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DOODLE GROUPS

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Abstract. A doodle is a finite number of closed curves without triple intersections on an oriented surface. There is a "fundamental" group, naturally associated with a doodle. In this paper we study these groups, in particular, we show that fundamental groups of some doodles are automatic and give examples of doodles whose fundamental groups have non-trivial center.

0. Introduction

A doodle is a collection of piecewise-linear closed curves without triple intersections on a closed oriented surface. Two doodles are equivalent if they can be connected by a finite sequence of local moves as shown in Figure 4, that is, by a PL-homotopy without creating triple intersections at any time.

It is possible to transfer the basic concepts of link theory to the theory of doodles. The twin group, defined in §1, plays the role of the braid group. Each doodle has a group, called the fundamental group, which resembles the fundamental group of a link complement.

In this paper, we construct examples of doodles on the two-sphere whose fundamental groups have nontrivial center. Also, for some special types of doodles, we prove that their fundamental groups are automatic. The proof uses a theorem of Gersten and Short [GS1], [GS2] that the fundamental group of a 2-complex of non-positive curvature, modelled on equilateral triangles, is automatic.

The concept of a doodle is due to Fenn and Taylor [FT]. Their definition differs from ours by the extra condition that all curves are simple, that is, lack self-intersections. The results of this paper indicate that it is natural to omit this condition.

We will assume that all manifolds and maps between them are piecewise-linear.

1. Twin group

Take a Euclidean plane $\mathbb{R}^2 = \{(x, y)|x, y \in \mathbb{R}\}$ and consider two parallel lines $y = 0$ and $y = 1$ on it. Pick n points on the line $y = 1$, say, points $(1, 1), ..., (n, 1)$. Take n points $(1, 0), ..., (n, 0)$ on the line $y = 0$. Consider configurations of n arcs in $\mathbb{R} \times [0, 1]$ that connect points $(1, 1), ..., (n, 1)$ with points $(1, 0), ..., (n, 0)$ in some order. We require that

(i) the projection of each arc on the y-coordinate is a homeomorphism onto $[0, 1]$ *i.e.* arcs are monotonic.
(ii) no three arcs have a point in common.

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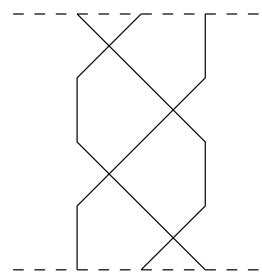

FIGURE 1

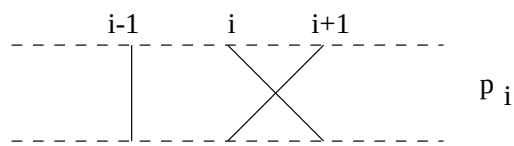

FIGURE 2

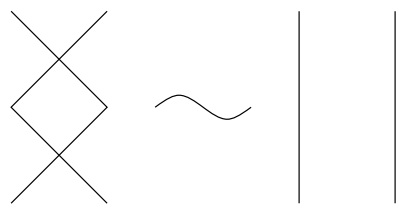

FIGURE 3

Figure 1 gives an example.

Two configurations satisfying (i) and (ii) are called *equivalent* if one can be deformed into the other by a homotopy of arcs in $\mathbb{R} \times [0,1]$ such that conditions (i) and (ii) hold and the ends of the arcs are fixed throughout the homotopy.

Definition 1.1. A twin is an equivalence class of configurations satisfying (i) and (ii).

The product of two twins on the same number of arcs is defined by putting one on top of the other and squeezing the interval $[0,2]$ into $[0,1]$. This operation turns the set of twins on a fixed number of arcs into an associative semigroup. The unit element is given by a configuration whose arcs do not intersect.

Let p_i be a twin with only one double point (see Figure 2). Obviously, any twin can be written as a product of p_i's. Note that $p_i p_i = 1$, because p_i^2 can be homotoped to a configuration without intersections without producing triple points (see Figure 3). Thus p_i, and consequently every twin, has an inverse. Geometrically, the inverse of a twin is its reflection with respect to the horizontal line $y = \frac{1}{2}$. Hence, the set of twins with n arcs is a group. We call it *the twin group on n arcs* and denote it by TW_n.

Proposition 1.1. *TW_n is generated by $p_i, i = 1, ..., n-1$, with defining relations*

$$\begin{aligned} p_i^2 =& 1, \quad i = 1, 2, ..., n, \\ p_i p_j =& p_j p_i, \quad |i-j| > 1, \quad i, j = 1, ..., n. \end{aligned} \tag{1.1}$$

Definition 1.2. *The pure twin group on n arcs,* denoted TW_n^0, is a subgroup of the twin group TW_n consisting of twins with arcs connecting pairs of points $(i, 1)$ and $(i, 0)$, $1 \leq i \leq n$.

The pure twin group TW_n^0 is the kernel of a natural homomorphism from the twin group TW_n to the group of permutations of the set $\{1, ..., n\}$. The homomorphism sends the twin p_i to the transposition $(i, i+1)$.

Consider the space

$$\mathbf{X}_n = \mathbf{R}^n \setminus \{(x_1, ..., x_n) \in \mathbf{R}^n | x_i = x_j = x_k, i \neq j \neq k \neq i, i, j, k = 1, ..., n\},$$

$\mathbf{X}_n$ is $\mathbf{R}^n$ minus the triple diagonals $x_i = x_j = x_k$.

The pure twin group on n arcs is isomorphic to the fundamental group of the space $\mathbf{X}_n$, which is a $K(\pi, 1)$-space. This is proven in [K]. Björner and Welker [BW] showed (in a more general setting) that every $H^i(\mathbf{X}_n, \mathbf{Z})$ is free, $H^i(\mathbf{X}_n, \mathbf{Z}) \neq 0$ iff $0 \leq i \leq \frac{n}{3}$, and $H^1(\mathbf{X}_n, \mathbb{Z})$ has rank $\sum_{i=3}^{n} \binom{n}{i} \binom{i-1}{2}$.

Remark. The twin group TW_{n+2} is isomorphic to the Grothendieck n-dimensional cartographical group as defined by Voevodsky [V] (also see [SV] for the case $n = 2$).

2. Doodles

Definition 2.1. A *doodle* Δ is a collection of piecewise-linear closed curves $(C_1, ..., C_n)$ without triple intersections on a closed oriented surface.

