is a commuting compact, remember that the Weyl spectrum is invariant under compact purturbations, giving again (11.1), while the Browder spectrum is invariant under commuting compact perturbations, giving (11.3). □

This may fail if K is not assumed to commute with T, even if K both compact and nilpotent:

12. Example. *If*

$$T = \begin{pmatrix} u & 1-uv \\ 0 & v \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 & 1-uv \\ 0 & 0 \end{pmatrix}, \tag{12.1}$$

then K is a rank one nilpotent, T is unitary, and Browder's theorem does not hold for $T-K$.

Proof. It is clear that K is both rank one and square zero. The operator T is unitary (essentially the bilateral shift), so that Weyl's theorem holds for T; we saw in Example 6 that Browder's theorem fails for $T-K$. □

Berberian ([1]; [10], Definition 2) called the operator $T \in BL(X,X)$ *isoloid* if the isolated points of its spectrum are always eigenvalues:

13. Definition. $T \in BL(X,X)$ is called *isoloid* if

$$\lambda \in \text{iso}\ \sigma(T) \Longrightarrow (T-\lambda I)^{-1}(0) \neq \{0\}, \tag{13.1}$$

and will be called *reguloid* if each isolated point of its spectrum is a regular point, in the sense that there is a generalized inverse:

$$\lambda \in \text{iso}\ \sigma(T) \Longrightarrow T-\lambda I = (T-\lambda I)S_\lambda(T-\lambda I) \text{ with } S_\lambda \in BL(X,X). \tag{13.2}$$

We shall call $T \in BL(X,X)$ *normaloid* if it satisfies the growth condition, that for all $\lambda \in \mathbb{C} \setminus \sigma(T)$

$$||(T-\lambda I)^{-1}||\text{dist}(\lambda, \sigma(T)) \leq 1. \tag{13.3}$$

We are guilty of an abuse of language in (13.3). For "reguloid" operators Weyl's theorem and Browder's theorem are equivalent:

14. Theorem. *If $T \in BL(X,X)$ is reguloid then Browder's theorem for T implies Weyl's theorem for T. There is the implication, for arbitrary $T \in BL(X,X)$,*

$$T \textit{ normaloid} \implies T \textit{ reguloid} \implies T \textit{ isoloid}. \tag{14.1}$$

Proof. We claim that the condition (2.3) holds for reguloid operators: for if $T-\lambda I$ has a generalized inverse and if $\lambda \in \partial\sigma(T)$ is in the boundary of the spectrum then $T-\lambda I$ has ([5]; [6], Theorem 7.3.4) an invertible generalized inverse. If therefore T is reguloid, then $T-\lambda I$ has an invertible generalized inverse for each $\lambda \in \pi_0^{\text{left}}(T)$, and hence ([6], (3.8.6.1))

$$(T-\lambda I)^{-1}(0) \cong X/(T-\lambda I)(X). \tag{14.2}$$

Since also

$$\dim\ (T-\lambda I)^{-1}(0) < \infty, \tag{14.3}$$

it follows that $T-\lambda I$ is Weyl, and hence $\lambda \in \sigma(T) \setminus \omega_{\text{ess}}(T)$. The same argument gives the second implication of (14.1), for if in (14.3) the operator $T-\lambda I$ is one-one then by (14.2) it is invertible. Towards the first implication we may write T in place of $T-\lambda I$ and hence assume $\lambda = 0$; then using the spectral projection at $0 \in \mathbb{C}$ we can represent T as a 2×2 operator matrix:

$$T = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}, \tag{14.4}$$

where $\sigma(T_0) = \{0\}$ and $\sigma(T_1) = \sigma(T) \setminus \{0\}$. Now we can borrow an argument of Stampfli ([12], Theorem C): take $0 < \epsilon \leq \frac{1}{2}\text{dist}(0, \sigma(T) \setminus \{0\})$ and argue

$$T_0 = \frac{1}{2\pi i} \int_{|z|=\epsilon} z(T - zI)^{-1} dz,$$

using the growth condition (13.3) to see that

$$||T_0|| \leq \frac{1}{2\pi} \int_{|z|=\epsilon} |z| \; ||(T - zI)^{-1}|| \; |dz| \leq \frac{1}{2\pi} \epsilon \frac{1}{\epsilon} 2\pi\epsilon = \epsilon, \tag{14.5}$$

which tends to 0 with ϵ. It follows that $T_0 = 0$ and hence that

$$T = \begin{pmatrix} 0 & 0 \\ 0 & T_1 \end{pmatrix} = TST \text{ with } S = \begin{pmatrix} 0 & 0 \\ 0 & T_1^{-1} \end{pmatrix}$$

has a generalized inverse. □

Coburn's argument ([3], Corollary 3.2) also shows that if Weyl's theorem holds for T, if T is normaloid (in the proper sense of the word, norm equals spectral radius), and if $\pi_0^{\text{left}}(T) = \emptyset$, then T is *extremally non-compact* in the sense [3] that

$$||T|| = \inf\{||T + K|| : K \text{ compact in } BL(X, X)\}; \tag{14.6}$$

for we can argue

$$||T|| = |T|_\sigma = |T|_\omega = |T + K|_\omega \leq |T + K|_\sigma \leq ||T + K||, \tag{14.7}$$

where $|\cdot|_\sigma$ and $|\cdot|_\omega$ denote the spectral radius and the "Weyl spectral radius".

Hyponormal operators on Hilbert space are well known to satisfy the growth condition (14.3), and hence are also "reguloid". It is not difficult to see that any "cohyponormal" operator is also reguloid, and hence direct sums of hypo- and cohyponormal operators. Example 6, however, shows that Browder's theorem need not hold for a reguloid operator T, while Example 3 shows that we cannot replace "reguloid" by "isoloid" in the first part of Theorem 14.

References

1. S.K. Berberian, *An extension of Weyl's theorem to a class of not necessarily normal operators*, Michigan Math. Jour. **16** (1969), 273–279. MR **40:**3335
2. S.K. Berberian, *The Weyl spectrum of an operator*, Indiana Univ. Math. Jour. **20** (1970), 529–544. MR **43:**5344
3. L.A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. Jour. **13** (1966), 285–288. MR **30:**1846
4. R.E. Harte, *Fredholm, Weyl and Browder theory*, Proc. Royal Irish Acad. **85A** (1985), 151–176. MR **87h:**47029
5. R.E. Harte, *Regular boundary elements*, Proc. Amer. Math. Soc. **99** (1987), 328–330. MR **88d:**46088
6. R.E. Harte, *Invertibility and singularity for bounded linear operators*, Dekker, New York, 1988. MR **89d:**47001
7. W.Y. Lee and H.Y. Lee, *On Weyl's theorem*, Math. Japon. **39** (1994), 545–548. MR **95a:**47008
8. W.Y. Lee and S.H. Lee, *A spectral mapping theorem for the Weyl spectrum*, Glasgow Math. Jour. **38** (1996), 61–64. CMP 1996:8
9. K.K. Oberai, *On the Weyl spectrum*, Illinois J. Math. **18** (1974), 208–212. MR **48:**12086
10. K.K. Oberai, *On the Weyl spectrum II*, Illinois J. Math. **21** (1977), 84–90. MR **55:**1102

11. C.M. Pearcy, *Some recent developements in operator theory*, CBMS Regional Conf. Ser. Math., vol. 36, Amer. Math. Soc., Providence, RI, 1978. MR **58:**7120
12. J.G. Stampfli, *Hyponormal operators and spectral density*, Trans. Amer. Math. Soc. **117** (1965), 469–476. MR **33:**4686

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