HERZ-SCHUR MULTIPLIERS 
AND WEAKLY ALMOST PERIODIC FUNCTIONS 
ON LOCALLY COMPACT GROUPS

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Abstract. For a locally compact group $G$ and $1 < p < \infty$, let $A_p(G)$ be 
the Herz–Figà-Talamanca algebra and $B_p(G)$ the Herz-Schur multipliers of $G$, 
and $MA_p(G)$ the multipliers of $A_p(G)$. Let $W(G)$ be the algebra of continuous 
weakly almost periodic functions on $G$. In this paper, we show that (1), if $G$ is a 
noncompact nilpotent group or a noncompact [IN]-group, then $W(G)/B_p(G)^-$ contains 
a linear isometric copy of $l^\infty(N)$; (2), for a noncommutative free group $F$, $B_p(F)$ is a proper subset of $MA_p(F) \cap W(F)$.

1. Introduction

Let $G$ be a locally compact group, $C(G)$ the space of bounded continuous 
functions on $G$ with the sup norm. For a subset $S$ of $C(G)$, $S^-$ denotes the uniform closure of $S$ in $C(G)$. Let $A_p(G)$ be the Herz–Figà-Talamanca algebra of $G$ and $B_p(G)$ the algebra of Herz-Schur multipliers, with $1 < p < \infty$. Note that $A_2(G) = A(G)$ is the Fourier algebra of $G$, introduced by Eymard [12], and $B_2(G)$ is the completely bounded multipliers $M_0 A(G)$ of $A(G)$, as was shown by Bożejko and Fendler [5]. The Fourier-Stieltjes algebra $B(G)$ of $G$ is the space of coefficients of strongly continuous unitary representations of $G$. It is known that $B(G) \subseteq M_0 A(G)$, and they are equal if $G$ is amenable. Also, $M_0 A(G) \subseteq B_p(G)$ for every $1 < p < \infty$ (see [1], [15]). Let $W(G)$ be the algebra of continuous weakly almost periodic functions on $G$. Then it can be shown that $B_p(G) \subseteq W(G)$ for every $1 < p < \infty$. In answering a question raised by Eberlein, i.e., whether for an abelian group $G$, $B(G)^- = W(G)$, Rudin [32] showed that $B(G)^- \subseteq W(G)$ if $G$ is abelian and contains a discrete subgroup which is not of bounded order, and Ramirez [31] later showed that Rudin's conclusion holds for all noncompact abelian groups. More general results on this topic were obtained by Chou [7]. He extended the Rudin-Ramirez result to include many nonabelian groups: if $G$ is either a noncompact nilpotent group or a noncompact [IN]-group, then $W(G)/B(G)^-$ contains a linear isometric copy of $l^\infty(N)$, in particular $B(G)^- \subseteq W(G)$. In the first part of this paper, we are able to replace $B(G)$ by some larger spaces. More precisely, we have the following result: for every $1 < p < \infty$, $W(G)/B_p(G)^-$ contains a linear isometric copy of $l^\infty(N)$, if $G$ is a noncompact nilpotent group or a noncompact [IN]-group. This generalizes Chou's result mentioned above. This will be the contents of sections 3 and 4.
If $G$ is a locally compact group, we denote by $MA_p(G)$ the space of multipliers of $A_p(G)$, $1 < p < \infty$. When $G$ is amenable, we have the equality $B_p(G) = MA_p(G)$. It was shown by Bo̧żejko [2], [4] that for a noncommutative free group $F$, $B_p(F)$ is a proper subset of $MA_p(F)$. In fact he constructed a function $\phi$ in [4] such that $\phi \in MA_p(F)$ but $\phi \notin W(F)$, hence $\phi \notin B_p(F)$. It is therefore interesting to decide whether $B_p(F) = MA_p(F) \cap W(F)$. In section 5, by constructing a Leinert set and using the discussions in section 3, we are able to show that $B_p(F)$ is a proper subset of $MA_p(F) \cap W(F)$ for a free group $F$ on at least two generators.

2. Preliminaries

Let $G$ be a locally compact group with a fixed left Haar measure and $L^p(G)$, $1 \leq p \leq \infty$, the usual Lebesgue spaces on $G$ with the norm $\| \cdot \|_p$.

Suppose that $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The Herz–Figà-Talamanca algebra $A_p(G)$ is the space of continuous functions $u$ which can be represented as

$$u = \sum_{i=1}^{\infty} f_i \ast \tilde{g}_i,$$

where $f_i \in L^q(G)$, $g_i \in L^p(G)$ ($\tilde{g}_i(x) = g_i(x^{-1})$) and $\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty$, with norm the infimum of the last expression over all such representations of $u$. $A_p(G)$ is a Banach algebra with pointwise multiplication. Note that $A_p(G)$ is contained in $C_0(G)$, the subspace of $C(G)$ consisting of functions vanishing at infinity, and for every $u \in A_p(G)$, $\|u\|_\infty \leq \|u\|_{A_p}$.