A triple point is either a common point of three curves, or a triple self-intersection point of a curve, or a self-intersection point of a curve which lies on another curve. Two doodles Δ and Δ^0 on a surface M are called *equivalent* iff there exists a homotopy in M from the collection of curves representing Δ to the collection of curves representing Δ^0 with no triple intersections throughout the homotopy.

Figure 4 gives two elementary transformations of doodle diagram, that can be understood as local moves. Doodles are equivalent iff they can be connected by a finite sequence of the moves shown in Figure 4.

An oriented doodle is a doodle with an orientation of each component. The closure of a twin is a doodle on a two-sphere. The closure operation is given by Figure 5.

Theorem 2.1. *Every oriented doodle on a two-sphere is the closure of a twin.*

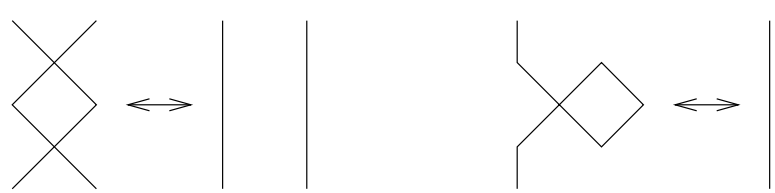

Figure 4

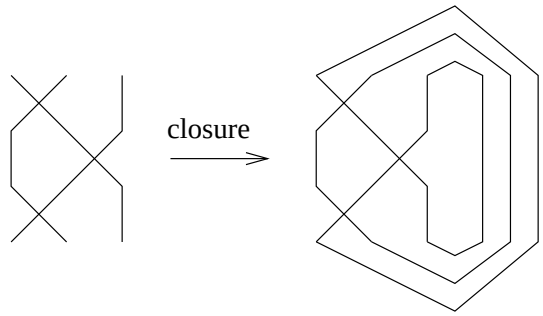

FIGURE 5

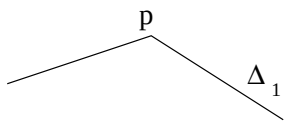

FIGURE 6

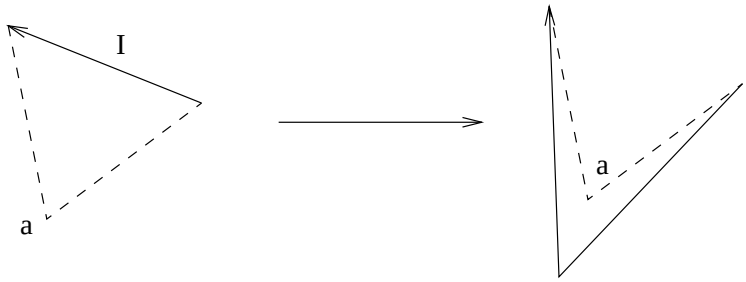

FIGURE 7

Proof. Represent $\mathbb{S}^2$ as $\mathbf{R}^2 \cup \infty$. Let $a \in \mathbf{R}^2$ and Δ be a doodle. We want to deform Δ to lie in $\mathbb{R}^2 \setminus a$ so that each segment is oriented clockwise around a. If such a deformation exists, the result of cutting $\mathbf{R}^2 \setminus a$ along a ray emanating from a would be a twin whose closure is Δ. Choose a diagram Δ_1 of Δ such that the following conditions are satisfied:

(1) $\Delta_1 \in \mathbf{R}^2 \setminus \{a\}$.
(2) No double point or angle point of Δ_1 is collinear with a and another angle or double point.

Here by an *angle point* of Δ_1 we mean a vertex of an arc of Δ_1 viewed as a polygon (Figure 6).

Consider each straight line segment I of Δ_1 separately.

(i) If I is oriented clockwise with respect to a, we will do nothing.
(ii) If I is oriented counter-clockwise, we will change the segment into a configuration of clockwise segments.
(a) If there are no double points of Δ_1 inside the triangle formed by the segment I and the double point a, we change I into 2 clockwise segments as in Figure 7.

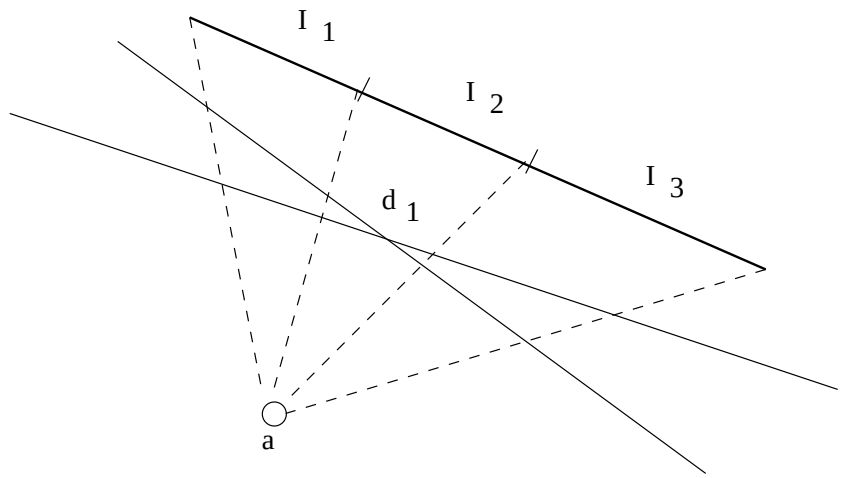

FIGURE 8

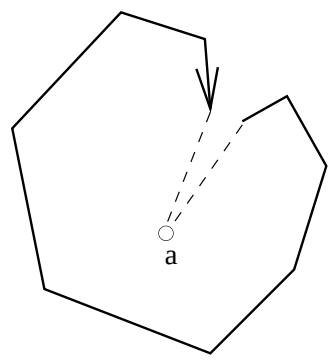

FIGURE 9

(b) Suppose that the triangle formed by I and a contains the double points $d_1, ..., d_k$. Denote this triangle by $T(I, a)$. We cut I into $2k + 1$ segments $I_1, ..., I_{2k+1}$ such that

(i) The triangles $T(I_{2i+1}, a)$ do not contain any double points .

(ii) There is only one double point d_i in the segment of the plane bounded by the two rays connecting a with the ends of I_{2i}, and d_i lies in the triangle formed by a and I_{2i} (Figure 8, case $i = 1$).