Denote $MA_p(G) = \{u \in C(G) : uv \in A_p(G) \text{ for all } v \in A_p(G)\}$ with the norm $\|u\|_M = \sup\{\|uv\|_{A_p} : v \in A_p(G), \|v\|_{A_p} \leq 1\}$. It is called the space of multipliers of $A_p(G)$.

Let $V_p(G) = \{\psi : G \times G \rightarrow \mathbb{C} : \psi F \in L^p(G) \otimes L^q(G) \text{ for all } F \in L^p(G) \otimes L^q(G)\}$. It is the space of pointwise multipliers of the projective tensor product $L^p(G) \otimes L^q(G)$. The norm on $V_p(G)$ is the operator norm on $L^p(G) \otimes L^q(G)$.

Let $\phi : G \rightarrow \mathbb{C}$ be a function. Define $M\phi : G \times G \rightarrow \mathbb{C}$ by

$$M\phi(x, y) = \phi(xy^{-1})$$

for all $x, y \in G$. The space of Herz–Schur multipliers is defined to be

$$B_p(G) = \{\phi : G \rightarrow \mathbb{C} : M\phi \in V_p(G)\}.$$ 

The norm $\|\phi\|_{B_p}$ is given by $\|\phi\|_{B_p} = \|M\phi\|_{V_p}$. Elements of $B_p(G)$ are continuous, and $\|u\|_\infty \leq \|u\|_{B_p}$ for every $u \in B_p(G)$.

For each $1 < p < \infty$, let $B_p$ denote the category of $p$-spaces (see [21]). It is a subcategory of the category of Banach spaces. The following characterisation of the space $B_p(G)$ is due to Fendler [13, Theorem 4.4] (see also Pisier [29, Theorem 2.1] for a general treatment): a function $\phi$ on $G$ is in $B_p(G)$ if and only if there exist $B \in B_p$ and (continuous) bounded maps $a : G \rightarrow B$ and $b : G \rightarrow B^*$ such that

$$\phi(xy^{-1}) = \langle a(x), b(y) \rangle$$

for all $x, y \in G$.

If $f \in C(G)$ and $x \in G$, then $\lambda(x)f$, the left translate of $f$ by $x$, is defined by $\lambda(x)f(y) = f(x^{-1}y)$. $f \in C(G)$ is said to be a weakly almost periodic function (w.a.p. for short) if the set $\{\lambda(x)f : x \in G\}$ is relatively compact with respect to the weak topology of $C(G)$. We denote by $W(G)$ the space of w.a.p. functions. It is known that $W(G)$ has a unique translation-invariant mean $m_G$. 
Finally, we point out that the inclusion $B_p(G) \subseteq W(G)$ ($1 < p < \infty$) follows from the description of $B_p(G)$ and the Grothendieck criterion, which says that $f \in C(G)$ is w.a.p. if and only if whenever $\{x_n\}$ and $\{y_m\}$ are two sequences in $G$ and $\lim_n \lim_m f(x_ny_m)$ and $\lim_m \lim_n f(x_ny_m)$ exist, then they are equal.

3. Discrete Groups

Throughout this section, we will assume that $G$ is a discrete group.

First, let us give an alternative description of $B_p(G)$. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, and let $END_p(G)$ be the Banach algebra of bounded linear operators on $l^p(G)$. Every element $k \in END_p(G)$ can be identified with a function $k : G \times G \to \mathbb{C}$ such that

$$\|k\|_{END_p} = \sup \left\{ \left| \sum_{x,y \in G} k(x,y)u(y)v(x) \right| : \|u\|_p \leq 1, \|v\|_q \leq 1 \right\}$$

is a finite number, where $u \in l^p(G)$ and $v \in l^q(G)$. The algebra of Herz-Schur multipliers $B_p(G)$ is the space of functions $\phi$ such that

$$M\phi \cdot END_p(G) \subseteq END_p(G),$$

where $M\phi \cdot k$ is the pointwise multiplication for $k \in END_p(G)$. The norm $\|\phi\|_{B_p}$ is given by

$$\|\phi\|_{B_p} = \sup \{ \|M\phi \cdot k\|_{END_p} : \|k\|_{END_p} \leq 1 \}.$$

Let $X_p$ be the completion of $l^1(G)$ with respect to the norm

$$\|f\|_{X_p} = \sup \left\{ \left| \sum_{x \in G} f(x)\phi(x) \right| : \phi \in B_p(G), \|\phi\|_{B_p} \leq 1 \right\}.$$

Then $X_p^* = B_p(G)$, as was shown in [4], [13].

In [28], Picardello introduced the concept of weak Sidon sets, which was later made use of in [7]. In our situation, we need the following

**Definition 3.1.** A subset $S \subseteq G$ is said to be a $B_p$-Sidon set, if given any $f \in l^{\infty}(G)$ there exists $u \in B_p(G)$ such that $f|_S = u|_S$.

If $g$ is a function defined on a subset $S$ of $G$, we can regard $g$ as a function on $G$ by setting its values to be zero outside of $S$. Thus, it is natural to identify $l^p(S)$ as a closed subspace of $l^p(G)$, for $1 \leq p \leq \infty$.

**Proposition 3.2.** Let $S$ be a subset of the discrete group $G$. Then the following conditions are equivalent:

1. $S$ is a $B_p$-Sidon set;
2. $l^1(S)$ is closed in $X_p$;
3. $\|\cdot\|_1$ and $\|\cdot\|_{X_p}$ are equivalent on $l^1(S)$.

**Proof.** (1) $\Rightarrow$ (2). Suppose that $S$ is a $B_p$-Sidon set; then

$$B_p(G) \xrightarrow{T} l^\infty(S), \quad u \mapsto u|_S,$$

is continuous and surjective; hence $T$ is an open mapping. Therefore, there is a $\delta > 0$ such that

$$Ball_{l^\infty(S)}(0, \delta) \subseteq T(Ball_{B_p}(0, 1)).$$
So for any \( f \in l^\infty(G) \), there exists \( u_f \in B_p(G) \) with \( f|_S = u_f|_S \) and \( \|f|_S\|_\infty \geq \frac{3}{2}\|u_f\|_{B_p} \).

Suppose that \( \{g_n\} \) is a sequence in \( l^1(S) \) that converges in the norm \( \|\cdot\|_{X_p} \). For every \( f \in L^\infty(S) \) with \( \|f\|_\infty = 1 \), we have

\[
\|g_n - g_m, f\| = \|g_n - g_m, u_f|_S\| \\
\leq \|g_n - g_m\|_{X_p} \|u_f\|_{B_p} \\
\leq \frac{2}{\delta}\|g_n - g_m\|_{X_p}.
\]

So, \( \{g_n\} \) is a Cauchy sequence in the norm \( \|\cdot\|_1 \), and hence \( l^1(S) \) is closed in \( X_p \).

\( (2) \Rightarrow (1) \). Let \( f \in l^\infty(S) = l^1(S)^* \). Since \( l^1(S) \) is closed in \( X_p \), we can get an extension \( T \in X_p^* \) of \( f \). Note that we can identify \( T \) with a function \( u \in B_p(G) \) by setting

\[
u(x) = T(\delta_x),
\]

where \( \delta_x \) is the function on \( G \) which is 1 at \( x \) and 0 elsewhere. It is easy to see that \( u(x) = f(x) \) holds for \( x \in S \).

\( (2) \Rightarrow (3) \). Note that \( \|\cdot\|_{X_p} \leq \|\cdot\|_1 \). So (3) is a consequence of the open mapping theorem.

\( (3) \Rightarrow (2) \). Trivial. \( \square \)

We give another useful criterion of \( B_p \)-Sidon sets, similar to the Lemma 3.11 of [7].

**Corollary 3.3.** A subset \( S \) of \( G \) is a \( B_p \)-Sidon set if and only if there is a positive constant \( c < 1 \) such that for every \( f \in l^\infty(S) \) with \( \|f\|_\infty = 1 \) there exists a \( u \in B_p(G) \) with \( \|f - u|_S\|_\infty \leq c \).

**Proof.** One direction is trivial.

Now suppose that \( S \) is not a \( B_p \)-Sidon set. Then \( \|\cdot\|_1 \) and \( \|\cdot\|_{X_p} \) are not equivalent on \( l^1(S) \) by the above proposition. Let us choose \( g_1 \in l^1(S) \) with finite support \( F_1 \), and \( \|g_1\|_1 = 1, \|g_1\|_{X_p} < 1 \). Note that \( B_p(G) \) contains all functions with finite support; hence \( S \setminus F_1 \) is again not a \( B_p \)-Sidon set. Therefore, we can choose \( g_2 \in l^1(S \setminus F_1) \) with finite support \( F_2 \), and \( \|g_2\|_1 = 1, \|g_2\|_{X_p} < \frac{1}{2} \). Continuing this procedure, we can get a sequence of functions \( \{g_n\} \) in \( l^1(S) \) with disjoint supports \( F_n \), and \( \|g_n\|_1 = 1, \|g_n\|_{X_p} < \frac{1}{n} \), for \( n = 1, 2, \ldots \).