Such a subdivision of I is possible because of condition (2) on the diagram Δ_1. Varying i from 0 to k, we deform I_{2i+1} into 2 segments going clockwise around a as in Figure 7. We deform I_{2i+1} so that no double points appear in any of the $k - i$ triangles bounded by a and $I_{2j+1}, i < j \leq k$.

By moving one of the points of I_{2i} through ∞, we change $I_{2i}, 1 \leq i \leq k$, into the configuration in Figure 9. Note that each of the new segments is oriented clockwise relative to a.

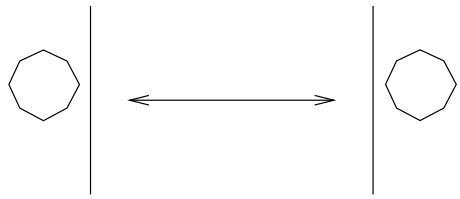

FIGURE 10

We deformed I into a union of segments, each oriented clockwise. New double points could appear after that deformation. A small perturbation produces a new diagram satisfying conditions (1) and (2). The number of counter-clockwise segments in the new diagram is one less than in the diagram Δ_1. Repeating the procedure several times produces a diagram with all segments oriented clockwise with respect to a. This establishes the theorem. □

The minimal number of double points of a doodle diagram is a simply computable invariant.

Theorem 2.2. *A doodle has a unique (up to the transformation in Figure 10) diagram with a minimal number of vertices (intersection points). This diagram can be constructed from any other doodle diagram by applying only those moves in Figure 4 that reduce the number of intersection points.*

Proof. We start with

Definition 2.2. A doodle Δ is *rigid* if Δ does not have a diagram such that one of the components is a simple curve, which does not intersect other components and which bounds an empty disk in $\mathbb{S}^2$ (Figure 10 shows part of a non-rigid doodle).

Denote the local moves in Figure 4 by $\pm 1, \pm 2$ depending on the number of double points that appear/disappear. Thus, a $+$ move creates 1 or 2 double points, while a $-$ move annihilates 1 or 2 double points.

If Δ', Δ'' are diagrams representing the same doodle, there is a sequence of diagrams $\Delta' = \Delta_1, \Delta_2, ..., \Delta_k = \Delta''$ such that any two consecutive diagrams are connected by one of $\pm 1, \pm 2$ moves.

Lemma 2.1. *Let Δ be a rigid doodle. Then for any two diagrams Δ', Δ'' of Δ, there is a sequence of diagrams $\Delta^0 = \Delta_1, \Delta_2, ..., \Delta_k = \Delta^1$ connected by $\pm 1, \pm 2$ moves and with no $+$ move preceding a $-$ move. That is, for some j with $1 \leq j \leq k$,*

$$|\Delta_1| > |\Delta_2| > ... > |\Delta_j| < |\Delta_{j+1}| < ... < |\Delta_k|.$$

Here $|\Delta_s|$ denotes the number of double points of the diagram Δ_s.

Proof of Lemma 2.1. Start with any sequence of diagrams $\Delta' = \Delta_1, \Delta_2, ..., \Delta_n = \Delta''$, connecting diagrams Δ' and Δ''. Suppose that the move m_i from Δ_i to Δ_{i+1} is a $+$ move, $m_{i+1} : \Delta_{i+1} \to \Delta_{i+2}$ is a $-$ move. The move m_i creates one or two double points. If m_{i+1} does not destroy at least one double point created by m_i, we can change the order and do m_{i+1} first and then m_i. All other cases are listed below.

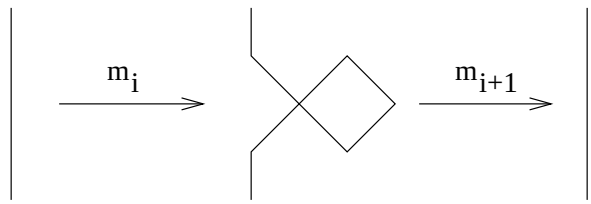

FIGURE 11

(i) m_i is a $+1$ move, m_{i+1} is a -1 move. Because the doodle Δ is rigid, there is only one possibility, depicted in Figure 11.

Thus, m_i and m_{i+1} cancel.

(ii) m_i is a $+1$ move, m_{i+1} is a -2 move. Then the composition $m_{i+1} \circ m_i$ is a -1 move.

(iii) m_i is a $+2$ move, m_{i+1} is a -1 move. Then $m_{i+1} \circ m_i$ is a $+1$ move.

(iv) m_i is a $+2$ move, m_{i+1} is a -2 move. These two moves cancel.

In each case, m_i and m_{i+1} cancel or they can be replaced by a single move. Lemma 2.1 is established by induction on n. □

Lemma 2.1 implies that a rigid doodle has a unique diagram with a minimal number of double points and that any other diagram can be obtained from the minimal diagram using only + moves. This proves Theorem 2.2 for rigid doodles. For nonrigid doodles, the same analysis applies except that the minimal diagram is defined only up to the move in Figure 10. □

[F, Corollary 2.8.9] presents a result similar to Theorem 2.2.

3. Doodles and 2-complexes

By a *2-dimensional complex* or a *2-complex* we mean a topological space homeomorphic to a two-dimensional finite CW-complex.

Consider a doodle Δ on a closed oriented surface M. We will associate a 2–dimensional complex $R(\Delta^1)$ to any diagram Δ^1 of a doodle Δ. This 2-complex will admit cell decompositions, but, in general, no canonical cell decompositions, and this is why we use the term *2-complex* rather than *2-dimensional CW-complex.* The simple homotopy type of this complex will not depend on the choice of diagram. Hence, it will be an invariant of Δ.

Let Δ^1 be a diagram of Δ. Denote the double points of Δ^1 by $\mathrm{pt}_1, ..., \mathrm{pt}_d$, the edges of Δ^1 by $\mathrm{edg}_1, ..., \mathrm{edg}_q$ and the regions of Δ^1 (connected components of $M \setminus \mathrm{edg}_1 \cup ... \cup \mathrm{edg}_q$) by $\mathrm{reg}_1, ..., \mathrm{reg}_s$.

The 2-complex $R(\Delta^1)$ will consist of a surface PL-homeomorphic to M with 1-dimensional cells and 2-dimensional compact surfaces with boundary attached. Take d 1-cells $p_1, ..., p_d$ (one 1-cell for every double point of Δ^1) and s surfaces $r_1, ..., r_s$ where surface r_j is homeomorphic to the region $\mathrm{reg}_j, 1 \leq j \leq s$.