Define

\[
f(x) = \begin{cases} 
\frac{g_n(x)}{|g_n(x)|}, & x \in F_n \text{ for some } n, \\
0, & x \notin \bigcup_{n=1}^{\infty} F_n.
\end{cases}
\]

Then \( f \in l^\infty(S) \) and \( \|f\|_\infty = 1 \). Note that for every \( u \in B_p(G) \),

\[
\|f - u|_S\|_\infty \geq |\langle f - u|_S, g_n \rangle| \\
\geq |\langle f, g_n \rangle| - |\langle u, g_n \rangle| \\
\geq 1 - \|u\|_{B_p} \|g_n\|_{X_p} \\
\geq 1 - \frac{\|u\|_{B_p}}{n}.
\]

As \( n \) can be arbitrarily large, \( \|f - u|_S\|_\infty = 1 \), and the condition in the statement is not satisfied. \( \square \)
A subset $C$ of $G$ is called an $n$-square if $C = AB$ where $A, B \subseteq G$ and $|A| = |B| = n$ and $|C| = n^2$ ( $|X|$ denotes the cardinality of the set $X$ ). A subset $S$ of $G$ is said to contain large squares if for each positive integer $k$, $S$ contains a $k$-square.

**Proposition 3.4.** Suppose $S \subseteq G$ contains large squares. Then $\| \cdot \|_1$ and $\| \cdot \|_{X_p}$ are not equivalent on $l^1(S)$, i.e., $S$ is not a $B_{p}$-Sidon set.

**Proof.** For each integer $n > 0$, choose an $n$-square $C = \{a_1, \ldots, a_n\}\{b_1, \ldots, b_n\}$.

It was shown by Bennett [1, Proposition 3.2] that there exist an $n \times n$ matrix $A = (a_{ij})$ all of whose entries are $\pm 1$, and a constant $D$, which is independent of $n$, such that the norm of the linear operator $A : l^p(Z_n) \rightarrow l^p(Z_n)$, where $Z_n = \{1, \ldots, n\}$, satisfies

$$\|A\|_{p,p} \leq D_{\max}\{n^\frac{1}{p}, n^\frac{1}{q}\}.$$  

Let

$$g = \sum_{i,j=1}^{n} a_{ij} \delta_{a_i b_j};$$

then $g \in l^1(S)$ and $\|g\|_1 = n^2$.

Now let us estimate $\|g\|_{X_p}$. By the definition,

$$\|g\|_{X_p} = \sup \left\{ \left| \sum_{x \in G} g(x) \phi(x) \right| : \phi \in B_p(G), \|\phi\|_{B_p} \leq 1 \right\}.$$  

For $\phi \in B_p(G)$ with $\|\phi\|_{B_p} \leq 1$, we have

$$\left| \sum_{x \in G} g(x) \phi(x) \right| = \left| \sum_{i,j=1}^{n} a_{ij} \phi(a_i b_j) \right|.$$  

Let $k \in END_p(G)$ be defined as

$$k(x,y) = \begin{cases} a_{ij}, & \text{if } (x,y) = (a_i, b_j^{-1}), \\ 0, & \text{otherwise}; \end{cases}$$

then

$$\|k\|_{END_p} = \|A\|_{p,p} \leq D_{\max}\{n^\frac{1}{p}, n^\frac{1}{q}\}.$$  

Let $k_1 = M \phi \cdot k$ and $u = \sum_{j=1}^{n} \delta_{b_j^{-1}}$; then for every $v \in l^q(G)$,

$$\left| \sum_{x,y \in G} k_1(x,y) u(y) v(x) \right| \leq \|k_1\|_{END_p} \|u\|_p \|v\|_q,$$

i.e.,

$$\left| \sum_{i,j=1}^{n} a_{ij} \phi(a_i b_j) v(a_i) \right| \leq n^\frac{1}{p} \|k_1\|_{END_p} \|v\|_q.$$  

Therefore, we get

$$\left( \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} \phi(a_i b_j) \right)^p \right)^{\frac{1}{p}} \leq n^\frac{1}{p} \|k_1\|_{END_p}.$$
Hence
\[
\left| \sum_{i,j=1}^{n} a_{ij} \phi(a_i b_j) \right| \leq \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} \phi(a_i b_j) \right|
\]
\[
\leq n^{\frac{1}{p}} \left( \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} \phi(a_i b_j) \right|^p \right)^{\frac{1}{p}}
\]
\[
\leq n \| k_1 \|_{END_p}
\]
\[
\leq n \| k \|_{END_p} \| \phi \|_{B_p}
\]
\[
\leq D_{\text{max}} \left\{ n^{1+\frac{1}{p}}, n^{\frac{1}{q}} \right\}.
\]
So,
\[
\| g \|_{X_p} \leq D_{\text{max}} \left\{ n^{1+\frac{1}{p}}, n^{1+\frac{1}{q}} \right\}.
\]
Since \( n \) can be arbitrarily large, we conclude that \( \| \cdot \|_1 \) and \( \| \cdot \|_{X_p} \) are not equivalent.