First step: For $i = 1, ..., d$ we glue both ends of the 1-cell p_i to the double point $\mathrm{pt}_i \in M$ (see Figure 12). Denote this complex by $PR(\Delta^1)$ and the image of p_i in $PR(\Delta^1)$ by the same symbol p_i. Fix an orientation of p_i for every i.

Second step: For $j = 1, ..., s$ we glue the surface r_j to $PR(\Delta^1)$ as follows, see Figures 13-16. Suppose first that r_j is a disk. Moving in the clockwise direction

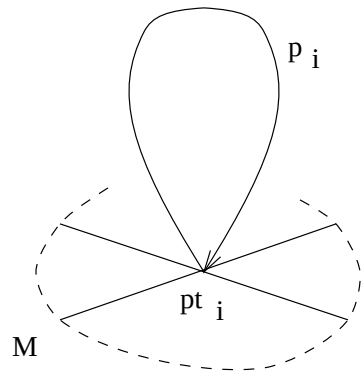

FIGURE 12

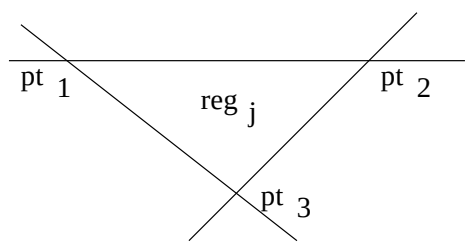

FIGURE 13

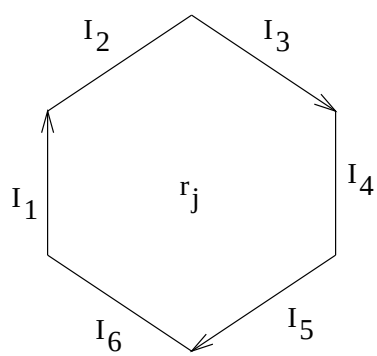

FIGURE 14

along the boundary of the 2-cell $\mathrm{reg}_j \in M$ we meet some double points and edges of Δ^1. For simplicity denote them $\mathrm{pt}_1, \mathrm{edg}_1, \mathrm{pt}_2, ..., \mathrm{pt}_k, \mathrm{edg}_k$ (the order is unique up to permutation). Figure 13 gives an example with $k = 3$.

Separate the boundary of the 2-cell r_j into $2k$ intervals, denote them (in the clockwise order) $I_1, ..., I_{2k}$, and orient intervals $I_1, I_3, ..., I_{2k-1}$ clockwise (Figure 14).

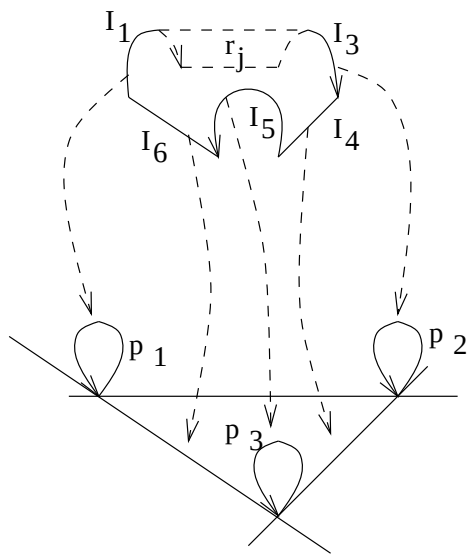

FIGURE 15

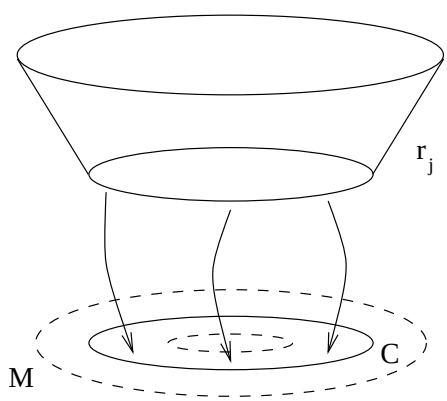

FIGURE 16

Then we identify oriented segments I_1 and p_1, I_3 and p_2,..., I_{2k-1} and p_k. At last identify segments I_2 and edg_1, I_4 and edg_2,..., I_{2k} and edg_k. These operations are given in Figure 15 ($k = 3$).

Dashed arrows connecting the upper half of the picture with the lower half show how we glue r_j along its boundary to the previously constructed complex $PR(\Delta^1)$. If r_j is not a disk, then it has more than one boundary component, and we glue r_j in a similar manner to $PR(\Delta^1)$ along each boundary component. If a diagram Δ^1 has a component C without double points, none of the 1-cells $p_1, ..., p_d$ are attached to C and part of the boundary of the corresponding r_j's is glued homeomorphically to C as depicted in Figure 16.

Denote by $R(\Delta^1)$ the complex that we obtained by gluing each of the surfaces $r_1, ..., r_s$ to $PR(\Delta^1)$ as explained above. $R(\Delta^1)$ contains the surface M as a subcomplex. Define $\bar{R}(\Delta^1)$ as $R(\Delta^1)$ with M contracted to a point, i.e. $\bar{R}(\Delta^1) \cong R(\Delta^1)/\sim$

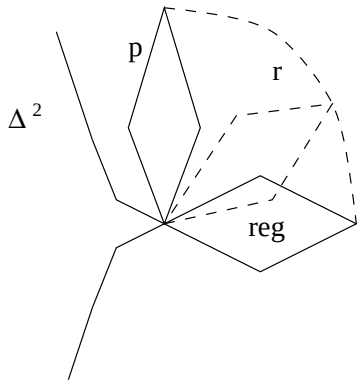

FIGURE 17

where for $x_1, x_2 \in R(\Delta^1), x_1 \sim x_2$ iff $x_1 = x_2$ or $x_1, x_2 \in M$. The topology on $\bar{R}(\Delta^1)$ is that induced from $R(\Delta^1)$.

We will call $R(\Delta^1)$ *the geometric realization* of the diagram Δ^1 and $\bar{R}(\Delta^1)$ *the reduced geometric realization* of Δ^1.

Theorem 3.1. *If Δ^1 and Δ^2 are two diagrams of a doodle Δ, then the 2–complex $R(\Delta^1)$ is simple homotopy equivalent to $R(\Delta^2)$ and the 2–complex $\bar{R}(\Delta^1)$ is simple homotopy equivalent to $\bar{R}(\Delta^2)$*

Proof. We have to check simple homotopy invariance of $R(\Delta^1)$ and $\bar{R}(\Delta^1)$ under the two elementary transformations of doodles (Figure 4). Let Δ^1 be a diagram and let Δ^2 be obtained from Δ^1 by adding a curl. Let p be the 1-cell of $R(\Delta^2)$ corresponding to the new double point. Denote by reg the region of M bounded by the curl and by r the corresponding disk of $R(\Delta^2)$ glued to p and to the boundary of reg.

In Figure 17 dashed lines show parts of the disk r.

Then $r \cup \text{reg}$ is a subcomplex of $R(\Delta^2)$ homeomorphic to a disk. The complex $R(\Delta^2)/(r \cup \text{reg})$ obtained by contracting $r \cup \text{reg}$ to a point is homeomorphic to $R(\Delta^1)$. The other move and the reduced case are treated similarly. □

Definition 3.1. The fundamental group of the 2-complex $R(\Delta^1)$ is called *the fundamental group* of the doodle Δ represented by the diagram Δ^1 and is denoted by $\pi_1(\Delta)$. The fundamental group of the 2-complex $\bar{R}(\Delta^1)$ is called *the reduced fundamental group* of the doodle Δ and is denoted by $\bar{\pi}_1(\Delta)$.

Theorem 3.1 implies that $\pi_1(\Delta)$ and $\bar{\pi}_1(\Delta)$ are invariants of doodle. If Δ is a doodle on a two-sphere, the groups $\pi_1(\Delta)$ and $\bar{\pi}_1(\Delta)$ are canonically isomorphic.

4. The fundamental group of a doodle

The fundamental group of a doodle Δ plays the role of the fundamental group of a link. The construction of the 2-complex $R(\Delta)$ in §3 translates to an algorithm that describes $\pi_1(\Delta)$ by generators and relations. The algorithm goes as follows (we restrict ourselves to the case of a doodle on the two-sphere):

Fix an orientation of $\mathbb{S}^2$. Let Δ be a doodle on $\mathbb{S}^2$.

Definition 4.1. A diagram Δ^1 of a doodle Δ is *a disk diagram* if the union of curves $(C_1, ..., C_n)$ which represent Δ^1 cuts the two–sphere into a union of disks.

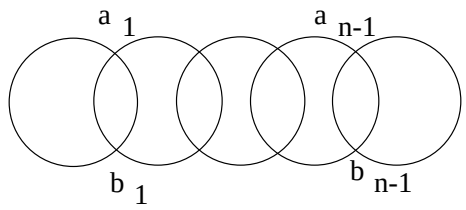

FIGURE 18

It is evident that disk diagrams exist for every doodle.

Take any disk diagram Δ_1 of Δ. Denote vertices of this diagram by $a_1, a_2, ..., a_k$. Abusing the notations, denote by $a_1, a_2, ..., a_k$ the generators of the doodle group $\pi_1(\Delta)$. Consider the regions $\text{reg}_1, ..., \text{reg}_p$ separated by Δ_1 on the sphere. To each region we associate a relation among $a_1, ..., a_k$. Let the vertices of reg_i taken in the counter-clockwise order be $a_{i_1}, a_{i_2}, ..., a_{i_t}$ (up to a cyclic permutation) . Then the relation associated to reg_i is

$$a_{i_1} a_{i_2} ... a_{i_t} = 1.$$

The fundamental group $\pi_1(\Delta)$ of Δ is a group with generators $a_1, ..., a_k$ and defining relations

$$a_{i_1} a_{i_2} ... a_{i_t} = 1$$

for all regions $\text{reg}_i, i = 1, 2, ..., p$, of Δ_1. Theorem 3.1 implies that $\pi_1(\Delta) \cong \pi_1(R(\Delta))$ does not depend on the choice of disk diagram Δ_1 of Δ.

Example. Let Δ be a trivial n-component doodle on the two-sphere. Consider the diagram of Δ in Figure 18. The $2n-2$ intersection points of the diagram yield $2n-2$ generators $a_1, b_1, ..., a_{n-1}$, b_{n-1} of $\pi_1(\Delta)$, and the $2n$ regions correspond to $2n$ relations that reduce to $n-1$ relations $a_1 b_1 = 1$, $a_2 b_2 = 1, ...,$ $a_{n-1} b_{n-1} = 1$. Thus, $\pi_1(\Delta)$ is a free group of rank $n-1$.

More generally, we have

Proposition 4.1. *Suppose that a doodle Δ_1 is obtained from a doodle Δ by adding a trivial component. Then $\pi_1(\Delta_1)$ is the free product of $\pi_1(\Delta)$ and $\mathbb{Z}$.*

Proof. Immediate. □

Example. Let Δ be a trivial (i.e. without self-intersections and bounding a disc) one-component doodle on a closed oriented surface M. Then the reduced fundamental group of Δ is isomorphic to the fundamental group of the surface M, and the fundamental group of Δ is isomorphic to $\pi_1(M) * \pi_1(M)$.

Example. Consider the doodle Δ depicted in Figure 19.

If we take generators a, b for $\pi_1(\Delta)$, we get the following defining relations

$$\langle a, b | a^2 b = b a^2, ab^2 = b^2 a, abab = baba \rangle.$$

The center of $\pi_1(\Delta)$ is generated by $b^2, a^2, b^{-2} a^{-2} abab$ and is isomorphic to $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_2$.

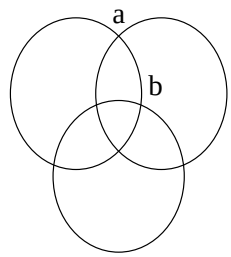

FIGURE 19

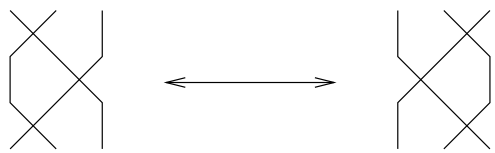

FIGURE 20

Proposition 4.2. *Let Δ be a doodle on an oriented closed surface M. Then the first homology groups $H_1(\pi_1(\Delta), \mathbf{Z})$ and $H_1(\bar{\pi}_1(\Delta), \mathbf{Z})$ of the fundamental group of Δ and of the reduced fundamental group of Δ depend only on the conjugacy classes of the components of Δ in the fundamental group of the surface M.*

Proof. The proposition follows from the invariance of $H_1(\pi_1(\Delta), \mathbf{Z})$ and $H_1(\bar{\pi}_1(\Delta), \mathbf{Z})$ under the triple point move depicted in Figure 20.