A subset \( T \subseteq G \) is said to be a \( t \)-set if \((T \cap Tx) \cup (T \cap xT)\) is finite for every \( x \in G \setminus \{ e \} \). It is known that if \( T \) is a \( t \)-set, then any \( f \in l^\infty(G) \) with \( \text{supp} f \subseteq T \) is in \( W(G) \); see [6], [7].

A consequence of Proposition 3.4 related to the concept of \( t \)-set is the following:

**Corollary 3.5.** If \( S \) is a countable \( B_p \)-Sidon set, then \( S \) is a finite union of \( t \)-sets.

**Proof.** Since \( S \) does not contain large squares, by Theorem 4.1 of [8] \( S \) is a finite union of \( t \)-sets.

The following result of Chou [7] will play a very important role in our proof of the main result of this section.

**Theorem (Chou).** If \( G \) is an infinite group, then there is a \( t \)-set \( T \) of \( G \) such that \( T = \bigcup_{n=1}^{\infty} S_n \) is a disjoint union and each \( S_n \) contains large squares.

By applying Proposition 3.4 and a device in Chou [7], we are able to show the following.

**Theorem 3.6.** Let \( G \) be an infinite discrete group. Then \( W(G)/B_p(G)^- \) contains a linear isometric copy of \( l^\infty(\mathbb{N}) \). In particular, \( B_p(G)^- \) is a proper subset of \( W(G) \).

**Proof.** Let \( T \) be the \( t \)-set as in Chou’s construction. So \( T = \bigcup_{n=1}^{\infty} S_n \) is a disjoint union of \( S_n \)’s with each \( S_n \) containing large squares. Thus each \( S_n \) is not a \( B_p \)-Sidon set, and hence by Corollary 3.3, there exists a function \( f_n \in l^\infty(G) \) with the following properties: \( \| f_n \|_\infty = 1, \text{supp} f_n \subseteq S_n \) and \( \| (f_n - u)|_{S_n} \|_{l^\infty(S_n)} \geq 1 \) for every \( u \in B_p(G) \).

Since \( S_n \) is a \( t \)-set, \( f_n \in W(G) \), \( n = 1, 2, \ldots \).

Define
\[
\xi : l^\infty(\mathbb{N}) \to W(G)/B_p(G)^-,
\]
\[
(c_n) \mapsto \sum_{i=1}^{\infty} c_n f_n + B_p(G)^-.
\]
It is not hard to see that \( \xi \) is an isometry.
4. Nilpotent Groups and \([\text{IN}]-\)Groups

First, let us recall that a locally compact group \(G\) is called an \([\text{IN}]-\)group if it has a compact neighborhood of the identity which is invariant under all inner automorphisms of \(G\).

Let \(H\) be a closed normal subgroup of \(G\) and \(\pi : G \to G/H\) be the canonical homomorphism. For \(1 < p < \infty\) and \(f \in B_p(G/H)\), there exist \(B \in B_p\) and continuous bounded maps \(a_0 : G/H \to B, b_0 : G/H \to B^*\) such that

\[
f(\pi(y)\pi(x)^{-1}) = \langle a_0(\pi(x)), b_0(\pi(y)) \rangle
\]

for all \(x, y \in G\). Let \(a = a_0 \circ \pi, b = b_0 \circ \pi\); then the continuous function \(f \circ \pi\) on \(G\) satisfies

\[
(f \circ \pi)(yx^{-1}) = \langle a(x), b(y) \rangle
\]

for all \(x, y \in G\). So, \(f \circ \pi \in B_p(G)\) and the map

\[
\Phi : B_p(G/H) \to B_p(G),
\]

\[
f \mapsto f \circ \pi,
\]

is an isometry from \(B_p(G/H)\) onto the subspace of \(B_p(G)\) consisting of functions that are constant on the left cosets of \(H\).

Fix \(x \in G\). For any function \(f\) on \(G\), define a function

\[
f_x : H \to \mathbb{C}
\]

by \(f_x(t) = f(xt), t \in H\). If \(f \in B_p(G)\) and

\[
f(yx^{-1}) = \langle a(x), b(y) \rangle
\]

for some space \(B \in B_p\) and bounded maps \(a : G \to B, b : G \to B^*\), then

\[
f_x(ts^{-1}) = f(xts^{-1}) = \langle a(s), b(xt) \rangle
\]

for \(s, t \in H\); so \(f_x \in B_p(H)\).

Let \(m_H\) be the unique invariant mean of \(W(H)\). For \(f \in B_p(G)\) and \(x \in G\); since \(f_x \in B_p(H) \subseteq W(H)\), we can define

\[
\phi(x) = m_H(f_x)
\]

for \(x \in G\).

**Proposition 4.1.** Let \(\phi\) be defined as above. Then \(\phi \in B_p(G)\) and \(\phi\) is constant on the left cosets of \(H\).

**Proof.** Since \(m_H\) is \(H\)-invariant, \(\phi\) is constant on left cosets of \(H\). The function \(\phi\) is continuous, since \(x \mapsto f_x\) is continuous.

By a result of Davis [10], there exists a net of open and relatively compact subsets \(\{U_\alpha\}\) of \(H\) such that

\[
m_H(k) = \lim_{\alpha} \lambda(U_\alpha)^{-1} \int_{U_\alpha} k(t)d\lambda(t),
\]

where \(k \in W(H)\) and \(\lambda\) is a fixed left Haar measure of \(H\).
Let $B \in B_p$. If $p : H \to B$ or $B^*$ is a continuous bounded map, $U \subseteq H$ is a relatively compact open set, the vector-valued integral
\[
\int_U p(t)d\lambda(t)
\]
exists, and $\|\int_U p(t)d\lambda(t)\| \leq \int_U \|p(t)\|d\lambda(t)$.

For fixed $x, y \in G$,
\[
\phi(yx^{-1}) = m_H(f_{yx^{-1}}) = \lim_\alpha \frac{1}{\lambda(U_\alpha)} \int_{U_\alpha} f(yx^{-1}t)d\lambda(t)
\]
\[
= \lim_\alpha \frac{1}{\lambda(U_\alpha)} \int_{U_\alpha} (a(t^{-1}x), b(y))d\lambda(t)
\]
\[
= \lim_\alpha (c_\alpha(x), b(y))
\]
where
\[
c_\alpha(x) = \frac{1}{\lambda(U_\alpha)} \int_{U_\alpha} a(t^{-1}x)d\lambda(t).
\]

Note that
\[
\|c_\alpha(x)\| \leq \frac{1}{\lambda(U_\alpha)} \int_{U_\alpha} \|a(t^{-1}x)\|d\lambda(t)
\]
\[
\leq \sup_{x \in G}\|a(x)\|,
\]
and that any space in $B_p$ is reflexive [21, Proposition 7], the net $\{c_\alpha(x)\}$ has a weak limit, say $c(x)$, in $B$. Clearly, $\|c(x)\| \leq \sup_{x \in G}\|a(x)\|$ and
\[
\phi(yx^{-1}) = \langle c(x), b(y) \rangle
\]
for all $x, y \in G$. So $\phi \in B_p(G)$.

Let $A = (a_{ij})_{n \times n}, B = (b_{ij})_{n \times n}$ be two matrices. The Schur product of $A$ and $B$ is the matrix
\[
A \ast B = (a_{ij}b_{ij})_{n \times n}.
\]

Let $\|A\|_{(p)} = \sup\\{\|A \ast B\|_{p,p} : \|B\|_{p,p} \leq 1\}$. Recall that $\|\cdot\|_{p,p}$ is the norm of a linear operator on $l^p(Z_n)$.

We will use the following characterisation of $B_p(G)$ due to Fendler [13]:

**Lemma.** A function $\phi$ on $G$ is in $B_p(G)$ if and only if $\phi$ is continuous and there is a constant $C$ such that for any finite set $\{x_1, \ldots, x_n\} \subseteq G$, $\|\phi(x_ix_j^{-1})\|_{n \times n} \leq C$.

In order to prove the main result of this section, we need the next lemma.

**Lemma 4.2.** Let $H$ be an open subgroup of $G$. Extend $f \in C(H)$ to $f^\circ \in C(G)$ by setting $f^\circ(x) = 0$ if $x \in G \setminus H$. If $f \in B_p(H)$, then $f^\circ \in B_p(G)$.

**Proof.** $f^\circ$ is clearly a continuous function on $G$. Since $f \in B_p(H)$, there exists a constant $C$ such that for any finite set $\{t_1, \ldots, t_k\} \subseteq H$, $\|(f(t_jt_i^{-1}))_{k \times k}\|_{(p)} \leq C$.