Corollary 4.3. *For a k-component doodle Δ on a 2-sphere $H_1(\pi_1(\Delta), \mathbf{Z}) \cong \mathbf{Z}^{k-1}$.*

□

5. Abelian subgroups of doodle groups

Consider a doodle Δ. Denote by Δ^{min} the diagram of Δ with the minimal possible number of double points. By Theorem 2.2 such a diagram is unique up to the move in Figure 10.

Proposition 5.1. *Let Δ be a doodle on the two-sphere. Suppose that Δ^{min} contains a subdiagram, depicted in Figure 21, such that the segments s_1, s_2 belong to the different components of Δ. Then $\pi_1(\Delta)$ contains a free abelian subgroup of rank two.*

Proof. Let a, b, c, d, e be the elements of $\pi_1(\Delta)$ associated with double points of the Figure 21 part of Δ^{min}. Four regions of Δ^{min} shown in Figure 21 give four relations:

$$eba = 1, \quad ecb = 1, \quad edc = 1, \quad ead = 1 \tag{5.1}$$

Excluding c, d and e, we get

$$abab = baba. \tag{5.2}$$

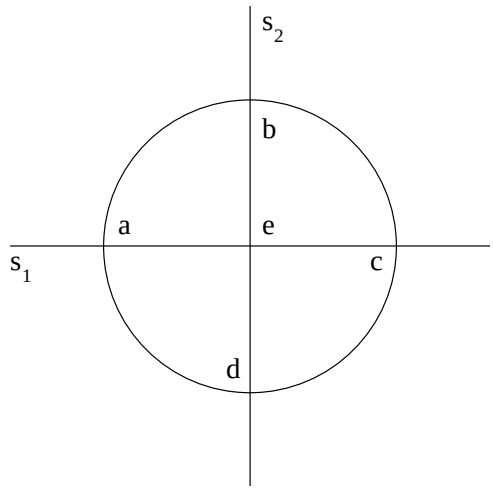

FIGURE 21

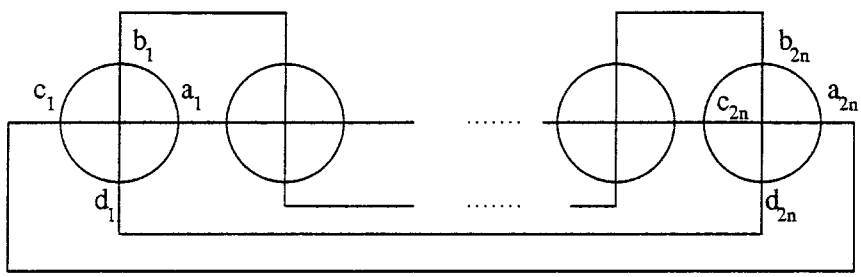

FIGURE 22

Note that (5.2) is equivalent to any of the two relations

$$[abab, a] = 1, \quad [abab, b] = 1. \tag{5.3}$$

Consider the subgroup G of $\pi_1(\Delta)$ generated by $(ab)^2, a$. The first of relations (5.3) tells us that G is abelian. The segments s_1, s_2 belong to different components of doodle Δ. This implies that the image of G in $H_1(\pi_1(\Delta), \mathbf{Z})$ has rank 2. Hence, G is a rank 2 abelian subgroup of $\pi_1(\Delta)$. □

Corollary 5.2. *If a doodle Δ satisfies the hypothesis of Proposition 5.1, then $\pi_1(\Delta)$ is not a word hyperbolic group.*

Proof. Word hyperbolic groups do not contain rank two abelian subgroups [G]. □

Using identities (5.3), we construct doodles on the two-sphere whose fundamental groups have nontrivial center. Let $\Delta(2n)$ be the doodle in Figure 22, and let $\Delta(2n-1)$ be the doodle in Figure 23. The doodle $\Delta(2n)$ has $2n+2$ components, while $\Delta(2n-1)$ has $2n+1$ components.

Proposition 5.3. *For $n \geq 1$, the fundamental groups of the doodles $\Delta(2n)$ and $\Delta(2n-1)$ have infinite center.*

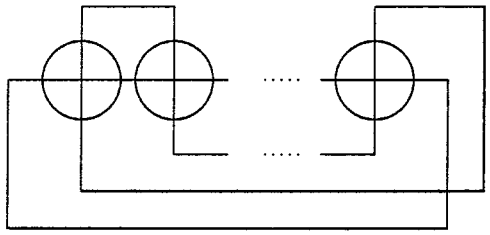

FIGURE 23

Proof. Let $a_1, b_1, c_1, d_1, ..., a_{2n}, b_{2n}, c_{2n}, d_{2n}$ be the elements of $\pi_1(\Delta(2n))$ associated with double crossings as in Figure 22. We claim that the element $(b_1a_1)^2$ is in the center of the fundamental group of $\Delta(2n)$. Indeed, by Proposition 5.1, for $i = 1, ..., 2n$, element $(b_ia_i)^2$ commutes with each of the four elements a_i, b_i, c_i, d_i. Also, we have

$$c_2b_2b_1a_1 = 1. \tag{5.4}$$

Therefore,

$$(b_1a_1)^2 = (c_2b_2)^{-2}. \tag{5.5}$$

Note that

$$(c_2b_2)^2 = (b_2c_2)^2 = (a_2b_2)^2 = (b_2a_2)^2. \tag{5.6}$$

Hence,

$$(b_1a_1)^2 = (b_2a_2)^{-2}. \tag{5.7}$$

Similarly,

$$(b_ia_i)^2 = (b_{i+1}a_{i+1})^{-2}, \qquad i = 1, ..., 2n-1. \tag{5.8}$$

Thus,

$$(b_1a_1)^2 = (b_ia_i)^{\mp 2}, \qquad i = 1, ..., 2n-1, \tag{5.9}$$

$\mp$ depending on the parity of i. It follows that $(b_1a_1)^2$ commutes with a_i, b_i, c_i, d_i, $i = 1, ..., 2n$. Therefore, $(b_1a_1)^2$ is central in $\pi_1(\Delta(2n))$. The image of $(b_1a_1)^2$ in the first homology group of $\pi_1(\Delta(2n))$ is nontrivial. Hence, $(b_1a_1)^2$ has infinite order in $\pi_1(\Delta(2n))$. We proved Proposition 5.3 for $\Delta(2n)$. The case of $\Delta(2n-1)$ is worked out similarly. □

Remark. Our method also applies to another infinite family of doodles typified by the one in Figure 24. The fundamental groups of these doodles also have infinite centers.