Consider now a finite set $\{x_1, \ldots, x_n\} \subseteq G$ of cardinality $n$. Since the norm of a matrix remains the same after interchanging any two rows or any two columns, we may assume that $x_1, \ldots, x_l$ belong to a right coset of $H$ and $x_{l+1}, \ldots, x_n$ belong to another right coset of $H$ (the proof for the case of more cosets is similar).
Let $A = (a_{ij})$ be an $n \times n$ matrix with $\|A\|_{p,p} \leq 1$. Then

$$(f^\circ (x_i x_j^{-1}))_{n \times n} * A = \begin{pmatrix} T_1 * A_1 & 0 \\ 0 & T_2 * A_2 \end{pmatrix}$$

where

$$A_1 = \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & \ddots & \vdots \\ a_{l1} & \cdots & a_{ll} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{l+1,l+1} & \cdots & a_{l+1,n} \\ \vdots & \ddots & \vdots \\ a_{n,l+1} & \cdots & a_{nn} \end{pmatrix},$$

and

$$T_1 = \begin{pmatrix} f(x_1 x_1^{-1}) & \cdots & f(x_1 x_l^{-1}) \\ \vdots & \ddots & \vdots \\ f(x_l x_1^{-1}) & \cdots & f(x_l x_l^{-1}) \end{pmatrix}, \quad T_2 = \begin{pmatrix} f(x_{l+1} x_{l+1}^{-1}) & \cdots & f(x_{l+1} x_n^{-1}) \\ \vdots & \ddots & \vdots \\ f(x_n x_{l+1}^{-1}) & \cdots & f(x_n x_n^{-1}) \end{pmatrix}.$$ 

Note that

$$\|A_1\|_{p,p} \leq 1, \quad \|A_2\|_{p,p} \leq 1,$$

so

$$\|T_1 * A_1\|_{p,p} \leq C, \quad \|T_2 * A_2\|_{p,p} \leq C.$$ 

Since $(f^\circ (x_i x_j^{-1}))_{n \times n} * A$ is in diagonal form, we have the following equality:

$$\|(f^\circ (x_i x_j^{-1}))_{n \times n} * A\|_{p,p} = \max\{\|T_1 * A_1\|_{p,p}, \|T_2 * A_2\|_{p,p}\}.$$ 

Since $A$ can be arbitrary, we conclude that

$$\|(f^\circ (x_i x_j^{-1}))_{n \times n}\|_{(p)} \leq C.$$ 

Hence, by the above lemma, $f^\circ \in B_p(G)$.

Applying Proposition 4.1, Lemma 4.2 and Theorem 3.6, a proof similar to that of Theorem 4.5 of Chou [7] gives us

**Theorem 4.3.** Let $G$ be a noncompact nilpotent group or a noncompact $[IN]$-group. Then $W(G)/B_p(G)^-$ contains a linear copy of $l^\infty(\mathbb{N})$. In particular, $B_p(G)^-$ is a proper subset of $W(G)$.

## 4. Free Groups

It is well known that for an amenable locally compact group $G$, and $1 < p < \infty$, $MA_p(G) = B_p(G)$, and in particular, $MA(G) = M_0A(G) = B(G)$. Losert [26] showed that $MA(G) = B(G)$ implies the amenability of $G$ (the discrete case was due to Nebbia [27]), and for a discrete group $G$, Bożejko [3] showed that $M_0A(G) = B(G)$ implies the amenability of $G$. He also obtained in [2], [4] the following result: for a noncommutative free group $F$, $B_p(F) \subsetneq MA_p(F)$. The proof in [4] gives a function $\phi$ with $\phi \in MA(F)$ but $\phi \notin W(F)$, hence $\phi \notin B_p(F)$. Thus it is natural to ask whether $B_p(F) = MA_p(F) \cap W(F)$. In this section, we show that this is not the case.

Recall that a subset $E$ of a discrete group $G$ is a **Leinert set** if there is $C > 0$ such that for every $f \in l^2(E)$

$$\|f\|_{VN} = \sup\{\|f * g\|_2 : \|g\|_2 = 1\} \leq C\|f\|_2,$$
or, equivalently,
\[ \chi_{E\mathcal{A}(G)} = l^2(E). \]

It was shown by Bożejko that \( E \) is a Leinert set if and only if \( l^\infty(E) \subseteq \mathcal{M}A(G) \), and if \( E \) is a Leinert set then \( l^\infty(E) \subseteq \mathcal{M}A_p(G) \), for \( 1 < p < \infty \); see [2]. Now we are ready for the main result of this section.