Suppose that part of the minimal diagram of a doodle Δ on a two-sphere is as depicted in Figure 25. We denote elements of $\pi_1(\Delta)$ associated with the double points in Figure 25, by $a, b, c_1, ..., c_n, d_1, ..., d_n, x_1, ..., x_n$. Let us read off the elements of $\pi_1(\Delta)$ associated with the double points of the component l starting with c_1 and going counter-clockwise along l. We obtain an element of $\pi_1(\Delta)$

$$c_1c_2...c_nbd_1...d_na, \tag{5.10}$$

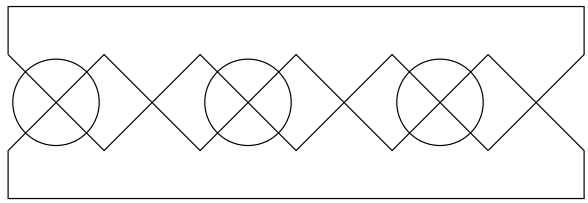

FIGURE 24

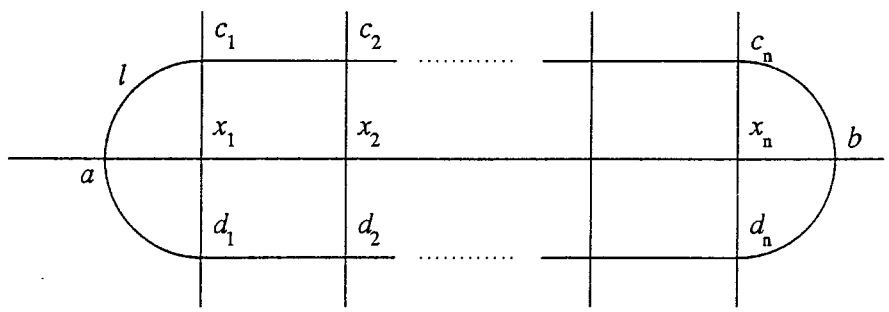

FIGURE 25

which we denote by long(l) (the "longitude" of the component l). It is easy to see that

$$\text{long}(l) = x_1^{-1}x_2^{-1}...x_n^{-1}x_n^{-1}x_{n-1}^{-1}...x_1^{-1}. \tag{5.11}$$

$2n+2$ regions of Δ inside l give us $2n+2$ relations on $a, b, c_1, ..., c_n, d_1, ..., d_n$, $x_1, ..., x_n$. Using these relations to exclude all generators but $a, x_1, ..., x_n$, we end up with just one relation:

$$ax_1x_2...x_nx_nx_{n-1}...x_1 = x_1x_2...x_nx_nx_{n-1}...x_1a. \tag{5.12}$$

(5.13) is equivalent to

$$[a, \text{long}(l)] = 1. \tag{5.13}$$

Thus, if the image of the subgroup generated by $a, \text{long}(l)$ has rank 2 in $H_1(\Delta, \mathbf{Z})$, then the fundamental group of the doodle Δ contains an abelian subgroup of rank 2. That generalizes Proposition 5.1 and Corollary 5.2.

6. Reduced fundamental groups

For $n \geq 1$, let Δ_n be the doodle on the torus shown in Figure 26.

The solid lines denote the doodle, while torus is shown as a rectangle. The reduced fundamental group of Δ_n is isomorphic to

$$\langle a, b | (bab^{-1}a^{-1})^2 = 1\rangle \text{ for odd } n,$$
$$\langle a, c | a^{-1}c^{1-n}ac^{n+1} = 1\rangle \text{ for even } n.$$

Note that, for even n, $\bar{\pi}_1(\Delta_n)$ is isomorphic to the Baumslag-Solitar group $G_{n+1,n-1}$ and is not automatic [ECHLPT]. From Theorem 2.2 we obtain that the doodle Δ_n

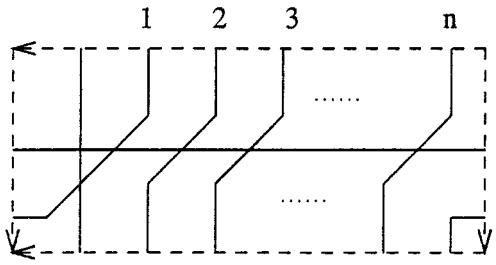

FIGURE 26

is equivalent to Δ_m if and only if $n = m$. However, we see that for odd n and m

$$\bar{\pi}_1(\Delta_n) \cong \bar{\pi}_1(\Delta_m).$$

Moreover, $\Delta_1, \Delta_3, ...$ is an infinite sequence of pairwise inequivalent doodles with the same reduced fundamental group.

These are indications that, for doodles on surfaces of genus greater than 0, the reduced fundamental group is a "bad" group. To draw a parallel with links, define *the reduced fundamental group* of a link l in a handlebody H_g of genus $g > 0$ as follows:

Let $\phi : \partial H_g \to H_g \setminus l$ be the embedding of the boundary of H_g to $H_g \setminus l$. Let ϕ_* be the induced mapping of fundamental groups,

$$\phi_* : \pi_1(\partial H_g) \to \pi_1(H_g \setminus l).$$

Define the reduced fundamental group, $\bar{\pi}_1(l)$, as the quotient of $\pi_1(H_g \setminus l)$ by the normal closure of $\phi_*(\pi_1(\partial H_g))$. The reduced fundamental group of a typical link in a handlebody of genus > 0 is a very bad object. Similarly, the reduced fundamental groups of doodles on surfaces of genus > 0 are expected to be of little interest. The natural object is the fundamental group.

7. Curvature of doodle groups

We use the following definitions from [GS1],[GS2]:

Definition 7.1. An $\mathbf{A}_2$ *complex* is a 2–dimensional CW–complex equipped with a metric with all 2–cells isometric to equilateral triangles.