**Theorem 5.1.** Let \( F \) be the free group on \( k \) generators with \( k > 1 \). Then \( B_p(F) \) is a proper subset of \( \mathcal{M}A_p(F) \cap W(F) \).

**Proof.** First, let us consider the case that \( k = \infty \). Let \( E = \{ x_1, x_2, \cdots \} \) be the set of free generators of \( F \). By the Haagerup convolution theorem [18] (see also [14]), we conclude that \( E^2 = \{ x_i x_j : i, j = 1, 2, \cdots \} \) is a Leinert set.

For an integer \( k > 0 \), define
\[ T_k = \{ x_i x_j \in E^2 : 2^{k-1} \leq i, j < 2^k \} \]
and set
\[ T = \bigcup_{k=1}^{\infty} T_k. \]

Since \( T \) contains large squares, by Proposition 3.4 it is not a \( B_p \)-Sidon set. Therefore, by Corollary 3.3, we can find a function \( \phi \in l^\infty(F) \) such that \( \text{supp} \phi \subseteq T \), \( \| \phi \|_\infty = 1 \) and \( \| \phi - u/T \|_{l^\infty(T)} \geq 1 \) for every \( u \in B_p(F) \). In particular, \( \phi \notin B_p(F) \).

We claim that \( \phi \in \mathcal{M}A_p(F) \cap W(F) \).

\( \phi \in \mathcal{M}A_p(F) \), since \( \text{supp} \phi \subseteq T \) and \( T \) is a Leinert set, being a subset of the Leinert set \( E^2 \).

To show that \( \phi \in W(F) \), it suffices to show that \( T \) is a \( t \)-set. Indeed, let \( x \in F \setminus \{ e \} \) and \( x = x_{i_1}^{u_1} \cdots x_{i_n}^{u_n} \) be the reduced form, where \( u_i = \pm 1, i = 1, \cdots, n \). If \( y \in T \cap xT \), then
\[ y = x_i x_j \in T_k \]
for some positive integer \( k \), and
\[ y = x_{i_1}^{u_1} \cdots x_{i_n}^{u_n} x_{u_v} \]
for some \( x_{u_v} \in T \).

Comparing the two forms of \( y \), and noticing that \( x \neq e \), we get \( i = i_1 \). Moreover, \( x \) can take the forms \( x_{i_1}^{u_1} x_{i_2}^{u_2} x_{i_3}^{u_3} \) and \( x_{i_1}^{u_1} x_{i_3}^{u_3} x_{i_2}^{u_2} \). In the first case, we have at most one choice of \( y \), namely \( y = x_{i_1} x_{i_2} \) provided \( u_1 = u_2 = 1 \) and \( x_{i_3} x_{i_2} x_{i_1} x_{i_3} = e \).

In the second case, let \( k \) be \( \lceil \log_2 i_1 \rceil + 1 \); then we have at most \( 2^{k-1} \) choices of \( y \), namely \( y = x_{i_1} x_{i_2} \) with \( j \geq 2^{k-1}, 2^{k-1} + 1, \cdots, 2^k - 1 \), provided \( u_2 = -1, i_2 = u \). So
\[ |T \cap xT| \leq i_1 + 1 < \infty. \]

Similarly, \( |T \cap xT| < \infty. \)

Now let \( F \) be the free group on \( k \) generators with \( k > 1 \). We can find a subgroup \( H \) of \( F \) with \( H \cong F^\infty \). Therefore, there exists a function \( \phi \in W(H) \cap \mathcal{M}A_p(H) \), but \( \phi \notin B_p(H) \), as in the proof above.

Let us extend \( \phi \) to a function \( \phi^0 \) on \( F \) by setting \( \phi^0(x) = 0 \) for \( x \notin H \). Using the definition of Herz-Schur multipliers, we can check that \( \phi^0 \notin B_p(F) \). Also, by Lemma 4.1 of [7], \( \phi^0 \in W(F) \). Notice that our \( \phi \) has support in a Leinert set \( E \) of \( H \). To show \( \phi \in \mathcal{M}A_p(F) \), it suffices to show that \( E \) is a Leinert set of \( F \). By a theorem of Herz [22], \( u|_H \in A(H) \) whenever \( u \in A(F) \). Let \( u \in A(F) \) since \( E \) is a
Leinert set of $H$, $(\chi_E u)|_H \in l^2(H)$. Hence $\chi_E u \in l^2(F)$, which shows that $E$ is a Leinert set of $F$.

So, $\phi^0 \in MA_p(F) \cap W(F) \setminus B_p(F)$, and the proof is complete.

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REFERENCES

[18] U. Haagerup, An example of a nonnuclear $C^*$-algebra which has the metric approximation property, Invent. Math. 50 (1979), 279-293. MR 80j:46094


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