Definition 7.2. An $\mathbf{A}_2$ complex X has *non-positive curvature* if every cycle without backtracking in the link of the vertex has length greater than or equal to 6.

Gersten and Short proved ([GS1],[GS2])

Theorem 7.1. *The fundamental group of a finite* $\mathbf{A}_2$ *complex of non-positive curvature is automatic.* □

Using this theorem, we will prove that fundamental groups of some doodles are automatic. For simplicity we restrict ourselves to the case of doodles on the two-sphere.

Definition 7.3. A doodle Δ on the two-sphere is *reducible* if it can be represented as the *disjoint union* of two doodles (Figure 27). Otherwise it is *irreducible*.

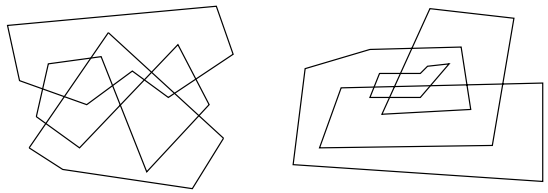

FIGURE 27

Note that the disjoint union of two doodles is not uniquely defined – it depends on the choice of diagrams of doodles and two regions along which we glue two-spheres together.

Recall that a doodle is rigid if it does not contain a free component. Thus, every irreducible doodle is rigid. By Theorem 2.2 an irreducible doodle has a unique minimal diagram Δ^{min}.

Definition 7.4. A doodle Δ on the two-sphere is called *thick* if it is irreducible and each cycle of even length of Δ^{min} without backtracking has length greater than or equal to 6.

Theorem 7.2. *The fundamental group of any thick doodle Δ can be realized as the fundamental group of a finite $\mathbf{A}_2$ complex of non-negative curvature, and (Theorem 7.1) is automatic.*

Proof. Recall that for doodles on the two-sphere, the fundamental group is isomorphic to the reduced fundamental group. We start with an irreducible doodle Δ. Consider the reduced geometric realization $\bar{R}(\Delta^{min})$ of the minimal diagram of Δ (see §3). Note that the minimal diagram of an irreducible doodle is a disk diagram (as in Definition 4.1). The reduced geometric realization $\bar{R}(\Delta^{min})$ is a 2-dimensional complex. It has one 0-cell, while the 1-cells are in one-to-one correspondence with the double points of Δ^{min}, and the 2-cells are in one-to-one correspondence with the regions of Δ^{min}. Because Δ^{min} is the minimal diagram, each of its regions is bounded by at least 3 edges; otherwise we can apply one of $-1, -2$ moves (as defined in §2) and the diagram is not minimal.

To make an $\mathbf{A}_2$ complex out of $\bar{R}(\Delta^{min})$ we try to triangulate each of the 2-cells of $\bar{R}(\Delta^{min})$ and make all triangles equilateral. The triangulations we consider do not introduce new 0–cells, thus, if the region of Δ^{min} (=2-cell of $\bar{R}(\Delta^{min})$) is an n-gon ($n \geq 3$), the triangulation is a partition of this n-gon into $n-2$ triangles. If we triangulate all of the 2-cells of $\bar{R}(\Delta^{min})$ like that, there is no obstruction to make all triangles equilateral.

Fix an arbitrary such triangulation of the 2-cells of $\bar{R}(\Delta^{min})$. Denote the resulting $\mathbf{A}_2$ complex by $X(\Delta^{\min})$ (in this notation we suppress the dependence on triangulations of the 2-cells). Then $X(\Delta^{\min})$ has only one vertex.

The link of the vertex of $X(\Delta^{\min})$ is described as follows. It is a 1-dimensional CW-complex with two 0-cells v^+ and v^- for each double point v of the diagram Δ^{min}. If two double points v_1 and v_2 of Δ^{min} are connected by an arc, there are two 1-cells connecting v_1^+ with v_2^- and v_2^+ with v_1^-.

Thus, cycles without backtracking in the link of the only vertex of $X(\Delta^{min})$ are in one-to-one correspondence with the cycles of even length without backtracking of the diagram Δ^{min}, where Δ^{min} is considered as a 4-valent plane graph.

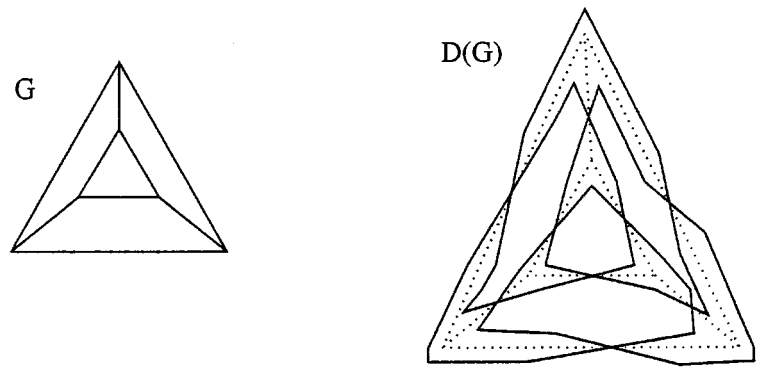

FIGURE 28

Hence, if a doodle Δ is thick, for any triangulation as above of the 2-cells of $\bar{R}(\Delta^{min})$, the resulting $\mathbf{A}_2$ complex $X(\Delta^{min})$ has non-negative curvature. Also $\pi_1(\Delta) \cong \pi_1(X(\Delta^{min}))$. Therefore, the fundamental group of a thick doodle is automatic. $\square$

We now give examples of thick doodles.

Let G be a trivalent graph without loops on the two-sphere. To such a graph one can associate a doodle as follows. Pick a point on each edge of G. Connect two points by an arc if the corresponding edges of G share a common point. If two edges of G have two points in common, we connect the points corresponding to the edges by two arcs.

This gives us a 4-valent graph on the sphere. Denote this graph by $D(G)$. An example is given in Figure 28. The graph $D(G)$ represents a doodle which we also denote by $D(G)$. The following proposition is immediate.

Proposition 7.3. *If G is a trivalent graph on the two-sphere without cycles of length less than 5, then the associated doodle $D(G)$ is thick.*

Therefore, if G is a graph as in Proposition 7.3, the fundamental group of the doodle $D(G)$ is automatic. It is easy to construct examples of trivalent graphs on the two-sphere without cycles of length less than 5.

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